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Integral Inequalities for Some Convex Functions

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Abstract

In this paper, we established some new integral inequalities for different kinds of convex functions by using some classical inequalities.

Keywords: Convex functions, m −convex functions, s −convex functions, (α, m) −convex functions, log −convex functions.

1. Introduction

We recall following definitions.

The functions $f:[a,b]\to\mathbb{R}$, is said to be convex, if we have

$$f(tx+(1-t)y)\le tf(x)+(1-t)f(y)$$

for all $x,y\in[a,b]$ and $t\in[0,1]$. We can define starshaped functions on $[0,b]$ which satisfy the condition

$$f(tx)\le tf(x)$$

for $t\in[0,1]$. TOADER (1984) defined the concept of m −convexity as the following:

Definition 1. The function $f:[a,b]\to\mathbb{R}$ is said to be m −convex, where $m\in[0,1]$, if for every $x,y\in[a,b]$ and $t\in[0,1]$, we have:

$$f(tx+m(1-t)y)\le tf(x)+m(1-t)f(y).$$

Denote by $K_m(b)$ the set of the m −convex functions on $[0,b]$ for which $f(0)\le 0$.

Some interesting and important inequalities for m −convex functions can be found in our references.

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HUDZIK and MALIGRANDA (1994) considered among others the class of functions which are s −convex in the second sense.

Definition 2. A function $f:\mathbb{R}^+\to\mathbb{R}$ where $\mathbb{R}^+=[0,\infty)$, is said to be s −convex in the second sense if

$$f(\alpha x+\beta y)\le\alpha^s f(x)+\beta^s f(y)$$

for all $x,y\in[0,\infty)$, $\alpha,\beta\ge 0$ with $\alpha+\beta=1$ and for some fixed $s\in(0,1]$. This class of s −convex functions in the second sense is usually denoted by K_s^2.

s −convexity introduced by BRECKNER (1978) as a generalization of convex functions. Also, BRECKNER (1993) proved the fact that the set valued map is s −convex only if the associated support function is s −convex function.

DRAGOMIR and FITZPATRICK (1999) proved the following Hadamard type integral inequality:

Theorem 1. Suppose that $f:[0,\infty)\to[0,\infty)$ is an s −convex function in the second sense, where $s\in(0,1]$ and let $a,b\in[0,\infty)$, $a<b$. If $f\in L_1[0,1]$, then the following inequalities hold:

(1.1)

$$2^{s-1}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{s+1}$$

The constant $k=\frac{1}{s+1}$ is the best possible in the second inequality in (1.1). The above inequalities are sharp.

Several properties of s −convexity in the first sense are discussed in the paper that is written by HUDZIK and MALIGRANDA (1994). Obviously, s −convexity means just convexity when $s=1$. Some new Hermite Hadamard type inequalities based on concavity and s −convexity established by KIRMACI *et al. (2007)*. For related results see the papers DRAGOMIR and FITZPATRICK (1999) and KIRMACI *et al (2007).*

DRAGOMIR (2002) proved the following theorem.

Theorem 2. Let $f:[0,\infty)\to\mathbb{R}$ be an m −convex function with $m\in(0,1]$ and $0\le a\le b$. If $f\in L_1[0,1]$, then one has the inequalities

(1.2)

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b\frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx$$

$$\le\frac{m+1}{4}\left[\frac{f(a)+f(b)}{2}+m\,\frac{f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)}{2}\right]$$

MIHEŞAN (1993) gave definition of (α,m) −convexity as following;

Definition 3. The function $f:[0,b)\to\mathbb{R}$, $b>0$ is said to be (α,m) −convex, where $(\alpha,m)\in[0,1]^2$, if we have

$$f(tx+m(1-t)y)\le t^a f(x)+m(1-t^a)f(y)$$

for all $x,y\in[0,b]$ and $t\in[0,1]$.

Denote by $K_m^a(b)$ the class of all (α,m) −convex functions on $[0,b]$ for which $f(0)\le 0$. If we choose $(\alpha,m)=(1,m)$, it can be easily seen that (α,m) −convexity reduces to m −convexity and for $(\alpha,m)=(1,1)$. We have ordinary convex functions on $[0,b]$. For the recent results based on the above definition see the papers BAKULA *et al.* (2006), BAKULA *et al.* (2008), ÖZDEMİR *et al.* (2010), SARIKAYA *et al.* (2011), SET *et al.* (2009), ÖZDEMİR *et al.* (2011).

Definition 4. (See PECARIC *et al.* (1992)) A function $f:I\to[0,\infty)$ is said to be log −convex or multiplicatively convex if $logf$ is convex or equivalently if for all $x,y\in I$ and $t\in[0,1]$ one has the inequality:

(1.3)

$$f(tx+(1-t)y)\le[f(x)]^t[f(y)]^{1-t}$$

We note that a log −convex function is convex, but the converse may not necessarily be true.

Theorem 3. (ÖZDEMİR *et al.* (2010)) Let $f,g:[a,b]\to\mathbb{R}$ be real valued non-negative convex functions and $F(x,y)(t),G(x,y)(t):[0,1]\to\mathbb{R}^+$ are defined as the following;

$$F(x,y)(t)=\tfrac{1}{2}\left[f(tx+(1-t)y)+f\big((1-t)x+ty\big)\right]$$

$$G(x,y)(t)=\tfrac{1}{2}[g(tx+(1-t)y)+g((1-t)x+ty)]$$

for all $t\in[0,1]$, we have

(1.4)

$$\frac{1}{b-a}\int_a^b F\left(x,\frac{a+b}{2}\right)(t)G\left(x,\frac{a+b}{2}\right)(t)\,dx$$

$$\le\frac{1}{4(b-a)}\int_a^b f(x)g(x)\,dx+\frac{3}{16}[M(a,b)+N(a,b)]$$

And

(1.5)

$$\frac{2}{(b-a)^2}\int_a^b\int_a^b F(x,y)(t)G(x,y)(t)\,dx\,dy$$

$$\le\frac{1}{b-a}\int_a^b f(x)g(x)\,dx+\frac{1}{4}[M(a,b)+N(a,b)]$$

where

$$M(a,b)=f(a)g(a)+f(b)g(b)$$

$$N(a,b)=f(a)g(b)+f(b)g(a).$$

The main purpose of this paper is to prove some new inequalities as above, but now for m −convex and s −convex functions by modified the mappings $F(x,y)(t)$ and $G(x,y)(t)$.

2. *Main Results*

Theorem 4. Let $f,g:[0,\infty)\to\mathbb{R}^+$ be m −convex functions with $m\in(0,1]$, $0\le a<b$ and $f,g,fg\in L_1[a,b]$ $F(x,y)(t),G(x,y)(t):[0,1]\to\mathbb{R}^+$ are defined as the followings:

$$F(x,y)(t)=\tfrac{1}{2}\left[f(tx+m(1-t)y)+f\big((1-t)x+mty\big)\right]$$

$$G(x,y)(t)=\tfrac{1}{2}\left[g(tx+m(1-t)y)+g\big((1-t)x+mty\big)\right]$$

for all $t\in[0,1]$, we have

(2.1)

$$\int_a^b F\left(x,\frac{a+b}{2}\right)(t)G\left(x,\frac{a+b}{2}\right)(t)dx\le\frac{1}{4}\int_a^b f(x)g(x)\,dx$$

$$+\frac{m^2}{4}(b-a)\mu_1\mu_2+\frac{m}{4}\left[\mu_1\int_a^b f(x)\,dx+\mu_2\int_a^b g(x)\,dx\right]$$

where

$$\mu_1=\frac{m+1}{4}\left(\frac{g(a)+g(b)}{2}+m\frac{g\left(\frac{a}{m}\right)+g\left(\frac{b}{m}\right)}{2}\right)$$

$$\mu_2 = \frac{m+1}{4}\left(\frac{f(a)+f(b)}{2} + m\frac{f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)}{2}\right)$$

and

(2.2)

$$\frac{2}{(b-a)^2}\int_a^b\int_a^b F(x,y)(t)G(x,y)(t)\,dx\,dy$$
$$\leq \frac{m^2+1}{2}\int_a^b f(x)g(x)\,dx + \frac{m}{b-a}\int_a^b f(x)\,dx\int_a^b g(y)\,dy.$$

Proof. Since f and g are $m-$convex functions, we can write

$$F(x,y)(t)$$
$$\leq \frac{1}{2}[\,tf(x)+m(1-t)f(y)+(1-t)f(x)+mtf(y)]$$
$$=\tfrac{1}{2}[\,f(x)+mf(y)],$$

(2.3)

$$F\left(x,\frac{a+b}{2}\right)(t) \leq \frac{1}{2}\left[f(x)+mf\left(\frac{a+b}{2}\right)\right]$$

and analogously, if we set $x = x$ and $y = \frac{a+b}{2}$, we can write

$$G(x,y)(t)$$
$$\leq \frac{1}{2}[tg(x)+m(1-t)g(y)+(1-t)g(x)+mtg(y)]$$
$$=\tfrac{1}{2}[\,g(x)+mg(y)],$$

(2.4)

$$G\left(x,\frac{a+b}{2}\right)(t) \leq \frac{1}{2}\left[g(x)+mg\left(\frac{a+b}{2}\right)\right]$$

By multiplying the inequalities (2.3) and (2.4), we get

(2.5)

$$F\left(x,\frac{a+b}{2}\right)(t)\,G\left(x,\frac{a+b}{2}\right)(t)$$
$$\leq \frac{1}{4}\left[f(x)+mf\left(\frac{a+b}{2}\right)\right]\left[g(x)+mg\left(\frac{a+b}{2}\right)\right]$$
$$=\left[\frac{f(x)g(x)+mf\left(\frac{a+b}{2}\right)g(x)+\,mg\left(\frac{a+b}{2}\right)f(x)+m^2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)}{4}\right].$$

Integrating the above inequality with respect to x on $[a,b]$, we obtain the following inequality:

(2.6)

$$\int_a^b F\left(x,\frac{a+b}{2}\right)(t)\,G\left(x,\frac{a+b}{2}\right)(t)\,dx$$
$$\leq \frac{1}{4}\left\{\int_a^b f(x)g(x)\,dx + \int_a^b mf\left(\frac{a+b}{2}\right)g(x)\,dx\right.$$
$$\left.+\int_a^b mg\left(\tfrac{a+b}{2}\right)f(x)\,dx + m^2\int_a^b f\left(\tfrac{a+b}{2}\right)g\left(\tfrac{a+b}{2}\right)dx\right\}$$

Using the inequalities in (1.2) and by rewriting the (2.6), the proof is completed.

Remark 1. If we choose $m = 1$, inequalities (2.1) and (2.2) reduces to (1.4) and (1.5) respectively.

Theorem 5. Let $f,g\colon[0,\infty)\to\mathbb{R}$ be $s-$convex functions in the second sense and $F(x,y)(t), G(x,y)(t)\colon[0,1]\to\mathbb{R}^+$ are defined as the following:

$$F(x,y)(t) = \tfrac{1}{2}[f(t^s x+(1-t)^s y)+f((1-t)^s x+t^s y)]$$

$$G(x,y)(t) = \tfrac{1}{2}[g(t^s x+(1-t)^s y)+f((1-t)^s x+t^s y)]$$

If $f,g,fg \in L_1[a,b]$, for all $t\in[0,1]$, we have

$$\int_0^1 [F(a,b)(t)+\;G(a,b)(t)]dt$$
$$\leq \frac{f(a)+f(b)+g(a)+g(b)}{s+1}$$

and

$$\int_0^1 F(a,b)(t)\,G(a,b)(t)\,dt$$
$$\leq\;[M(a,b)+N(a,b)]\left[\frac{1}{2(2s+1)}+\;\frac{1}{2}\beta(s+1\,,s+1)\right]$$

where

$$M(a,b) = f(a)g(a)+f(b)g(b)$$
$$N(a,b)\;=f(a)g(b)+\;f(b)g(a)$$

and the Euler Beta function is defined by

$$\beta(x,y) =\;\int_0^1 t^{x-1}\,(1-t)^{y-1}\;dt, \qquad x,y\;>0.$$

Proof. Since f and g are $s-$convex functions in the second sense, we can write

(2.7)

$$F(x,y)(t)\leq\;\frac{t^s f(x)+(1-t)^s f(y)+(1-t)^s f(x)+t^s f(y)}{2}$$

(2.8)

$$G(x,y)(t) \leq \frac{t^s g(x)+(1-t)^s g(y)+(1-t)^s g(x)+t^s g(y)}{2}$$

If we set $x=a, y=b$ in the above inequalities and by addition, then by integrating with respect to t over [0,1], we get:

$$\int_0^1 [F(a,b)(t)G(a,b)(t)]dt$$
$$\leq \left[\frac{f(a)+f(b)+g(a)+g(b)}{2}\right]\left[\int_0^1 t^s dt + \int_0^1 (1-t)^s dt\right]$$
$$= \frac{f(a)+f(b)+g(a)+g(b)}{s+1}$$

This completes the proof of the first inequality.

For the proof of the second inequality, if we multiply the inequalities (2.7) and (2.8) for $x=a, y=b$ and by integrating with respect to t over [0,1], we have

$$\int_0^1 [F(a,b)(t)G(a,b)(t)]dt$$
$$= [M(a,b)+N(a,b)]\left[\frac{1}{2(2s+1)} + \frac{1}{2}\beta(s+1,s+1)\right]$$

The proof is completed.

Theorem 6. Let $f,g:[0,\infty) \to \mathbb{R}^+$ be (α,m) −convex fonctions with $(\alpha,m) \in (0,1]^2$, $0 \leq a < b$ and $f,g,fg \in L_1[a,b]$. $F(x,y)(t), G(x,y)(t): [0,1] \to \mathbb{R}^+$ are defined as the followings:

$$F(x,y)(t) = \frac{1}{2}[f(tx + m(1-t)y) + f(m(1-t)x + ty)]$$

$$G(x,y)(t) = \frac{1}{2}[g(tx + m(1-t)y) + g(m(1-t)x + ty)]$$

for all $t \in [0,1]$, we have

(2.9)

$$\int_0^1 [F(a,b)(t)+G(a,b)(t)]dt$$
$$\leq \frac{1}{2}\left[\frac{1+ma}{1+a}\right][f(a)+f(b)+g(a)+g(b)]$$

and

(2.10)

$$\int_0^1 [F(a,b)(t)G(a,b)(t)]dt$$
$$\leq \frac{1}{4}[M(a,b)+N(a,b)]\left(\frac{a(2m+m^2}{a+1} + \frac{m^2+1}{2a+1}\right)$$

where $M(a,b)$ and $N(a,b)$ as in Theorem 5.

Proof. Since f and g are (α,m) −convex functions, we can write

$$F(x,y)(t) \leq \frac{t^a f(x)+m(1-t^a)f(y)+t^a f(y)+m(1-t^a)f(x)}{2}$$

If we set $x=a$ and $y=b$, we get

(2.11)

$$F(a,b)(t) \leq \frac{1}{2}[(f(a)+f(b))(t^a + m(1-t^a))]$$

and analogously, we have

(2.12)

$$G(a,b)(t) \leq \frac{1}{2}[(g(a)+g(b))(t^a + m(1-t^a))]$$

By adding the inequalities (2.11) and (2.12), we get

(2.13)

$$F(a,b)(t)+G(a,b)(t)$$
$$\leq \frac{1}{2}[(f(a)+f(b)+g(a)+g(b))(t^a + m(1-t^a))]$$

Integrating the above inequality with respect to t on [0,1], we obtain the inequality (2.9). For the proof of the inequality (2.10), by multiplying the inequalities (2.11) and (2.12), we have

$$F(a,b)(t)G(a,b)(t)$$
$$\leq \frac{1}{4}[M(a,b)+N(a,b)][t^{2a} + 2m\,t^a(1-t^a) + m^2(1-t^a)^2]$$

By integrating the above inequality with respect to t over [0,1], we get the inequality (2.10).

Theorem 7. Let $f,g:[0,\infty) \to \mathbb{R}^+$ be logarithmically convex functions on $[0,\infty)$ and $f,g,fg \in L_1[a,b]$. $F(x,y)(t), G(x,y)(t): [0,1] \to \mathbb{R}^+$ are defined as in Theorem 3, then the following inequalities hold;

(2.14)

$$\int_0^1 F(a,b)(t)G(a,b)(t)dt$$
$$\leq \frac{1}{2}\left[L(f(a)g(a), f(b)g(b)) + L(f(a)g(b), f(b)g(a))\right]$$

for all $t \in [0,1]$, where

$$L(f(a)g(a), f(b)g(b)) = \frac{f(a)g(a)-f(b)g(b)}{ln f(a)g(a) - ln\, f(b)g(b)}$$

and

$$L\big(f(a)g(b), f(b)g(a)\big) = \frac{f(a)g(b) - f(b)g(a)}{ln f(a)g(b) - ln\, f(b)g(a)}.$$

Proof. Since f, g are log −convex functions on $[a, b] \subseteq [0, \infty)$, we can write

$$F(x,y)(t) \leq \frac{1}{2}[f^t(x) + f^{(1-t)}(y) + f^{(1-t)}(x) + f^t(y)]$$

and

$$G(x,y)(t) \leq \frac{1}{2}[g^t(x) + g^{(1-t)}(y) + g^{(1-t)}(x) + g^t(y)].$$

If we set $x = a$ and $y = b$, we have

(2.15)

$$F(x,y)(t) \leq \frac{1}{2}[f^t(a) + f^{(1-t)}(b) + f^{(1-t)}(a) + f^t(b)]$$

and

(2.16)

$$G(a,b)(t) \leq \frac{1}{2}[g^t(a) + g^{(1-t)}(b) + g^{(1-t)}(a) + g^t(b)]$$

By multiplying the inequalities (2.15) and (2.16), we get

$$F(a,b)(t)G(a,b)(t)$$
$$\leq \frac{1}{4}[f^t(a) + f^{(1-t)}(b) + f^{(1-t)}(a) + f^t(b)]$$
$$\times [g^t(a) + g^{(1-t)}(b) + g^{(1-t)}(a) + g^t(b)]$$

By integrating the above inequality with respect to t on [0,1], we obtain the inequality (2.14).

REFERENCES

DRAGOMIR, S.S., (2002). *On Some New Inequalities of Hermite-Hadamard Type For* m *−Convex Functions*, Tamkang Journal of Mathematics, 33 (1).

BAKULA, M.K., PECARIC, J., RIBICIC, M., (2006). *Companion inequalities to Jensen's inequality for* m *−convex and* (α, m) *−convex functions*, J. Inequal. Pure Appl. Math., 7. Article 194.

ÖZDEMIR, M.E., AVCI, M. and SET, E., (2010). *On some inequalities of Hermite Hadamard type via* m *−convexity*, Appl. Math. Lett., 23 1065-1070.

BAKULA, M.K., ÖZDEMİR, M.E. and PECARIC, J., (2008). *Hadamard type inequalities for* m *−convex and* (α, m) *−convex functions*, J. Inequal. Pure Appl. Math., 9. Article 96.

HUDZIK, H. and MALIGRANDA, L., (1994). *Some remarks on* s *−convex functions*, Aequationes Math., (48) 100-111.

DRAGOMIR S.S. and TOADER, G., (1993). *Some inequalities for* m *−convex functions*, Studia University Babes Bolyai, Mathematica, 38 (1), 21-28.

TOADER, G., (1984). Some generalization of the convexity, Proc. Colloq. Approx. Opt., Cluj-Napoja, 329-338.

TOADER, G., (1998). *On a generalization of the convexity*, Mathematica, 30 (53), 83-87.

DRAGOMIR S.S., (2002). *On some new inequalities of Hermite-Hadamard type for* m *−convex functions*, Tamkang Journal of Mathematics, 33 (1).

BRECKNER, W.W., (1978). *Stetingkeitsaussagen fur eine Klasse verallgemeinerter konvexer funktionen in topologischen linearel Raumen*, Pupl.Inst.Math., 23, 13-20.

BRECKNER, W.W., (1993). *Continuity of generalized convex and generalized concave set valued functions*, Rev Anal., Number Thkor. Approx., 22, 39-51.

DRAGOMIR, S.S. and FITZPATRICK, S., (1999). *The Hadamard's inequality for s-convex functions in the second sense*, Demonstratio Math., 32 (4), 687-696.

KIRMACI, U.S., BAKULA, M.K., ÖZDEMİR, M.E. and PECARIC, J., (2007). *Hadamard-type inequalities for* s *−convex functions*, Applied Mathematics and Computation, 193, 26-35.

ÖZDEMİR, M.E., SET, E. and SARIKAYA, M.Z., (2010). *Konveks Fonksiyonlar Üzerine Notlar*, Atatürk University.

SARIKAYA, M.Z., SET, E. and ÖZDEMİR, M.E., (2011). *Some new Hadamard's type inequalits for co-ordinated* m *−convex and* (α, m) *−convex functions*, Hacettepe J. of. Math. and St., 40 219-229.

MIHEŞAN, V.G., (1993). *Generalization of the convexity, Seminar of Functional Equations*, Approx. and convex, Cluj-Napoca (Romania).

SET, E., SARDARI, M., ÖZDEMIR, M.E. and ROOIN, J., (2009). *On generalizations of the Hadamard inequality for* (α, m) *−convex functions*, RGMIA Res. Rep. Coll., 12 (4), Article 4.

ÖZDEMIR, M.E., KAVURMACI, H. and SET, E., (2010). *Ostrowski's type inequalities for* (α, m) *−convex functions*, Kyungpook Math. J. 50, 371-378.

ÖZDEMIR, M.E., AVCI, M. and KAVURMACI, H., (2011). *Hermite-Hadamard- type inequalities via* (α, m) *−convextiy*, Computers and Mathematics with Applications, 61, 2614-2620.

PECARIC, J., PROSCHAN, F. and TONG, Y.L., (1992). *Convex Functions, Partial Orderings and Statistical Applications*, Acedemic Press, Inc.