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STUDY AND SUPPRESSION OF SINGULARITIES IN WAVE-TYPE EVOLUTION EQUATIONS ON NON-CONVEX DOMAINS WITH CRACKS

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Abstract. One of the objectives of this paper is to establish the exact controllability for wave-type evolution equations on non-convex and/or cracked domains with non-concurrent support crack lines. Admittedly, we know that according to the work of Grisvard P., in domains with corners or cracks, the formulas of integrations by parts are subject to geometric conditions: the lines of cracks or their supports must be concurrent. In this paper, we have established the exact controllability for the wave equation in a domain with cracks without these additional geometric conditions.

1. Introduction

The presence of a crack in equipment (especially under pressure) requires, for obvious safety reasons, to know precisely its degree of harmfulness. When this crack propagates, under cyclic loading, it is important to evaluate and to quickly control the evolution of this degree of harmfulness and more concretely the residual life of the cracked structure.
In the works of the pioneers and precursors, not least Kondratiev [1], Grisvard [2], Moussaoui [3] and Niane [4], the control and removal of singularities were established in domains with corners or cracks.
Indeed, when these cracks propagate, under cyclic loading, it is important to evaluate and to quickly control the evolution of this degree of harmfulness and more concretely the residual life of the cracked structure. Thin plates and shells are widely used in aeronautics. Due to the significant stresses to which the structure

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of an aircraft is subjected in flight, for example, the appearance of small cracks is inevitable. Depending on the situation, these cracks are more or less dangerous; thus, certain cracks do not propagate, on the other hand, others present a certain risk.The risks alluded to earlier must, consequently, be curbed. So, once a crack has been detected, it is important to know if it can be dangerous or not? The safety of persons and that of the goods involved means repairing the work first and foremost. Notwithstanding, repairing all the cracks won't be necessary as, if the crack is not dangerous, it is no good repairing as it will be costly.

Accordingly, it is important to figure out whether or not the crack is dangerous, and whether it can be spread. Apart from extreme cases (very small or very large cracks), this diagnosis is not easy to make because even a small crack can spread brutally. It is very clear that the accuracy of this diagnosis is very important.

More recently, Seck [5], Bayili [6], taking inspiration from the exact controllability in Lipschitzian domains by Costabel [7], Niane [8] and Lions [9, 10], established results of exact controllability of the wave equation in non-regular Sobolev spaces. But, in all these works, the domains admit a crack or a corner or even cracks with condition of control: the lines of cracks are concurrent (or the supports of the lines of cracks are concurrent).

In this paper, without making additional assumptions and conditions on the crack lines and their supports, an exact controllability result was established for wave equation.

2. Reminders of Fundamental Results

2.1. **Problem position.** We denote by Ω an open polygonal uncracked, non convex and bounded of $\mathbb{R}^2$ and for $T > 0$, we denote by $Q_T = \Omega \times]0, T[$.

Let Γ the boundary of Ω, $\nu(x)$ the external unit normal at all points x (apart from the vertices) of Γ and Σ_T the lateral border of the cylinder Q_T.

Γ is the union of a finite number of closed line segments; the corresponding open segments are denoted Γ_j, $0 \leq j \leq N$ and S_{ij} the end common to Γ_j and Γ_i if it exists. We denote by ω_{ij} the measure of the angle made by Γ_j and Γ_i in S_{ij} towards the interior of Ω.

We denote by ν_j the unit normal vector outside Γ_j and τ_j the unit vector tangent to Γ_j and directed towards the vertex S_i. For x_0 any point of $\mathbb{R}^2$, we consider the function $m(x) = x - x_0$ and a partition of the border as follows:

$$\Gamma_0 = \{x \in \Gamma; m(x) \cdot \nu \geq 0\}, \quad \Gamma_0^* = \{x \in \Gamma; m(x) \cdot \nu < 0\},$$

and

$$\Sigma_0^* = \Gamma_0^* \times]0, T[.$$

Let $||.||$ be the Euclidean norm in $\mathbb{R}^2$ and introduce the following constants

$$R_0 = R(x_0) = \max_{x \in \overline{\Omega}} ||x - x_0||, \text{ and } T_0 = 2R(x_0).$$

Let also $f \in L^2(\Omega)$ and $y \in H_0^1(\Omega)$ be the unique solution of the homogeneous Dirichlet problem

$$(P1) \begin{cases} -\triangle y = f, \\ y_{|\Gamma} = 0. \end{cases} \tag{1}$$

In the space $H = L^2(\Omega)$, we consider A the operator defined by:

$$D(A) = \Big\{ y \in H_0^1(\Omega); -\triangle y \in L^2(\Omega) \Big\},$$
$$\forall A \in D(A), Ay = -\triangle y.$$

A: is a compact positive inverse self-adjoint operator see Brezis [11] and Hormander [12].
y is solution of (1)$(P1) \Longrightarrow y \in D(A)$.
Let $m+1$ be the number of non-convex angles of the $\partial\Omega$ boundary of the domain Ω having $m+1$ vertices $(S_i)_{0 \leq i \leq m}$.
It has been proved in Niane [4] that if $\bar{\omega}$ is an arbitrarily small part of Ω not meeting any vertex of cracks, there exist regular functions $(g_i)_{1 \leq i \leq m}$ with compact support in $\bar{\omega}$ such that for all $f \in L^2(\Omega)$, if $(\lambda_i)_{1 \leq i \leq m}$ are the coefficients of singularities of the problem $(P1)$ then the problem

$$(P2) \begin{cases} -\triangle \tilde{y} = f + u, \\ \tilde{y}_{|\Gamma} = 0. \end{cases} \tag{2}$$

admits a solution $\tilde{y} \in H^2(\Omega)$, with $u = -\sum_{i=0}^{m} \lambda_i h_i, \quad \lambda_i = \int_\Omega f w_i dx$ where w_i the singular functions Cf. Grisvard [2] and $< g_i, w_j >= \delta_{ij}$ Moussaoui [3] and Niane [4].
Let for $i \in \{0, ..., m\}, (r_i, \theta_i)$ represent the polar coordinates of a point M of Ω relatively to the vertex S_i with $r_i = \| \overrightarrow{S_i M} \|$ Gilbert [6].

Remark 1. *The singular functions w_i are harmonic*

$$\begin{cases} -\triangle w_i = 0 \ sur \ \Omega, \\ \omega_i = 0 \qquad \partial\Omega \setminus \{x_i\}. \end{cases}$$

2.2. **Internal control of the homogeneous waves equation on a non-convex domain.** Let y: solution of the following homogeneous wave equation

$$(EOH): \begin{cases} y'' - \Delta y = 0 \qquad in \ Q_T, \\ \\ y = 0, \qquad in \ \Sigma_T, \\ \\ y(0) = y_0 \ \ y'(0) = y_1 \qquad in \ \Omega. \end{cases} \tag{3}$$

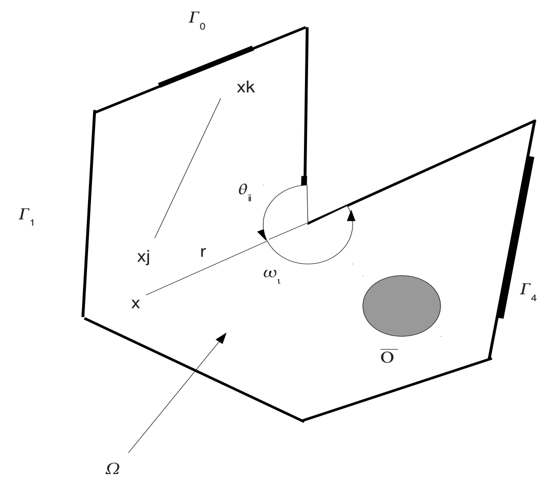

FIGURE 1. Non-convex cracked domain

$$(EOH) \Longleftrightarrow (EOH)' \begin{cases} -\Delta y = -y'' & in\ Q_T, \\ y = 0, & in\ \Sigma_T, \\ y(0) = y_0\ \ y'(0) = y_1 & in\ \Omega. \end{cases} \tag{4}$$

Let $(y_o, y_1) \in D(A) \times H_0^1(\Omega) \Longrightarrow$ the solution y of the equation (EOH) (3) verified $y \in C(O, T; D(A)) \cap C^1(0, T; H_0^1(\Omega)) \cap C^2(0, T; L^2(\Omega))$.

In addition, in Grisvard [2], the solution can be decomposed as folloow:
$y = y_R + \sum_{i=1}^m \lambda_i(t) S_i(t)$ with:
$\lambda_i(t) = \int_\Omega (-y'') w_i(t) dt$ and $S_i(t) = r^{\alpha_i} sin(\alpha_i \theta_i)$ with α_i : the singularity exponent defined by $\alpha_i = \frac{\pi}{w_i}, w_i$: the aperture angle at the vertex S_i.
As in the first part, we can, for any $t > 0$, add an internal check $u(t) = -\sum_{i=1}^m \lambda_i(t) g_i(t)$

of such that if $\hat{y}$ is the regularized solution of the equation

$$\begin{cases} -\Delta\tilde{y} = -\tilde{y}'' + u(t) \qquad in\ Q_T, \\ \qquad \tilde{y} = 0 \qquad in\ \Sigma_T, \\ \tilde{y}(0) = \tilde{y}_0,\ \ \tilde{y}'(0) = \tilde{y}_1 \qquad in\ \Omega. \end{cases} \tag{5}$$

then $\hat{y} \in H^2(\Omega)$.
In fact, $\hat{y} = 0$ on the edge Σ_T, the solution $\hat{y} \in C(0,T; H^2(\Omega) \cap H_0^1(\Omega))$.
Let $\tilde{V}$ be a subspace of $H^1(\Omega)$ of admissible solutions for the problem $(EOH)'$ defined by

$$\tilde{V} = \{\hat{y} \in H^1(\Omega)/\ \hat{y}_{|\Sigma_T} = 0\}. \tag{6}$$

For continuity, let us state the following proposition:

Proposition 1. *The problem $(EOH)'$ (4) admits an unique solution $\hat{y}$ in the space $\tilde{V}$ and there exist a constant $C_T > 0$ such that*

$$||\hat{y}||_{C(0,T;H_0^1(\Omega))} \leq C_T \left[||\tilde{y_0}||_{H_0^1(\Omega)} + ||\tilde{y_1}||_{L^2(\Omega)}\right]^{\frac{1}{2}}. \tag{7}$$

Proof. Let A be the unbounded operator of $L^2(\Omega)$ previously defined. According to Spectral Theory and by Fourier transform, A is diagonalizable and there exists a countable Hilbertian basis of $L^2(\Omega)$ made up of eigenvectors $(z_k)_{k\in\mathbb{N}^*} \subset D(A)$ such that the sequence of eigenvalues $(\lambda_k)_{k\geq 1}$ of associated eigenvalues verify: $(\lambda_k) \nearrow +\infty$ and $\lambda_1 > 0$.

$$z_k \in H_0^1(\Omega), -\Delta z_k = \lambda_k z_k \tag{8}$$

The family $Z = (z_k)_{k\geq 1}$ Hilbert base of $L^2(\Omega)$ ie $\hat{y} \in L^2(\Omega) \Longrightarrow \hat{y} = \sum_{k\geq 1} \hat{y}_k z_k$ with $\hat{y}_k =< \hat{y}, z_k >_{L^2(\Omega)}$ and $\sum_{k\geq 1} z_k^2 < +\infty$. What's more $\|\hat{y}\|_{L^2(\Omega)} = \left(\sum_{k=1}^{+\infty} \hat{y}_k^2\right)^{\frac{1}{2}}$.

$$\hat{y} \in H_0^1(\Omega) \Longleftrightarrow \hat{y} = \sum_{k\geq 1} \hat{y}_k z_k,\ \sum_{k\geq 1} \lambda_k \hat{y}_k^2 < +\infty \ and\ ||\hat{y}||_{H_0^1(\Omega)} = \left(\sum_{k\geq 1} \lambda_k \hat{y}_k^2\right)^{\frac{1}{2}}. \tag{9}$$

So, if $\hat{y}$ is solution of $(EOH)'$ (4) then

$$\begin{cases} \hat{y}(t,x) = \sum_{k\geq 1} \hat{y}_k(t) z_k(x), \\ \hat{y}_{0k}(x) = \sum_{k\geq 1} \hat{y}_{0k} z_k(x), \\ \hat{y}_{1k}(x) = \sum_{k\geq 1} \hat{y}_{1k} z_k(x), \\ \sum_{k\geq 1} \left(\hat{y}_k''(t) - \lambda_k \hat{y}_k(t)\right) z_k(x) = 0. \end{cases} \tag{10}$$

We multiply the relation (10) by the eigenfunctions z_k and integrate on the cylinder Q_T

$$\begin{cases} \hat{y}_k^{''}(t) - \lambda_k \hat{y}_k(t) = 0, \\ \quad \hat{y}_k(0) = \hat{y}_{0k}, \\ \quad \hat{y}_k(1) = \hat{y}_{1k}. \end{cases} \tag{11}$$

And, for all $k \geq 1$, the solution of (11) (see Lions [9, 10])is under the form

$$\hat{y}_k(t) = \hat{y}_{0k} cos(\sqrt{\lambda_k} t) + \hat{y}_{1k} \frac{sin(\sqrt{\lambda_k} t)}{\sqrt{\lambda_k}}, \tag{12}$$

So

$$\hat{y}_k(t,x) = \sum_{k \geq 1} \left(\hat{y}_{0k} cos(\sqrt{\lambda_k} t) + \hat{y}_{1k} \frac{sin(\sqrt{\lambda_k} t)}{\sqrt{\lambda_k}} \right) z_k(x). \tag{13}$$

Assume

$$\begin{aligned} ||\hat{y}||^2_{C(0,T;H_0^1(\Omega))} &= sup_{t \in [0,T]} ||\hat{y}(t,.)||^2_{H_0^1(\Omega)} \\ &= sup_{t \in [0,T]} \sum_{k \geq 1} |\lambda_k| |\hat{y}_k(t)|^2 \end{aligned}$$

$\Longrightarrow$

$$||\hat{y}||^2_{C(0,T;H_0^1(\Omega))} \leq \sum_{k \geq 1} |\lambda_k| sup_{t \in [0,T]} |\hat{y}_k(t)|^2 \tag{14}$$

Based on the relationship (13)

$$\begin{aligned} ||\hat{y}||^2_{C(0,T;H_0^1(\Omega))} &\leq 2. \sum_{k>1} \lambda_k \left[\hat{y}_{0k}^2 + \frac{\hat{y}_{1k}^2}{\lambda_k} \right] \\ &\leq 2. \sum_{k>1} \lambda_k \left[\hat{y}_{0k}^2 + \hat{y}_{1k}^2 \right] \end{aligned}$$

let's remember that

$$\hat{y}_0 \in H_0^1(\Omega) \Longleftrightarrow \begin{cases} \hat{y}_0(x) = \sum_{k>1} \hat{y}_{0k} z_k(x), \\ \sum_{k>1} \lambda_k \tilde{y}_{0k}^2 < +\infty \ and \\ ||\hat{y}_0||^2_{H_0^1(\Omega)} = \sum_{k>1} \lambda_k \hat{y}_{0k}^2. \end{cases} \tag{15}$$

and

$$\hat{y}_1 \in L^2(\Omega) \Longleftrightarrow \begin{cases} \hat{y}_1(x) = \sum_{k>1} \hat{y}_{1k} z_k(x), \\ \sum_{k>1} \lambda_k \hat{y}_{1k}^2 < +\infty \ and \\ ||\hat{y}_1||^2_{H_0^1(\Omega)} = \sum_{k>1} \lambda_k \hat{y}_{1k}^2. \end{cases} \tag{16}$$

Therefore, we get that $\hat{y} \in C(0,T;H_0^1(\Omega))$ (1) with

$$||\hat{y}||_{C(0,T;H_0^1(\Omega))} \leq C_T \left(||\hat{y}_0||_{H_0^1(\Omega)} + ||\hat{y}_1||_{L^2(\Omega)} \right) \tag{17}$$

□

2.3. **Application to the removal of singularities.** Let $\tilde{y}$ regularized solution of the equation

$$(EOS): \begin{cases} \tilde{y}'' - \Delta \tilde{y} + \sum_{i=1}^m g_i \int_\Omega (\tilde{y}'') w_i dx = 0 \qquad in \ Q_T, \\ \tilde{y} = 0, \qquad in \ \Sigma_T, \\ \tilde{y}(0) = \tilde{y}_0, \ \ \tilde{y}'(0) = \tilde{y}_1 \qquad in \ \Omega. \end{cases} \tag{18}$$

It will then be a matter of showing that the solution $\tilde{y}$ of the equation (EOH) (3) is in $C(0,T;H^2(\Omega) \cap H_0^1(\Omega))$?
In general, it was proved in Grisvard [2] that the following wave equation

$$(EOS)_2: \begin{cases} \varphi'' - \Delta \varphi = f \in L^1(0,T;H_0^1(\Omega)), \\ \varphi = 0 \qquad in \ \Sigma_T, \\ \varphi(0) = \varphi_0, \ \ \varphi'(0) = \varphi_1 \qquad in \ \Omega, \\ (\varphi_0, \varphi_1) \in D(A) \times D(A^{\frac{1}{2}}). \end{cases} \tag{19}$$

admit a solution $\varphi \in C(0,T;D(A)) \cap C^1(0,T;H^1(\Omega)) \cap C(0,T;L^2(\Omega))$ and that this solution verifies the inequality:

$$||\varphi||_{C(0,T;D(A))} \leq K \left(||\varphi_0||_{D(A)} + ||\varphi_1||_{D(A^{\frac{1}{2}})} + ||f||_{H_0^1(\Omega)} \right), \tag{20}$$

called continuous dependence of the solution compared to the initial conditions and to the second member.
Let us apply this Grisvard result to the equation (EOS) (18); For this consider for

$\zeta \in C^2(0,T;L^2(\Omega))$, $\tilde{y} = y(\zeta)$ is solution of the equation

$$(EOS)_3 : \begin{cases} \tilde{y}''(\zeta) - \Delta\tilde{y}(\zeta) = -\sum_{i=1}^m g_i \int_\Omega \zeta'' w_i dx \qquad in\ Q_T, \\ \\ \tilde{y}(\zeta) = 0 \qquad in\ \Sigma_T, \\ \\ \tilde{y}(\zeta)(0) = \tilde{y}(\zeta_0),\ \ \tilde{y}(\zeta)'(0) = \tilde{y}(\zeta_1) \qquad in\ \Omega. \end{cases}$$

$(EOS)_3$ and the inequality (20) implies a priori that $y(\zeta) \in C(0,T;D(A))$ and that

$$||y(\zeta)||_{C(0,T;D(A))} \leq K_1 \left(||\sum_{i=1}^m g_i \int_\Omega \zeta'' ||_{L^1(\Omega)} \right). \tag{21}$$

Consider the application $\Lambda : \zeta \longmapsto y(\zeta)$; Let us show that Λ is contracting ?

Let $\zeta_1 \longmapsto y(\zeta_1),\ \zeta_2 \longmapsto y(\zeta_2)$ and $\zeta = \zeta_1 - \zeta_2 \longmapsto y(\zeta)$.
Applying it to the equation $(EOS)_3$ we get

$$(EOS)_4 : \begin{cases} \tilde{y}''(\zeta) - \Delta\tilde{y}(\zeta) = -\sum_{i=1}^m g_i \left(\int_\Omega \zeta'' w_i dx\right) \qquad in\ Q_T, \\ \\ \tilde{y}(\zeta) = 0 \qquad in\ \Sigma_T. \end{cases}$$

More $y(\zeta_1)(0) = y(\zeta_2)(0) = 0$ ($y(\zeta_1)$ and $y(\zeta_2)$ have the same initial conditions as y_0 and y_1).
From inequality (21) we deduce

$$\begin{aligned} ||y(\zeta)||_{C(0,T;D(A))} &\leq K_1 \left(||\sum_{i=1}^m g_i \int_\Omega \zeta'' w_i ||_{L^1(\Omega)} dx \right), & (22) \\ &\leq K_2 \left(\sum_{i=1}^m ||g_i||.||w_i||.|| \int_\Omega \zeta'' dx|| \right), & (23) \\ &\leq K_3 \left(\sum_{i=1}^m ||g_i||.||w_i||||\zeta'||_{L^1(\Omega)}.mes(\Omega) \right), & (24) \\ &\leq K_4 \left(\sum_{i=1}^m ||g_i||_{H_0^1(\Omega)} ||w_i||_{L^1(\Omega)} ||\zeta||_{L^1(\Omega)} \right), & (25) \\ &\leq K_5 ||\zeta||_{L^1(\Omega)}. & (26) \end{aligned}$$

With the constant $K_5 = \sum_{i=1}^m ||g_i||_{H_0^1(\Omega)} ||w_i||_{L^1(\Omega)}$.
Let us show that $0 < K_5 < 1$ ie Λ is contracting ?

We know that the dual singular functions are such that:

$w_i = r^{-\alpha_i} sin(\alpha_i \theta_i)\eta_i + \zeta_i$ *with* $\alpha_i = \frac{\pi}{\omega_i}$ *and* $\omega_i > \pi$, η_ia truncation function in the neighborhood of the vertices of x_i and $\zeta_i \in H_0^1(\Omega)$ for all $i \in \{0, ..., m\}$.

The application Λ is Lipschitzian, let us show that it is contracting ie $0 < K_5 < 1$?

$$\begin{aligned} ||w_i|| &= ||r^{-\alpha_i} sin(\alpha_i\theta_i)\eta_i + \zeta_i||, & (27)\\ &\leq \frac{1}{r^{\alpha_i}}||sin(\alpha_i\theta_i)\eta_i|| + ||\zeta_i||, & (28)\\ &\leq \frac{1}{r^{\alpha_0}} + ||\zeta||. & (29) \end{aligned}$$

where $\alpha_0 = min_{i\in\{1,...,m\}}\alpha_i$ thus $\frac{1}{r^{\alpha_i}} < \frac{1}{r^{\alpha_0}}$.
The functions $(g_i)_{1\leq i\leq m}$ are compact support on $\bar{\omega}$ which is compact, so there is $g_0 = max_{1\leq i\leq m} g_i$ on $\bar{\omega}$ such that $||g_i|| \leq ||g_0||$ for all i. Therefore

$$\sum_{i=1}^{m} ||g_i||||w_i|| \leq m^2||g_0||\frac{1}{r^{\alpha_0}} + C_1 \ with\ C_1 > 1\ a\ constant.$$

As a result,

$$0 < K_5 < m^2||g_0||\frac{1}{r^{\alpha_0}} + C_1.$$

A sufficient condition for Λ to be contracting is that

$$m^2||g_0||\frac{1}{r^{\alpha_0}} + C_1 < 1 \Longleftrightarrow r \geq e^{\frac{1}{\alpha_0}log\left(\frac{m^2||g_0||}{1-C_1}\right)}. \tag{30}$$

Remember that

$$r = ||\vec{S_iM}|| = ||x - x_i||$$

ie $M \neq S_i$, $\forall i \in \{1, ..., m\}$ on $\bar{\omega}$.
Hence if M is far from the top of the crack ie $r >> 1$ the application Λ is contracting. Thereby,

$$||y(\zeta)||_{C(0,T;D(A))} \leq K_5||\zeta||_{C^2(0,T;L^2(\Omega))} \tag{31}$$

Therefore, if (30) holds then the application Λ is contracting and according to the Fixed Point Theorem $y(\zeta) = y(\zeta_1) - y(\zeta_2) = 0$ and y being continuous so ζ is unique.
Hence the equation

$$(EOS)_3 : \begin{cases} \tilde{y}''(\zeta) - \Delta\tilde{y}(\zeta) = \underbrace{-\sum_{i=1}^{m} g_i \int_\Omega (\tilde{y}''(\zeta))w_i dx}_{u(t)} & in\ Q_T, \\ \tilde{y}(\zeta) = 0 & in\ \Sigma_T, \\ \tilde{y}(\zeta)(0) = \tilde{y}(\zeta)_0,\ \ \tilde{y}(\zeta)'(0) = \tilde{y}(\zeta)_1 & in\ \Omega. \end{cases} \tag{32}$$

admits a unique solution $\tilde{y} \in C(0,T; H^2(\Omega) \cap H_0^1(\Omega))$.

Proposition 2. *The solution $\tilde{y}$ is the regularized solution, therefore the singularity coefficient $\tilde{\lambda}$ associated with it is null.*

Proof. Let $\tilde{\lambda}$ the singularity coefficient associated with $\tilde{y}$. By definition,

$$\begin{aligned}\tilde{\lambda} &= \int_\Omega u(t) w_i dx, &(33)\\ &= \int_\Omega \left(-\sum_{i=1}^m g_i \int_\Omega \tilde{y}'' w_i dx \right) .w_i dx, &(34)\\ &= -\int_\Omega \int_\Omega \sum_{i=1}^m g_i \tilde{y}'' w_i .w_i dx dx, &(35)\\ &= -\int_\Omega \int_\Omega \sum_{i=1}^m < g_i, w_i > \Delta \tilde{y} w_i dx dx. &(36)\end{aligned}$$

so $< g_i, w_i >= 1 \Longrightarrow$

$$\begin{aligned}\tilde{\lambda} &= -\int_\Omega \int_\Omega \sum_{i=1}^m \Delta \tilde{y} w_i dx dx, &(37)\\ &= -\int_\Omega \int_\Omega \sum_{i=1}^m \tilde{y} \Delta w_i dx dx &(38)\end{aligned}$$

because $\tilde{y}_{|\Sigma_T} = 0$.
We also know that the dual singular functions are harmonic ie $\Delta w_i = 0$ hence $\tilde{\lambda} = 0$ □

Remark 2. *The corrective term or internal control $u(t)$ depends on $\tilde{y}''$, therefore $\tilde{y}$.*

3. Use in the Implementation of the Hum Method

3.1. **Preliminaries.** Let y solution of wave equation

$$(EOH) : \begin{cases} y'' - \Delta y = 0 & in\ Q_T, \\ y = 0 & in\ \Sigma_T, \\ y(0) = y_0, \ \ y'(0) = y_1 & in\ \Omega. \end{cases}$$

For initial data y_0 and y_1 belonging respectively to $H_0^1(\Omega$ and $L^2(\Omega)$. Let also be the energy of (EOH) defined by

$$E_0 = \frac{1}{2} \left(||y_0||_{H_0^1(\Omega)} + ||y_1||_{H_0^1(\Omega)} \right). \tag{39}$$

We know that in a polygonal domain with corner, $(x-x_0).\nu\frac{\partial\varphi}{\partial\nu}$ is not always a square integrable on the edge near of corner. Grisvard [2] got around this difficulty by imposing drastic geometric conditions. And, in Seck [5] this result has been generalized with less constraints in non-regular Sobolev spaces. Also Niane [4] have shown, without geometric conditions, the exact controllability of the wave equation by combining a boundary control and an internal control on a small part whose support is in the vicinity of a vertex crack.

3.2. **Implementation of the HUM method.** Let us return to the equation of the following waves

$$(EOS)_5 : \begin{cases} \tilde{\varphi}'' - \Delta\tilde{\varphi} = u(\tilde{\varphi}) \qquad in\ Q_T, \\ \\ \tilde{\varphi} = 0 \qquad in\ \Sigma_T, \\ \\ \tilde{\varphi}(0) = \tilde{\varphi}_0,\ \ \tilde{\varphi}'(0) = \tilde{\varphi}_1 \qquad in\ \Omega. \end{cases} \tag{40}$$

From the above, with $u(\tilde{\varphi}) = \sum_{i=1}^m g_i\left(\int_\Omega \varphi'' w_i dx\right)$, the solution $\tilde{\varphi} \in H^2(\Omega)$. Indeed, we multiply the equation $(EOS)_5$ (40) by $m\nabla\tilde{y}$ and integrate by parts:

$$\int_{Q_T} (\tilde{\varphi}'' - \Delta\tilde{\varphi}) m\nabla\tilde{y} dxdt = \int_{Q_T} m\nabla\tilde{y} u(\tilde{\varphi}) dxdt, \tag{41}$$

$$= -m\nabla\tilde{y}\sum_{i=1}^m g_i\left(\int_{Q_T}\varphi'' w_i dxdt\right). \tag{42}$$

Assume

$$I = \int_{Q_T} (\tilde{\varphi}'' - \Delta\tilde{\varphi}) m\nabla\tilde{y} dxdt \tag{43}$$

$$= \underbrace{\int_{Q_T} \tilde{\varphi}'' m\nabla\tilde{y} dxdt}_{I_1} - \underbrace{\int_{Q_T} \Delta\tilde{\varphi} m\nabla\tilde{y} dxdt}_{I_2} \tag{44}$$

3.3. **Some integrations by parts.**

3.3.1. *First Term I_1.*

$$\begin{aligned} I_1 &= \int_{Q_T} \tilde{\varphi}'' m\nabla\tilde{y} dxdt, \\ &= \int_0^T \int_\Omega \tilde{\varphi}'' m(x)\nabla\tilde{y} dxdt, \\ &= \int_\Omega \tilde{\varphi}' m(x)\nabla\tilde{y} dx|_0^T - \int_0^T \int_\Omega \frac{\partial\tilde{\varphi}}{\partial t} m_k \frac{\partial^2\tilde{y}}{\partial t\partial x_k} dtdx, \end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \tilde{\varphi}' m(x)\nabla\tilde{y}dx|_0^T - \int_0^T\left(\int_\Omega\left(\frac{\partial\tilde{\varphi}}{\partial t}m_k\frac{\partial}{\partial t}(\frac{\partial\tilde{y}}{\partial x_k})\right)dx\right)dt,\\
&= \int_\Omega \tilde{\varphi}' m(x)\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T - \int_0^T\int_\Omega\left(\frac{\partial\tilde{\varphi}}{\partial t}m_k\frac{\partial^2\tilde{y}}{\partial t\partial x_k}\right)dxdt,\\
&= \int_\Omega \frac{\partial\tilde{\varphi}}{\partial t}m_k\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T - \int_0^T\int_\Omega m_k\frac{\partial^2\tilde{\varphi}}{\partial t\partial x_k}\frac{\partial\tilde{y}}{\partial t}dxdt.
\end{aligned}
$$

Noting that: $N = 2, divm = \sum_{k=1}^2 \frac{\partial m_k}{\partial x_k} = 2$ and applying Green again we have:

$$
\begin{aligned}
I_1 &= \int_\Omega \frac{\partial\tilde{\varphi}}{\partial t}m_k\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T - \int_0^T\left[-\int_\Omega\frac{\partial m_k}{\partial x_k}\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial t}dx + \underbrace{\int_{\partial\Omega}m_k\tilde{\varphi}_k\frac{\partial\tilde{y}}{\partial t}d\sigma}_{=0}\right]dt,\\
&= \int_\Omega \frac{\partial\tilde{\varphi}}{\partial t}m_k\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T + \int_{Q_T} divm\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial t}dxdt.
\end{aligned}
$$

3.3.2. *Second Term I_2.*

$$
\begin{aligned}
I_2 &= \int_{Q_T}\Delta\tilde{\varphi}m\nabla\tilde{y}dxdt,\\
&= \int_0^T\int_\Omega\Delta\tilde{\varphi}m\nabla\tilde{y}dxdt,\\
&= \int_0^T\left[\int_\Omega\nabla\tilde{\varphi}.\nabla(m_k\frac{\partial\tilde{\varphi}}{\partial x_k})dx - \int_{\partial\Omega}\frac{\partial\tilde{\varphi}}{\partial n}m_k\frac{\partial\tilde{y}}{\partial x_k}d\sigma\right]dt,\\
&= \int_0^T\left[\int_\Omega\frac{\partial\tilde{\varphi}}{\partial x_i}(m_k\frac{\partial\tilde{y}}{\partial x_k})dx - \int_{\partial\Omega}\frac{\partial\tilde{\varphi}}{\partial n}m_k\frac{\partial\tilde{y}}{\partial x_k}d\sigma\right]dt,\\
&= \int_0^T\left[\int_\Omega\frac{\partial m_k}{\partial x_i}.\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\tilde{y}}{\partial x_k}dx + \underbrace{\int_\Omega\frac{\partial\tilde{\varphi}}{\partial x_i}m_k\frac{\partial^2\tilde{y}}{\partial x_i\partial x_k}dx}_{J} - \int_{\partial\Omega}\frac{\partial\tilde{\varphi}}{\partial n}m_k\frac{\partial\tilde{y}}{\partial x_k}d\sigma\right]dt.
\end{aligned}
$$

Let's study the integral J:

$$
\begin{aligned}
J &= \int_\Omega\frac{\partial\tilde{\varphi}}{\partial x_i}m_k\frac{\partial^2\tilde{y}}{\partial x_i\partial x_k}dx,\\
&= \int_\Omega m_k\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial^2\tilde{y}}{\partial x_i\partial x_k}dx,\\
&= \int_\Omega\frac{\partial}{\partial x_k}(m_k\frac{\tilde{\varphi}}{\partial x_i})\frac{\partial\tilde{y}}{\partial x_i}dx - \int_{\partial\Omega}m_k n_k\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_i}d\sigma.
\end{aligned}
$$

By grouping together we get:

$$\begin{aligned} I_2 &= \int_0^T \Bigg[\int_\Omega \frac{\partial m_k}{\partial x_i}.\frac{\partial \tilde{\varphi}}{\partial x_i}\frac{\tilde{y}}{\partial x_k}dx + \int_\Omega \frac{\partial}{\partial x_k}(m_k\frac{\tilde{\varphi}}{\partial x_i})\frac{\partial \tilde{y}}{\partial x_i}dx \\ &- \int_{\partial\Omega} m_k n_k \frac{\partial \tilde{\varphi}}{\partial x_i}\frac{\partial \tilde{y}}{\partial x_i}d\sigma - \int_{\partial\Omega}\frac{\partial \tilde{\varphi}}{\partial n}m_k\frac{\partial \tilde{y}}{\partial x_k}d\sigma\Bigg]dt, \\ &= \int_{Q_T} \frac{\partial m_k}{\partial x_i}.\frac{\partial \tilde{\varphi}}{\partial x_i}\frac{\tilde{y}}{\partial x_k}dxdt + \int_{Q_T}\frac{\partial}{\partial x_k}(m_k\frac{\tilde{\varphi}}{\partial x_i})\frac{\partial \tilde{y}}{\partial x_i}dxdt \\ &- \int_{\Sigma_T} m.n\frac{\partial \tilde{\varphi}}{\partial x_i}\frac{\partial \tilde{y}}{\partial x_i}d\sigma dt - \int_{\Sigma_T}\frac{\partial \tilde{\varphi}}{\partial n}m_k\frac{\partial \tilde{y}}{\partial x_k}d\sigma dt. \end{aligned}$$

Back to $I = I_1 + I_2$ (43) and (45):

$$I = \int_\Omega \frac{\partial \tilde{\varphi}}{\partial t}m_k\frac{\partial \tilde{y}}{\partial x_k}dx|_0^T + \int_{Q_T} divm\frac{\partial \tilde{\varphi}}{\partial t}\frac{\partial \tilde{y}}{\partial t}dxdt + \int_{Q_T}\frac{\partial m_k}{\partial x_i}.\frac{\partial \tilde{\varphi}}{\partial x_i}\frac{\tilde{y}}{\partial x_k}dxdt$$
$$+ \int_{Q_T}\frac{\partial}{\partial x_k}(m_k\frac{\tilde{\varphi}}{\partial x_i})\frac{\partial \tilde{y}}{\partial x_i}dxdt - \int_{\Sigma_T} m.n\frac{\partial \tilde{\varphi}}{\partial x_i}\frac{\partial \tilde{y}}{\partial x_i}d\sigma dt - \underbrace{\int_{\Sigma_T}\frac{\partial \tilde{\varphi}}{\partial n}m_k.n_k\frac{\partial \tilde{y}}{\partial x_k}d\sigma dt}_{L}.$$

Also

$$\begin{aligned} L &= \frac{1}{2}\int_{\Sigma_T}\frac{\partial \tilde{\varphi}}{\partial n}m.n\nabla\tilde{y}d\sigma dt, \\ &= \frac{1}{2}\int_{\Sigma_T}\left(\frac{\partial \tilde{\varphi}}{\partial n}\right)m.nd\sigma dt. \end{aligned}$$

The two equations have the same initial and boundary conditions.
Let's study L ?

$$\frac{\partial \tilde{\varphi}}{\partial x_i} = \frac{\partial \tilde{\varphi}}{\partial n}.n_i + \frac{\partial \tilde{\varphi}}{\partial \tau_i},$$

Decomposition according to the normal and the tangential. However

$$\frac{\partial \tilde{\varphi}}{\partial n_i} = \frac{\partial \tilde{\varphi}}{\partial n}.n_i \Rightarrow \sum_i \frac{\partial \tilde{\varphi}}{\partial n_i} = \sum_i \frac{\partial \tilde{\varphi}}{\partial n}.n_i \Rightarrow \nabla\tilde{\varphi} = \frac{\partial \tilde{\varphi}}{\partial n}.n.$$

So we deduce that:

$$\begin{aligned} I = &\int_\Omega \frac{\partial \tilde{\varphi}}{\partial t}m_k\frac{\partial \tilde{y}}{\partial x_k}dx|_0^T + \int_{Q_T} divm\frac{\partial \tilde{\varphi}}{\partial t}\frac{\partial \tilde{y}}{\partial t}dxdt + \int_{Q_T}\frac{\partial m_k}{\partial x_i}.\frac{\partial \tilde{\varphi}}{\partial x_i}\frac{\tilde{y}}{\partial x_k}dxdt + \\ &\int_{Q_T}\frac{\partial}{\partial x_k}(m_k\frac{\tilde{\varphi}}{\partial x_i})\frac{\partial \tilde{y}}{\partial x_i}dxdt - \int_{\Sigma_T} m.n\frac{\partial \tilde{\varphi}}{\partial x_i}\frac{\partial \tilde{y}}{\partial x_i}d\sigma dt - \frac{1}{2}\int_{\Sigma_T}(\frac{\partial \tilde{\varphi}}{\partial n})^2 m.nd\sigma dt. \quad (45) \end{aligned}$$

3.3.3. *Third Term* I_3.

$$
\begin{aligned}
I_3 &= \int_{Q_T}\left\{-m\nabla\tilde{y}\sum_{i=1}^{m} g_i\left(\int_\Omega \varphi'' w_i dx\right)dxdt\right\},\\
&= \int_0^T\int_\Omega\left\{-m\nabla\tilde{y}\sum_{i=1}^{m} g_i\left(\int_\Omega \varphi'' w_i dx\right)dxdt\right\},\\
&= -\int_0^T\int_\Omega\int_\Omega m\nabla\tilde{y}\underbrace{\sum_{i=1}^{m} < g_i, w_i >}_{\delta_{ii}=1}\tilde{\varphi}'' dxdxdt,\\
&= -\int_0^T\int_\Omega\int_\Omega m\nabla\tilde{y}\tilde{\varphi}'' dxdxdt,\\
&= -\int_0^T\int_\Omega\int_\Omega m\nabla\tilde{y}\nabla\tilde{\varphi} dxdxdt - \underbrace{\int_{\Sigma_T}\int_{\partial\Omega} m\tilde{y}\tilde{\varphi}(\sigma)d\sigma dt}_{=0},\\
&= -\int_0^T\int_\Omega\int_\Omega m\nabla\tilde{y}\nabla\tilde{\varphi} dxdxdt.
\end{aligned}
$$

Let's recap $I = I_3(46) \Longleftrightarrow$

$$
\begin{aligned}
&\int_\Omega \frac{\partial\tilde{\varphi}}{\partial t} m_k \frac{\partial\tilde{y}}{\partial x_k} dx|_0^T + \int_{Q_T} divm \frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial t} dxdt + \int_{Q_T}\frac{\partial m_k}{\partial x_i}.\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\tilde{y}}{\partial x_k} dxdt\\
&+ \int_{Q_T}\frac{\partial}{\partial x_k}(m_k\frac{\tilde{\varphi}}{\partial x_i})\frac{\partial\tilde{y}}{\partial x_i} dxdt - \int_{\Sigma_T} m.n\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_i} d\sigma dt - \frac{1}{2}\int_{\Sigma_T}(\frac{\partial\tilde{\varphi}}{\partial n})^2 m.nd\sigma dt\\
&= -\int_0^T\int_\Omega\int_\Omega m\nabla\tilde{y}\nabla\tilde{\varphi} dxdxdt \qquad (46)
\end{aligned}
$$

$\Longleftrightarrow$

$$
\begin{aligned}
\frac{1}{2}\int_{\Sigma_T}(\frac{\partial\tilde{\varphi}}{\partial n})^2 m.nd\sigma dt &= \int_\Omega \frac{\partial\tilde{\varphi}}{\partial t} m_k \frac{\partial\tilde{y}}{\partial x_k} dx|_0^T + \int_{Q_T} divm \frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial t} dxdt\\
&+ \int_{Q_T}\frac{\partial m_k}{\partial x_i}.\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\tilde{y}}{\partial x_k} dxdt + \int_{Q_T}\frac{\partial}{\partial x_k}(m_k\frac{\tilde{\varphi}}{\partial x_i})\frac{\partial\tilde{y}}{\partial x_i} dxdt\\
&- \int_{\Sigma_T} m.n\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_i} d\sigma dt + \int_0^T\int_\Omega\int_\Omega m\nabla\tilde{y}\nabla\tilde{\varphi} dxdxdt. \qquad (47)
\end{aligned}
$$

3.4. **Getting started with the HUM method.** For $x_0 \in \mathbb{R}^2$, assume

$$\Sigma_T^{0*} = \Gamma_0^* \times]0, T[\ ,\ \Sigma_T^{1*} = \Gamma_0 \times]0, T[.$$

Let $||.||$ the Euclidean in $\mathbb{R}^2$ and introduce the following constants.

$$R_0 = R(x_0) = \max_{x\in\overline{\Omega}} ||x - x_0||,\ T_0 = 2R(x_0).$$

Let us define in the same way the energies (see Lions [9, 10]) associated respectively with the systems $(EOS)_5$, (40) and (EOH) :

$$E(t,\tilde{\varphi}_0,\tilde{\varphi}_1)=\frac{1}{2}\left[\int_\Omega ||\nabla\tilde{\varphi}(t)||^2_{\mathbb{R}^2}dx+\int_\Omega(\frac{\partial\tilde{\varphi}}{\partial t}(t))^2dx\right],$$

$$E(t,\tilde{y}_0,\tilde{y}_1)=\frac{1}{2}\left[\int_\Omega ||\nabla\tilde{y}(t)||^2_{\mathbb{R}^2}dx+\int_\Omega(\frac{\partial\tilde{y}}{\partial t}(t))^2dx\right].$$

3.4.1. *Direct Inequality.* Back to the relationship (46)

$$\begin{aligned}\frac{1}{2}\int_{\Sigma_T}(\frac{\partial\tilde{\varphi}}{\partial n})^2m.nd\sigma dt=&\int_\Omega\frac{\partial\tilde{\varphi}}{\partial t}m_k\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T+\int_{Q_T}divm\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial t}dxdt+\\&+\int_{Q_T}\frac{\partial m_k}{\partial x_i}.\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\tilde{y}}{\partial x_k}dxdt\int_{Q_T}\frac{\partial}{\partial x_k}(m_k\frac{\tilde{\varphi}}{\partial x_i})\frac{\partial\tilde{y}}{\partial x_i}dxdt+\\&-\int_{\Sigma_T}m.n\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_i}d\sigma dt+\int_0^T\int_\Omega\int_\Omega m\nabla\tilde{y}\nabla\tilde{\varphi}dxdxdt\end{aligned}\tag{48}$$

$$\begin{aligned}\Longleftrightarrow\frac{1}{2}\int_{\Sigma_T}(\frac{\partial\tilde{\varphi}}{\partial n})^2m.nd\sigma dt-\int_0^T\int_\Omega\int_\Omega m\nabla\tilde{y}\nabla\tilde{\varphi}dxdxdt=&\int_\Omega\frac{\partial\tilde{\varphi}}{\partial t}m_k\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T\\&+\int_{Q_T}divm\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial t}dxdt+\int_{Q_T}\frac{\partial m_k}{\partial x_i}.\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\tilde{y}}{\partial x_k}dxdt\\&+\int_{Q_T}\frac{\partial}{\partial x_k}(m_k\frac{\tilde{\varphi}}{\partial x_i})\frac{\partial\tilde{y}}{\partial x_i}dxdt-\int_{\Sigma_T}m.n\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_i}d\sigma dt.\end{aligned}\tag{49}$$

We know that:

$$\int_\Omega\frac{\partial\tilde{\varphi}}{\partial t}m_k\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T\leq R_0.\int_\Omega\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T\tag{50}$$

and noticing that: $|ab|\leq\frac{1}{2}(a^2+b^2)$ we have:

$$\int_\Omega\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial x_k}dx\leq\frac{1}{2}\int_{Q_T}\left[(\frac{\partial\tilde{\varphi}}{\partial t})^2+(\frac{\partial\tilde{y}}{\partial x_k})^2\right]dx.\tag{51}$$

Therefore:

$$\int_\Omega\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T\leq\frac{T}{2}\int_{Q_T}\left[(\frac{\partial\tilde{\varphi}}{\partial t})^2+(\frac{\partial\tilde{y}}{\partial x_k})^2\right]dxdt.\tag{52}$$

Assume

$$\Sigma_T=\Sigma_T^{0*}\cup\Sigma_T^{1*},\qquad M_1=max_{1\leq i,k\leq 2}max_{x\in\bar{\mathcal{B}}_i}|\frac{\partial x_k}{\partial x_i}(x)|.$$

Consider an open ball $\mathcal{B}_i$ which does not meet any crack vertex ie $h\equiv\eta h$ (In the general case we can recover the domain Ω by a finite union of $\mathcal{B}_i$ ie $\Omega=\cup_{i=1}^{Ns}\mathcal{B}_i$).

$$\int_{Q_T}\frac{\partial m_k}{\partial x_i}.\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_k}dxdt\quad=\quad\int_{B_i\times]0,T[}\frac{\partial m_k}{\partial x_i}.\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_k}dxdt,$$

$$\leq \frac{M_1}{2}\int_{B_i\times]0,T[}\left[(\frac{\partial\tilde{\varphi}}{\partial x_i})^2+(\frac{\partial\tilde{y}}{\partial x_k})^2\right]dxdt,$$

$$\leq \frac{M_1}{2}\int_{B_i\times]0,T[}\left[||\nabla\tilde{\varphi}||^2_{L^2(\mathbb{R}^2)}+||\nabla\tilde{y}||_{L^2(\mathbb{R})^2}\right]dxdt \quad (53)$$

Relationships (50), (51), and (53), we deduce:

$$\frac{1}{2}\int_{\Sigma_T}(\frac{\partial\tilde{\varphi}}{\partial n})^2 m.nd\sigma dt-\sum_{i=1}^{N_s}\int_0^T\int_{B_i}\int_{B_i}m\nabla\tilde{y}\nabla\tilde{\varphi}dxdxdt\leq R_0.\sum_{i=1}^{N_s}\int_{B_i}\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T+$$

$$\frac{T}{2}\int_{Q_T}\left[(\frac{\partial\tilde{\varphi}}{\partial t})^2+(\frac{\partial\tilde{y}}{\partial x_k})^2\right]dxdt+\sum_{i=1}^{N_s}\frac{M_1}{2}\int_{B_i\times]0,T[}\left[||\nabla\tilde{\varphi}||^2_{L^2(\mathbb{R}^2)}+||\nabla\tilde{y}||_{L^2(\mathbb{R})^2}\right]dxdt \quad (54)$$

$$\begin{aligned}\Longrightarrow\quad \frac{1}{2}\int_{\Sigma_T}(\frac{\partial\tilde{\varphi}}{\partial n})^2 m.nd\sigma dt &\leq R_0.\sum_{i=1}^{N_s}\int_{B_i}\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T\\ &+\frac{T}{2}\int_{Q_T}\left[(\frac{\partial\tilde{\varphi}}{\partial t})^2+(\frac{\partial\tilde{y}}{\partial x_k})^2\right]dxdt\\ &+\sum_{i=1}^{N_s}\frac{M_1}{2}\int_{B_i\times]0,T[}\left[||\nabla\tilde{\varphi}||^2_{L^2(\mathbb{R}^2)}+||\nabla\tilde{y}||_{L^2(\mathbb{R})^2}\right]dxdt\\ &+\sum_{i=1}^{N_s}\int_0^T\int_{B_i}\int_{B_i}m\nabla\tilde{y}\nabla\tilde{\varphi}dxdxdt. \quad (55)\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{i=1}^{N_s}\int_0^T\int_{B_i}\int_{B_i}m\nabla\tilde{y}\nabla\tilde{\varphi}dxdxdt &\leq R_0\sum_{i=1}^{N_s}\int_0^T\int_{B_i}\int_{B_i}\nabla\tilde{y}\nabla\tilde{\varphi}dxdxdt\\ &\leq R_0\sum_{i=1}^{N_s}\int_0^T\int_{B_i}\int_{B_i}\left[(\frac{\partial\tilde{\varphi}}{\partial x_k})^2+(\frac{\partial\tilde{y}}{\partial t})^2\right],\\ &\leq R_0\sum_{i=1}^{N_s}\int_{B_i}\int_0^T\int_{B_i}\left[(\frac{\partial\tilde{\varphi}}{\partial x_k})^2+(\frac{\partial\tilde{y}}{\partial t})^2\right],\\ &\leq R_0\sum_{i=1}^{N_s}mes(B_i)\left[E(t,\tilde{\varphi}_0,\tilde{\varphi}_1)+E(t,\tilde{y}_0,\tilde{y}_1)\right]. \quad (56)\end{aligned}$$

Starting from the fact that the energy associated with the wave equation is constant, we obtain:

$$\sum_{i=1}^{N_s}\int_0^T\int_{B_i}\int_{B_i}m\nabla\tilde{y}\nabla\tilde{\varphi}dxdxdt\leq 2R_0\sum_{i=1}^{N_s}mes(B_i)E(t,\tilde{\varphi}_0,\tilde{\varphi}_1). \quad (57)$$

So the relation (53) implies

$$\begin{aligned}\frac{1}{2}\int_{\Sigma_T}(\frac{\partial\tilde{\varphi}}{\partial n})^2 m.nd\sigma dt &\leq R_0.\sum_{i=1}^{N_s}\int_{B_i}\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T\\ &+ \frac{T}{2}\sum_{i=1}^{N_s}\int_{B_i}\left[(\frac{\partial\tilde{\varphi}}{\partial t})^2+(\frac{\partial\tilde{y}}{\partial x_k})^2\right]dxdt\\ &+ \sum_{i=1}^{N_s}\frac{M_1}{2}\int_{B_i\times]0,T[}\left[||\nabla\tilde{\varphi}||^2_{L^2(\mathbb{R}^2)}+||\nabla\tilde{y}||_{L^2(\mathbb{R})^2}\right]dxdt\\ &+ 2R_0\sum_{i=1}^{N_s}mes(B_i)E(t,\tilde{\varphi}_0,\tilde{\varphi}_1), \qquad (58)\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\int_{\Sigma_T}(\frac{\partial\tilde{\varphi}}{\partial n})^2 m.nd\sigma dt &\leq \frac{T.R_0}{2}.\sum_{i=1}^{N_s}\int_{B_i}\left[(\frac{\partial\tilde{\varphi}}{\partial t})^2+(\frac{\partial\tilde{y}}{\partial x_k})^2\right]dxdt\\ &+ \frac{T}{2}\sum_{i=1}^{N_s}\int_{B_i}\left[(\frac{\partial\tilde{\varphi}}{\partial t})^2+(\frac{\partial\tilde{y}}{\partial x_k})^2\right]dxdt\\ &+ \sum_{i=1}^{N_s}\frac{M_1}{2}\int_{B_i\times]0,T[}\left[||\nabla\tilde{\varphi}||^2_{L^2(\mathbb{R}^2)}+||\nabla\tilde{y}||_{L^2(\mathbb{R})^2}\right]dxdt\\ &+ 2R_0\sum_{i=1}^{N_s}mes(B_i)E(t,\tilde{\varphi}_0,\tilde{\varphi}_1), \qquad (59)\end{aligned}$$

$$\frac{1}{2}\int_{\Sigma_T}(\frac{\partial\tilde{\varphi}}{\partial n})^2 m.nd\sigma dt \leq \frac{T.R_0+1}{2}.\sum_{i=1}^{N_s}\int_{B_i}\left[(\frac{\partial\tilde{\varphi}}{\partial t})^2+(||\nabla\tilde{\varphi}||_{L^2(\mathbb{R}^2)})^2\right]dxdt+$$
$$\sum_{i=1}^{N_s}\frac{M_1}{2}\int_{B_i\times]0,T[}\left[(\frac{\partial\tilde{y}}{\partial t})^2+(||\nabla\tilde{y}||_{L^2(\mathbb{R}^2)})^2\right]dxdt+2R_0\sum_{i=1}^{N_s}mes(B_i)E(t,\tilde{\varphi}_0,\tilde{\varphi}_1), \quad (60)$$

$$\begin{aligned}\frac{1}{2}\int_{\Sigma_T}(\frac{\partial\tilde{\varphi}}{\partial n})^2 m.nd\sigma dt &\leq \left(\frac{T.R_0+1}{2}+\frac{M_1}{2}+2R_0\sum_{i=1}^{N_s}mes(B_i)\right)E(t,\tilde{\varphi}_0,\tilde{\varphi}_1)\\ &\leq C_T^0(\Omega)E(t,\tilde{\varphi}_0,\tilde{\varphi}_1.) \qquad (61)\end{aligned}$$

From

$$\frac{1}{2}||\frac{\partial\tilde{\varphi}}{\partial n}||^2_{L^2(\Omega)} \leq C_T^0(\Omega)E(t,\tilde{\varphi}_0,\tilde{\varphi}_1), \qquad (62)$$

$$\text{where} \qquad C_T^0(\Omega)=\left(\frac{T.R_0+1}{2}+\frac{M_1}{2}+2R_0\sum_{i=1}^{N_s}mes(B_i)\right). \qquad (63)$$

3.4.2. *Inverse Inequality.* Feedback on the relationship (47)

$$\begin{aligned}\frac{1}{2}\int_{\Sigma_T}(\frac{\partial\tilde{\varphi}}{\partial n})^2 m.nd\sigma dt &= \int_\Omega \frac{\partial\tilde{\varphi}}{\partial t}m_k\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T + \int_{Q_T} divm\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial t}dxdt\\ &+ \int_{Q_T}\frac{\partial m_k}{\partial x_i}.\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_k}dxdt + \int_{Q_T}\frac{\partial}{\partial x_k}(m_k\frac{\tilde{\varphi}}{\partial x_i})\frac{\partial\tilde{y}}{\partial x_i}dxdt\\ &- \int_{\Sigma_T} m.n\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_i}d\sigma dt + \int_0^T\int_\Omega\int_\Omega m\nabla\tilde{y}\nabla\tilde{\varphi}dxdxdt. \quad (64)\end{aligned}$$

$$\begin{aligned}\int_{Q_T}\frac{\partial m_k}{\partial x_i}.\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_k}dxdt &= \sum_{i=1}^{Ns}\int_{B_i}\int_0^T\frac{\partial m_k}{\partial x_i}.\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_k}dxdt,\\ &\le \frac{1}{2}\sum_{i=1}^{Ns}\int_{B_i}\int_0^T\frac{\partial m_k}{\partial x_i}\left[(\frac{\partial\tilde{\varphi}}{\partial x_i})^2+(\frac{\partial\tilde{y}}{\partial x_k})^2\right]dxdt,\\ &\le \frac{1}{2}\left[||\nabla\tilde{\varphi}||^2_{\mathbb{R}^2}+||\nabla\tilde{y}||^2_{\mathbb{R}^2}\right]\sum_{i=1}^{Ns}\int_{B_i}\int_0^T\frac{\partial m_k}{\partial x_i},\\ &\le \frac{1}{2}\left[||\nabla\tilde{\varphi}||^2_{\mathbb{R}^2}+||\nabla\tilde{y}||^2_{\mathbb{R}^2}\right]Ns.\int_0^T m(x)dt,\\ &\le \frac{T}{2}\left[||\nabla\tilde{\varphi}||^2_{\mathbb{R}^2}+||\nabla\tilde{y}||^2_{\mathbb{R}^2}\right]Ns.m(x),\\ &\le \frac{T.R_0Ns}{2}\left[||\nabla\tilde{\varphi}||^2_{\mathbb{R}^2}+||\nabla\tilde{y}||^2_{\mathbb{R}^2}\right].\end{aligned}$$

We deduce that:

$$-\sum_{i=1}^{Ns}\int_{B_i}\int_0^T\frac{\partial m_k}{\partial x_i}.\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_k}dxdt \ge -\frac{T.R_0Ns}{2}\left[||\nabla\tilde{\varphi}||^2_{\mathbb{R}^2}+||\nabla\tilde{y}||^2_{\mathbb{R}^2}\right]. \quad (65)$$

In addition, let us pose $M_2 = min_{x\in\bar{\Omega}}||m(x)||^2_{\mathbb{R}^2}$:

$$\int_\Omega\frac{\partial\tilde{\varphi}}{\partial t}m_k\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T \ge M_2\int_\Omega\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T,$$

therefore

$$\int_\Omega\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T dx \le \frac{1}{2}\int_{Q_T}\left[(\frac{\partial\tilde{\varphi}}{\partial t})^2+(\frac{\partial\tilde{y}}{\partial x_k})^2\right]dxdt \Rightarrow$$

$$-\frac{1}{2}\int_{Q_T}\left[(\frac{\partial\tilde{\varphi}}{\partial t})^2+(\frac{\partial\tilde{y}}{\partial x_k})^2\right]dxdt \le -\int_\Omega\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T dx,$$

So

$$\int_\Omega\frac{\partial\tilde{\varphi}}{\partial t}m_k\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T \ge -\frac{M_2.T}{2}\int_{Q_T}\left[(\frac{\partial\tilde{\varphi}}{\partial t})^2+(\frac{\partial\tilde{y}}{\partial x_k})^2\right]dxdt. \quad (66)$$

Also, from the relation (65) we deduce

$$-\sum_{i=1}^{N_s}\int_0^T\int_{B_i}\int_{B_i} m\nabla\tilde{y}\nabla\tilde{\varphi}dxdxdt \geq -2R_0\sum_{i=1}^{N_s} mes(B_i)E(t,\tilde{\varphi}_0,\tilde{\varphi}_1). \qquad (67)$$

We also know that,

$$\int_{Q_T}\frac{\partial}{\partial x_k}(m_k\frac{\partial\tilde{\varphi}}{\partial x_i})\frac{\partial\tilde{y}}{\partial x_i}dxdt = \int_{Q_T}\frac{\partial m_k}{\partial x_k}\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_i}dxdt + \int_{Q_T} m_k\frac{\partial^2\tilde{\varphi}}{\partial x_i^2}\frac{\partial\tilde{y}}{\partial x_i}dxdt$$

$$\int_{\Sigma_T} m.n\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_i}d\sigma dt = \underbrace{\int_{\Sigma_T^{0*}} m.n\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_i}d\sigma dt}_{m.n<0} + \underbrace{\int_{\Sigma_T^{1*}} m.n\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_i}d\sigma dt}_{m.n>0}$$

By grouping and reducing simultaneously we have:

$$\begin{aligned}\frac{1}{2}||\frac{\partial\tilde{\varphi}}{\partial n}||_{L^2(\Sigma_T)} \geq & -\frac{M_2.T}{2}\int_{Q_T}\left[(\frac{\partial\tilde{\varphi}}{\partial t})^2+(\frac{\partial\tilde{y}}{\partial x_k})^2\right]dxdt \\ & - \frac{T.R_0Ns}{2}\left[||\nabla\tilde{\varphi}||^2_{\mathbb{R}^2}+||\nabla\tilde{y}||^2_{\mathbb{R}^2}\right] \\ & - 2R_0\sum_{i=1}^{N_s} mes(B_i)E(t,\tilde{\varphi}_0,\tilde{\varphi}_1) + \int_{Q_T} divm\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial t}dxdt \\ & + \int_{Q_T}\frac{\partial}{\partial x_k}(m_k\frac{\partial\tilde{\varphi}}{\partial x_i})\frac{\partial\tilde{y}}{\partial x_i}dxdt - \int_{\Sigma_T} m.n\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_i}d\sigma dt \qquad \Longrightarrow\end{aligned}$$

$$\frac{1}{2}||\frac{\partial\tilde{\varphi}}{\partial n}||_{L^2(\Sigma_T)} \geq \left(-\frac{M_2.T}{2}-\frac{T.R_0Ns}{2}-2R_0\sum_{i=1}^{N_s} mes(B_i)+2.\frac{M_2.T}{2}\right)E(t,\tilde{\varphi}_0,\tilde{\varphi}_1), \qquad (68)$$

$$\geq \left(\frac{M_2.T}{2}-\frac{T.R_0Ns}{2}-2R_0\sum_{i=1}^{N_s} mes(B_i)\right)E(t,\tilde{\varphi}_0,\tilde{\varphi}_1). \qquad (69)$$

By posing

$$C_T^1(\Omega) = \left(\frac{M_2.T}{2}-\frac{T.R_0Ns}{2}-2R_0\sum_{i=1}^{N_s} mes(B_i)\right), \qquad (70)$$

$$\frac{1}{2}||\frac{\partial\tilde{\varphi}}{\partial n}||_{L^2(\Sigma_T)} \geq C_T^1(\Omega)E(t,\tilde{\varphi}_0,\tilde{\varphi}_1). \qquad (71)$$

3.5. **Exact Controllability Result.** Either the operator $\Lambda : H_0^1(\Omega)\times L^2(\Omega)$ Lions [9] defined by:

$$\Lambda(\tilde{\varphi}_0,\tilde{\varphi}_1) = (\tilde{y}'(0),-\tilde{y}(0)). \qquad (72)$$

Indeed, we know that Grisvard [2]:

$$\Lambda \in \mathcal{L}\left[H_0^1(\Omega)\times L^2(\Omega), H^{-1}(\Omega)\times L^2(\Omega)\right] \qquad \text{and} \qquad (73)$$

$$||\Lambda(\tilde{\varphi}_0, \tilde{\varphi}_1)||^2_{H^{-1}(\Omega)} = ||\tilde{y}(0)||^2_{L^2(\Omega)} + ||\tilde{y}'(0)||^2_{H^{-1}(\Omega)}. \tag{74}$$

Considering $\tilde{\varphi}_n \in C(0,T;D(A)) \cap C^1(0,T;D(A^{\frac{1}{2}})) \cap C^2(0,T;L^2(\Omega))$, and also $\tilde{\varphi}_{0n} \in D(A), \tilde{\varphi}_{1n} \in D(A^{\frac{1}{2}})$.
Assume $\zeta_n = \left[u(\tilde{\varphi})\right]\chi_{\bar{O}} = \left[\sum_{i=1}^m g_i\left(\int_\Omega \varphi'' w_i dx\right)\right]\chi_{\bar{O}}$ where O is an arbitrarily small part of the domain Ω not meeting any vertex of cracks.

Let $z_n \in C(0,T;H_0^2(\Omega)) \cap C^1(0,T;L^2(\Omega))$ solution of the following equation

$$(EOS)_6 : \begin{cases} z_n'' - \Delta z_n = \zeta_n & in\ Q_T, \\ (z_n).\chi_{\bar{O}} = 0 & on\ \Sigma_T, \\ z_n(T) = z_n'(T) = 0 & in\ \Omega. \end{cases}$$

So we have (3.5):

$$\langle \Lambda(\tilde{\varphi}_{0n}, \tilde{\varphi}_{1n}), (z_{0n}, z_{1n}) \rangle = \langle z_n'(0), \tilde{\varphi}_{0n} \rangle - \langle z_n(0), \tilde{\varphi}_{1n} \rangle. \tag{75}$$

By multiplying the equation $(EOS)_6$ by $\tilde{\varphi}_n$ and the equation (EOS) (18) by z_n we get:

$$-\int_{Q_T}(z_n'' - \Delta z_n)\tilde{\varphi}_n dxdt + \int_{Q_T}(\tilde{\varphi}_n'' - \Delta\tilde{\varphi}_n)z_n dxdt = \int_{Q_T}\sum_{i=1}^m ||g_i||\left(\int_\Omega ||\tilde{\varphi}''||.||w_i||dx\right)dt.$$

In particular on the open O:

$$\begin{aligned} -\int_{O\times]0,T[}(z_n'' - \Delta z_n)\tilde{\varphi}_n dxdt &+ \int_{O\times]0,T[}(\tilde{\varphi}_n'' - \Delta\tilde{\varphi}_n)z_n dxdt \\ &= \int_{O\times]0,T[}\sum_{i=1}^m ||g_i||\left(\int_O ||\tilde{\varphi}''||.||w_i||dx\right)dt, \end{aligned}$$

which is also written

$$\begin{aligned} \int_{O\times]0,T[}\sum_{i=1}^m ||g_i||\left(\int_O ||\tilde{\varphi}''||.||w_i||dx\right)dt &= -\langle \tilde{\varphi}_n, z_n' \rangle |_0^T + \langle \tilde{\varphi}_n', z_n \rangle |_0^T \\ &- \underbrace{\int_{\partial O\times]0,T[}\left(\frac{\partial\tilde{\varphi}_n}{\partial\nu}\right)^2 \tilde{\varphi}_n(\sigma)d\sigma dt}_{=0} \\ &- \underbrace{\int_{\partial O\times]0,T[}\left(\frac{\partial z_n}{\partial\nu}\right)^2 z_n(\sigma)d\sigma dt}_{=0}. \end{aligned}$$

Therefore

$$\begin{aligned}\int_{O\times]0,T[}\sum_{i=1}^{m}||g_i||\left(\int_O||\tilde{\varphi}''||.||w_i||dx\right)dt &= -\left\langle\tilde{\varphi}_n, z_n'\right\rangle|_0^T + \left\langle\tilde{\varphi}_n', z_n\right\rangle|_0^T,\\ &= \left\langle\Lambda(\tilde{\varphi}_{0n},\tilde{\varphi}_{1n}),(\tilde{\varphi}_{0n},\tilde{\varphi}_{1n})\right\rangle.\end{aligned}$$

Passing to the limit,

$$\langle\Lambda(\tilde{\varphi}_0,\tilde{\varphi}_1),(\tilde{\varphi}_0,\tilde{\varphi}_1)\rangle = \int_{O\times]0,T[}\sum_{i=1}^{m}||g_i||\left(\int_O||\tilde{\varphi}''||.||w_i||dx\right)dt.$$

But we also know that

$$\begin{aligned}\int_{O\times]0,T[}\sum_{i=1}^{m}||g_i||\left(\int_O||\tilde{\varphi}''||.||w_i||dx\right)dt &= \int_{O\times]0,T[}\sum_{i=1}^{m}\int_O||\tilde{\varphi}''||.||g_i||.||w_i||dxdt,\\ &\geq \int_{O\times]0,T[}.||\sum_{i=1}^{m}\int_O\tilde{\varphi}''dx||.|<g_i,w_i>|dxdt,\\ &\geq 2m\left[\frac{1}{2}\int_O||\tilde{\varphi}'|||_0^T dx\right].\end{aligned}$$

By covering the domain Ω by a disjoint finite union of openings O_i *ie* $\Omega = \bigcup_{i=1}^m O_i$ and $O_i \cap O_j = \emptyset$ if $i \neq j$.
Consequently, we deduce that:

$$\langle\Lambda(\varphi_0,\varphi_1),(\varphi_0,\varphi_1)\rangle \geq K_1(T-T_0)E_0. \tag{76}$$

Λ being linear, continuous and coercive on $H_0^1(\Omega)\times L^2(\Omega)$ for $T > T_0$, then according to a Classical Controllability Theorem, Λ is an isomorphism of $H_0^1(\Omega)\times L^2(\Omega)$ in $L^2(\Omega)\times H^{-1}(\Omega)$.
Let $(z_0, z_1)\in L^2(\Omega)\times H^{-1}(\Omega)$, the following equation

$$\Lambda(\tilde{\varphi}_0,\tilde{\varphi}_1) = (z_1, -z_0),$$

admit a unique solution $(\tilde{\varphi}_0,\tilde{\varphi}_1)\in H_0^1(\Omega)\times L^2(\Omega)$ for all $T > T_0$.
Let us now consider $\tilde{\varphi}$ and z respective solutions of the equations $(EOS)_5$ and $(EOS)_6$ with as initial conditions:

$$\begin{aligned} z_0 &= \tilde{\varphi}_0,\\ z_1 &= \tilde{\varphi}_1,\\ \zeta_n &= \left(\sum_{i=1}^{m} g_i\left(\int_\Omega\tilde{\varphi}''w_i dx\right)\right)\chi_O,\\ &\quad and\\ \varphi &= \begin{cases}\tilde{\varphi}.\chi_O & on\ \Sigma_T^{*0},\\ 0 & on\ \Sigma_T^{*1}.\end{cases}\end{aligned}$$

By a uniqueness of solutions theorem, we deduce that: $\tilde{\varphi} = z$ so therefore $z(T) = z'(T) = 0$.

Hence the result of exact controllability.

Remark 3. *This result does not depend on any geometrical condition: consequently the crack lines may not be concurrent; and, the exact controllability result has been proven.*

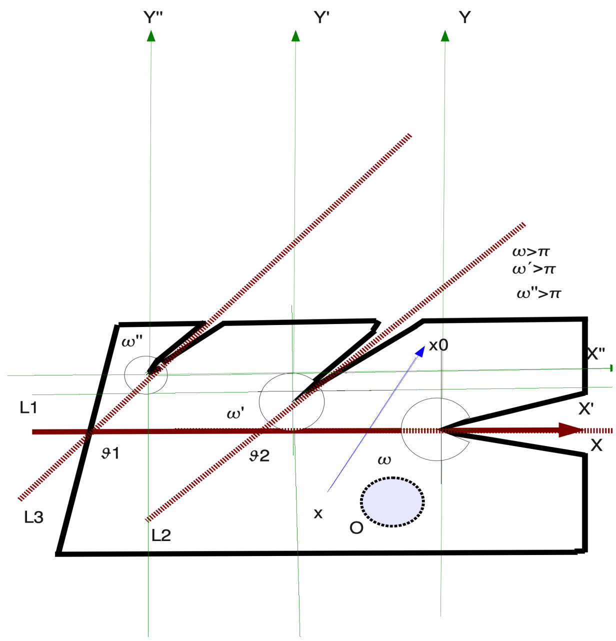

FIGURE 2. Non-convex domain with non-concurrent cracks

4. Conclusion and Perspectives

The presence of cracks, corners or angles in a mechanical device or materials always leads to the appearance of singularities. And, once the diagnosis of these cracks (desired or not) has been made, it is necessary to try to control them without

major geomeric constraints.
One of the objectives that we set ourselves, within the framework of this research paper, was assess the exact controllability of the wave equation in the cracked domains without constraints on the cracks. If anything, the formulas of integrations by parts (formulas of Green in the fields with corners and/or cracks) could be done (to our knowledge) only if the lines of cracks or their support were concurrent.
Based on recent work by Dauge [13, 14], Dauge [15] and Costabel [16], we were able to establish, without additional assumptions on the nature of the cracks or their support, the exact controllability of the wave equation with more cracks. Consequently its results were obtained on a non-convex polygonal domain with non-concurrent crack lines. From the results obtained in this paper, certain questions naturally emerge. Our goal is to no longer have constraining geometric conditions ("Closer" to reality).
When it comes to the perspectives, we have a double goal that we plan on achieving in the near future. Firstly, generalize in higher dimension the results obtained in this paper. And,secondly, make numerical simulations to support its theoretical results.

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