Turk. J. Math. Comput. Sci.
14(1)(2022) 16–23
© MatDer
DOI : 10.47000/tjmcs.975585

Weighted Statistical Limit Supremum-Infimum

Maya Altinok

Department of Natural and Mathematical Sciences, Faculty of Engineering, Tarsus University, 33400, Mersin, Turkey.

Received: 28-07-2021 • Accepted: 07-04-2022

Abstract. In this paper, by using weight g-statistical density we introduce weight g-statistical supremum-infimum for real valued sequences. We also define weight g-statistical limit supremum-infimum with the help of above new concepts. In addition, we shall establish some results about weight g-statistical core.

2010 AMS Classification: 40A05, 46A45

Keywords: Weighted density, weighted statistical limit supremum-infimum, weighted statistical core.

1. Introduction

The idea of statistical convergence was formerly described under the name "almost convergence" by Zygmund in the first edition of his celebrated monograph published in Warsaw in 1935 [24]. The concept was formally introduced by Fast [12] and was later reintroduced by Schoenberg [20] and also, independently, by Buck [6]. Statistical convergence becomes one of the most popular areas of studies in summability theory [13, 16, 21]. Some significant results on statistical convergence of real valued sequences can be seen from [7–9, 14, 15, 17, 19, 22, 23].

In [5, 10], the authors proposed a modified version of density by replacing n by n^α where $0 < \alpha \leq 1$. In [4], the authors defined a more general kind of density by replacing n^α by a function $g : \mathbb{N} \to [0, \infty)$ with $\lim_{n\to\infty} g(n) = \infty$.

Firstly, we remind some basic notions related to statistical convergence and weight g-statistical convergence of sequences.

Let A be a subset of $\mathbb{N}$. $A(n)$ denotes the set of elements of the set A which are less or equal to $n \in \mathbb{N}$. The natural density of the set A is defined by $d(A) := \lim_{n\to\infty} \frac{|A(n)|}{n}$ if the limit exists where $|A(n)|$ denotes the number of elements of $A(n)$.

A sequence $x = (x_n)$ is statistical convergent to x_0 (denoted by $st - \lim x_n = x_0$) if for every $\varepsilon > 0$, the set

$$\{k \in \mathbb{N} : |x_k - x_0| \geq \varepsilon\}$$

has zero asymptotic density. The set of all statistical convergent sequences is denoted by the symbol c^{st}.

Let $g : \mathbb{N} \to [0, \infty)$ be a function with $\lim_{n\to\infty} g(n) = \infty$. The function g is called weight function and g density of a set $A \subseteq \mathbb{N}$ defined by the formula

$$d_g(A) = \lim_{n\to\infty} \frac{|A(n)|}{g(n)},$$

if the limit exists [4, 11] under the condition $\frac{n}{g(n)} \nrightarrow 0$, $n \to \infty$.

Now let us remember the definition of weight g-statistical convergence.

Email address: mayaaltinok@tarsus.edu.tr (M. Altınok)

Definition 1.1. Let $\tilde{x} = (x_n)$ be a real valued sequence. $\tilde{x}$ is weight g-statistical convergent to the number x_0 if for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{|\{k \leq n : |x_k - x_0| \geq \varepsilon\}|}{g(n)} = 0$$

holds. In this case we write $st_g - \lim x_n = x_0$. c_g^{st} denotes the set of all weight g-statistical convergent sequences [1].

2. Some Definitions and Main Results

In this section, weighted analogue of statistical upper and statistical lower bound, introduced and studied in [2, 3], will be given by considering g density.

g-statistical lower and upper bound can be defined for $\tilde{x} = (x_n)$ as follows:

Definition 2.1. Let $\tilde{x} = (x_n)$ be a real valued sequence.

(i) Weight g-Statistical Lower Bound: A number $l \in \mathbb{R}$ is weight g-statistical lower bound of the sequence $\tilde{x} = (x_n)$, if

$$d_g(\{k : x_k < l\}) = 0$$

holds.

(ii) Weight g-Statistical Upper Bound: A number $u \in \mathbb{R}$ is weight g-statistical upper bound of the sequence $\tilde{x} = (x_n)$, if

$$d_g(\{k : x_k > u\}) = 0 \tag{2.1}$$

holds.

The set of weight g-statistical lower bounds and the set of weight g-statistical upper bounds of the sequence $\tilde{x} = (x_n)$ are denoted by $L_{st_g}(\tilde{x})$ and $U_{st_g}(\tilde{x})$, respectively.

Also, the set of all usual lower bounds and upper bounds of the sequence $\tilde{x} = (x_n)$ are denoted by $L(\tilde{x}) := \{l \in \mathbb{R} : l \leq x_n$ holds for all $n \in \mathbb{N}\}$ and $U(\tilde{x}) := \{u \in \mathbb{R} : x_n \leq u$ holds for all $n \in \mathbb{N}\}$, respectively.

Let us note that $L(\tilde{x}) \subset L_{st_g}(\tilde{x})$ holds for any weight function g.

Actually, let $l \in L(\tilde{x})$ be an arbitrary element. Then, $\{k : x_k < l\} = \emptyset$ and $d_g(\{k : x_k < l\}) = 0$ holds. Therefore, $l \in L_{st_g}(\tilde{x})$. By the same way using (2.1) it can be obtained easily that $U(\tilde{x}) \subset U_{st_g}(\tilde{x})$ holds for any weight function g.

The converses are not true. Let us consider the sequence $(x_n) = (-\frac{1}{n})$ and take $l = -\frac{1}{3} \in \mathbb{R}$. $l = -\frac{1}{3}$ is a weight g-statistical lower bound because $d_g(\{k : x_k < -\frac{1}{3}\}) = d_g(\{1, 2\}) = 0$, but it is clear that $-\frac{1}{3}$ is not usual lower bound for given sequence.

Also, let us consider the sequence $\tilde{x} = (x_n) = (\frac{1}{n})$ and take $u = \frac{1}{3} \in \mathbb{R}$. $u = \frac{1}{3}$ is a weight g-statistical upper bound because $d_g(\{k : x_k > \frac{1}{3}\}) = d_g(\{1, 2\}) = 0$, but it is clear that $\frac{1}{3}$ is not usual upper bound for given sequence.

Following result is clearly obtained from Definition 2.1.

Corollary 2.2. *(i) If $l \in \mathbb{R}$ is a weight g-statistical lower bound of the sequence $\tilde{x} = (x_n)$, then any number less than l is also weight g-statistical lower bound.*

(ii) If $u \in \mathbb{R}$ is a weight g-statistical upper bound of the sequence $\tilde{x} = (x_n)$, then any number greater than u is also weight g-statistical upper bound.

Proof. (i) Let us assume that $l \in \mathbb{R}$ is a weight g-statistical lower bound of the sequence $\tilde{x} = (x_n)$ and $l' < l$ be an arbitrary real number. From the assumption the set $\{k : x_k < l\}$ has weight g density 0. Since $l' < l$, the inclusion

$$\{k : x_k < l'\} \subset \{k : x_k < l\}$$

holds. So, we have

$$0 \leq d_g(\{k : x_k < l'\}) \leq 0.$$

This gives that l' is a weight g-statistical lower bound. Because of the arbitrariness of l', desired result holds.

(ii) Let us assume that $u \in \mathbb{R}$ is a weight g-statistical upper bound of the sequence $\tilde{x} = (x_n)$ and $u < u'$ be an arbitrary real number. From the assumption the set $\{k : x_k > u\}$ has weight g density 0. Since $u < u'$, the inclusion

$$\{k : x_k > u'\} \subset \{k : x_k > u\}$$

holds. So, we have

$$0 \leq d_g(\{k : x_k \leq u'\}) \leq 0.$$

This gives that u' is a weight g-statistical upper bound. Because of the arbitrariness of u', desired result holds. □

Now, let us give the main definitions that we will focus on.

Definition 2.3. (i) Weight g-Statistical Infimum ($st_g - \inf$): If $l^* \in \mathbb{R}$ is the supremum of $L_{st_g}(\tilde{x})$, then it is called weight g-statistical infimum of the sequence $\tilde{x} = (x_n)$. That is, $st_g - \inf x_n := \sup L_{st_g}(\tilde{x})$.

(ii) Weight g-Statistical Supremum ($st_g - \sup$): If $u^* \in \mathbb{R}$ is the infimum of $U_{st_g}(\tilde{x})$, then it is called weight g-statistical supremum of the sequence $\tilde{x} = (x_n)$. That is, $st_g - \sup x_n := \inf U_{st_g}(\tilde{x})$.

Theorem 2.4. *Following inequalities*

$$\inf x_n \leq st_g - \inf x_n \leq st_g - \sup x_n \leq \sup x_n$$

hold for any real valued sequence $\tilde{x} = (x_n)$ and any weight function g.

Proof. Definition of usual infimum gives

$$d_g(\{k : x_k < \inf x_n\}) = d_g(\emptyset) = 0.$$

So, $\inf x_n \in L_{st_g}(x)$. Since $st_g - \inf x_n = \sup L_{st_g}(x)$, then we have $st_g - \inf x_n \geq \inf x_n$. Definition of usual supremum gives

$$d_g(\{k : \sup x_n < x_k\}) = d_g(\emptyset) = 0.$$

So, $\sup x_n \in U_{st_g}(x)$. Since $st_g - \sup x_n = \inf U_{st_g}(x)$, we have

$$st_g - \sup x_n \leq \sup x_n.$$

To complete the proof it is enough to show that

$$l \leq u \tag{2.2}$$

holds for an arbitrary $l \in L_{st_g}(x)$ and $u \in U_{st_g}(x)$. Let us assume that the inverse of (2.2) holds. So, there exist $l' \in L_{st_g}(x)$ and $u' \in U_{st_g}(x)$ such that $u' < l'$ is satisfied. Since u' is a weight g-statistical upper bound, from Corollary 2.2 (ii), l' is also weight g-statistical upper bound of the sequence. This is a contradiction of the assumption on l'. That is, (2.2) is true and desired result holds. □

Lemma 2.5. *Let $\tilde{x} = (x_n)$ be a real valued sequence and $l^* \in \mathbb{R}$. Then, $st_g - \inf x_n = l^*$ if and only if for an arbitrary $\varepsilon > 0$*

$$(i)\ d_g(\{k \ : \ x_k < l^* - \varepsilon\}) = 0,$$

and

$$(ii)\ d_g(\{k \ : \ x_k < l^* + \varepsilon\}) \neq 0$$

hold.

Proof. "⇒" Let us assume that $st_g - \inf x_n = l^*$. That is, $\sup L_{st_g}(\tilde{x}) = l^*$. So, we have

$$(a)\ l \leq l^*, \forall\, l \in L_{st_g}(\tilde{x}),$$

and

$$(b)\ \forall\, \varepsilon > 0\ \exists\, l' \in L_{st_g}(\tilde{x}) \text{ such that } l^* - \varepsilon < l'.$$

Corollary 2.2-(i) and (b) imply $l^* - \varepsilon$ is a weight g-statistical lower bound. So, (i) is hold. Now suppose that (ii) is not true. That is, there exists ε_0 such that $d_g(\{k \ : \ x_k < l^* + \varepsilon_0\}) = 0$. It means that, $l^* + \varepsilon_0 \in L_{st_g}(\tilde{x})$. Since $l^* < l^* + \varepsilon_0$, it contradicts to $l^* = \sup L_{st_g}(\tilde{x})$.

"⇐" Now assume that (i) and (ii) are hold for all positive $\varepsilon > 0$. It is clear that $l^* - \varepsilon \in L_{st_g}(\tilde{x})$ and $l^* + \varepsilon \notin L_{st_g}(\tilde{x})$. So, $L_{st_g}(\tilde{x}) = (-\infty, l^* - \varepsilon]$ for all $\varepsilon > 0$. Thus, $\sup L_{st_g}(\tilde{x}) = l^*$ holds. □

Lemma 2.6. *Let $\tilde{x} = (x_n)$ be a real valued sequence and $u^* \in \mathbb{R}$. Then, $st_g - \sup x_n = u^*$ if and only if for an arbitrary $\varepsilon > 0$*

$$(i)\ d_g(\{k \ : \ x_k > u^* + \varepsilon\}) = 0,$$

and

$$(ii)\ d_g(\{k \ : \ x_k > u^* - \varepsilon\}) \neq 0$$

hold.

Proof. "$\Rightarrow$" Let us assume that $st_g - \sup x_n = u^*$. That is, $\inf U_{st_g}(\tilde{x}) = u^*$. So, we have

$$(a)\ u^* \leq u, \forall\, u \in U_{st_g}(x),$$

and

$$(b)\ \forall \varepsilon > 0\ \exists\, u' \in U_{st_g}(\tilde{x}) \text{ such that } u' < u^* + \varepsilon.$$

Corollary 2.2-(ii) and (b) imply $u^* + \varepsilon$ is a weight g-statistical upper bound. So, (i) is hold. Now suppose that (ii) is not true. That is, there exists $\varepsilon_0 > 0$ such that $d_g(\{k\ :\ u^* - \varepsilon_0 < x_k\}) = 0$. It means that, $u^* - \varepsilon_0 \in U_{st_g}(\tilde{x})$, it contradicts to $u^* = \inf U_{st_g}(\tilde{x})$.

"$\Leftarrow$" Now assume that (i) and (ii) are hold for every $\varepsilon > 0$. It is clear that $u^* + \varepsilon \in U_{st_g}(\tilde{x})$ and $u^* - \varepsilon \notin U_{st_g}(\tilde{x})$. So, $U_{st_g}(\tilde{x}) = [u^* + \varepsilon, \infty)$ for all $\varepsilon > 0$. Thus, $\inf U_{st_g}(\tilde{x}) = u^*$ holds. □

Theorem 2.7. *Let $\tilde{x} = (x_n)$ be a real valued sequence. If $\tilde{x} = (x_n)$ is monotone increasing (or decreasing), then $st_g - \inf x_n = \sup x_n$ (or $st_g - \sup x_n = \inf x_n$) holds.*

Proof. We shall prove only first case. The other case can be obtained by to follow same steps. Now, suppose that $\tilde{x} = (x_n)$ is monotone increasing and

$$\sup x_n < \infty.$$

So,

$$x_k \leq \sup x_n$$

holds for all $k \in \mathbb{N}$. Also, there exists $k_0 \in \mathbb{N}$ such that

$$\sup x_n - \varepsilon < x_{k_0}$$

for every $\varepsilon > 0$. From the first inequality above, $\sup x_n \notin L_{st_g}(\tilde{x})$. From the second inequality we have

$$\{k : x_k < \sup x_n - \varepsilon\} = \{1, 2, 3, ..., k_0\}.$$

Since $d_g(\{1, 2, 3, ..., k_0\}) = 0$, then $\sup x_n - \varepsilon \in L_{st_g}(\tilde{x})$. So, Corollary 2.2-(i) gives that

$$L_{st_g}(\tilde{x}) = (-\infty, \sup x_n - \varepsilon)$$

for all $\varepsilon > 0$. Thus,

$$st_g - \inf x_n = \sup L_{st_g}(\tilde{x}) = \sup x_n.$$

Now, let us assume that

$$\sup x_n = \infty.$$

It means that for all $l \in \mathbb{R}$ there exists $k_0 \in \mathbb{N}$ such that $l \leq x_{k_0}$ and $x_{k_0} \leq x_k$ holds for every $k \geq k_0$. Thus, we have

$$\{k : x_k < l\} \subset \{1, 2, 3, ..., k_0\}.$$

Since $d_g(\{1, 2, 3, ..., k_0\}) = 0$, then for an arbitrary point l, $l \in L_{st_g}(\tilde{x})$. So,

$$L_{st_g}(\tilde{x}) = (-\infty, \infty) \text{ and } \sup L_{st_g}(\tilde{x}) = \infty.$$

So, the proof is completed. □

Corollary 2.8. *Let $\tilde{x} = (x_n)$ be a bounded real valued sequence. If $\tilde{x} = (x_n)$ is monotone decreasing (or increasing) then,*

$$\lim_{n\to\infty} x_n = st_g - \sup x_n\ (= st_g - \inf x_n).$$

Theorem 2.9. *If $\lim_{n\to\infty} x_n = x_0$, then $st_g - \sup x_n = st_g - \inf x_n = x_0$.*

Proof. Let us assume that $\lim_{n\to\infty} x_n = x_0$. That is, for any $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$|x_n - x_0| < \varepsilon, \tag{2.3}$$

holds for all $n \geq n_0$. Thus, (2.3) implies

$$\{k : x_k < x_0 - \varepsilon\} \subset \{1, 2, ..., n_0\},\ \{k : x_k > x_0 + \varepsilon\} \subset \{1, 2, ..., n_0\}. \tag{2.4}$$

So, from (2.4) we have

$$d_g(\{k : x_k < x_0 - \varepsilon\}) = 0,\ d_g(\{k : x_k > x_0 + \varepsilon\}) = 0,$$

respectively. This discussion gives

$$x_0 - \varepsilon \in L_{st_g}(\tilde{x}),\ \ x_0 + \varepsilon \in U_{st_g}(\tilde{x})$$

for all $\varepsilon > 0$. Also, from Corollary 2.2-(i) and (ii)

$$L_{st_g}(\tilde{x}) = (-\infty, x_0) \text{ and } U_{st_g}(\tilde{x}) = (x_0, \infty)$$

hold, respectively. Thus,

$$st_g - \inf x_n = \sup(-\infty, x_0) = x_0,$$

and

$$st_g - \sup x_n = \inf(x_0, \infty) = x_0$$

are obtained. □

Remark 2.10. The inverse of the Theorem 2.9 is not true.

Example 2.11. Let us consider the function $g(n) = n$ and the sequence $\tilde{x} = (x_n)$ as

$$x_n = \begin{cases} 3, & n = m^2, m = 1, 2, ..., \\ 0, & \text{otherwise}. \end{cases}$$

It is clear that $st_g - \inf x_n = st_g - \sup x_n = 0$ but this sequence does not converge to 0.

Theorem 2.12. $st_g - \lim\limits_{n\to\infty} x_n = x_0$ *if and only if* $st_g - \sup x_n = st_g - \inf x_n = x_0$.

Proof. "⟹" Let us assume that $st_g - \lim\limits_{n\to\infty} x_n = x_0$. From the hypothesis,

$$\lim_{n\to\infty} \frac{1}{g(n)} |\{k : k \le n,\ |x_k - x_0| \ge \varepsilon\}| = 0 \tag{2.5}$$

holds for every $\varepsilon > 0$. Also, we have

$$\{k : k \le n,\ |x_k - x_0| \ge \varepsilon\} = \{k : k \le n,\ x_k \ge x_0 + \varepsilon\} \cup \{k : k \le n,\ x_k \le x_0 - \varepsilon\}.$$

By using last equality and (2.5), we obtain

$$d_g\left(\{k : x_k > x_0 + \varepsilon\}\right) = 0, \tag{2.6}$$

and

$$d_g\left(\{k : x_k < x_0 - \varepsilon\}\right) = 0. \tag{2.7}$$

From (2.6), $x_0 + \varepsilon$ is a weight g-statistical upper bound, also from (2.7), $x_0 - \varepsilon$ is a weight g-statistical lower bound. Thus,

$$L_{st_g}(\tilde{x}) = (-\infty, x_0),\ \ U_{st_g}(\tilde{x}) = (x_0, \infty)$$

hold for all $\varepsilon > 0$. Therefore, we obtain

$$st_g - \inf x_n = \sup L_{st_g}(\tilde{x}) = x_0, \quad st_g - \sup x_n = \inf U_{st_g}(\tilde{x}) = x_0.$$

"⟸" Let us assume that

$$st_g - \sup x_n = st_g - \inf x_n = x_0.$$

Namely,

$$x_0 = \sup L_{st_g}(\tilde{x}) = \inf U_{st_g}(\tilde{x}).$$

From the definitions of usual supremum and infimum, for every $\varepsilon > 0$, there exists at least one element $l \in L_{st_g}(\tilde{x})$ and $u \in U_{st_g}(\tilde{x})$ such that the inequalities

$$x_0 - \varepsilon < l,\ \ u < x_0 + \varepsilon$$

hold.

Since u is a weight g-statistical upper bound, then the following inclusion

$$\{k : x_k \ge x_0 + \varepsilon\} \subset \{k : x_k > u\}$$

holds. Thus, we have

$$d_g(\{k : x_k \ge x_0 + \varepsilon\}) = 0. \tag{2.8}$$

Since l is a weight g-statistical lower bound, then the following inclusion

$$\{k : x_k \le x_0 - \varepsilon\} \subset \{k : x_k < l\}$$

holds. Thus, we have

$$d_g(\{k : x_k \leq x_0 - \varepsilon\}) = 0. \tag{2.9}$$

From (2.8), (2.9) and following equation

$$\{k : |x_k - x_0| \geq \varepsilon\} = \{k : x_k \geq x_0 + \varepsilon\} \cup \{k : x_k \leq x_0 - \varepsilon\},$$

we have

$$d_g(\{k : |x_k - x_0| \geq \varepsilon\}) = 0.$$

Thus, the sequence $\tilde{x} = (x_n)$ is weight g-statistical convergent to $x_0 \in \mathbb{R}$. □

3. Weighted g-Statistical Core Theorem

In this section, we will define weight g-statistical limit supremum and weight g-statistical limit infimum for real valued sequences. Then, we will express weight g-statistical core theorem.

Definition 3.1. Let $\tilde{x} = (x_n)$ be a real valued sequence. Weight g-statistical limit supremum and weight g-statistical limit infimum of $\tilde{x}$ defined as

$$st_g - \limsup x_n := \lim_{n\to\infty}(st_g - \sup_{k\geq n} x_k)$$

and

$$st_g - \liminf x_n := \lim_{n\to\infty}(st_g - \inf_{k\geq n} x_k),$$

respectively.

Lemma 3.2. *Let $\tilde{x} = (x_n)$ be a real valued sequence.*
(i) If $\alpha_n := st_g - \sup_{k\geq n} x_k$ for all $n \in \mathbb{N}$, then $(\alpha_n)_{n\in\mathbb{N}}$ is a constant sequence and so

$$st_g - \limsup x_n = st_g - \sup x_n.$$

(ii) If $\beta_n := st_g - \inf_{k\geq n} x_k$ for all $n \in \mathbb{N}$, then $(\beta_n)_{n\in\mathbb{N}}$ is a constant sequence and so

$$st_g - \liminf x_n = st_g - \inf x_n.$$

As a result of Definition 3.1 and Theorem 2.4 we have following result.

Theorem 3.3. *For any real valued sequence $\tilde{x}$,*

$$\liminf x_n \leq st_g - \liminf x_n \leq st_g - \limsup x_n \leq \limsup x_n \tag{3.1}$$

hold.

Definition 3.4. The real number sequence $\tilde{x} = (x_n)$ is said to be weight g-statistically bounded if there is a number M such that $d_g(\{k : |x_k| > M\}) = 0$.

Theorem 3.5. *Weight g-statistical bounded sequence $\tilde{x} = (x_n)$ is weight g-statistical convergent if and only if $st_g - \liminf x_n = st_g - \limsup x_n$.*

Proof. Let us assume that for simplicity that $st_g - \liminf x_n := l^*$ and
$st_g - \limsup x_n := u^*$.

(⇒) Let $st_g - \lim x_n = x_0$. So, $d_g(\{k : |x_k - x_0| \geq \varepsilon\}) = 0$ holds for every $\varepsilon > 0$. Thus, $d_g(\{k : x_k > x_0 + \varepsilon\}) = 0$ which implies that $u^* \leq x_0$. Also, we have from the definition of g-statistical convergence that $d_g(\{k : x_k < x_0 - \varepsilon\}) = 0$ which implies that $x_0 \leq l^*$. Therefore, $l^* = u^*$.

(⇐) Conversely, assume that $l^* = u^*$ and choose $x_0 := l^* = u^*$. If $\varepsilon > 0$ then from Theorem 2.5, Theorem 2.6 and Definition 3.4 we have $d_g(\{k : x_k > x_0 + \varepsilon\}) = 0$ and $d_g(\{k : x_k < x_0 - \varepsilon\}) = 0$. Therefore, $st_g - \lim x_n = x_0$. □

In [18], Knopp defined core of a sequence and proved well-known Core Theorem. In this paper, we will use weight g-statistical limit infimum and weight g-statistical limit supremum instead of limit points to obtain an analogue of Knopp's core.

Definition 3.6. Let $\tilde{x} = (x_n)$ be a weight g-statistical bounded sequence. Then, weight g-statistical core of $\tilde{x}$ is the closed interval $[st_g - \liminf x_n, st_g - \limsup x_n]$ and it is denoted by $st_g - core\{\tilde{x}\}$. If $\tilde{x}$ is not weight g-statistical bounded, then $st_g - core\{\tilde{x}\}$ is defined as either $[st_g - \liminf x_n, \infty)$, $(-\infty, st_g - \limsup x_n]$ or $(-\infty, \infty)$.

$K - core\{\tilde{x}\}$ is the usual core of $\tilde{x}$. It is clear from (3.1) that for any real valued sequence $\tilde{x}$

$$st_g - core\{\tilde{x}\} \subseteq K - core\{\tilde{x}\}.$$

Theorem 3.7. *Let g_1 and g_2 be two weight function and $\tilde{x} = (x_n)$ be a real valued sequence. If we have following limit condition*

$$\frac{g_1(n)}{g_2(n)} = 1, \quad (n \to \infty), \tag{3.2}$$

then

$$st_{g_1} - core\{\tilde{x}\} = st_{g_2} - core\{\tilde{x}\} \tag{3.3}$$

holds.

Proof. Let $\tilde{x} = (x_n)$ be a weight g_1-statistical bounded sequence. Then, $st_{g_1} - core\{\tilde{x}\} = [st_{g_1} - \liminf x_n, st_{g_1} - \limsup x_n]$. For brevity, let us take $l^* = st_{g_1} - \liminf x_n$ and $u^* = st_{g_1} - \limsup x_n$. So, from Theorem 2.5

$$d_{g_1}(\{k : x_k < l^* - \varepsilon\}) = 0 \quad \text{and} \quad d_{g_1}(\{k : x_k < l^* + \varepsilon\}) \neq 0$$

hold for every $\varepsilon > 0$. Thus, from (3.2)

$$\frac{|\{k : x_k < l^* - \varepsilon\}|}{g_2(n)} = \frac{|\{k : x_k < l^* - \varepsilon\}|}{g_2(n)} \cdot \frac{g_1(n)}{g_1(n)} = \frac{|\{k : x_k < l^* - \varepsilon\}|}{g_1(n)} \cdot \frac{g_1(n)}{g_2(n)}$$

holds. If we take limit when $n \to \infty$, then from hypothesis we obtain

$$d_{g_2}(\{k : x_k < l^* - \varepsilon\}) = 0.$$

Moreover, by the same way we obtain

$$d_{g_2}(\{k : x_k < l^* + \varepsilon\}) \neq 0.$$

Therefore, $l^* = st_{g_2} - \liminf x_n$. Considering that $u^* = st_{g_1} - \limsup x_n$ from Theorem 2.6 we obtain $u^* = st_{g_2} - \limsup x_n$. So, $st_{g_1} - core\{\tilde{x}\} = st_{g_2} - core\{\tilde{x}\}$. Now, let us assume that $\tilde{x} = (x_n)$ is not weight g_1-statistical bounded sequence. Then, one of the last three cases in the Definition 3.6 is provided. In this case, (3.3) can be obtained by the similar calculations above. □

4. Conclusion

In this paper, Theorem 2.9 and Theorem 2.12 are given as an application of $st_g - \sup$ and $st_g - \inf$. It is seen that equality of $st_g - \sup$ and $st_g - \inf$ necessary but not sufficient for existence of classical limit but necessary and sufficient for weighted g statistical limit.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors have contributed sufficiently to the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

References

[1] Adem, A.A., Altınok, M., *Weighted statistical convergence of real valued sequences*, Facta Universitatis, Series: Mathematics and Informatics, **35**(3)(2020), 887–898.
[2] Altınok, M., Küçükaslan, M., *Statistical supremum-infimum and statistical convergence*, The Aligarh Bulletin of Mathematics, **32**(2013), 1–16.
[3] Altınok, M., Küçükaslan, M., *A-statistical supremum-infimum and A-statistical convergence*, Azerbaijan Journal of Mathematics, **4**(2)(2014), 31–42.
[4] Balcerzak, M., Das, P., Filipczak, M., Swaczyna, J., *Generalized kinds of density and the associated ideals*, Acta Math. Hungar., **147**(1)(2015), 97–115.
[5] Bhunia, S., Das, P., Pal, S.K., *Restricting statistical convergence*, Acta Mathematica Hungarica, **134**(1-2)(2012), 153–161.
[6] Buck, C., *Generalized asymptotic density*, Amer. J. Math. **75**(1953), 335–346.
[7] Cakalli, H., *A new approach to statistically quasi Cauchy sequences*, Maltepe Journal of Mathematics, **1**(1)(2019), 1–8.
[8] Connor, J., *The statistical and strong p-Cesàro convergence of sequences*, Analysis, **8**(1988), 47–63.
[9] Connor, J., *On strong matrix summability with respect to a modulus and statistical convergence*, Canad. Math. Bull., **32**(1989), 194–198.
[10] Çolak, R., Statistical Convergence of Order α, Modern Methods in Analysis and Its Applications, Anamaya Pub., New Delhi, India, 2010.

[11] Das, P., Savaş, E., *On generalized statistical and ideal convergence of metric-valued sequences*, Reprinted in Ukrainian Math. J., **68**(12)(2017), 1849–1859. Ukrain. Mat. Zh., **68**(12)(2016), 1598–1606.

[12] Fast, H., *Sur la convergence statistique*, Colloq. Math., **2**(1951), 241–244.

[13] Freedman, A.P., Sember, J.J., *Densities and summability*, Pasific J. Math., **95**(1981), 293–305.

[14] Fridy, J.A., *On statistical convergence*, Analysis, **5**(1985), 1301–1313.

[15] Fridy, J.A., *Statistical limit points*, Proc. Amer. Math. Soc., **118**(1993), 1187–1192.

[16] Hardy, G.H., Divergent Series, Oxford Univ. Press, London, 1949.

[17] Sengul Kandemir, H., *On I-Deferred statistical convergence in topological groups*, Maltepe Journal of Mathematics, **1**(2)(2019), 48–55.

[18] Knopp, K., *Zur theorie der limitierungsverfahren (Erste Mitteilung)*, Math. Zeit., **31**(1930), 115–127.

[19] Kostyrko, P, Macaj, M., Salat, T., Strauch, O., *On statistical limit points*, Proc. Amer. Math. Soc., **120**(2000), 2647–2654.

[20] Schoenberg, I.J., *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, **66** (5)(1959), 361–375.

[21] Steinhaus, H., *Sur la convergence ordinate et la convergence asymptotique*, Colloq. Math., **2**(1951), 73–84.

[22] Taylan, I., *Abel statistical delta quasi Cauchy sequences of real numbers*, Maltepe Journal of Mathematics, **1**(1)(2019), 18–23.

[23] Tok, N., Basarir, M., *On the lambda alpha h statistical convergence of the functions defined on the time scale*, Proceedings of International Mathematical Sciences, **1**(1)(2019), 1–10.

[24] Zygmund, A., Trigonometric Series, Cambridge Univ. Press, Cambridge, UK, 1979.