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Hurewicz and Poincaré Theorems for Simplicial Modules

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Abstract

We will give the simplicial analogues of Hurewicz and Poincaré theorems as an application of simplicial homology and homotopy.

1. Introduction

Let X be a topological space. Then we have a singular simplicial set. $C_*(X)$which is chain complex is obtained with its singular homology $H_*(X;\mathbb{Z})$. Any singular homology of X can be get from $S_*(X)$. So the concept of simplicial sets was defined as combinatorial models of spaces. In the following diagram, one can see relations among simplicial sets and spaces:

$$\begin{array}{ccc} \boxed{\text{Simplical sets}} & \xrightarrow{|.|} & \boxed{\text{CW- complexes}} \\ S_* \circ |.| \downarrow & & \downarrow |.| \circ S_* \\ \boxed{\text{Simplical sets}} & \xleftarrow{S_*} & \boxed{\text{Topological spaces}} \end{array}$$

In [1], J.C. Moore defined simplicial groups. Author also gave the isomorphism

$$\pi_*(|\mathscr{G}|) \cong H_*(N\mathscr{G}),$$

where $N\mathscr{G}$ is Moore chain complex of and$|\mathscr{G}|$ is geometrical realization of $\mathscr{G}$. J.W. Milnor [2] shown that a loop space is homotopy equivalent to the geometric realization of any simplicial group. Hence, the homotopy groups of any space is defined as the homology of a Moore chain complex.

The simplicial modules and simplicial algebras are developed by M. André [3] and D. Quillen [4]. They constructed ways of building simplicial resolutions of algebras and defined a homology and cohomology of commutative algebras. Also Z. Arvasi [5, 6], analyses the Higher order Peiffer elements of simplicial algebras.

In this work, firstly we will give some preliminaries for simplicial modules and their homology and homotopy. Then we will proof the main theorem called as Hurewicz Theorem and also its corollary called as Poincaré Theorem. These theorems are applications for homology and homotopy of simplicial modules.

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2. Δ-sets

For details of this section you can see [7].

A Δ-*set*, $X = \{X_n\}_{n\geq 0}$, is a sequence with the maps, $d_i : X_n \to X_{n-1}$, satisfying the Δ-identity:

$$d_i d_j = d_j d_{i+1}$$

for $i \geq j$ and $0 \leq i \leq n$.

Remark: We can write Δ-idenitity:

$$d_i : (r_0, r_1, \cdots, r_n) \mapsto (r_0, \cdots, r_{i-1}, r_{i+1}, \cdots r_n).$$

A Δ-set $M = \{M_n\}_{n\geq 0}$ is defined a Δ-module satisfying M_n is a module, and d_i is a module homomorphism. Given any category $\mathscr{C}$, a Δ-object is a sequence of objects in $\mathscr{C}$, with faces as morphisms in $\mathscr{C}$.

We have a category $\mathscr{O}^+$ with the objects which are finite order sets, morphisms which are monoton functions. We can write the objects as $n \geq 0, [n] = \{0, 1, \cdots, n\}$, and the morphisms generated by $d^i : [n-1] \to [n]$ such that

$$d^i(j) = \begin{cases} j & j < i \\ j+1 & j \geq i\,, \\ \end{cases}$$

$0 \leq i \leq n$.

Corollary 2.1. Δ-*sets has one to one correspondence to contravarient fuctors from* $\mathscr{O}^+$ *to S.*

A Δ-*map* is a sequence of $f := f_n(X_n \to Y_n)$ satisfying the following commutative diagram, that is $n \geq 0$, $f_0 d_i = d_i f$.

$$\begin{array}{ccc} X_n & \xrightarrow{f} & Y_n \\ \downarrow & & \downarrow \\ X_{n-1} & \xrightarrow[f]{} & Y_{n-1} \end{array}$$

A Δ-*subset* of X is any sequece of $Y_n \subseteq X_n$ satisfying

$$d_i(Y_n) \subseteq Y_{n-1}$$

where X is a Δ-set, $0 \leq i \leq n < \infty$. Suppose X and Y are Δ-set. If there exists a bijective Δ-map between X and Y, then X is isomorphic to Y.

2.1. Geometric realization of Δ-sets

Suppose A is a Δ-set. A geometrical realization of A, $|A|$, is determined as

$$|A| = \coprod_{\substack{x \in A_n \\ n \geq 0}} (\Delta^n, x)/\sim = \coprod_{n=0}^{\infty} \Delta^n \times A_n/\sim$$

where $\sim$ is obtained by $(z, d_i x) \sim (d^i z, x)$ for $x \in A_n$, $z \in \Delta^{n-1}$ is labeled by $d_i x$.

2.2. Homology of Δ-sets

It is well known that a chain complex is a collection of $C = \{C_n\}$ with diferantial $\partial_n : C_n \to C_{n-1}$ which is satisfy $\mathrm{Im}(\partial_{n+1}) \subseteq Ker(\partial_n)$, namely $\partial_n\ \partial_{n+1}$ is trivial. Then the homology can be defined as

$$H_n(C) = Ker(\partial_n)/Im(\partial_{n+1}).$$

Proposition 2.2 ([7])**.** *For a* Δ-*abelian group G, G is a chain complex with* ∂_* *where*

$$\partial_n : \sum_{i=0}^{n} (-1)^i d_i : G_n \longrightarrow G_{n-1}.$$

Proof. We must show that $\partial_n \circ \partial_{n+1}$ is trivial.

$$\begin{aligned}
\partial_{n-1}\circ\partial_n &= \sum_{i=0}^{n-1}(-1)^i d_i \sum_{j=0}^{n}(-1)^j d_j, \\
&= \sum_{0\le i<j\le n}^{n}(-1)^{i+j}d_i d_j + \sum_{0\le j<i\le n-1}^{n}(-1)^{i+j}d_i d_j, \\
&= \sum_{0\le i<j\le n}^{n}(-1)^{i+j}d_i d_j + \sum_{0\le j<i+1\le n}^{n}(-1)^{i+j}d_j d_{i+1}, \\
&= \sum_{0\le i<j\le n}^{n}(-1)^{i+j}d_i d_j + \sum_{0\le i<j<n}^{n}(-1)^{i+j-1}d_j d_i, \\
&= 0.
\end{aligned}$$

□

For a given Δ-set X , the homology $H_*(X;G)$ of X with coefficients in an abelian group G can be defined as

$$H_*(X;G) = H_*(\mathbb{Z}(X)\otimes G, \partial_*).$$

Here $\mathbb{Z}(X_n)$ is a free abelian group with generator X_n, $\mathbb{Z}(X) = \{\mathbb{Z}(X_n)\}_{n\ge 0}$.

3. Simplicial modules

Let R be a fixed commutative ring. Fore more details about simplicial modules and algebras, we refer to [5, 6], [8]-[10].

A *simplicial R-module* (shotrly simplicial module) is a Δ-module M with degeneracies and faces satisfying the following identities:

$$d_j d_i = d_{i-1}d_j, \text{ for } j<i,$$

$$s_j s_i = s_{i+1}s_j, \text{ for } j\le i,$$

also

$$d_j s_i = \begin{cases} s_{i-1}d_j & j<i \\ id & j=i,i+1 \\ s_i d_{j-1} & j>i+1. \end{cases}$$

These are defined as simplicial identities.

A *simplicial module homomorphism* $f : M \to M'$is a sequence of module homomorphisms $f_n : M_n \to M'_n$ $(n\ge 0)$ satisfying the following commutative diagram, i.e $f_{n-1}\ d_i = d_i f_n$ and $f_n s_i = s_i f_{n+1}$:

$$\begin{array}{ccccc}
M_{n+1} & \xleftarrow{s_i} & M_n & \xrightarrow{d_i} & M_{n-1} \\
f_{n+1}\downarrow & & f_n\downarrow & & \downarrow f_{n-1} \\
M'_{n+1} & \xleftarrow[s_i]{} & M'_n & \xrightarrow[d_i]{} & M'_{n-1}
\end{array}$$

M is defined as *simplicial submodule* of M' if each M_n is a submodule of M'_n. A simplicial module M is said to be *isomorphic* to a simplicial module M', if a bijective simplicial module homomorphism $f : M \to M'$exists.

3.1. Geometric realization of simplicial modules

The standart n-simplex Δ^n is

$$\Delta^n = \{(r_0, r_1, \cdots r_n) \mid r_i \ge 0 \text{ and } \sum_{i=0}^{n} r_i = 1\}$$

where $d^i : \Delta^{n-1} \longrightarrow \Delta^n$ and $s^i : \Delta^{n+1} \longrightarrow \Delta^n$ are given as

$$\begin{aligned}
d^i(r_0, r_1, \cdots, r_{n-1}) &= (r_0, \cdots, r_{i-1}, 0, r_i, \cdots, r_{n-1}), \\
s^i(r_0, r_1, \cdots, t_{n+1}) &= (r_0, \cdots, r_{i-1}, r_i + r_{i+1}, \cdots, r_{n+1}),
\end{aligned}$$

where $0 \leq i \leq n$.

Suppose M is a simplicial module. Its geometric realization $|M|$ is a CW-complex such that

$$|M| = \bigsqcup_{\substack{x \in M_n \\ n \geq 0}} (\Delta^n, x)/\sim = \bigsqcup_{n=0}^{\infty} (\Delta^n \times M_n)/\sim .$$

Here (Δ^n, x) is Δ^n associated with $x \in M_n$, $\sim$ is generated by

$$(z, d_i x) \sim (d^i z, x)$$

$x \in M_n$, $z \in \Delta^{n-1}$ associated with $d_i x$,

$$(z, s_i x) \sim (s^i z, x)$$

$x \in M_n$, $z \in \Delta^{n+1}$ associated with $s_i x$.

3.2. Homotopy and fibrant simplicial modules

Suppose $f, g : M \to N$ are simplicial module homomorphisms. If we have a simplicial module homomorphism $F : M \times I \to N$ satisfying $F|_{M\times 0} = f$, $F|_{M\times 1} = g$, then we can say that f *homotopic* to g and can be written as $f \simeq g$. Suppose that X is any simplicial submodule of M, $f; g : M \to N$ are simplicial module homomorphisms satisfying$f|_X = g|_X$. If we have a homotopy $F : M \times I \to N$ satisfying $F|_{M\times 0} = f$ and $F|_{M\times 1} = g$ and $F|_{X\times I} = f$, then we say that f homotopic to g relative to X, and can be shown as $f \simeq g$ rel X.

The image of $f_{x_0} : \Delta[0] \to M$ is a simplicial submodule of M which has only element $f_{x_0}(0, 0, \cdots, 0) = S_I(x_0)$ for each dimension where M is any simplicial module and $x_0 \in M_0$. So a basepoint $*$ of M is a sequence of $\{f_{x_0}(\overbrace{0, 0 \cdots, 0}^{n+1})\}_{n\geq 0}$ correspond to $x_0 \in M_0$.

A *pointed simplicial module* is a simplicial module with basepoint. Suppose M and N are pointed simplicial modules. A *pointed simplicial module homomorphism* $f : M \to N$ is a simplicial module homomorphism which preserve the basepoints. We usually use $*$ for defining the basepoint.

For given pointed simplicial module homomorphisms $f, g : M \to N$, pointed homotopy means that f and g are homotopic rel $*$.

We should assume that there is a homotopy relation $\simeq$ on the module of simplicial module homomorphisms from M to $N$where N is a fibrant simplicial module. So, we will define fibrant simplicial module.

Given a simplicial module M, if $d_j x_k = d_k x_{j+1}$, where $j \geq k$; $k, j+1 \neq i$, then the elements $a_0, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n \in M_{n-1}$ are called matching faces w.r.t i.

If the simplicial module M provides the homotopy extension condition, then it is called *fibrant*. Suppose the elements $a_0, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n \in M_{n-1}$ are matching faces w.r.t i, we have an element $w \in M_n$ such that $d_j w = a_j$ for $j \neq i$. This condition is called homotopy extension condition.

3.3. Homotopy modules

The homotopy module $\pi_n(M)$ is defined by

$$\pi_n(M) = [S^n, M]$$

and so $\pi_n(M) = \pi_n(|M|)$ where M is a pointed fibrant simplicial module.

An element $x \in M_n$ satisfying the condition $d_i x = *$ for all $0 \leq i \leq n$, is named *spherical*. For a spherical element $x \in M_n$, the map $f_x : \Delta[n] \to M$ sends to quotient simplicial module $S^n = \Delta[n]/\partial\Delta[n]$. In contrast, a simplicial map $f : S^n \to M$ gets a spherical element $f(\sigma_n) \in M_n$, where σ_n is a nondegenerate element in S^n. So we have one to one correspondence such as

$$\boxed{\text{Spherical element in } M_n} \longleftrightarrow \boxed{\text{Simplicial module homomorphism } S^n \longrightarrow M.}$$

Theorem 3.1. *(Homotopy Addition Theorem) For pointed fibrant simplicial module M and spherical elements $x_i \in M_n$, the equation in $\pi_n(M)$*

$$[x_0]-[x_1]+[x_2]\cdots+(-1)^{n+1}[x_{n+1}]=0$$

satisfies if and only if there is $x \in K_{n+1}$ such that $d_i x = x_i$ where $0 \leq i \leq n+1$.

Proof. See [7], for details. □

Suppose M is a fibrant simplicial module. If f_x, f_y are homotopic relative to $\partial\Delta[n]$, then $x,y \in M_n$ is $x \simeq y$. So a fibrant simplicial module M is named *minimal* if it satisfies $x \simeq y \Rightarrow x = y$,

3.4. Homology of simplicial modules

For a simplicial module M, we define

$$N_nM = \bigcap_{j=1} Ker(d_j : M_n \longrightarrow M_{n-1}),$$

such that $x \in N_nM$, i.e $x \in M_n$ such that $d_j x = 1$ for $j > 0$. That is,

$$d_k(d_0x) = d_0d_{k+1}x = 1$$

for any $0 \leq k \leq n-1$.

A *chain complex* (C,∂) consists of modules and module homomorphisms

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

satisfying $\mathrm{Im}(\partial_{n+1}) \subseteq Ker(\partial_n)$, i.e $\partial_n \circ \partial_{n+1}$ is trivial. The *homology* $H_n(C,\partial)$ is written as $Ker(\partial_n)/\mathrm{Im}(\partial_{n+1})$.

Proposition 3.2. *Given a simplicial module M, if*

$$\partial_n =: \sum_{i=0}^{n}(-1)^i d_i : M_n \longrightarrow M_{n-1},$$

then $\partial_{n-1} \circ \partial_n = 0$, i.e M is a chain complex.

Proof. Similar to proposition 2.2. □

Remark The homology $H_*(M;A)$ of M with coefficients in a $\mathbb{Z}$-module A is defined by

$$H_*(M;A) = H_*(\mathbb{Z}(M) \otimes_{\mathbb{Z}} A, \partial_*),$$

where M is a simplicial module, $\mathbb{Z}(M) = \{\mathbb{Z}(M_n)\}_{n\geq 0}$ and $\mathbb{Z}(M_n)$ is the free $\mathbb{Z}$-module generated by M_n.

The *Moore chain complex* of simplicial R-module M, denoted NM, is the sequence of R-modules

$$\cdots \longrightarrow N_{n+1}M \xrightarrow{d_0} N_nM \xrightarrow{d_0} N_{n-1}M \longrightarrow \cdots$$

The elements in Z_nM, are called Moore *cycles* and the elements B_nM are called Moore *boundaries*.
By definition,

$$\begin{aligned} H_n(NM,d_0) &= Ker(d_0)/d_0(N_{n+1}M), \\ &= \bigcap_{j=0}^{n} Ker(d_j)/B_nM, \\ &= Z_nM/B_nM \\ &= \pi_n(M). \end{aligned}$$

So, for the simplicial module M, we can write

$$H_n(NM,d_0) \cong \pi_n(M) \cong \pi_n(|M|).$$

The significiance of this corollary is that the homology modules can be defined as the homology of chain complex.

4. Hurewicz and Poincaré theorems for simplicial modules

Suppose M is a simplicial module and $\mathbb{Z}(M)=\{\mathbb{Z}(M_n)\}_{n\geq 0}$ is a sequence of the free $\mathbb{Z}$-module generated by M_n. By using $d_i : M_n \longrightarrow M_{n-1}$, $s_i : M_n \longrightarrow M_{n+1}$, we can write

$$d_i^{\mathbb{Z}(M)} : \mathbb{Z}(M_n) \longrightarrow \mathbb{Z}(M_{n-1})$$

and the degeneracies

$$s_i^{\mathbb{Z}(M)} : \mathbb{Z}(M_n) \longrightarrow \mathbb{Z}(M_{n+1}).$$

Hence $\mathbb{Z}(M)$ is a simplicial $\mathbb{Z}$-module. The homology of M is defined by

$$H_*(M)=H_*(\mathbb{Z}(M))\cong H_*(\mathbb{Z}(M),\partial).$$

Clearly, a simplicial module homomorphism $f : M \longrightarrow M'$ induces a simplicial $\mathbb{Z}$-module morphism $\mathbb{Z}(f) : \mathbb{Z}(M) \longrightarrow \mathbb{Z}(M')$. So we have a functor such that $M \longmapsto \mathbb{Z}(M)$, $f \longmapsto \mathbb{Z}(f)$. If $f \simeq g : M \longrightarrow M'$ (suppose that M' is fibrant), we have $\mathbb{Z}(f)\simeq\mathbb{Z}(g) : \mathbb{Z}(M) \longrightarrow \mathbb{Z}(M')$. Hence if $M\simeq M'$ with M and M' fibrant, we get $\mathbb{Z}(M)\simeq\mathbb{Z}(M')$ and so $H_*(M)\cong H_*(M')$.

As the geometric realization of any simplicial module is Δ-complex, the homology $H_*(M)=H_*(|M|)$ is the simplicial homology of the Δ-complex $|M|$. So, if $|M|\simeq|M'|$, then $H_*(M)\cong H_*(|M'|)$.

Thus the homology $H_*(M;A)$ with coefficients in A is defined by

$$H_*(M;A)=\pi_*(\mathbb{Z}(M)\otimes_{\mathbb{Z}}A)\cong H_*(\mathbb{Z}(M)\otimes_{\mathbb{Z}}A,\partial),$$

where A is a free $\mathbb{Z}$-module, $\mathbb{Z}(M)\otimes_{\mathbb{Z}}A=\{\mathbb{Z}(M)\otimes_{\mathbb{Z}}A\}_{n\geq 0}$.

As using redued homology, one can obtain a single relation on $\mathbb{Z}(M)$. Suppose $\mathbb{Z}[M]$ is the quotient $\mathbb{Z}$-module of $\mathbb{Z}(M)$ the simplicial submodule with the basepoint $*$. So the reduced integral homology $\overline{H_*}(M)$can be defined as

$$\overline{H_*}(M)=\pi_*(\mathbb{Z}[M])\cong H_*(\mathbb{Z}[M],\partial).$$

The reuced homology with coefficients in A is defined by

$$\overline{H_*}(M;A)=\pi_*(\mathbb{Z}[M]\otimes_{\mathbb{Z}}A)\cong H_*(\mathbb{Z}[M]\otimes_{\mathbb{Z}}A,\partial).$$

The inclusion $j : M \hookrightarrow \mathbb{Z}(M)$ is a simplicial module homomorphism and the composite

$$\overline{j} : M \hookrightarrow \mathbb{Z}(M) \twoheadrightarrow \mathbb{Z}[M]$$

is pointed simplicial module homomorphism. (Note that the basepoint in M is $*$ and the basepoint in $\mathbb{Z}(M)$ is 0.) The map $\overline{j}$ induces a $\mathbb{Z}$-module homomorphism

$$h_n=\overline{j_*} : \pi_n(M) \longrightarrow \pi_n(\mathbb{Z}[M])=\overline{H_n}(M)$$

where M is a fibrant simplicial module and $n\geq 1$, then this homomorphism is called *Hurewicz homomorphism*.

Theorem 4.1. *(Hurewicz Theorem) Suppose M is any fibrant simplicial module with $\pi_i(M)=0$ for $i<n$ with $n\geq 2$. Then $\overline{H_i}(M)=0$ for $i<n$ and $h_n : \pi_n(M) \longrightarrow \overline{H_n}(M)$ is an isomorphism.*

Proof. Suppose M is a minimal simplicial module. Think that

$$\overline{j} : M \longrightarrow \mathbb{Z}[M].$$

We can write $M_q=*$ for $q<n$ and $M_n=\pi_n(M)$, since M is minimal. Hence $\mathbb{Z}[M]_q=\{0\}$ for $q<n$ and $\mathbb{Z}[M]_n=\mathbb{Z}[M_n]$.

(1). h_n is onto: As the following diagram is commutative

$$\begin{array}{ccc} M_n & \hookrightarrow & \mathbb{Z}[M_n] \\ \| & & \downarrow \\ \pi_n(M) & \longrightarrow & \overline{H_n}(M) \end{array}$$

$\overline{H_n}(M)$ is generated by M_n as a $\mathbb{Z}$-module. As h_n is a $\mathbb{Z}$-module homomorphism and every generator of $\overline{H_n}(M)$ is in its image, it should be onto.

(2). $Ker(h_n) = \{0\}$: Assume that $x \in Ker(h_n)$. As x is 0 in

$$\overline{H_n}(M) = H_n(\mathbb{Z}[M], \partial),$$

we have an element $c \in \mathbb{Z}[M]_{n+1}$ such that $\partial(y) = x$ in $\mathbb{Z}[M]_n$. Let $y = \sum_{j=1}^{t} n_j y_j$ with $n_j \in \mathbb{Z}$ and $y_j \in M_{n+1}$. Then $\phi : \mathbb{Z}[M_n] \longrightarrow \pi_n(M)$ is the $\mathbb{Z}$-module homomorphism such that $\phi|_{M_n} : M_n \longrightarrow \pi_n(M) = M_n$ is the identity map seen as the commutative diagram

$$\begin{array}{ccccc} \mathbb{Z}[M_{n+1}] & \longrightarrow & \mathbb{Z}[M_n] & & \\ \uparrow & & \uparrow & \searrow & \\ M_{n+1} & & M_n & = & \pi_n(M). \end{array}$$

For $y_j \in M_{n+1}$, we get

$$\begin{aligned} \phi \circ \partial(y_j) &= \phi(\sum_{i=0}^{n+1} (-1)^i d_i y_j \\ &= \sum_{i=0}^{n+1} (-1)^i \phi(d_i y_j) \\ &= \sum_{i=0}^{n+1} (-1)^i d_i y_j \ (\because d_i y_j \in M_n) \end{aligned}$$

in $\pi_n(M)$. By using Homotopy Addition Theorem, we can get $\phi(\partial(y_j)) = 0$ for each j. So

$$x = \phi \overline{j_n}(x) = \phi(\partial(y)) = 0$$

in $\pi_n(M)$ and i.e $Ker(h_n) = \{0\}$. □

Theorem 4.2. *(Poincaré Theorem) Suppose M is a connected (that is $\pi_0(|M|) = 0$) fibrant simplicial module. Then there is an isomorphism*

$$h' : \pi_1(M)/[\pi_1(M), \pi_1(M)] \longrightarrow \overline{H_1}(M)$$

induced by $h_1 : \pi_1(M) \longrightarrow \overline{H_1}(M)$.

Proof. Assume that M is a minimal simplicial module. By similar way, one can show that h' is onto. To proof that that h' is one to one, let $\phi : \mathbb{Z}[M_1] \longrightarrow \pi_1(M)/[\pi_1(M), \pi_1(M)]$ be the $\mathbb{Z}$-module homomorphism such that $\phi|_{M_1} : M_1 = \pi_1(M) \longrightarrow \pi_1(M)/[\pi_1(M), \pi_1(M)]$ is the quotient homomorphism consider the commutative diagram

$$\begin{array}{ccccc} \mathbb{Z}[M_2] & \longrightarrow & \mathbb{Z}[M_1] & & \\ \uparrow & & \uparrow & \searrow & \\ M_2 & & M_1 & \twoheadrightarrow & \pi_1(M)/[\pi_1(M), \pi_1(M)]. \end{array}$$

From after, one can continue the proof by same way of Hurewicz Theorem. □

5. Conclusion

By using simplicial theory, we give applications for simplicial homology and simplicial homotopy. Also, we proof the Hurewicz and Poincaré Theorems for simplicial modules.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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