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$\alpha-$SASAKIAN, $\beta-$KENMOTSU AND TRANS-SASAKIAN STRUCTURES ON THE TANGENT BUNDLE

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Abstract. This paper consists of two main sections. In the first part, we give some general information about the almost contact manifold, $\alpha-$Sasakian, $\beta-$Kenmotsu and Trans-Sasakian Structures on the manifolds. In the second part, these structures were expressed on the tangent bundle with the help of lifts and the most general forms were tried to be obtained.

1. Introduction

1.1. Lifts of Vector Fields.

Definition 1. *Let M^n be an $n-$dimensional differentiable manifold of class C^∞ and let $T_p(M^n)$ be the tangent space of M^n at a point p of M^n. Then the set [12]*

$$T(M^n) = \underset{p\in M^n}{\cup} T_p(M^n) \tag{1}$$

is called the tangent bundle over the manifold M^n.

For any point $\tilde{p}$ of $T(M^n)$, the correspondence $\tilde{p} \to p$ determines the bundle projection $\pi : T(M^n) \to M^n$, Thus $\pi(\tilde{p}) = p$, where $\pi : T(M^n) \to M^n$ defines the bundle projection of $T(M^n)$ over M^n. The set $\pi^{-1}(p)$ is called the fibre over $p \in M^n$ and M^n the base space.

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1.1.1. *Vertical Lifts.* If f is a function in M^n, we write f^v for the function in $T(M^n)$ obtained by forming the composition of $\pi : T(M^n) \to M^n$ and $f : M^n \to R$, so that

$$f^v = fo\pi. \tag{2}$$

Thus, if a point $\tilde{p} \in \pi^{-1}(U)$ has induced coordinates (x^h, y^h), then

$$f^v(\tilde{p}) = f^v(\sigma, \theta) = fo\pi(\tilde{p}) = f(p) = f(\sigma). \tag{3}$$

Thus the value of $f^v(\tilde{p})$ is constant along each fibre $T_p(M^n)$ and equal to the value $f(p)$. We call f^v the vertical lift of f [12].

Let $\sigma \in \Im_0^1(T(M^n))$ be such that $\sigma f^v = 0$ for all $f \in \Im_0^0(M^n)$. Then we say that σ is a vertical vector field. Let $\begin{bmatrix} \sigma^h \\ \sigma^{\bar{h}} \end{bmatrix}$ be components of σ with respect to the induced coordinates. Then σ is vertical if and only if its components in $\pi^{-1}(U)$ satisfy

$$\begin{bmatrix} \sigma^h \\ \sigma^{\bar{h}} \end{bmatrix} = \begin{bmatrix} 0 \\ \sigma^{\bar{h}} \end{bmatrix}. \tag{4}$$

Suppose that $\sigma \in \Im_0^1(M^n)$, so that is a vector field in M^n. We define a vector field σ^v in $T(M^n)$ by

$$\sigma^v(\iota\ \zeta) = (\zeta\sigma)^v \tag{5}$$

ζ being an arbitrary $1-$form in M^n. We cal σ^v the vertical lift of σ [12].

Let $\zeta \in \Im_1^0(T(M^n))$ be such that $\zeta(\sigma)^v = 0$ for all $\sigma \in \Im_0^1(M^n)$. Then we say that ζ is a vertical $1-$form in $T(M^n)$. We define the vertical lift ζ^v of the $1-$form ζ by

$$\zeta^v = (\zeta_i)^v (dx^i)^v \tag{6}$$

in each open set $\pi^{-1}(U)$, where $(U; x^h)$ is coordinate neighbourhood in M^n and ζ is given by $\zeta = \zeta_i dx^i$ in U. The vertical lift ζ^v with local expression $\zeta = \zeta_i dx^i$ has components of the form

$$\zeta^v : (\zeta^i, 0) \tag{7}$$

with respect to the induced coordinates in $T(M^n)$.

Vertical lift has the following formulas [10, 12]:

$$\begin{aligned} (f\sigma)^v &= f^v\sigma^v,\ I^v\sigma^v = 0,\ \eta^v(\sigma^v) = 0, \\ (f\eta)^v &= f^v\eta^v,\ [\sigma^v, \theta^v] = 0,\ \varphi^v\sigma^v = 0, \\ \sigma^v f^v &= 0,\ \sigma^v f^v = 0 \end{aligned} \tag{8}$$

hold good, where $f \in \Im_0^0(M_n^n)$, $\sigma, \theta \in \Im_0^1(M_n^n)$, $\eta \in \Im_1^0(M_n^n)$, $\varphi \in \Im_1^1(M_n^n)$, $I = id_{M_n^n}$.

1.1.2. *Complete Lifts.* If f is a function in M^n, we write f^c for the function in $T(M^n)$ defined by

$$f^c = \iota(df) \tag{9}$$

and call f^c the comple lift of f. The complete lift f^c has the local expression

$$f^c = y^i \partial_i f = \partial f \tag{10}$$

with respect to the induced coordinates in $T(M^n)$, where ∂f denotes $y^i \partial_i f$.

Suppose that $\sigma \in \Im_0^1(M^n)$. We define a vector field σ^c in $T(M^n)$ by

$$\sigma^c f^c = (\sigma f)^c, \tag{11}$$

f being an arbitrary function in M^n and call σ^c the complete lift of σ in $T(M^n)$ [3, 12]. The complete lift σ^c with components x^h in M^n has components

$$\sigma^c = \begin{pmatrix} \sigma^h \\ \partial \sigma^h \end{pmatrix} \tag{12}$$

with respect to the induced coordinates in $T(M^n)$.

Suppose that $\zeta \in \Im_1^0(M^n)$, then a 1−form ζ^c in $T(M^n)$ defined by

$$\zeta^c(\sigma^c) = (\zeta\sigma)^c \tag{13}$$

σ being an arbitrary vector field in M^n. We call ζ^c the complete lift of ζ. The complete lift ζ^c of ζ with components ζ_i in M^n has components of the form

$$\zeta^c : (\partial \zeta_{i,} \zeta_i) \tag{14}$$

according to the induced coordinates in $T(M^n)$ [3].

$$\begin{aligned} \sigma^c f^v &= (\sigma f)^v,\ \eta^v(\sigma^c) = (\eta(\sigma))^v, \\ (f\sigma)^c &= f^c \sigma^v + f^v \sigma^c = (\sigma f)^c, \\ \sigma^v f^c &= (\sigma f)^v,\ \varphi^v \sigma^c = (\varphi\sigma)^v, \\ \varphi^c \sigma^v &= (\varphi\sigma)^v,\ (\varphi\sigma)^c = \varphi^c \sigma^c, \\ \eta^v(\sigma^c) &= (\eta(\sigma))^c,\ \eta^c(\sigma^v) = (\eta(\sigma))^v, \\ [\sigma^v, \theta^c] &= [\sigma, \theta]^v,\ I^c = I,\ I^v \sigma^c = \sigma^v,\ [\sigma^c, \theta^c] = [\sigma, \theta]^c. \end{aligned} \tag{15}$$

1.2. **Almost Contact Manifolds.** An almost contact manifold is an odd-dimensional C^∞ manifold whose structural group can be reduced to $U(x) \times 1$. This is equivalent to the existence of a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1−form η satisfying $\phi^2 = -I + \eta \otimes \xi$ and $\eta(\xi) = 1$. From these conditions one can deduce that $\phi\xi = 0$ and $\eta \circ \phi = 0$. A Riemannian metric g is compatible with these structure tensors if

$$g(\phi\sigma, \phi\theta) = g(\sigma, \theta) - \eta(\sigma)\eta(\theta) \tag{16}$$

and we refer to an almost contact metric structure (ϕ, ξ, η, g). Note also that $\eta(\sigma) = g(\sigma, \xi)$.

Let M^n be an almost contact manifold and define an almost complex structure J on $M^n \times R$ by

$$J(\sigma, f\frac{d}{dt}) = (\phi\sigma - f\xi, \eta(\sigma)\frac{d}{dt}). \tag{17}$$

A Sasakian manifold is a normal contact metric manifold. It is well known that the Sasakian condition may be expressed as an almost contact metric structure satisfying

$$(\nabla_\sigma \phi)\theta = g(\sigma, \theta)\xi - \eta(\theta)\sigma, \tag{18}$$

again see e.g. [1].

2. $\alpha-$Sasakian and $\beta-$Kenmotsu structures on the tangent bundle

A $\alpha-$Sasakian structure [6] which may be defined by the requirement

$$(\nabla_\sigma \phi)\theta = \alpha(g(\sigma, \theta)\xi - \eta(\theta)\sigma), \tag{19}$$

where α is a non-zero constant. Setting $\theta = \xi$ in this formula, one readily obtains

$$\nabla_\sigma \xi = -\alpha\phi\sigma \tag{20}$$

Theorem 1. *Let a vector field ξ, ϕ be a tensor field of type $(1,1)$, $1-$form η satisfying $\phi^2 = -I + \eta \otimes \xi$ i.e. $\eta(\xi) = 1$, $\phi\xi = 0$ and $\eta \circ \phi = 0$. A $\alpha-$Sasakian structure on tangent bundle defined by*

$$(\nabla^c_{\sigma^c}\phi^c)\theta^c = \alpha((g(\sigma,\theta))^V\xi^C + (g(\sigma,\theta))^C\xi^V - (\eta(\theta))^C\sigma^V - (\eta(\theta))^V\sigma^C),$$

where g is a Riemannian metric, α is a non-zero constant. In addition, if we put $\theta = \xi$, we get

$$\nabla^C_{\sigma^C}\xi^C = -\alpha\phi^C\sigma^C.$$

Proof. From (19) , we get the $\alpha-$Sasakian structure on the bundle

$$\begin{aligned}
(\nabla^c_{\sigma^c}\phi^c)\theta^c &= \nabla^C_{\sigma^C}\phi^c\theta^C - \phi^C\,\nabla^C_{\sigma^C}\,\theta^C \\
&= \alpha((g(\sigma,\theta))^V\xi^C + (g(\sigma,\theta))^C\xi^V - (\eta(\theta))^C\sigma^V - (\eta(\theta))^V\sigma^C).
\end{aligned}$$

If we put $\theta = \xi$, we get

$$\begin{aligned}
(\nabla^C_{\sigma^C}\phi^C)\xi^C &= \nabla^C_{\sigma^C}\phi^C\xi^C - \phi^C\,\nabla^C_{\sigma^C}\,\xi^C \\
&= -\phi^C\,\nabla^C_{\sigma^C}\,\xi^C \\
&= \alpha(\eta(\sigma)^V\xi^C + (\eta(\sigma))^C\xi^V - (\eta(\xi))^C\sigma^V - (\eta(\xi)^V\sigma^C) \\
&= \alpha((\eta(\sigma))^V\xi^C + (\eta(\sigma))^C\xi^V - \sigma^C) \\
-\phi^C\,\nabla^C_{\sigma^C}\,\xi^C &= \alpha(\phi^C)^2\sigma^C \\
\nabla^C_{\sigma^C}\xi^C &= -\alpha\phi^C\sigma^C
\end{aligned}$$

□

In particular the almost contact metric structure in this case satisfies

$$(\bigtriangledown_{\sigma}\phi)\theta = g(\phi\sigma, \theta)\xi - \eta(\theta)\phi\sigma \tag{21}$$

and an almost contact metric manifold satisfying this condition is called a Kenmotsu manifold [6,7]. Again one has the more general notion of a $\beta-$Kenmotsu structure [6] which may be defined by

$$(\bigtriangledown_{\sigma}\phi)\theta = \beta(g(\phi\sigma, \theta)\xi - \eta(\theta)\phi\sigma), \tag{22}$$

where β is a non-zero constant. From the condition one may readily deduce that

$$\bigtriangledown_{\sigma}\xi = \beta(\sigma - \eta(\sigma)\xi). \tag{23}$$

Theorem 2. *Let ϕ be a tensor field of type $(1,1)$, a vector field ξ, $1-$form η satisfying $\phi^2 = -I + \eta \otimes \xi$ i.e. $\eta(\xi) = 1$, $\phi\xi = 0$ and $\eta \circ \phi = 0$. A $\beta-$Kenmotsu structure on tangent bundle defined by*

$$((\bigtriangledown_{\sigma}\phi)\theta)^C = \beta((g(\phi\sigma,\theta))^V\xi^C + (g(\phi\sigma,\theta))^C\xi^V - (\eta(\theta))^C(\phi\sigma)^V - (\eta(\theta))^V(\phi\sigma)^C),$$

where g is a Riemannian metric, β is a non-zero constant. In addition, if we put $\theta = \xi$, we get

$$\bigtriangledown^C_{\sigma^C}\xi^C = \beta(\sigma^C - ((\eta(\sigma))\xi)^C). \tag{24}$$

Proof. From (22) , we get the $\beta-$Kenmotsu structure on the bundle

$$\begin{aligned} ((\bigtriangledown_{\sigma}\phi)\theta)^C &= \bigtriangledown^C_{\sigma^C}\phi^C\theta^C - \phi^C\bigtriangledown^C_{\sigma^C}\theta^C \\ &= \beta((g(\phi\sigma,\theta))^V\xi^C + (g(\phi\sigma,\theta))^C\xi^V - (\eta(\theta))^C(\phi\sigma)^V - (\eta(\theta))^V(\phi\sigma)^C) \end{aligned}$$

If we put $\theta = \xi$, we get

$$\begin{aligned} -\phi^C\bigtriangledown^C_{\sigma^C}\xi^C &= \beta((\eta(\phi\sigma))^V\xi^C + (\eta(\phi\sigma))^C\xi^V - (\eta(\xi))^C(\phi\sigma)^V - (\eta(\xi))^V(\phi\sigma)^C) \\ -\phi^C\bigtriangledown^C_{\sigma^C}\xi^C &= \beta(-(\phi\sigma)^C + (\eta(\phi\sigma))^V\xi^C + (\eta(\phi\sigma))^C\xi^V) \\ \phi^2\bigtriangledown^C_{\sigma^C}\xi^C &= \beta((\phi\sigma)^C - (\eta(\phi\sigma))^V\xi^C - (\eta(\phi\sigma))^C\xi^V) \\ \phi\bigtriangledown^C_{\sigma^C}\xi^C &= \beta(\phi^C\sigma^C - \eta^V(\phi^C\sigma^C)\xi^C - \eta^C(\phi^C\sigma^C)\xi^V) \\ \bigtriangledown^C_{\sigma^C}\xi^C &= \beta(\sigma^C - (\eta^V\sigma^C)\xi^C - (\eta^C\sigma^C)\xi^V) \\ \bigtriangledown^C_{\sigma^C}\xi^C &= \beta(\sigma^C - (\eta\sigma)^V\xi^C - (\eta\sigma)^C\xi^V) \\ \bigtriangledown^C_{\sigma^C}\xi^C &= \beta(\sigma^C - ((\eta(\sigma))\xi)^C) \end{aligned}$$

□

3. Trans-Sasakian manifolds on the tangent bundle

An almost contact metric structure (ϕ, ξ, η, g) on M^n is trans-Sasakian [9] if $(M^n \times R, J, G)$ belongs to the class W_4, where J is the almost complex structure on $M^n \times R$ defined by (17) and G is the product metric on $M^n \times R$. This expressed by the condition

$$(\bigtriangledown_{\sigma}\phi)\theta = \alpha(g(\sigma,\theta)\xi - \eta(\theta)\sigma) + \beta(g(\phi\sigma,\theta)\xi - \eta(\theta)\phi\sigma) \tag{25}$$

for functions α and β on M^n, and we shall say that the trans-Sasakian structure is of type (α, β); in particular, it is normal and it generalizes both $\alpha - Sasakian$ and $\beta - Kenmotsu$ structures. From the formula one obtain

$$\triangledown_\sigma \xi = -\alpha\phi\sigma + \beta(\sigma - \eta(\sigma)\xi), \tag{26}$$

$$(\triangledown_\sigma \eta)(\theta) = -\alpha g(\phi\sigma, \theta) + \beta(g(\sigma, \theta) - \eta(\sigma)\eta(\theta)), \tag{27}$$

$$(\triangledown_\sigma \phi)(\theta, Z) = \alpha(g(\sigma, Z)\eta(\theta) - g(\sigma, \theta)\eta(Z)) - \beta(g(\sigma, \phi Z)\eta(\theta) - g(\sigma, \phi\theta)\eta(Z)), \tag{28}$$

where ϕ is the fundamental $2-$form of the structure, given by $\phi(\sigma, \theta) = g(\sigma, \phi\theta)$.

Theorem 3. *Let ϕ be a tensor field of type $(1,1)$, a vector field ξ, $1-$form η satisfying $\phi^2 = -I + \eta \otimes \xi$ i.e. $\eta(\xi) = 1$, $\phi\xi = 0$ and $\eta \circ \phi = 0$. A trans-Sasakian structure on tangent bundle defined by*

$$(\triangledown^C_{\sigma^C} \phi^C)\theta^C = \alpha((g(\sigma, \theta))^V \xi^C + (g(\sigma, \theta))^C \xi^V - (\eta(\theta))^C \sigma^V - (\eta(\theta))^V \sigma^C)$$
$$+\beta((g(\phi\sigma, \theta))^V \xi^C + (g(\phi\sigma, \theta))^C \xi^V - (\eta(\theta))^C (\phi\sigma)^V - (\eta(\theta))^V (\phi\sigma)^C),$$

where g is a Riemannian metric, α, β are non-zero constants. In addition, if we put $\theta = \xi$, we get

$$\triangledown^C_{\sigma^C} \xi^C = -\alpha\phi^C\sigma^C + \beta(\sigma^C - ((\eta(\sigma))\xi)^C)$$

Proof. From (25), we get the trans-Sasakian structure on the bundle

$$(\triangledown^C_{\sigma^C} \phi^C)\theta^C = \triangledown^C_{\sigma^C} \phi^C\theta^C - \phi^C \triangledown^C_{\sigma^C} \theta^C$$
$$= \alpha((g(\sigma, \theta))^V \xi^C + (g(\sigma, \theta))^C \xi^V - (\eta(\theta))^C \sigma^V - (\eta(\theta))^V \sigma^C)$$
$$+\beta((g(\phi\sigma, \theta))^V \xi^C + (g(\phi\sigma, \theta))^C \xi^V - (\eta(\theta))^C (\phi\sigma)^V - (\eta(\theta))^V (\phi\sigma)^C).$$

If we put $\theta = \xi$ and using the formulas of (8),(15), similarly we get

$$\triangledown^C_{\sigma^C} \xi^C = -\alpha\phi^C\sigma^C + \beta(\sigma^C - ((\eta(\sigma))\xi)^C).$$

□

Theorem 4. *Let a vector field ξ, ϕ be a tensor field of type $(1,1)$, $1-$form η satisfying $\phi^2 = -I + \eta \otimes \xi$ i.e. $\eta(\xi) = 1$, $\phi\xi = 0$ and $\eta \circ \phi = 0$. The term $(\triangledown^C_{\sigma^C} \eta^C)\theta^C$ in a trans-Sasakian structure on tangent bundle defined by*

$$(\triangledown^C_{\sigma^C} \eta^C)\theta^C = -\alpha g^C(\phi^C\sigma^C, \theta^C) + \beta g^C(\phi^C\sigma^C, \phi^C\theta^C),$$

where g is a Riemannian metric, α, β is a non-zero constant.

Proof. From (27) , we get

$$\begin{aligned}
(\triangledown^C_{\sigma^C} \eta^C)\theta^C &= \triangledown^C_{\sigma^C} \eta^C\theta^C - \eta^C \triangledown^C_{\sigma^C} \theta^C \\
&= \triangledown^C_{\sigma^C} (g(\theta, \xi))^C - (g(\triangledown_\sigma \theta, \xi))^C \\
&= \triangledown^C_{\sigma^C} g^C)(\theta^C, \xi^C) + g^C(\triangledown^C_{\sigma^C} \theta^C, \xi^C) + g^C(\theta^C, \triangledown^C_{\sigma^C} \xi^C) \\
&\quad -g^C(\triangledown^C_{\sigma^C} \theta^C, \xi^C)
\end{aligned}$$

$$\begin{aligned}
&= g^C(\theta^C, \nabla^C_{\sigma^C}\xi^C) = g^C(\theta^C, -\alpha\phi^C\sigma^C + \beta(\sigma^C((\eta(\sigma))\xi)^C) \\
&= -\alpha g^C(\theta^C, \phi^C\sigma^C) + \beta g^C(\theta^C, \sigma^C - ((\eta(\sigma))\xi)^C) \\
&= -\alpha g^C(\phi^C\sigma^C, \theta^C) + \beta g^C(\theta^C, \sigma^C - (\eta(\sigma))^V\xi^C - (\eta(\sigma))^C\xi^V) \\
&= -\alpha g^C(\phi^C\sigma^C, \theta^C) + \beta(g^C(\theta^C, \sigma^C) - (\eta(\sigma))^V g^C(\theta^C, \xi^C) \\
&\quad -(\eta(\sigma))^C g^C(\theta^C, \xi^V)) \\
&= -\alpha g^C(\phi^C\sigma^C, \theta^C) + \beta g^C(\phi^C\sigma^C, \phi^C\theta^C),
\end{aligned}$$

where $g^C(\theta^C, \xi^V) = (\eta(\theta))^V$ and $g^C(\phi^C\sigma^C, \phi^C\theta^C) = g^C(\sigma^C, \theta^C) - (\eta(\sigma))^C(\eta(\theta))^V - (\eta(\sigma))^V(\eta(\theta))^C$. □

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