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d-Gaussian Pell-Lucas Polynomials and Their Matrix Representations

Engin Özkan [1], Mine Uysal[2,*]

[1]Department of Mathematics, Faculty of Arts and Sciences, Erzincan Binali Yıldırım University, 24100, Erzincan, Turkey.
[2]Department of Mathematics, Graduate School of Natural and Applied Sciences, Erzincan Binali Yıldırım University, 24100, Erzincan, Turkey.

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Abstract. We define a new generalization of Gaussian Pell-Lucas polynomials. We call it $d-$Gaussian Pell-Lucas polynomials. Then we present the generating function and Binet formula for the polynomials. We give a matrix representation of $d-$Gaussian Pell-Lucas polynomials. Using the Riordan method, we obtain the factorizations of Pascal matrix involving the polynomials.

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1. Introduction

Number sequences and their polynomials have attracted the attention of many scientists for many years, as they find application in nature and in many sciences. Fibonacci numbers are the best known of the sequences of numbers [4, 16, 18]. Many generalizations of number sequences were then described and studied [1, 8, 10–14]. Mikkawy et al. gave a new family of $k-$Fibonacci numbers [7]. Özkan et al. defined Gaussian Fibonacci polynomials, Gaussian Lucas polynomials and gave some properties for these polynomials [9].

Now, let's give basic definitions for this paper.
The Pell numbers P_n are defined by

$$P_n = 2P_{n-1} + P_{n-2}, n \geq 3$$

with $P_1 = 1$ and $P_2 = 2$ [6].
Similarly, the Pell-Lucas numbers Q_n,

$$Q_n = 2Q_{n-1} + Q_{n-2}, n \geq 3$$

with $Q_1 = 1$ and $Q_2 = 3$ [6].
The Pell polynomials are defined by

$$P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x)$$

with $P_0(x) = 0$ and $P_1(x) = 1$ [4]. The Pell-Lucas polynomials are defined by

$$Q_{n+2}(x) = 2xQ_{n+1}(x) + Q_n(x)$$

with $Q_0(x) = 2$ and $Q_1(x) = 2x$ [4].

*Corresponding Author
Email addresses: eozkan@erzincan.edu.tr (E. Özkan), mine.uysal@erzincan.edu.tr (M. Uysal)

Definition 1.1. Let $p_i(x)$ be a polynomials with real coefficients for $i = 1, ..., d+1$. Then,

$$F_{n+1}(x) = p_1(x)F_n(x) + p_2(x)F_{n-1}(x) + \cdots + p_{d+1}(x)F_{n-d}(x)$$

with $F_n(x) = 0$ for $n \leq 0$ and $F_1(x) = 1$ [15].

In [2], the authors defined the Gaussian Pell and Gaussian Pell-Lucas numbers and examined their properties. Then, Halıcı et al. gave the Gaussian Pell polynomials [3]. Halıcı et al. defined the Gaussian Pell-Lucas polynomials examined some properties of them [19]. Shapiro et al. described Riordan matrices and the Riordan group as a set of matrices $M = (m_{ij})$, $i, j \geq 0$ 0 whose elements are complex numbers [17]. Sadaoui et al. have recently been studying $d-$Fibonacci and $d-$Lucas polynomials [15].

We present $d-$Gaussian Pell-Lucas polynomials. Then, we find the generating function and Binet formula for the polynomials. We give a matrix representation of d-Gaussian Pell-Lucas polynomials. We also give the factorizations of Pascal matrix involving the polynomials with the help of the Riordan method. In addition, we introduce the inverse of matrices for these polynomials.

2. Generalization of Gaussian Pell-Lucas Polynomials

Now, we present a new generalization of Gaussian Pell-Lucas polynomials. Let $p_i(x)$ be a real coefficient for $i = 1, ..., d+1$. Then, d-Gaussian Pell-Lucas polynomials are defined by

$$GPL_n(x) = p_1(x)GPL_{n-1}(x) + p_2(x)GPL_{n-2}(x) + \ldots + p_{d+1}(x)GPL_{n-d-1}(x)$$

with $GPL_0(x) = 2 - 2p_1(x)i$ and $GPL_n = 0$ for $n < 0$. We give a few terms of d-Gaussian Pell-Lucas polynomials in Table 1.

n	$GPL_n(x)$
0	$2 - 2p_1(x)i$
1	$2p_1(x) - 2p_1^2(x)i$
2	$2p_1^2(x) - 2p_1(x)^3 i + 2p_2(x) - 2p_1(x)p_2(x)i$
3	$2p_1^3(x) - 2p_1^4(x)i + 4p_1(x)p_2(x) - 4p_1^2(x)p_2(x)i + 2p_3(x) - 2p_1(x)p_3(x)$
4	$2p_1^4(x) - 2p_1^5(x)i + 6p_1^2(x)p_2(x) - 6p_1^3(x)p_2(x)i + 4p_1(x)p_3(x) - 2p_1^2(x)p_2(x)i + 2p_2^2(x) + 2p_4(x) - 2p_1(x)p_4(x)i$

Table 1. Some values of d-Gaussian Pell-Lucas polynomials

Theorem 2.1. *The generating function of $GPL_n(x)$ is given as follows;*

$$G(x,t) = \sum_{n=0}^{\infty} GPL_n(x)t^n = \frac{2 - 2p_1(x)i}{(1 - p_1(x)t - p_2(x)t^2 - ... - p_{d+1}(x)t^{d+1})}.$$

Proof. We have

$$\begin{aligned} G(x,t) &= \sum_{n=0}^{\infty} GPL_n(x)t^n \\ &= GPL_0(x) + GPL_1(x)t^1 + GPL_2(x)t^2 + ... + GPL_n(x)t^n. \end{aligned} \tag{2.1}$$

Multiplying equation (2.1) by $p_1(x)t, p_2(x)t^2, ..., p_{d+1}(x)t^{d+1}$, we obtain the following equations, respectively,

$$\begin{aligned} G(x,t) =& GPL_0(x) + GPL_1(x)t + GPL_2(x)t^2 + ... + GPL_n(x)t^n + ..., \\ p_1(x)tG(x,t) =& p_1(x)tGPL_0(x) + p_1(x)t^2GPL_1(x) + p_1(x)t^3GPL_2(x) + ..., \\ p_2(x)t^2G(x,t) =& p_2(x)t^2GPL_0(x) + p_2(x)t^3GPL_1(x) + p_2(x)t^4GPL_2(x) + ..., \\ &\vdots \\ p_{d+1}(x)t^{d+1}G(x,t) =& p_{d+1}(x)t^{d+1}GPL_0(x) + p_{d+1}(x)t^{d+2}GPL_1(x) + p_{d+1}(x)t^{d+3}GPL_2(x) + ... \quad . \end{aligned}$$

If the necessary calculations are made, we obtain the following equation

$$G(x,t)(1-p_1(x)t-p_2(x)t^2-...-p_{d+1}(x)t^{d+1})=GPL_0(x)$$
$$\Longrightarrow G(x,t)=\frac{2-2p_1(x)i}{(1-p_1(x)t-p_2(x)t^2-...-p_{d+1}(x)t^{d+1})}.$$

□

Binet formula of $GPL_n(x)$ is as follows;

$$GPL_n(x)=\sum_{i=1}^{d+1}R_i(x)[\alpha_i(x)]^n.$$

Let's write the following equations for each value of n.

$$\begin{aligned}GPL_0(x)&=\sum_{i=1}^{d+1}R_i(x),\\ GPL_1(x)&=\sum_{i=1}^{d+1}R_i(x)[\alpha_i(x)],\\ GPL_2(x)&=\sum_{i=1}^{d+1}R_i(x)[\alpha_i(x)]^2,\\ &\vdots\\ GPL_n(x)&=\sum_{i=1}^{d+1}R_i(x)[\alpha_i(x)]^n.\end{aligned}$$

If we multiplying last equations by $1,t,t^2,...,t^n$, then we get the following equations, respectively;

$$\begin{aligned}GPL_0(x)&=\sum_{i=1}^{d+1}R_i(x),\\ tGPL_1(x)&=\sum_{i=1}^{d+1}R_i(x)[\alpha_i(x)]t,\\ t^2GPL_2(x)&=\sum_{i=1}^{d+1}R_i(x)[\alpha_i(x)]^2t^2,\\ &\vdots\\ t^nGPL_n(x)&=\sum_{i=1}^{d+1}R_i(x)[\alpha_i(x)]^nt^n.\end{aligned}$$

So, we have

$$\sum_{n=0}^{\infty}GPL_n(x)t^n=\sum_{i=1}^{d+1}R_i(x)\Big(1+\alpha_i(x)t+\alpha_i(x)^2t^2+\cdots\Big)=\sum_{i=1}^{d+1}\frac{R_i(x)}{\Big(1-\alpha_i(x)t\Big)}.$$

From Theorem 2.1, we obtain

$$\frac{2-2p_1(x)i}{(1-p_1(x)t-p_2(x)t^2-...-p_{d+1}(x)t^{d+1})}=\sum_{i=1}^{d+1}\frac{R_i(x)}{(1-\alpha_i(x)t)}.$$

More precisely, the coefficients $R_i(x)$ allow us to give the explicit form of d-Gaussian Pell-Lucas polynomials.

Theorem 2.2. *For $n \geq 0$, the following equality is true;*

$$GPL_n(x) = (2-2p_1(x)i)\sum_{\substack{n_1,n_2,\ldots,\,n_{d+1}\\ n_1+2n_2+\cdots+(d+1)n_{d+1}=n}}\left[\binom{n_1+n_2+\cdots+\,n_{d+1}}{n_1,n_2,\ldots,\,n_{d+1}} {p_1}^{n_1}(x)\,{p_2}^{n_2}(x)\ldots {p_{d+1}}^{n_{d+1}}(x)\right].$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} GPL_n(x)t^n &= \frac{2-2p_1(x)i}{(1-p_1(x)t-p_2(x)t^2-\ldots-p_{d+1}(x)t^{d+1})}\\
&= (2-2p_1(x)i)\sum_{n=0}^{\infty}\left(p_1(x)t+p_2(x)t^2+\cdots+p_{d+1}(x)t^{d+1}\right)^n\\
&= (2-2p_1(x)i)\sum_{n=0}^{\infty}\left[\sum_{n_1+n_2+\cdots+n_{d+1}=n}^{\infty}\binom{n}{n_1,n_2,\cdots,n_{d+1}} p^{n_1}(x){p_2}^{n_2}(x)\ldots {p_{d+1}}^{n_{d+1}}(x)\right]t^{n_1+2n_2+\cdots+(d+1)n_{d+1}}\\
&= (2-2p_1(x)i)\sum_{n=0}^{\infty}\left[\sum_{\substack{n_1,n_2,\cdots,n_{d+1}\\ n_1+2n_2+\cdots+(d+1)n_{d+1}=n}}\binom{n_1+n_2+\ldots+n_{d+1}}{n_1,n_2,\cdots n_{d+1}}{p_1}^{n_1}(x){p_2}^{n_2}(x)\cdots {p_{d+1}}^{n_{d+1}}(x)\right]t^n.
\end{aligned}$$

Thus, the proof is obtained. □

Theorem 2.3. *Let $SGPL_n(x)$ be sum of the $d-$Gaussian Pell-Lucas polynomials. Then, we have*

$$SGPL_n(x) = \sum_{n=0}^{\infty} GPL_n(x) = \frac{2-2p_1(x)i}{1-p_1(x)-p_2(x)-\cdots-p_{d+1}(x)}.$$

Proof. We get the following equation:.

$$SGPL_n(x) = \sum_{n=0}^{\infty} GPL_n(x) = GPL_0(x)+GPL_1(x)+GPL_2(x)+\ldots GPL_n(x)+\cdots.$$

If we multiply the last equation by $p_1(x), p_2(x), \ldots, p_{d+1}(x)$, respectively, then we obtain

$$\begin{aligned}
p_1(x)\ SGPL_n(x) &= p_1(x)GPL_0(x)+p_1(x)GPL_1(x)+\cdots+p_1(x)GPL_n(x)+\cdots,\\
p_2(x)SGPL_n(x) &= p_2(x)GPL_0(x)+p_2(x)GPL_1(x)+\cdots+p_2(x)GPL_n(x)+\cdots,\\
&\vdots\\
p_{d+1}(x)SGPL_n(x) &= p_{d+1}(x)GPL_0(x)+p_{d+1}(x)GPL_1(x)+\cdots+p_{d+1}(x)GPL_n(x)+\cdots.
\end{aligned}$$

If the necessary operations are done, we get

$$SGPL_n(x)(1-p_1(x)-p_2(x)-\cdots-p_{d+1}(x)) = GPL_0(x).$$

Thus, we obtain

$$SGPL_n(x) = \sum_{n=0}^{\infty} GPL_n(x) = \frac{2-2p_1(x)i}{1-p_1(x)-p_2(x)-\cdots-p_{d+1}(x)}.$$

□

The $d-$Gaussian Pell-Lucas polynomials matrix PL_d is given by

$$PL_d = (2-2p_1(x)i)Q_d = \begin{pmatrix} 2p_1(x)-2p_1^2(x)i & 2p_2(x)-2p_1(x)p_2(x) & \cdots & 2p_{d+1}(x)-2p_1(x)p_{d+1}(x)\\ 2-2p_1(x)i & 0 & & 0\\ 0 & \ddots & & \\ 0 & 0 & 2-2p_1(x)i & 0 \end{pmatrix}, \tag{2.2}$$

where

$$Q_d = \begin{pmatrix} p_1(x) & p_2(x) & \cdots & p_{d+1}(x) \\ 1 & 0 & & 0 \\ 0 & \ddots & & \\ & \ddots & & \\ 0 & & 0 \quad 1 & 0 \end{pmatrix}.$$

Now, we can give matrix representation for $GPL_n(x)$ in the next theorem.

Theorem 2.4. *The matrix representation for $GPL_n(x)$ has the form*

$$PL_d^n = \begin{pmatrix} GPL_n(x) & p_2(x)GPL_{n-1}(x) + \cdots + p_{d+1}(x)GPL_{n-d}(x) & \cdots & p_{d+1}(x)GPL_{n-1}(x) \\ GPL_{n-1}(x) & p_2(x)GPL_{n-2}(x) + \cdots + p_{d+1}(x)GPL_{n-d-1}(x) & \cdots & p_{d+1}(x)GPL_{n-2}(x) \\ \vdots & \vdots & \vdots & \vdots \\ GPL_{n-d}(x) & p_2(x)GPL_{n-d-1}(x) + \cdots + p_{d+1}(x)GPL_{n-2d}(x) & \cdots & p_{d+1}(x)GPL_{n-d-1}(x) \end{pmatrix}, \tag{2.3}$$

where $PL_d^{n+1} = PL_d^n Q_d$.

Proof. To prove the theorem, let's use mathematical induction on n.
If we take $n = 1$ in equation (2.3), we get the following matrix:

$$PL_d = \begin{pmatrix} 2p_1(x) - 2p_1^2(x)i & 2p_2(x) - 2p_1(x)p_2(x) & \cdots & 2p_{d+1}(x) - 2p_1(x)p_{d+1}(x) \\ 2 - 2p_1(x)i & 0 & & 0 \\ 0 & \ddots & & \\ 0 & 0 & 2 - 2p_1(x)i & 0 \end{pmatrix}. \tag{2.4}$$

From the recurrence relation of $GPL_n(x)$, it will be seen that the matrices in (2.2) and (2.4) are equal.
Assume that the equation (2.3) satisfies for n. That is, we have

$$PL_d^n = \begin{pmatrix} GPL_n(x) & p_2(x)GPL_{n-1}(x) + \cdots + p_{d+1}(x)GPL_{n-d}(x) & \cdots & p_{d+1}(x)GPL_{n-1}(x) \\ GPL_{n-1}(x) & p_2(x)GPL_{n-2}(x) + \cdots + p_{d+1}(x)GPL_{n-d-1}(x) & \cdots & p_{d+1}(x)GPL_{n-2}(x) \\ \vdots & \vdots & \vdots & \vdots \\ GPL_{n-d}(x) & p_2(x)GPL_{n-d-1}(x) + \cdots + p_{d+1}(x)GPL_{n-2d}(x) & \cdots & p_{d+1}(x)GPL_{n-d-1}(x) \end{pmatrix}.$$

Let us show that, it is true for $n + 1$. So, we have

$$\begin{aligned} PL_d^{n+1} &= PL_d^n Q_d \\ &= \begin{pmatrix} GPL_n(x) & p_2(x)GPL_{n-1}(x) + \cdots + p_{d+1}(x)GPL_{n-d}(x) & \cdots & p_{d+1}(x)GPL_{n-1}(x) \\ GPL_{n-1}(x) & p_2(x)GPL_{n-2}(x) + \cdots + p_{d+1}(x)GPL_{n-d-1}(x) & \cdots & p_{d+1}(x)GPL_{n-2}(x) \\ \vdots & \vdots & \vdots & \vdots \\ GPL_{n-d}(x) & p_2(x)GPL_{n-d-1}(x) + \cdots + p_{d+1}(x)GPL_{n-2d}(x) & \cdots & p_{d+1}(x)GPL_{n-d-1}(x) \end{pmatrix} \\ &\times \begin{pmatrix} p_1(x) & p_2(x) & \cdots & p_{d+1}(x) \\ 1 & 0 & & 0 \\ 0 & \ddots & & \\ & \ddots & & \\ 0 & & 0 \quad 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} GPL_{n+1}(x) & p_2(x)GPL_n(x) + \cdots + p_{d+1}(x)GPL_{n+1-d}(x) & \cdots & p_{d+1}(x)GPL_n(x) \\ GPL_n(x) & p_2(x)GPL_{n-1}(x) + \cdots + p_{d+1}(x)GPL_{n-d}(x) & \cdots & p_{d+1}(x)GPL_{n-1}(x) \\ \vdots & \vdots & \vdots & \vdots \\ GPL_{n+1-d}(x) & p_2(x)GPL_{n-d}(x) + \cdots + p_{d+1}(x)GPL_{n+1-2d}(x) & \cdots & p_{d+1}(x)GPL_{n-d}(x) \end{pmatrix}. \end{aligned}$$

□

Corollary 2.5. *For $n, m \geq 0$, the following equality is provided:*
$\left(2-2p_1(x)i\right)GPL_{n+m}(x)$

$$\begin{aligned}
&= GPL_{n+1}(x)GPL_{m+1}(x) + p_2(x)(GPL_{n-1}(x)GPL_{m-1}(x) + p_3(x)(GPL_{n-2}(x)GPL_{m-1}(x)\\
&+ GPL_{n-1}(x)GPL_{m-2}(x)) + p_4(x)(GPL_{n-3}(x)GPL_{m-1}(x) + GPL_{n-2}(x)GPL_{m-2}(x)\\
&+ GPL_{n-1}(x)GPL_{m-3}(x) + \cdots + p_{d+1}(x)(GPL_{n-d+1}(x)GPL_{m-1}(x)\\
&+ \cdots + GPL_{n-1}(x)GPL_{m-d}(x)).
\end{aligned}$$

Proof. From the product of matrices PL_d^n and PL_d^m, we get

$$PL_d^n PL_d^m = PL_d^{m+n}.$$

The result is the first row and column of matrix PL_d^{m+n}. □

Lemma 2.6. *For $n \geq 1$, the following equality is true:*

$$GPL_{n-1}(x) = \left(2-2p_1(x)\,i\right)F_n(x)$$

where the $F_n(x)$ polynomials are $d-$Fibonacci polynomials.

Proof. The proof can be easily seen by induction on n **.** □

3. The Infinite $d-$Gaussian Pell-Lucas Polynomial Matrix

The $d-$Gaussian Pell-Lucas polynomials matrix is denoted by

$$G\mathcal{PL}(x) = [G\mathcal{PL}_{P_1,P_2,\ldots,P_{d+1},i,j}(x)]$$

and defined as follows;

$$\begin{aligned}
G\mathcal{PL}(x) &= \begin{pmatrix}
2-2p_1(x)i & 0 & 0 & \cdots\\
2-p_1(x)-2p_1^2(x)i & 2-2p_1(x)i & 0 & \cdots\\
2p_1^2(x)-2p_1^3(x)i+2p2_2(x)+2p_1(x)p_2(x)i & 2p_1(x)-2p_1^2(x)i & \ddots & \cdots\\
l_1(x) & l_2(x) & & \cdots\\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}\\
&= \left(g_{G\mathcal{PL}(x)}(t), f_{G\mathcal{PL}(x)}(t)\right),
\end{aligned}$$

where $l_1(x) = 2{p_1}^3(x) - 2{p_1}^4(x)\,i + 4p_1(x)\,p_2(x) - 4{p_1}^2(x)\,p_2(x)\,i + 2p_3(x) - 2p_1(x)\,p_3(x)$ and $l_2(x) = 2{p_1}^2(x) - 2p_1(x)^3 i + 2p_2(x) - 2p_1(x)\,p_2(x)\,i$.

The Gaussian Pell-Lucas polynomial matrix can also be written as

$$G\mathcal{PL}(x) = \begin{pmatrix}
GPL_0(x) & 0 & 0 & \cdots\\
GPL_1(x) & GPL_0(x) & 0 & \cdots\\
GPL_2(x) & GPL_1(x) & GPL_0(x) & \cdots\\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$

Note that $G\mathcal{PL}(x)$ is a Riordan matrix.

Theorem 3.1. *The first column of matrix $G\mathcal{PL}(x)$ is*

$$\left(2-2p_1(x)\,i,\ 2p_1(x)-2{p_1}^2(x)\,i, 2{p_1}^2(x)-2p_1(x)^3 i+2p_2(x)-2p_1(x)\,p_2(x)\,i, l_1(x),\ldots\right)^T.$$

According to the Riordan array, the generator function of the first column is as follows;

$$g_{G\mathcal{PL}(x)}(t) = \sum_{n=0}^{\infty} G\mathcal{PL}_{P_1,P_2,\ldots,P_{d+1},i,j}(x)t^n = \frac{2-2p_1(x)\,i}{(1-p_1(x)\,t-p_2(x)\,t^2-\cdots-p_{d+1}(x)\,t^{d+1})}.$$

Proof. Let's write generating function of the first column of the matrix $GPL(x)$ as follows

$$2-2p_1(x)i+\left(2p_1(x)-2{p_1}^2(x)i\right)t+\left(2{p_1}^2(x)-2p_1(x)^3 i+2p_2(x)-2p_1(x)p_2(x)i\right)t^2+\cdots$$
$$=GPL_0(x)+GPL_1(x)t+GPL_2(x)t^2+\cdots \quad .$$

From the generator function of $GPL_n(x)$, we have

$$\begin{aligned} G(x,t) &= GPL_0(x)+GPL_1(x)t+GPL_2(x)t^2+\cdots+GPL_n(x)t^n+\ldots \\ &= \frac{2-2p_1(x)i}{(1-p_1(x)t-p_2(x)t^2-\cdots-p_{d+1}(x)t^{d+1})}. \end{aligned}$$

Thus, the desired expression is obtained. □

So, we get

$$f_{G\mathcal{PL}(x)}(t)=t.$$

Then, we write $G\mathcal{PL}(x)$ as follows;

$$G\mathcal{PL}(x) = \left(g_{G\mathcal{PL}(x)}(t), f_{G\mathcal{PL}(x)}(t)\right)=\left(\frac{2-2p_1(x)i}{1-p_1(x)t-p_2(x)t^2-\cdots-p_{d+1}(x)t^{d+1}}, t\right).$$

If the Gaussian Pell-Lucas polynomials matrix $G\mathcal{PL}(x)$ is finite, then the matrix $G\mathcal{PL}_f(x)$ is

$$G\mathcal{PL}_f(x)=\begin{pmatrix} GPL_0(x) & 0 & 0 & \cdots \\ GPL_1(x) & GPL_0(x) & 0 & \cdots \\ GPL_2(x) & GPL_1(x) & GPL_0(x) & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ GPL_n(x) & GPL_{n-1}(x) & \cdots & GPL_0(x) \end{pmatrix},$$

where $detG\mathcal{PL}_f(x)=\left|G\mathcal{PL}_f(x)\right|=(GPL_0(x))^n=(2-2p_1(x)i)^n$.

Now, we give two factorizations of Pascal matrix including the $d-$Gaussian Pell-Lucas Polynomial matrix. We need to find two matrices for these factorizations. Firstly, we define an infinite matrix $C(x)=\frac{1}{2-2p_1(x)i}c_{i,j}(x)$, where

$$c_{i,j}(x)=\binom{i-1}{j-1}-p_1(x)\binom{i-2}{j-1}-p_2(x)\binom{i-3}{j-1}-...-p_{d+1}(x)\binom{i-d-2}{j-1}.$$

So, we obtain

$$C(x)=\begin{pmatrix} \frac{1}{2-2p_1(x)i} & 0 & 0 & 0 & \cdots \\ \frac{1-p_1(x)}{2-2p_1(x)i} & \frac{1}{2-2p_1(x)i} & 0 & 0 & \cdots \\ \frac{1-p_1(x)-p_2(x)}{2-2p_1(x)i} & \frac{2-p_1(x)}{2-2p_1(x)i} & \frac{1}{2-2p_1(x)i} & 0 & \cdots \\ \frac{1-p_1(x)-p_2(x)-p_3(x)}{2-2p_1(x)i} & \frac{3-2p_1(x)-p_2(x)}{2-2p_1(x)i} & \frac{3-p_1(x)}{2-2p_1(x)i} & \frac{1}{2-2p_1(x)i} & \cdots \\ \vdots & \vdots & \frac{6-3p_1(x)-p_2x}{2-2p_1(x)i} & \frac{4-p_1(x)}{2-2p_1(x)i} & \ddots \\ k_1(x) & k_2(x) & & & \\ k_3(x) & k_4(x) & & & \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{3.1}$$

where $k_1(x)=\frac{(1-p_1(x)-p_2(x)-\cdots-p_d(x))}{2-2p_1(x)i}$, $k_2(x)=\frac{(d-(d-1)p_1(x)-(d-2)p_2(x)-\cdots-p_{d-1}(x))}{2-2p_1(x)i}$, $k_3(x)=\frac{(1-p_1(x)-p_2(x)-\cdots-p_{d+1}(x))}{2-2p_1(x)i}$ and $k_4(x)=\frac{((d+1)-dp_1(x)-(d-1)p_2(x)-\cdots-p_d(x))}{2-2p_1(x)i}$.

By using the infinite $d-$Gaussian Pell-Lucas matrix and the infinite matrix $C(x)$ as in (3.1), we can introduce the first factorization of the infinite Pascal matrix with the following theorem.

Theorem 3.2. *The factorization of the infinite Pascal matrix $P(x)$ is as follows;*

$$P(x)=G\mathcal{PL}(x)*C(x).$$

Proof. From the definitions of infinite Pascal matrix and the infinite $d-$Gaussian Pell-Lucas polynomials matrix, we have the following Riordan representing

$$P=\left(\frac{1}{1-t},\frac{t}{1-t}\right),\ G\mathcal{PL}(x)=\left(\frac{2-2p_1(x)\,i}{1-p_1(x)\,t-p_2(x)\,t^2-\cdots-p_{d+1}(x)\,t^{d+1}},t\right).$$

Now, we can obtain the Riordan representation the infinite matrix $C(x)$ as follows;

$$\begin{aligned} C(x) &= \left(g_{C(x)}(t),f_{C(x)}(t)\right)\\ &=\begin{pmatrix} \frac{1}{2-2p_1(x)i} & 0 & 0 & 0 & \cdots\\ \frac{1-p_1(x)}{2-2p_1(x)i} & \frac{1}{2-2p_1(x)i} & 0 & 0 & \cdots\\ \frac{1-p_1(x)-p_2(x)}{2-2p_1(x)i} & \frac{2-p_1(x)}{2-2p_1(x)i} & \frac{1}{2-2p_1(x)i} & 0 & \cdots\\ \frac{1-p_1(x)-p_2(x)-p_3(x)}{2-2p_1(x)i} & \frac{3-2p_1(x)-p_2(x)}{2-2p_1(x)i} & \frac{3-p_1(x)}{2-2p_1(x)i} & \frac{1}{2-2p_1(x)i} & \cdots\\ \vdots & \vdots & \frac{6-3p_1(x)-p_2x}{2-2p_1(x)i} & \frac{4-p_1(x)}{2-2p_1(x)i} & \ddots\\ k_1(x) & k_2(x) & & & \\ k_3(x) & k_4(x) & & & \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \end{aligned}$$

where $k_1(x)=\frac{(1-p_1(x)-p_2(x)-\cdots-p_d(x))}{2-2p_1(x)i}$, $k_2(x)=\frac{(d-(d-1)p_1(x)-(d-2)p_2(x)-\cdots-p_{d-1}(x))}{2-2p_1(x)i}$, $k_3(x)=\frac{(1-p_1(x)-p_2(x)-\cdots-p_{d+1}(x))}{2-2p_1(x)i}$ and $k_4(x)=\frac{((d+1)-dp_1(x)-(d-1)p_2(x)-\cdots-p_d(x))}{2-2p_1(x)i}$.

From the first column of the matrix $C(x)$, we obtain

$$g_{C(x)}(t)=\left(\frac{1}{2-2p_1(x)\,i}\right)\left(\frac{1-p_1(x)\,t-p_2(x)\,t^2-\cdots-p_{d+1}(x)\,t^{d+1}}{1-t}\right).$$

From the rule of the matrix $C(x)$, we have

$$f_{C(x)}(t)=\frac{t}{1-t},$$

$$C(x)=\left(g_{C(x)}(t),f_{C(x)}(t)\right),$$

which completes the proof. □

Now, we define a matrix $D(x)=\frac{1}{2-2p_1(x)i}(d_{i,j}(x))$ as follows;

$$d_{i,j}(x)=\binom{i-1}{j-1}-p_1(x)\binom{i-2}{j}-p_2(x)\binom{i-3}{j+1}-...-p_{d+1}(x)\binom{i-1}{j+d}.$$

We give the infinite matrix $D(x)$ as

$$D(x)=\begin{pmatrix} \frac{1}{2-2p_1(x)i} & 0 & 0 & 0 & \cdots\\ \frac{1-p_1(x)}{2-2p_1(x)i} & \frac{1}{2-2p_1(x)i} & 0 & 0 & \cdots\\ \frac{1-2p_1(x)-p_2(x)}{2-2p_1(x)i} & \frac{2-p_1(x)}{2-2p_1(x)i} & \frac{1}{2-2p_1(x)i} & 0 & \cdots\\ \frac{1-3p_1(x)-3p_2(x)-p_3(x)}{2-2p_1(x)i} & \frac{3-2p_1(x)-p_2(x)}{2-2p_1(x)i} & \frac{3-p_1(x)}{2-2p_1(x)i} & \frac{1}{2-2p_1(x)i} & \cdots\\ \vdots & \vdots & \frac{6-3p_1(x)-p_2x}{2-2p_1(x)i} & \frac{4-p_1(x)}{2-2p_1(x)i} & \ddots\\ l_1(x) & l_2(x) & & & \\ l_3(x) & l_4(x) & & & \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{3.2}$$

where $d_1(x)=\frac{2(1-dp_1(x)-\frac{d(d-1)}{2!}p_2(x)-\cdots-p_d(x))}{i}$, $d_2(x)=\frac{2(d-(d-1)p_1(x)-(d-2)p_2(x)-\cdots-p_{d-1}(x))}{i}$,
$d_3(x)=\frac{2(1-(d+1)p_1(x)-\frac{d(d+1)}{2!}p_2(x)-\cdots-p_d(x))}{i}$ and $d_4(x)=\frac{2((d+1)-dp_1(x)-(d-1)p_2(x)-\cdots-p_d(x))}{i}$.

Now, we present the other factorization of the Pascal matrix including the $d-$Gaussian Pell-Lucas Polynomials matrix.

Corollary 3.3. *The factorization of the infinite Pascal matrix is as follows;*

$$P(x) = G\mathcal{PL}(x) * D(x),$$

where $D(x)$ is the matrix as in (3.2).

Proof. The proof is similar to that of theorem 3.2. □

Now, we can find the inverses of the $d-$Gaussian Pell-Lucas polynomials matrix by using the Riordan representation given matrices as in [17].

Corollary 3.4. *The inverse of Gaussian Pell-Lucas polynomials matrix is given by*

$$G\mathcal{PL}^{-1}(x) = \left(\frac{1 - p_1(x)t - p_2(x)t^2 - \cdots - p_{d+1}(x)t^{d+1}}{2 - 2p_1(x)i}, t\right).$$

4. Conclusions

A new generalization of Gaussian Pell-Lucas polynomials has been introduction and studied. We gave the matrix representations of $d-$Gaussian Pell-Lucas polynomials. Using the Riordan method, we found the factorizations of the Pascal matrix involving these polynomials. Also, we gave the inverse of matrices of these polynomials.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors contributed to the study conception and design. Material preparation, data collection and analysis were performed by Mine UYSAL, Engin ÖZKAN. The first draft of the manuscript was written by Engin ÖZKAN and all authors commented on previous versions of the manuscript. All authors read and approved the published version of the manuscript.

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