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COEFFICIENTS OF RANDIĆ AND SOMBOR CHARACTERISTIC POLYNOMIALS OF SOME GRAPH TYPES

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Abstract. Let G be a graph. The energy of G is defined as the summation of absolute values of the eigenvalues of the adjacency matrix of G. It is possible to study several types of graph energy originating from defining various adjacency matrices defined by correspondingly different types of graph invariants. The first step is computing the characteristic polynomial of the defined adjacency matrix of G for obtaining the corresponding energy of G. In this paper, formulae for the coefficients of the characteristic polynomials of both the Randić and the Sombor adjacency matrices of path graph P_n, cycle graph C_n are presented. Moreover, we obtain the five coefficients of the characteristic polynomials of both Randić and Sombor adjacency matrices of a special type of $3-$regular graph R_n.

1. Introduction

Let $G = (V, E)$ be a simple graph with the number of n vertices and m edges. If two vertices v_i and v_j are connected with an edge e, then they are called adjacent vertices and they are expressed as $e = v_i v_j$ or $e = v_j v_i$. If a vertex v is a terminal point of edge e, then they are called incident. Degree of a vertex v_i is the number of edges that are incident to the vertex v_i and it is denoted by $d(v_i)$. A graph does not contain any cycle is called acyclic. If there is a way between all vertices of a graph, then it is called connected. Connected acyclic graph is called tree. Path graph is a tree that is in the form of straight line with degrees of two vertices are one and degrees of other vertices are two and it is denoted by P_n. Cycle graph is a graph that contains only one cycle through all vertices and degrees of all vertices are two. It is denoted by C_n. If degrees of all vertices of G are k, then it is called $k-$regular graph.
Let $A = [a_{ij}]_{n\times n}$ be a matrix. If v_i and v_j are adjacent vertices then a_{ij} and a_{ji} are

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1 or else 0, see [1]. A is called adjacency matrix of G. Analogous with linear algebra, $\det(\lambda \cdot I - A)$ is called the characteristic polynomial of G and we denoted it by $P_G(\lambda)$. Roots of $P_G(\lambda)$ are called eigenvalues of G and the energy of G is defined as the summation of absolute values of the eigenvalues of G, see [6]. Furthermore, there are many topological invariants used in several researches. In [16], Randić index is a molecular descriptor defined by Milan Randić and denoted by $\sum_{v_i v_j \in E} \frac{1}{\sqrt{d(v_i)d(v_j)}}$. In [9], another important molecular descriptor recently introduced by Ivan Gutman with the name Sombor index is $\sum_{v_i v_j \in E} \sqrt{(d(v_i))^2 + (d(v_j))^2}$. In addition to topological invariants, several adjacency matrix forms have been defined until today, for more details see [13]. With the help of various adjacency matrices defined by correspondingly different types of graph invariants, it is possible to study different types of graph energy such as laplacian energy, distance energy, Randić energy and Sombor energy, see for details [15]. Two of the well-known them are Randić and Sombor matrices that are related to the corresponding topological indices. Researchers have studied these notions from various aspects so far. Some studies on the subjects Randić and Sombor adjacency matrices and energies can be seen in [2, 4, 5, 8, 10–12, 14, 17]. The first step to obtaining the desired energy type of a graph G is to calculate the characteristic polynomials of the corresponding adjacency matrices. In this paper, we obtain formulae for each coefficient of both Randić and Sombor characteristic polynomials of path graph P_n and cycle graph C_n by using a well-known equation. Also, we present formulae for some coefficients of Randić and Sombor characteristic polynomials of a special type of $3-$regular graph.

2. Coefficients of Randić and Sombor Characteristic Polynomials of Path, Cycle and a Special Type of 3-Regular Graphs

Let $G = (V, E)$ be a simple graph with n vertices and m edges. The Randić matrix of G was mentioned in the substantial book [3] and the Sombor matrix was defined in [10]. We denote the Randić and Sombor adjacency matrices of G by $R(G)$ and $S(G)$, respectively. $R(G) = [r_{ij}]_{n\times n}$ and $S(G) = [s_{ij}]_{n\times n}$ are formed by using the adjacency of vertices as the following:

$$r_{ij} = \begin{cases} \frac{1}{\sqrt{d(v_i)d(v_j)}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

$$s_{ij} = \begin{cases} \sqrt{(d(v_i))^2 + (d(v_j))^2}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $R(G)$ and $S(G)$ are symmetric matrices with dimension $n \times n$.
Let us denote the ordinary characteristic polynomial of G as follows:

$$P_G(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n.$$

Let us denote the number of components in an elementary subgraph G' which are single edges and cycles as $\rho_0(G')$ and $\rho_1(G')$, respectively.

In [18], the formula for the coefficients of the ordinary characteristic polynomial are given by

$$c_k = \sum(-1)^{\rho_0(G')+\rho_1(G')}2^{\rho_1(G')}, \tag{1}$$

where the summation is taken over all elementary subgraphs G' with k vertices for $1 \leq k \leq n$. At the present time, the formula is called Sachs theorem, for details and history of the theorem see [1, 3, 7].

Let ψ_{ij} denote the nonzero value in the entry ij of the adjacency matrix of a vertex-degree-based topological index of a regular graph G. As a natural result of the Sachs theorem, it is clear that the formula for each coefficient c'_k of the characteristic polynomial of the adjacency matrix of this vertex-degree-based topological index is obtained by

$$c'_k = (\psi_{ij})^k \sum(-1)^{\rho_0(G')+\rho_1(G')}2^{\rho_1(G')},$$

where the summation is taken over all elementary subgraphs G' with k vertices for $1 \leq k \leq n$.

In this paper, we aim to obtain all coefficients of the Randić and Sombor characteristic polynomials of path graph P_n and regular graph C_n by using the numbers of elementary subgraphs. Similarly, we also aim to obtain some coefficients of the same characteristic polynomials of a special type of $3-$regular graph we call R_n. We begin with the Randić characteristic polynomial of P_n. Let us note that the Randić characteristic polynomial of P_2 is equal to $\lambda^2 - 1$. Moreover, let us denote the set of non-negative integer numbers and the set of positive integer numbers by $\mathbb{Z}^*$ and $\mathbb{Z}^+$, respectively.

Theorem 1. *Let $P_n = (V, E)$ be a path graph with n vertices and $n-1$ edges. Let $P_{P_n}(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c$ be the Randić characteristic polynomial of P_n, where $c_k \in \mathbb{R}, 1 \leq k \leq n-1$. The formulae for the coefficients c_ks of the Randić characteristic polynomial of P_n, where $n \geq 3$, are as follows:*

$$c_2 = (-1)^{\frac{k}{2}}(\tfrac{n-3}{4} + 1),$$

$$c_k = 0, \text{ where } k \in 2\mathbb{Z}^* + 1,$$

$$c_k = (-1)^{\frac{k}{2}} \Big[\tbinom{(n-1)-\frac{k}{2}}{\frac{k}{2}-1} \cdot (\tfrac{1}{\sqrt{2}})^2 \cdot (\tfrac{1}{2})^{2(\frac{k}{2}-1)} + \tbinom{(n-2)-\frac{k}{2}}{\frac{k}{2}-2} \cdot (\tfrac{1}{4}) \cdot (\tfrac{1}{2})^{2(\frac{k}{2}-2)} \cdot (\tfrac{1}{\sqrt{2}})^2$$

$$+ \textstyle\sum_{j=\frac{k}{2}-1}^{n-2-\frac{k}{2}} \tbinom{j}{\frac{k}{2}-1} \cdot (\tfrac{1}{2})^k + \sum_{j=\frac{k}{2}-2}^{n-3-\frac{k}{2}} \tbinom{j}{\frac{k}{2}-2} \cdot (\tfrac{1}{4}) \cdot (\tfrac{1}{2})^{k-2} \Big], \text{ where } k \geq 4, k \in 2\mathbb{Z}^+.$$

Proof. First of all, it is clear that $c_2 = (-1)^{\frac{k}{2}}(\frac{n-3}{4} + 1)$ for all $n \geq 3$. By the Eqn. 1, we know that c_k consists of the contributions of several elementary subgraphs of

G with k vertices. Also, since P_n does not have any cycle we take into account only edges that do not have any common vertex. At this point, we will apply a method that involves an edge removing and continue calculation of remaining part. Let us consider a path graph P_n with n vertices whose vertices are labelled by $1, 2, \cdots, n$. For calculation of c_k, if we remove the edge v_1v_2, then remaining part with $k-2$ vertices consists of number of

$$\binom{\frac{k}{2}-2}{\frac{k}{2}-2}+\binom{\frac{k}{2}-1}{\frac{k}{2}-2}+\cdots+\binom{n-\frac{k}{2}-3}{\frac{k}{2}-2}+\binom{n-\frac{k}{2}-2}{\frac{k}{2}-2}=\binom{n-1-\frac{k}{2}}{\frac{k}{2}-1}$$

combinations. Moreover, if we remove any edge v_iv_{i+1} which is not terminal edges of P_n, then remaining part consists one of the numbers

$$\binom{\frac{k}{2}-1}{\frac{k}{2}-1}, \binom{\frac{k}{2}}{\frac{k}{2}-1}, \cdots, \binom{n-2-\frac{k}{2}}{\frac{k}{2}-1}.$$

Hence, contributions of elementary subgraphs that are in the form of $v_1v_2, \cdots, v_iv_j$ can be $(\frac{1}{\sqrt{2}})^2 \cdot (\frac{1}{2})^2 \cdots (\frac{1}{2})^2 \cdot (\frac{1}{2})^2$ or $(\frac{1}{\sqrt{2}})^2 \cdot (\frac{1}{2})^2 \cdots (\frac{1}{2})^2 \cdot (\frac{1}{\sqrt{2}})^2$. Hereby, the contribution of the type subgraphs that contribute to c_k in the $(\frac{1}{\sqrt{2}})^2 \cdot (\frac{1}{2})^2 \cdots (\frac{1}{2})^2 \cdot (\frac{1}{2})^2$ form is obtained as $\left[\binom{(n-1)-\frac{k}{2}}{\frac{k}{2}-1} - \binom{(n-2)-\frac{k}{2}}{\frac{k}{2}-2}\right] \cdot (\frac{1}{\sqrt{2}})^2 \cdot (\frac{1}{2})^{2(\frac{k}{2}-1)}$. Moreover, the contribution of the other type subgraphs that contribute to c_k in the $(\frac{1}{\sqrt{2}})^2 \cdot (\frac{1}{2})^2 \cdots (\frac{1}{2})^2 \cdot (\frac{1}{\sqrt{2}})^2$ form is obtained as $\binom{(n-2)-\frac{k}{2}}{\frac{k}{2}-2} \cdot (\frac{1}{\sqrt{2}})^4 \cdot (\frac{1}{2})^{2(\frac{k}{2}-2)}$. Thus, the first part of the formula is obtained as $\binom{(n-1)-\frac{k}{2}}{\frac{k}{2}-1} \cdot (\frac{1}{\sqrt{2}})^2 \cdot (\frac{1}{2})^{2(\frac{k}{2}-1)} + \binom{(n-2)-\frac{k}{2}}{\frac{k}{2}-2} \cdot (\frac{1}{4}) \cdot (\frac{1}{2})^{2(\frac{k}{2}-2)} \cdot (\frac{1}{\sqrt{2}})^2$ by arranging the contribution statements above.

Furthermore, contributions of elementary subgraphs that are in the form of $v_av_b, \cdots, v_iv_j$ can be $(\frac{1}{2})^2 \cdot (\frac{1}{2})^2 \cdots (\frac{1}{2})^2 \cdot (\frac{1}{2})^2$ or $(\frac{1}{2})^2 \cdot (\frac{1}{2})^2 \cdots (\frac{1}{2})^2 \cdot (\frac{1}{\sqrt{2}})^2$, where $a \neq 1, b \neq 2$ or $a \neq 2, b \neq 1$. Similar to the previous part of the proof, two contribution equations of c_k are obtained as $\sum_{j=\frac{k}{2}-1}^{n-2-\frac{k}{2}} \binom{j}{\frac{k}{2}-1} \cdot (\frac{1}{2})^k$ and $\sum_{j=\frac{k}{2}-2}^{n-3-\frac{k}{2}} \binom{j}{\frac{k}{2}-2} \cdot (\frac{1}{4}) \cdot (\frac{1}{2})^{k-2}$, respectively. As a result, since there is no other elementary subgraph contribution type, the proof is completed by summing all the above subgraph contributions. $\square$

In the next corollary, we continue with the Sombor characteristic polynomial of P_n. Firstly, it is clear that the Sombor characteristic polynomial of P_2 is equal to $\lambda^2 - 2$.

Corollary 1. *Let $P_n = (V, E)$ be a path graph with n vertices and $n-1$ edges. Let $P_{P_n}(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c$ be the Sombor characteristic polynomial of P_n, where $c_k \in \mathbb{Z}, 1 \leq k \leq n-1$. The formulae for the coefficients c_ks of the Sombor characteristic polynomial of P_n, where $n \geq 3$, are as follows:*

$$c_2=(-1)^{\frac{k}{2}}(8(n-3)+10),$$

$$c_k=0,\ \ where\ k\in 2\mathbb{Z}^*+1,$$

$$c_k=(-1)^{\frac{k}{2}}\Big[\tbinom{(n-1)-\frac{k}{2}}{\frac{k}{2}-1}\cdot(\sqrt{5})^2\cdot(\sqrt{8})^{2(\frac{k}{2}-1)}-3\tbinom{(n-2)-\frac{k}{2}}{\frac{k}{2}-2}\cdot(\sqrt{5})^2\cdot(\sqrt{8})^{2(\frac{k}{2}-2)}$$

$$+\sum_{j=\frac{k}{2}-1}^{n-2-\frac{k}{2}}\tbinom{j}{\frac{k}{2}-1}\cdot(\sqrt{8})^{k}-3\sum_{j=\frac{k}{2}-2}^{n-3-\frac{k}{2}}\tbinom{j}{\frac{k}{2}-2}\cdot(\sqrt{8})^{(k-2)}\Big],\ \ where\ k\geq 4,k\in 2\mathbb{Z}^+.$$

Proof. Proof is the same with the proof of Thm. 1. Only difference originated from the difference between the Randić and Sombor adjacency matrices of P_n. □

Theorem 2. *Let $P_n=(V,E)$ be a path graph with n vertices and $n-1$ edges. Let $P_{P_n}(\lambda)=\lambda^n+c_1\lambda^{n-1}+c_2\lambda^{n-2}+\cdots+c_{n-1}\lambda+c_n$ be the Randić characteristic polynomial of P_n, where $c_k\in\mathbb{R}$. The formula for the coefficient c_n, where $n\geq 3$, of the Randić characteristic polynomial of P_n is as follows:*

$$c_k=0,\ \ where\ k\in 2\mathbb{Z}^*+1,$$

$$c_k=(-1)^{\frac{k}{2}}\Big[\tbinom{(n-1)-\frac{k}{2}}{\frac{k}{2}-1}\cdot(\tfrac{1}{\sqrt{2}})^2\cdot(\tfrac{1}{2})^{2(\frac{k}{2}-1)}+\tbinom{(n-2)-\frac{k}{2}}{\frac{k}{2}-2}\cdot(\tfrac{1}{4})\cdot(\tfrac{1}{2})^{2(\frac{k}{2}-2)}\cdot(\tfrac{1}{\sqrt{2}})^2\Big],\ \ otherwise.$$

Proof. First of all, it clear that $c_k=0$, where $k\in 2\mathbb{Z}^*+1$. Similarly to Thm. 1, let us consider a path graph P_n with n vertices whose vertices are labelled by $1,2,\cdots,n$. We keep in view elementary subgraphs with n vertices that consist of disjoint edges since $n=k$. At this point, we have only one choice and it is $v_1v_2,v_3v_4,\cdots,v_{n-1}v_n$. Thus, by the proof of Thm. 1, we know that the contribution of this subgraph to c_k is equal to $\tbinom{(n-1)-\frac{k}{2}}{\frac{k}{2}-1}\cdot(\frac{1}{\sqrt{2}})^2\cdot(\frac{1}{2})^{2(\frac{k}{2}-1)}+\tbinom{(n-2)-\frac{k}{2}}{\frac{k}{2}-2}\cdot(\frac{1}{4})\cdot(\frac{1}{2})^{2(\frac{k}{2}-2)}\cdot(\frac{1}{\sqrt{2}})^2$. Finally, by using Eqn. 1, we have the result as follow:

$$c_k\ \ =\ \ (-1)^{\frac{k}{2}}\Big[\tbinom{(n-1)-\frac{k}{2}}{\frac{k}{2}-1}\cdot(\tfrac{1}{\sqrt{2}})^2\cdot(\tfrac{1}{2})^{2(\frac{k}{2}-1)}+\tbinom{(n-2)-\frac{k}{2}}{\frac{k}{2}-2}\cdot(\tfrac{1}{4})\cdot(\tfrac{1}{2})^{2(\frac{k}{2}-2)}\cdot(\tfrac{1}{\sqrt{2}})^2\Big].$$

□

Corollary 2. *Let $P_n=(V,E)$ be a path graph with n vertices and $n-1$ edges. Let $P_{P_n}(\lambda)=\lambda^n+c_1\lambda^{n-1}+c_2\lambda^{n-2}+\cdots+c_{n-1}\lambda+c_n$ be the Sombor characteristic polynomial of P_n, where $c_k\in\mathbb{Z}$. The formula for the coefficient c_n, where $n\geq 3$, of the Sombor characteristic polynomial of P_n is as follows:*

$$c_k=0,\ \ where\ k\in 2\mathbb{Z}^*+1,$$

$$c_k=(-1)^{\frac{k}{2}}\Big[\tbinom{(n-1)-\frac{k}{2}}{\frac{k}{2}-1}\cdot(\sqrt{5})^2\cdot(\sqrt{8})^{2(\frac{k}{2}-1)}-3\tbinom{(n-2)-\frac{k}{2}}{\frac{k}{2}-2}\cdot(\sqrt{5})^2\cdot(\sqrt{8})^{2(\frac{k}{2}-2)}\Big],\ \ otherwise.$$

Proof. Proof is the same with the proof of Thm. 2. Only difference originate from the definitions of Randić and Sombor adjacency matrices of P_n. □

For the next theorem, we denote the number of elementary subgraphs with k vertices by $N(c_k)$.

Theorem 3. *Let $C_n = (V, E)$ be a cycle graph with $n \geq 3$ vertices and n edges. Let $P_{C_n}(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n$ be the Sombor characteristic polynomials of C_n, where $c_k \in \mathbb{R}$ and $1 \leq k \leq n$. The formulae for the coefficients c_k $(k = 2t, t \in \mathbb{Z}^+)$ of the Sombor characteristic polynomial of C_n are as follows:*

$$\begin{aligned}
c_2 &= -8n,\\
c_4 &= (8)^2\left(\binom{n-2}{2} + \binom{n-3}{1}\right),\\
c_6 &= -(8)^3\left(\binom{n-3}{3} + \binom{n-4}{2}\right),\\
c_8 &= (8)^4\left(\binom{n-4}{4} + \binom{n-5}{3}\right),\\
c_{10} &= -(8)^5\left(\binom{n-5}{5} + \binom{n-6}{4}\right),\\
&\vdots\\
c_k &= (-1)^{\frac{k}{2}} \cdot (8)^{\frac{k}{2}}\left(\binom{n-\frac{k}{2}}{\frac{k}{2}} + \binom{n-(\frac{k}{2}+1)}{\frac{k}{2}-1}\right),
\end{aligned}$$

in the case of $n = k$, then $c_n = c_k - 2 \cdot 8^{\frac{n}{2}}$, where c_k is as given above.

Proof. We know that c_k consist of the contributions of different elementary subgraphs of G with k vertices by Eqn. 1. For the coefficients c_k $(k = 2t, t \in \mathbb{Z}^+)$ of the Sombor characteristic polynomials of C_n, where $c_k \in \mathbb{R}$ and $1 \leq k \leq n-1$, we take into account only elementary subgraphs that consist of disjoint edges without any elementary subgraph that does not involve any cycle. Similarly to proof of Thm. 1, we apply edge removing method so that we get the number of elementary subgraphs for forming $c_4, c_6, c_8, c_{10}, \cdots, c_k$, where $c_k \in \mathbb{R}$, $1 \leq k \leq n-1$, by using combinations as follows:

$$\begin{aligned}
N(c_4) &= \sum_{i=1}^{n-3}\binom{i}{1} + \binom{n-3}{1},\\
N(c_6) &= \sum_{i=2}^{n-4}\binom{i}{2} + \binom{n-4}{2},\\
N(c_8) &= \sum_{i=3}^{n-5}\binom{i}{3} + \binom{n-5}{3},
\end{aligned}$$

$$N(c_{10}) \quad = \quad \sum_{i=4}^{n-6} \binom{i}{4} + \binom{n-6}{4},$$

$$\vdots$$

$$N(c_k) \quad = \quad \sum_{i=\frac{k}{2}-1}^{n-(\frac{k}{2}+1)} \binom{i}{\frac{k}{2}-1} + \binom{n-(\frac{k}{2}-1)}{\frac{k}{2}-1}.$$

As a result, we get the desired result by using combination properties and Eqn. 1. In addition, if $n = k$, then there exists one possibility of elementary subgraph that consists of the cycle C_n itself. Therefore, in this case result is obtained as $c_n = c_k - 2 \cdot 8^{\frac{n}{2}}$, where c_k is as given above. □

In a cycle graph C_n, it is trivial that if k is odd, then $c_k = 0$ whenever $0 \leq k \leq n-1$. In the next corollary, the last part of the previous theorem is presented with a more explicit statement.

Corollary 3. *Let $C_n = (V, E)$ be a cycle graph with $n \geq 3$ vertices and n edges. Let $P_{C_n}(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n$ be the Sombor characteristic polynomials of C_n, where $c_k \in \mathbb{R}$ and $1 \leq k \leq n$. The formula for the coefficient c_n of the Sombor characteristic polynomial of C_n is as follows:*

$$c_n = \begin{cases} -2^{\frac{3n+2}{2}}, & n = 2t+1, \textit{where } t \in \mathbb{Z}^+ \\ -2^{\frac{3n+4}{2}}, & n = 2t, \textit{where } t \in \{3, 5, 7, \cdots\} \\ 0, & n = 4t, \textit{where } t \in \mathbb{Z}^+. \end{cases}$$

Proof. Let us consider a cycle graph C_n. There are three possible cases of elementary subgraph of C_n with n vertices. The first case is $n = 2t+1$, where $t \in \mathbb{Z}^+$. For this case, we have just an elementary subgraph that consists of C_n itself and contribution of this subgraph is equal to $-2 \cdot (2\sqrt{2})^n$ by using Eqn. 1.

Second case is $n = 2t$, where $t \in \{3, 5, 7, \cdots\}$. At this point, there are 2 types of elementary subgraphs with n vertices. These elementary subgraphs can consist either just a cycle C_n or $\frac{n}{2}$ disjoint edges. Therefore, contribution of these subgraphs is equal to $-2 \cdot 8^{\frac{n}{2}} - 2 \cdot 8^{\frac{n}{2}}$ that is $-2^{\frac{3n+4}{2}}$. Third case is $n = 4t$, where $t \in \mathbb{Z}^+$. Similarly to second case, there are two possible elementary subgraphs of C_n with n vertices. These consist of either just a cycle C_n or $\frac{n}{2}$ disjoint edges. At this point, since $\frac{n}{2}$ is even number contribution of these subgraphs is equal to $2 \cdot 8^{\frac{n}{2}} - 2 \cdot 8^{\frac{n}{2}}$ that is 0 by Eqn. 1. □

Corollary 4. *Let $C_n = (V, E)$ be a cycle graph with $n \geq 3$ vertices and n edges. Let $P_{C_n}(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n$ be the Randić characteristic* *polynomials of* C_n*, where* $c_k \in \mathbb{R}$ *and* $1 \leq k \leq n$*. The formulae for the coefficients* c_k $(k = 2t, t \in \mathbb{Z}^+)$ *of the Randić characteristic polynomial of* C_n *are as follows:*

$$\begin{aligned}
c_2 &= -\frac{n}{4}, \\
c_4 &= (\frac{1}{4})^2 \left(\binom{n-2}{2} + \binom{n-3}{1} \right), \\
c_6 &= -(\frac{1}{4})^3 \left(\binom{n-3}{3} + \binom{n-4}{2} \right), \\
c_8 &= (\frac{1}{4})^4 \left(\binom{n-4}{4} + \binom{n-5}{3} \right), \\
c_{10} &= -(\frac{1}{4})^5 \left(\binom{n-5}{5} + \binom{n-6}{4} \right), \\
&\vdots \\
c_k &= (-1)^{\frac{k}{2}} \cdot (\frac{1}{4})^{\frac{k}{2}} \left(\binom{n-\frac{k}{2}}{\frac{k}{2}} + \binom{n-(\frac{k}{2}+1)}{\frac{k}{2}-1} \right),
\end{aligned}$$

in the case of $n = k$*, then* $c_n = c_k - 2 \cdot (\frac{1}{4})^{\frac{n}{2}}$*, where* c_k *is as given above.*

Proof. Proof can be followed by using Theorem 3. □

In the previous theorem, it is clear that if k is odd, then $c_k = 0$ as long as $0 \leq k \leq n-1$ for each cycle graph C_n. The case $n = k$ is presented in the next result.

Corollary 5. *Let* $C_n = (V, E)$ *be a cycle graph with* $n \geq 3$ *vertices and* n *edges. Let* $P_{C_n}(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n$ *be the Randić characteristic polynomials of* C_n*, where* $c_k \in \mathbb{R}$ *and* $1 \leq k \leq n$*. The formula for the coefficient* c_n *of the Randić characteristic polynomial of* C_n *is as follows:*

$$c_n = \begin{cases} -2^{1-n}, & n = 2t+1, \textit{where } t \in \mathbb{Z}^+ \\ -2^{2-n}, & n = 2t, \textit{where } t \in \{3, 5, 7, \cdots\} \\ 0, & n = 4t, \textit{where } t \in \mathbb{Z}^+. \end{cases}$$

Proof. Proof can be followed by using Corollary 3. □

Let us define a special regular graph that consists of n $(n \geq 4, n = 2t, t \in \mathbb{Z}^+)$ vertices, $\frac{3n}{2}$ edges and degrees of all vertices are 3. Also vertices intersect each others in a point. We denote it by R_n. Let us demonstrate the structures of graphs R_6 and R_8 in Figure 1.

Theorem 4. *Let* $R_n = (V, E)$ *be a* $3-$*regular graph with* n *vertices and* $\frac{3n}{2}$ *edges as shown in Fig. 1. Let* $P_{R_n}(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n$ *be the Randić characteristic polynomial of* R_n*, where* $c_k \in \mathbb{R}$*. The formulae for some*

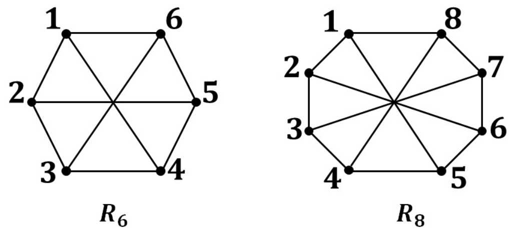

FIGURE 1. Graphs R_6 and R_8

coefficients of the Randić characteristic polynomial of R_n are as follows:

$$c_2 = -\tfrac{n}{6},$$

$$c_3 = 0,\ \textit{if}\ n=4,\ \textit{then}\ c_3 = -8\cdot(\tfrac{1}{3})^3,$$

$$c_5 = 0,\ \textit{if}\ n=8,\ \textit{then}\ c_5 = -16\cdot(\tfrac{1}{3})^5.$$

$$c_4 = \begin{cases} -3\cdot(\frac{1}{3})^4, & n=4 \\ 0, & n=6 \\ -(\frac{1}{3})^4 n + (\frac{1}{3})^4\left(\sum_{j=1}^{n-3} j + (n-3) + n\frac{n-4}{2} + \binom{\frac{n}{2}}{2}\right), & \textit{otherwise.} \end{cases}$$

$$c_6 = \begin{cases} 0, & n=4 \\ 0, & n=6 \\ -(\frac{1}{3})^6\left(\sum_{j=2}^{n-4}\binom{j}{2} + \binom{n-4}{2} + \binom{\frac{n}{2}}{3} + n\binom{\frac{n-4}{2}}{2} + n(\frac{n}{2}-3)(\frac{n}{2}-4) + n(\frac{n}{2}-3)\right. & \\ \left. + \frac{n}{2}(\frac{n}{2}-2)\right) + 2\cdot(\frac{1}{3})^6\left(n(\frac{n-4}{2}-1) + (\frac{n}{2}\frac{n-4}{2})\right) - 2\cdot(\frac{1}{3})^6\left(n+\frac{n}{2}\right), & n=10 \\ -(\frac{1}{3})^6\left(\sum_{j=2}^{n-4}\binom{j}{2} + \binom{n-4}{2} + \binom{\frac{n}{2}}{3} + n\binom{\frac{n-4}{2}}{2} + n(\frac{n}{2}-3)(\frac{n}{2}-4) + n(\frac{n}{2}-3)\right. & \\ \left. + \frac{n}{2}(\frac{n}{2}-2)\right) + 2\cdot(\frac{1}{3})^6\left(n(\frac{n-4}{2}-1) + (\frac{n}{2}\frac{n-4}{2})\right) - (\frac{1}{3})^6 n, & \textit{otherwise.} \end{cases}$$

Proof. It is clear that c_1 of $P_{R_n}(\lambda)$ is 0.

First of all, let us consider c_2. We know that the number of possible elementary subgraphs with 2 vertices is equal to the number of edges of R_n. Hence, since R_n is 3−regular, contribution of these elementary subgraphs to $c_2 = -(\frac{1}{3})^2\frac{3n}{2} = -\frac{n}{6}$ by Eqn. 1.

Secondly, it is clear that 3−cycles just exist in R_n when n is equal to 4. Thus, by Eqn. 1 if $n = 4$, then $c_3 = -(\frac{1}{3})^3 \cdot 2 \cdot 4$, otherwise $c_3 = 0$.

Thirdly, there exists 4 options for elementary subgraphs with 4 vertices. They can consist of 4−cycles that are in the form of cross labeling such as (1436) in R_6 in Fig. 1 and the number of possible elementary subgraphs in this form is $\frac{n}{2}$. The rest 3 options can be two disjoint edges that one belongs to C_n and other one is a diagonal edge, two disjoint edges that belong to C_n and lastly two disjoint edges that are diagonal edges, respectively. The number of possible elementary subgraphs in the form of second option is $n\frac{n-4}{2}$ because when we select an edge that belongs to C_n, we have $\binom{\frac{n-4}{2}}{1}$ possibility for an other diagonal edge. Since R_n has n vertices there are $n\frac{n-4}{2}$ elementary subgraphs in the second form. For the third option, the number of possible elementary subgraphs that are in the form of

$$\begin{aligned}
&\{v_1v_2, v_3v_4\}, \{v_1v_2, v_4v_5\}, \cdots, \{v_1v_2, v_{n-1}v_n\},\\
&\{v_2v_3, v_4v_5\}, \{v_2v_3, v_5v_6\}, \cdots, \{v_2v_3, v_{n-1}v_n\}, \{v_2v_3, v_nv_1\},\\
&\vdots\\
&\{v_{n-3}v_{n-2}, v_{n-1}v_n\}, \{v_{n-3}v_{n-2}, v_nv_1\}
\end{aligned}$$

is equal to $1+2+3+\cdots+(n-3)+(n-3)$. Also, it is clear that the number of possible elementary subgraphs of the last option is $\binom{\frac{n}{2}}{2}$. As a result, by using Eqn. 1 we get $c_4 = -2\cdot(\frac{1}{3})^4\frac{n}{2} + (\frac{1}{3})^4\left(\sum_{j=1}^{n-3} j + (n-3) + n\frac{n-4}{2} + \binom{\frac{n}{2}}{2}\right)$. However, additionally when n is equal to 4, for the first option we have one more possible elementary subgraph that is C_4 itself so we get the result as $-3\cdot(\frac{1}{3})^4$ by adding $-2\cdot(\frac{1}{3})^4$. Moreover, when n is equal to 6, for the first option, we have six more possible elementary subgraphs that are C_4 itself so we get result as 0 by adding $-12\cdot(\frac{1}{3})^4$.

Fourthly, there exists just one option for an elementary subgraph with 5 vertices that is a $5-$cycle C_5 itself and it can be possible only for R_n, where $n=8$. Therefore, c_5 is obtained as $-16\cdot(\frac{1}{3})^5$ by Eqn. 1.

Lastly, let us consider possible elementary subgraphs with 6 vertices, where $n \neq 6, 10$. One of the possible elementary subgraph types consisting of three edges that are in C_n are in the form

$$\begin{aligned}
&\{v_1v_2, v_3v_4, v_5v_6\}, \{v_1v_2, v_3v_4, v_6v_7\}, \cdots, \{v_1v_2, v_{n-3}v_{n-2}, v_{n-1}v_n\},\\
&\{v_2v_3, v_4v_5, v_6v_7\}, \{v_2v_3, v_4v_5, v_7v_8\}, \cdots, \{v_2v_3, v_{n-3}v_{n-2}, v_nv_1\}, \{v_2v_3, v_{n-2}v_{n-1}, v_nv_1\},\\
&\vdots\\
&\{v_{n-4}v_{n-3}, v_{n-2}v_{n-1}, v_nv_1\}.
\end{aligned}$$

Possible number of these types is equal to $\sum_{j=2}^{n-4}\binom{j}{2} + \binom{n-4}{2}$. An another type can consist of three diagonal edges whose possible number is $\binom{\frac{n}{2}}{3}$. Another type can consist of one edge that is in C_n and other two edges are diagonal edges. As explained before possible number of these elementary subgraphs is $n\binom{\frac{n-4}{2}}{2}$. For another type of elementary subgraphs that consist of two edges in C_n and one in diagonal edges, we get the possible number $n(\frac{n}{2}-3)(\frac{n}{2}-4) + n(\frac{n}{2}-3) + \frac{n}{2}(\frac{n}{2}-2)$ by using processes as mentioned above. The number of possible elementary subgraphs that consist of cross labeling C_4 and an edge in C_n is $n(\frac{n-4}{2}-1)$. Also, the number of possible elementary subgraphs that consist of cross labeling

C_4 and a diagonal edge is $(\frac{n}{2}\frac{n-4}{2})$. Moreover, the number of possible elementary subgraphs that consist of C_6 is $\frac{n}{2}$. Consequently, we get the formula by using Eqn. 1, where $n \neq 6, 10$. After all, additively when n is equal to 6, there is no possible elementary subgraph in the form of one edge that is in C_n and other two edges are diagonal edges. Therefore, for the $n = 6$ distinctively, we have $\sum_{j=2}^{2} \binom{j}{2} + \binom{2}{2} + \binom{\frac{6}{2}}{3} + 6(\frac{6}{2}-3)(\frac{6}{2}-4) + 6(\frac{6}{2}-3) + \frac{6}{2}(\frac{6}{2}-2)$ times possible elementary subgraphs that consist of disjoint edges of R_n and we have $(6 \cdot 0 + 3 \cdot 1)$ times possible elementary subgraphs that consist of one cross labeling C_4 and edge in R_6. Also, we have 6 possible elementary subgraphs that consist of C_6 and we have 6 possible elementary subgraphs consisting of an edge and a C_4 that is not cross labeling. As a consequence, privately for $n = 6$, we have the result $-6\cdot(\frac{1}{3})^6 + 6\cdot(\frac{1}{3})^6 - 12\cdot(\frac{1}{3})^6 + 12\cdot(\frac{1}{3})^6 = 0$ by using Eqn. 1. Finally, additively, if $n = 10$, there are n times more possible elementary subgraphs that consist of a C_6 so we have the formula by adding $-2 \cdot (\frac{1}{3})^6 n$ to the first formula. Thus, we complete the proof. $\square$

Corollary 6. *Let $R_n = (V, E)$ be a 3−regular graph with n vertices and $\frac{3n}{2}$ edges as shown in Fig. 1. Let $P_{R_n}(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n$ be the Sombor characteristic polynomial of R_n, where $c_k \in \mathbb{R}$. The formulae for some coefficients of the Sombor characteristic polynomial of R_n are as follows:*

$$
\begin{aligned}
c_2 &= -27n, \\
c_3 &= 0, \ \textit{if } n = 4, \ \textit{then } c_3 = -8 \cdot (\sqrt{18})^3, \\
c_5 &= 0, \ \textit{if } n = 8, \ \textit{then } c_5 = -16 \cdot (\sqrt{18})^5,
\end{aligned}
$$

Also, we get the equations as follows:

$$
c_4 = \begin{cases} -3 \cdot (\sqrt{18})^4, & n = 4 \\ 0, & n = 6 \\ -(\sqrt{18})^4 n + (\sqrt{18})^4 \left(\sum_{j=1}^{n-3} j + (n-3) + n\frac{n-4}{2} + \binom{\frac{n}{2}}{2}\right), & \textit{otherwise} \end{cases}
$$

$$
c_6 = \begin{cases} 0, & n = 4 \\ 0, & n = 6 \\ -(\sqrt{18})^6 \left(\sum_{j=2}^{n-4} \binom{j}{2} + \binom{n-4}{2} + \binom{\frac{n}{2}}{3} + n\binom{\frac{n-4}{2}}{2} + n(\frac{n}{2}-3)(\frac{n}{2}-4) + n(\frac{n}{2}-3)\right. & \\ \left. + \frac{n}{2}(\frac{n}{2}-2)\right) + 2 \cdot (\sqrt{18})^6 \left(n(\frac{n-4}{2}-1) + (\frac{n}{2}\frac{n-4}{2})\right) - 2 \cdot (\sqrt{18})^6 \left(n + \frac{n}{2}\right), & n = 10 \\ -(\sqrt{18})^6 \left(\sum_{j=2}^{n-4} \binom{j}{2} + \binom{n-4}{2} + \binom{\frac{n}{2}}{3} + n\binom{\frac{n-4}{2}}{2} + n(\frac{n}{2}-3)(\frac{n}{2}-4) + n(\frac{n}{2}-3)\right. & \\ \left. + \frac{n}{2}(\frac{n}{2}-2)\right) + 2 \cdot (\sqrt{18})^6 \left(n(\frac{n-4}{2}-1) + (\frac{n}{2}\frac{n-4}{2})\right) - (\sqrt{18})^6 n, & \textit{otherwise} \end{cases}
$$

Proof. The proof can be completed by simply replacing $\frac{1}{3}$ with $\sqrt{18}$ in the proof of the previous theorem. $\square$

3. Conclusion

The Randić and the Sombor characteristic polynomials of P_n and C_n were obtained. Additionally, the formulae of five coefficients of the Randić and Sombor characteristic polynomials of R_n were presented. The Randić and the Sombor energies of P_n and C_n can be studied by using these presented results. Furthermore, various characteristic polynomials of some similar adjacency matrices defined according to some vertex-degree-based topological invariants can be obtained by using the number of elementary subgraphs that we presented in the theorems and corollaries. Especially, this study can also be extended to the multiplicative Sombor index associated with the Sombor index.

Declaration of Competing Interests The author has no competing interest to declare.

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