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SPANNING SIMPLICIAL COMPLEXES OF n-CYCLIC GRAPHS WITH A COMMON EDGE

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Abstract. In this paper, we characterize some algebraic and combinatorial properties of spanning simplicial complex $\Delta_s(G_{t_1,t_2,\cdots,t_n})$ of the class of the n-cyclic graphs $G_{t_1,t_2,\cdots,t_n}$ with a common edge. We show that $\Delta_s(G_{t_1,t_2,\cdots,t_n})$ is pure simplicial complex of dimension $\sum\limits_{i=1}^{n} t_i - 2n$, and we also determine the Stanley-Reisner ideal $I_{\Delta_s(G_{t_1,t_2,\cdots,t_n})}$ of $\Delta_s(G_{t_1,t_2,\cdots,t_n})$ and its primary decomposition. Under the condition that the length of every cyclic graph G_{t_i} is t for $1 \leq i \leq n$, we give a formula for f-vector of $\Delta_s(G_{t_1,t_2,\cdots,t_n})$ and consequently a formula for Hilbert series of the Stanley-Reisner ring $k[\Delta_s(G_{t_1,t_2,\cdots,t_n})]$, where k is a field.

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1. Introduction

The note of spanning simplicial complex $\Delta_s(G)$ on edge set E of a graph $G = G(V, E)$ was introduced in [1], the set of its facets is exactly edge set $s(G)$ of all possible spanning trees of G, i.e.

$$\Delta_s(G) = \langle\, F_i \mid F_i \in s(G)\,\rangle.$$

Note that for a graph G, the problem of finding $s(G)$ is not always easy to handle. Anwar, Raza and Kashif [1] proved some algebraic and combinatorial properties of spanning simplicial complexes of the uni-cyclic graph U_n, where U_n is a connected graph on n vertices, which contains exactly one cycle of length n. In this paper, our goal is to characterize some algebraic and combinatorial properties of spanning simplicial complexes of a class of n-cyclic graphs $G_{t_1,t_2,\cdots,t_n}$ with a common edge, which is obtained by joining n cyclic graphs $G_{t_1}, G_{t_2}, \cdots, G_{t_n}$ of length $t_1, t_2, \cdots, t_n$ with a common edge. For $n = 2$ and $t_1 = t_2 = 4$, the graph of $G_{4,4}$ is shown in Figure 1 of Section 2.

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We give a brief overview of this paper. In Section 2, we recall some definitions and results from commutative algebra and algebraic combinatorics. In Section 3, we determine the Stanley-Reisner ideal $I_{\Delta_s(G_{t_1, t_2, \cdots, t_n})}$ of $\Delta_s(G_{t_1, t_2, \cdots, t_n})$ and its primary decomposition in Theorem 3.2. In Section 4, under the assumption that the length of cyclic graph G_{t_i} is t for every $1 \leq i \leq n$, we give a formula for f-vector of $\Delta_s(G_{t_1, t_2, \cdots, t_n})$ and consequently a formula for Hilbert series of the Stanley-Reisner ring $k[\Delta_s(G_{t_1, t_2, \cdots, t_n})]$, where k is a field.

2. Preliminaries

We firstly recall some definitions and basic facts about graph and simplicial complex to make this paper self-contained.

Definition 2.1. A *spanning tree* of a simple connected finite graph $G = G(V, E)$ is a subgraph of G, which is a tree and contains all vertices of G. We denote the collection of all edge sets of the spanning trees of G by $s(G)$, i.e.

$$s(G) = \{E(T_i) \subset E |\ T_i \text{ is a spanning tree of } G\}.$$

(See [3] for more details).

It is well known that for any simple connected finite graph, spanning trees always exist. One can find a spanning tree systematically by the cutting-down method, which says that a spanning tree is obtained by removing one edge from each cycle appearing in the graph. For example, for the following graph G, we obtain that

$$\begin{aligned} s(G) \quad = \quad & \{\{e_2, e_3, e_5, e_6, e_7\}, \{e_1, e_3, e_5, e_6, e_7\}, \{e_1, e_2, e_5, e_6, e_7\}, \{e_1, e_2, e_3, e_5, e_6\}, \\ & \{e_2, e_3, e_4, e_6, e_7\}, \{e_1, e_3, e_4, e_6, e_7\}, \{e_1, e_2, e_4, e_6, e_7\}, \{e_2, e_3, e_4, e_5, e_7\}, \\ & \{e_1, e_3, e_4, e_5, e_7\}, \{e_1, e_2, e_4, e_5, e_7\}, \{e_1, e_2, e_4, e_5, e_6\}, \{e_1, e_3, e_4, e_5, e_6\}, \\ & \{e_2, e_3, e_4, e_5, e_6\}, \{e_1, e_2, e_3, e_4, e_6\}, \{e_1, e_2, e_3, e_4, e_5\}\}. \end{aligned}$$

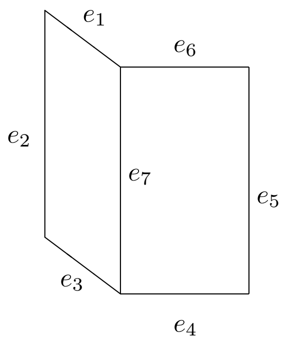

Figure 1. 2-cyclic graph with a common edge

Definition 2.2. A *simplicial complex* Δ on a set of vertices $[n] = \{1, 2, \ldots, n\}$ is a collection of subsets of $[n]$ such that

(1) $\{i\} \in \Delta$ for each $\{i\} \in [n]$;
(2) if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$.

An element of Δ is called a face of Δ, and the dimension of a face F of Δ is defined as $|F| - 1$, where $|F|$ is the number of vertices of F and denoted by dim F. The faces of dimension 0 and 1 are called vertices and edges, respectively, and dim $\emptyset = -1$.

The maximal faces of Δ under inclusion are called facets of Δ. The dimension of the simplicial complex Δ, which is denoted by dim Δ, is the maximal dimension of its facets, i.e.

$$\dim\ \Delta = \max\{\dim\ F |\, F \text{ is a facet of } \Delta\}.$$

We denote the simplicial complex Δ with facets $\{F_1, \cdots, F_q\}$ by

$$\Delta = \langle F_1, \cdots, F_q \rangle.$$

Definition 2.3. A simplicial complex Δ is *pure* if all of its facets have the same dimension.

Definition 2.4. Given a simplicial complex Δ of dimension d, we define its *f-vector* to be the $(d+1)$-tuple $f = (f_0, f_1, \ldots, f_d)$, where f_i is the number of i-dimensional faces of Δ.

Definition 2.5. For a simple connected finite graph $G = G(V, E)$ with $s(G) = \{E_1, \ldots, E_s\}$, we define a *simplicial complex* $\Delta_s(G)$ *on* E such that facets of $\Delta_s(G)$ are precisely the elements of $s(G)$, called the spanning simplicial complex of $G(V, E)$. In other words,

$$\Delta_s(G) = \langle E_1, \cdots, E_s \rangle.$$

For example, the spanning simplicial complex of the graph G with edge set $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ in Figure 1 is given by

$$\begin{aligned}\Delta_s(G) \;=\; & \langle \{e_2, e_3, e_5, e_6, e_7\}, \{e_1, e_3, e_5, e_6, e_7\}, \{e_1, e_2, e_5, e_6, e_7\}, \{e_1, e_2, e_3, e_5, e_6\}, \\ & \{e_2, e_3, e_4, e_6, e_7\}, \{e_1, e_3, e_4, e_6, e_7\}, \{e_1, e_2, e_4, e_6, e_7\}, \{e_2, e_3, e_4, e_5, e_7\}, \\ & \{e_1, e_3, e_4, e_5, e_7\}, \{e_1, e_2, e_4, e_5, e_7\}, \{e_1, e_2, e_4, e_5, e_6\}, \{e_1, e_3, e_4, e_5, e_6\}, \\ & \{e_2, e_3, e_4, e_5, e_6\}, \{e_1, e_2, e_3, e_4, e_6\}, \{e_1, e_2, e_3, e_4, e_5\} \rangle.\end{aligned}$$

Definition 2.6. *An n-cyclic graph $G_{t_1, t_2, \cdots, t_n}$ with a common edge* is a connected graph having $\sum\limits_{i=1}^{n} t_i - 2(n-1)$ vertices and $\sum\limits_{i=1}^{n} t_i - (n-1)$ edges, obtained by joining n cyclic graphs $G_{t_1}, G_{t_2}, \cdots, G_{t_n}$ with a common edge, where G_{t_i} denotes the cyclic graph of length t_i. We can assume that $t_1 \leq t_2 \leq \cdots \leq t_n$ and $t_i \geq 3$ for each $i \in \{1, 2, \ldots, n\}$.

3. Primary decomposition of the Stanley-Reisner ideal

In this section, we will determine the Stanley-Reisner ideal $I_{\Delta_s(G_{t_1,t_2,\cdots,t_n})}$ of $\Delta_s(G_{t_1,t_2,\cdots,t_n})$ and its primary decomposition.

We label the edge set of $G_{t_1,t_2,\cdots,t_n}$ such that $\{e_{i1}, e_{i2}, \cdots, e_{i\,t_i}\}$ is the edge set of cyclic graph G_{t_i} for every $1 \leq i \leq n$. By convention, $e_{1\,t_1} = e_{2\,t_2} = \cdots = e_{n\,t_n} = e$ is the common edge. First, we have the following proposition.

Proposition 3.1. *$\Delta_s(G_{t_1,t_2,\cdots,t_n})$ is a pure simplicial complex of dimension $\sum\limits_{i=1}^{n} t_i - 2n$.*

Proof. Let $E = \{e_{11}, e_{12}, \cdots, e_{1,t_1-1}, e_{21}, \cdots, e_{2,t_2-1}, \cdots, e_{n1}, \cdots, e_{n,t_n-1}, e\}$ be the edge set of n-cyclic graph $G_{t_1,t_2,\cdots,t_n}$, where e is the common edge. As $G_{t_1,t_2,\cdots,t_n}$ contains exactly n cycles of length t_1, t_2, $\cdots$, t_n, which has a common edge e, by the cutting-down method, its spanning trees are obtained by removing one edge from each cycle G_{t_i}, $1 \leq i \leq n$. Hence, the subset $E(T_i) \subset E$ is in $s(G_{t_1,t_2,\cdots,t_n})$ if and only if $E(T_i) = E \backslash \{e_{1\,i_1}, e_{2\,i_2}, \cdots, e_{n\,i_n}\}$ for some $i_j \in \{1, 2, \cdots, t_j\}$, where these e_{j,i_j}s are distinct and j runs from 1 to n, with convention $e_{1\,t_1} = e_{2\,t_2} = \cdots = e_{n\,t_n} = e$, i.e.

$$s(G_{t_1,t_2,\cdots,t_n}) = \{E \backslash \{e_{1\,i_1}, \cdots, e_{n\,i_n}\} \mid 1 \leq i_j \leq t_j, 1 \leq j \leq n, \text{ where } e_{j,i_j}\text{s are distinct}\}.$$

It is easily seen that each spanning tree of $\Delta_s(G_{t_1,t_2,\cdots,t_n})$ has $\sum\limits_{i=1}^{n} t_i - (n-1) - n = \sum\limits_{i=1}^{n} t_i - 2n + 1$ edges. Thus the result follows. $\square$

Let $E = \{e_{11}, e_{12}, \cdots, e_{1,t_1-1}, e_{21}, \cdots, e_{2,t_2-1}, \cdots, e_{n1}, \cdots, e_{n,t_n-1}, e\}$ be the edge set of n-cyclic graph $G_{t_1,t_2,\cdots,t_n}$, and let $\Delta_s(G_{t_1,t_2,\cdots,t_n})$ be the spanning simplicial complex of $G_{t_1,t_2,\cdots,t_n}$. We can assume that $S = k[x_{11}, \cdots, x_{1,t_1-1}, x_{21}, \cdots, x_{2,t_2-1}, \cdots, x_{n1}, \cdots, x_{n,t_n-1}, y]$ is a polynomial ring in $\sum\limits_{i=1}^{n} t_i - (n-1)$ variables over a field k, $I_{\Delta_s(G_{t_1,t_2,\cdots,t_n})}$ is the Stanley-Reisner ideal of $\Delta_s(G_{t_1,t_2,\cdots,t_n})$, which is a square-free monomial ideal. The standard graded algebra $k[\Delta_s(G_{t_1,t_2,\cdots,t_n})] = S/I_{\Delta_s(G_{t_1,t_2,\cdots,t_n})}$ is called the Stanley-Reisner ring of $\Delta_s(G_{t_1,t_2,\cdots,t_n})$. We can give a primary decomposition of ideal $I_{\Delta_s(G_{t_1,t_2,\cdots,t_n})}$, Hilbert series and h-vector of $k[\Delta_s(G_{t_1,t_2,\cdots,t_n})]$. We refer readers to [2] and [5] for detailed information about the Stanley-Reisner ideal, primary decomposition, Hilbert series and h-vector.

Now, we give a primary decomposition of the Stanley-Reisner ideal $I_{\Delta_s(G_{t_1,t_2,\cdots,t_n})}$ of $\Delta_s(G_{t_1,t_2,\cdots,t_n})$.

Theorem 3.2. *Let $\Delta_s(G_{t_1,t_2,\cdots,t_n})$ be the spanning simplicial complex of n-cyclic graph $G_{t_1,t_2,\cdots,t_n}$. Then the Stanley-Reisner ideal $I_{\Delta_s(G_{t_1,t_2,\cdots,t_n})}$ of $\Delta_s(G_{t_1,t_2,\cdots,t_n})$ is given by*

$$\begin{aligned} I_{\Delta_s(G_{t_1,t_2,\cdots,t_n})} &= \bigcap_{\substack{i_j\in\{1,\cdots,t_j-1\}\\ j\in\{1,\cdots,n\}}}(x_{1i_1},x_{2i_2},\cdots,x_{ni_n})\bigcap_{\substack{i_j\in\{1,\cdots,t_j-1\}\\ j\in\{1,\cdots,\hat{k},\cdots,n\}\\ k\in\{1,\cdots,n\}}}(x_{1i_1},\cdots,\hat{x}_{ki_k},\cdots,x_{ni_n},y)\\ &= (y\prod_{j=1}^{t_1-1}x_{1j},y\prod_{j=1}^{t_2-1}x_{2j},\cdots,y\prod_{j=1}^{t_n-1}x_{nj},\prod_{\substack{1\le i<j\le n\\ 1\le s\le t_i-1\\ 1\le l\le t_j-1}}x_{is}x_{jl}).\end{aligned}$$

Proof. As each of facets of $\Delta_s(G_{t_1,t_2,\cdots,t_n})$ is obtained by removing exactly one edge from each cycle G_{t_i}, $1\le i\le n$, from [5, Proposition 5.3.10], we get that

$$I_{\Delta_s(G_{t_1,t_2,\cdots,t_n})} = \bigcap_{\substack{i_j\in\{1,\cdots,t_j-1\}\\ j\in\{1,\cdots,n\}}}(x_{1i_1},\cdots,x_{ni_n})\bigcap_{\substack{i_j\in\{1,\cdots,t_j-1\}\\ j\in\{1,\cdots,\hat{k},\cdots,n\}\\ k\in\{1,\cdots,n\}}}(x_{1i_1},\cdots,\hat{x}_{ki_k},\cdots,x_{ni_n},y).$$

From [4, Proposition 1.2.1], we have that

$$\begin{aligned} &\bigcap_{\substack{i_j\in\{1,\cdots,t_j-1\}\\ j\in\{1,\cdots,n\}}}(x_{1i_1},x_{2i_2},\cdots,x_{ni_n}) = (x_{11},x_{21},\cdots,x_{n-1,1},x_{n1})\cap(x_{11},\cdots,x_{n-1,1},x_{n2})\\ &\quad\cap\cdots\cap(x_{11},x_{21},\cdots,x_{n-1,1},x_{n,t_n-1})\cap(x_{11},x_{21},\cdots,x_{n-1,2},x_{n1})\cap\cdots\\ &\quad\cap(x_{11},\cdots,x_{n-1,2},x_{n,t_n-1})\cap\cdots\cap(x_{1,t_1-1},x_{2,t_2-1},\cdots,x_{n-1,t_{(n-1)}-1},x_{n,t_n-1})\\ &= (x_{11},x_{21},\cdots,x_{n-1,1},\prod_{j=1}^{t_n-1}x_{nj})\cap(x_{11},x_{21},\cdots,x_{n-1,2},\prod_{j=1}^{t_n-1}x_{nj})\cap\cdots\\ &\quad\cap(x_{1,t_1-1},x_{2,t_2-1},\cdots,x_{n-1,t_{(n-1)}-1},\prod_{j=1}^{t_n-1}x_{nj})\end{aligned}$$

$$\begin{aligned} &= (x_{11},x_{21},\cdots,\prod_{j=1}^{t_{(n-1)}-1}x_{n-1,j},\prod_{j=1}^{t_n-1}x_{nj})\cap\cdots\cap(x_{1,t_1-1},x_{2,t_2-1},\cdots,\prod_{j=1}^{t_{(n-1)}-1}x_{n-1,j},\prod_{j=1}^{t_n-1}x_{nj})\\ &= (\prod_{j=1}^{t_1-1}x_{1j},\prod_{j=1}^{t_2-1}x_{2j},\cdots,\prod_{j=1}^{t_{(n-1)}-1}x_{n-1,j},\prod_{j=1}^{t_n-1}x_{nj}),\end{aligned}$$

and

$$\bigcap_{\substack{i_j \in \{1,\cdots,t_j-1\} \\ j \in \{1,\cdots,\hat{k},\cdots,n\} \\ k \in \{1,\cdots,n\}}} (x_{1i_1},\cdots,\hat{x}_{ki_k},\cdots,x_{ni_n},y) = (x_{11},x_{21},\cdots,x_{n-1,1},y) \cap (x_{11},x_{21},\cdots,x_{n-1,2},y)$$
$$\cap \cdots \cap (x_{11},x_{21},\cdots,x_{n-1,t_{(n-1)}-1},y) \cap (x_{11},x_{21},\cdots,y,x_{n1}) \cap (x_{11},x_{21},\cdots,y,x_{n2})$$
$$\cap \cdots \cap (x_{11},x_{21},\cdots,y,x_{n,t_n-1}) \cap \cdots \cap (y,x_{2,t_2-1},\cdots,x_{n,t_n-1})$$
$$= (x_{11},x_{21},\cdots,\prod_{j=1}^{t_{(n-1)}-1} x_{n-1,j},y) \cap (x_{11},x_{21},\cdots,y,\prod_{j=1}^{t_n-1} x_{nj}) \cap \cdots \cap (y,\prod_{j=1}^{t_2-1} x_{2j},\cdots,x_{n,t_n-1})$$
$$= (\prod_{j=1}^{t_1-1} x_{1j},\prod_{j=1}^{t_2-1} x_{2j},\cdots,\prod_{j=1}^{t_{(n-1)}-1} x_{n-1,j},y) \cap (\prod_{j=1}^{t_1-1} x_{1j},\prod_{j=1}^{t_2-1} x_{2j},\cdots,y,\prod_{j=1}^{t_n-1} x_{nj}) \cap \cdots$$
$$\cap (y,\prod_{j=1}^{t_2-1} x_{2j},\cdots,\prod_{j=1}^{t_n-1} x_{nj}).$$

Therefore,

$$I_{\Delta_s(G_{t_1,t_2,\cdots,t_n})} = \bigcap_{\substack{i_j \in \{1,\cdots,t_j-1\} \\ j \in \{1,\cdots,n\}}} (x_{1i_1},x_{2i_2},\cdots,x_{ni_n}) \bigcap_{\substack{i_j \in \{1,\cdots,t_j-1\} \\ j \in \{1,\cdots,\hat{k},\cdots,n\} \\ k \in \{1,\cdots,n\}}} (x_{1i_1},\cdots,\hat{x}_{ki_k},\cdots,x_{ni_n},y)$$
$$= (\prod_{j=1}^{t_1-1} x_{1j},\prod_{j=1}^{t_2-1} x_{2j},\cdots,\prod_{j=1}^{t_{(n-1)}-1} x_{n-1,j},\prod_{j=1}^{t_n-1} x_{nj}) \cap (\prod_{j=1}^{t_1-1} x_{1j},\prod_{j=1}^{t_2-1} x_{2j},\cdots,\prod_{j=1}^{t_{(n-1)}-1} x_{n-1,j},y)$$
$$\cap (\prod_{j=1}^{t_1-1} x_{1j},\prod_{j=1}^{t_2-1} x_{2j},\cdots,y,\prod_{j=1}^{t_n-1} x_{nj}) \cap \cdots \cap (y,\prod_{j=1}^{t_2-1} x_{2j},\cdots,\prod_{j=1}^{t_n-1} x_{nj})$$
$$= (y\prod_{j=1}^{t_1-1} x_{1j},y\prod_{j=1}^{t_2-1} x_{2j},\cdots,y\prod_{j=1}^{t_n-1} x_{nj},\prod_{\substack{1\le i<j\le n \\ 1\le s\le t_i-1 \\ 1\le l\le t_j-1}} x_{is}x_{jl}).$$

□

4. The computation of f-vector of $\Delta_s(G_{t_1,t_2,\cdots,t_n})$

In this section, we will give a formula for f-vector of $\Delta_s(G_{t_1,t_2,\cdots,t_n})$ and consequently a formula for Hilbert series of the Stanley-Reisner ring $k[\Delta_s(G_{t_1,t_2,\cdots,t_n})]$ under the assumption that the length of every cyclic graph G_{t_i} is t for $1 \le i \le n$. But before this we need the following proposition, its proof can be seen in Proposition 2.2 of [1].

Proposition 4.1. *For a simplicial complex Δ on $[n]$ of dimension d, if $f_t = \binom{n}{t+1}$ for some $t \le d$, then $f_i = \binom{n}{i+1}$ for all $0 \le i < t$.*

Now, under the assumption that the length of the cyclic graph G_{t_i} is t for every $1 \leq i \leq n$, we give the formula to compute the f-vector of $\Delta_s(G_{t_1, t_2, \cdots, t_n})$.

Theorem 4.2. *Let $t_i = t$ for every $1 \leq i \leq n$, and $b = n(t-1)+1$. Then the f-vector of $\Delta_s(G_{t_1, t_2, \cdots, t_n})$ is given by $f = (f_0, f_1, \ldots, f_d)$, where $d = n(t-2)$ and*

$$f_j = \begin{cases}
\binom{b}{j+1} & \text{if } 0 \leq j \leq t-2, \\
\binom{b}{j+1} - \binom{n}{1}\binom{b-t}{j-t+1} & \text{if } t-1 \leq j \leq \min\{2t-3, d+1\}-1, \\
\begin{array}{l} \binom{b}{j+1} - \binom{n}{1}\binom{b-t}{j-t+1} \\ -\binom{n}{2}\binom{b-(2t-2)}{j-(2t-2)+1} \end{array} & \text{if } j = \min\{2t-2, d+1\}-1, \\
\begin{array}{l} \binom{b}{j+1} + \sum_{i=1}^{2}(-1)^i\binom{n}{i}\binom{b-[it-(i-1)]}{j-[it-(i-1)]+1} \\ -\binom{n}{2}\binom{b-(2t-1)}{j-(2t-2)+1} \end{array} & \text{if } \min\{2t-2, d+1\} \leq j \leq \min\{3t-4, d+1\}-1, \\
\begin{array}{l} \binom{b}{j+1} + \sum_{i=1}^{2}(-1)^i\binom{n}{i}\binom{b-[it-(i-1)]}{j-[it-(i-1)]+1} \\ -\sum_{i=2}^{3}(-1)^i(i-1)\binom{n}{i}\binom{b-[it-(i-1)]}{j-(it-i)+1} \end{array} & \text{if } j = \min\{3t-3, d+1\}-1, \\
\begin{array}{l} \binom{b}{j+1} + \sum_{i=1}^{3}(-1)^i\binom{n}{i}\binom{b-[it-(i-1)]}{j-[it-(i-1)]+1} \\ -\sum_{i=2}^{3}(-1)^i(i-1)\binom{n}{i}\binom{b-[it-(i-1)]}{j-(it-i)+1} \end{array} & \text{if } \min\{3t-3, d+1\} \leq j \leq \min\{4t-5, d+1\}-1, \\
\begin{array}{l} \binom{b}{j+1} + \sum_{i=1}^{3}(-1)^i\binom{n}{i}\binom{b-[it-(i-1)]}{j-[it-(i-1)]+1} \\ -\sum_{i=2}^{4}(-1)^i(i-1)\binom{n}{i}\binom{b-[it-(i-1)]}{j-(it-i)+1} \end{array} & \text{if } j = \min\{4t-4, d+1\}-1, \\
\qquad\vdots & \qquad\vdots \\
\begin{array}{l} \binom{b}{j+1} + \sum_{i=1}^{m}(-1)^i\binom{n}{i}\binom{b-[it-(i-1)]}{j-[it-(i-1)]+1} \\ -\sum_{i=2}^{m}(-1)^i(i-1)\binom{n}{i}\binom{b-[it-(i-1)]}{j-(it-i)+1} \end{array} & \text{if } \min\{m(t-1), d+1\} \leq j \leq \min\{(m+1)(t-1)-1, d+1\}-1, \\
\begin{array}{l} \binom{b}{j+1} + \sum_{i=1}^{m}(-1)^i\binom{n}{i}\binom{b-[it-(i-1)]}{j-[it-(i-1)]+1} \\ -\sum_{i=2}^{m+1}(-1)^i(i-1)\binom{n}{i}\binom{b-[it-(i-1)]}{j-(it-i)+1} \end{array} & \text{if } j = \min\{(m+1)(t-1), d+1\}-1, \\
\qquad\vdots & \qquad\vdots \\
\begin{array}{l} \binom{b}{j+1} + \sum_{i=1}^{n-2}(-1)^i\binom{n}{i}\binom{b-[it-(i-1)]}{j-[it-(i-1)]+1} \\ -\sum_{i=2}^{n-1}(-1)^i(i-1)\binom{n}{i}\binom{b-[it-(i-1)]}{j-(it-i)+1} \end{array} & \text{if } j = \min\{(n-1)(t-1), d+1\}-1, \\
\begin{array}{l} \binom{b}{j+1} + \sum_{i=1}^{n-1}(-1)^i\binom{n}{i}\binom{b-[it-(i-1)]}{j-[it-(i-1)]+1} \\ -\sum_{i=2}^{n-1}(-1)^i(i-1)\binom{n}{i}\binom{b-[it-(i-1)]}{j-(it-i)+1} \end{array} & \text{if } \min\{(n-1)(t-1), d+1\} \leq j \leq d.
\end{cases}$$

Proof. Let $E = \{e_{11}, e_{12}, \cdots, e_{1,t-1}, e_{21}, \cdots, e_{2,t-1}, \cdots, e_{n1}, \cdots, e_{n,t-1}, e\}$ be the edge set of n-cyclic graph $G_{t_1,t_2,\cdots,t_n}$, where e is the common edge. By the definition of f-vector of $\Delta_s(G_{t_1,t_2,\cdots,t_n})$, f_j is the number of all those subsets of the edge set E of graph $G_{t_1,t_2,\cdots,t_n}$, with $j+1$ elements, that contain neither $\{e_{i1}, e_{i2}, \cdots, e_{i,t-1}, e\}$ for all $1 \le i \le n$ nor $\{e_{k1}, e_{k2}, \cdots, e_{k,t-1}, e_{l1}, e_{l2}, \cdots, e_{l,t-1}\}$ for $1 \le k < l \le n$.

Take any subset $F \subset E$ consisting of $t-1$ elements. For every $1 \le j \le n$ the edge set $\{e_{j1}, \cdots, e_{j,t-1}, e\}$ of the cyclic graph G_{t_j} has t elements, it is clear that $\{e_{j1}, e_{j2}, \cdots, e_{j,t-1}, e\}$ can not appear in F, so $F \in \Delta_s(G_{t_1,t_2,\cdots,t_n})$. It follows that $\Delta_s(G_{t_1,t_2,\cdots,t_n})$ contains all possible subsets of E with cardinality $t-1$, therefore $f_{t-2} = \binom{nt-(n-1)}{t-1} = \binom{b}{t-1}$. Thus, by Proposition 4.1, we have $f_j = \binom{b}{j+1}$ for all $0 \le j \le t-2$.

For $t-1 \le j \le \min\{2t-3, d+1\}-1$, we need to count all the subsets E with cardinality $j+1$ containing the edge set $\{e_{j1}, e_{j2}, \cdots, e_{j,t-1}, e\}$ of some G_{t_j} of the n-cyclic graph $G_{t_1,t_2,\cdots,t_n}$. The edge set E of the graph $G_{t_1,t_2,\cdots,t_n}$ has $b\,(= n(t-1)+1)$ elements, and there are $\binom{n}{1}\binom{b-t}{j-t+1}$ subsets of E with cardinality $j+1$ such that $\{e_{j1}, e_{j2}, \cdots, e_{j,t-1}, e\}$ is a part of it. In total, there are $\binom{b}{j+1}$ subsets of E with cardinality $j+1$, hence $f_j = \binom{b}{j+1} - \binom{n}{1}\binom{b-t}{j-t+1}$.

When $j = \min\{2t-2, d+1\}-1$, we need to compute all the subsets of E having the cardinality $j+1$ containing such edge sets $\{e_{k1}, \cdots, e_{k,t-1}, e_{l1}, \cdots, e_{l,t-1}\}$ for $1 \le k < l \le n$ of some two cyclic graphs G_{t_k} and G_{t_l} of n-cyclic graph $G_{t_1,t_2,\cdots,t_n}$, we get that there are $\binom{n}{2}\binom{b-(2t-2)}{j-(2t-2)+1}$ such subsets. It is clear that we also need to compute these subsets of E having the cardinality $j+1$ containing such edge set $\{e_{i1}, e_{i2}, \cdots, e_{i,t-1}, e\}$ of some cyclic graph G_{t_i} of n-cyclic graph $G_{t_1,t_2,\cdots,t_n}$. we get that there are $\binom{n}{1}\binom{b-t}{j-t+1}$ such subsets. In total, we have $\binom{b}{j+1}$ subsets of E with cardinality $j+1$. Therefore, $f_j = \binom{b}{j+1} - \binom{n}{1}\binom{b-t}{j-t+1} - \binom{n}{2}\binom{b-(2t-2)}{j-(2t-2)+1}$.

For $\min\{2t-2, d+1\} \le j \le \min\{3t-4, d+1\}-1$, on the one hand, we need to count all the subsets of E having the cardinality $j+1$ containing such edge set $\{e_{i1}, e_{i2}, \cdots, e_{i,t-1}, e_{j1}, e_{j2}, \cdots, e_{j,t-1}, e\}$ of some two cyclic graphs G_{t_i} and G_{t_j} of n-cyclic graph $G_{t_1,t_2,\cdots,t_n}$, by the inclusion exclusion principle, we get there are $\binom{n}{2}\binom{b-(2t-1)}{j-(2t-1)+1}$ such subsets. On the other hand, we need to compute all the subsets of E having the cardinality $j+1$ containing such edge set $\{e_{i1}, e_{i2}, \cdots, e_{i,t-1}, e\}$ of some cyclic graph G_{t_i} of n-cyclic graph $G_{t_1,t_2,\cdots,t_n}$, we get that there are $\binom{n}{1}[\binom{b-t}{j-t+1} - \binom{n-1}{1}\binom{b-(2t-1)}{j-(2t-1)+1}]$ such subsets. It is obvious that we also need to compute all the subsets of E having the cardinality $j+1$ containing $\{e_{k1}, e_{k2}, \cdots, e_{k,t-1}, e_{l1}, e_{l2}, \cdots, e_{l,t-1}\}$ for $1 \le k < l \le n$, there are $\binom{n}{2}[\binom{b-(2t-2)}{j-(2t-2)+1} - \binom{b-(2t-1)}{j-(2t-1)+1}] = \binom{n}{2}\binom{b-(2t-1)}{j-(2t-2)+1}$ such subsets. Therefore $f_j = \binom{b}{j+1} - \binom{n}{2}\binom{b-(2t-1)}{j-(2t-1)+1} - \binom{n}{1}[\binom{b-t}{j-t+1} - \binom{n-1}{1}\binom{b-(2t-1)}{j-(2t-1)+1}] - \binom{n}{2}\binom{b-(2t-1)}{j-(2t-2)+1} = \binom{b}{j+1} + \sum_{i=1}^{2}(-1)^i\binom{n}{i}\binom{b-[it-(i-1)]}{j-[it-(i-1)]+1} - \binom{n}{2}\binom{b-(2t-1)}{j-(2t-2)+1}$.

When $j = \min\{3t-3, d+1\}-1$, on the one hand, we not only need to compute all the subsets of E having the cardinality $j+1$ containing such edge set

$\{e_{k1}, e_{k2}, \cdots, e_{k,\,t-1}, e_{l1}, e_{l\,2}, \cdots, e_{l,\,t-1}, e_{s1}, e_{s\,2}, \cdots, e_{s,\,t-1}\}$ for $1 \le k < l < s \le n$ of some three cyclic graphs G_{t_k}, G_{t_l} and G_{t_s} of n-cyclic graph $G_{t_1, t_2, \cdots, t_n}$, we get that there are $\binom{n}{3}[\binom{b-(3t-3)}{j-(3t-3)+1} - \binom{b-(3t-2)}{j-(3t-2)+1}] = \binom{n}{3}\binom{b-(3t-2)}{j-(3t-3)+1}$ such subsets. we also need to compute all the subsets of E having the cardinality $j+1$ containing $\{e_{k1}, e_{k2}, \cdots, e_{k,\,t-1}, e_{l1}, e_{l\,2}, \cdots, e_{l,\,t-1}\}$ for $1 \le k < l \le n$, by the inclusion exclusion principle, we obtain that there are $\binom{n}{2}[\binom{b-(2t-1)}{j-(2t-2)+1} - \binom{n-2}{1}\binom{b-(3t-2)}{j-(3t-3)+1}]$ such subsets. On the other hand, we not only need to count all the subsets of E having the cardinality $j+1$ containing $\{e_{k1}, e_{k2}, \cdots, e_{k,\,t-1}, e_{l1}, e_{l\,2}, \cdots, e_{l,\,t-1}\}$ for $1 \le k < l \le n$, but also need to compute all the subsets of E having the cardinality $j+1$ containing $\{e_{k1}, e_{k2}, \cdots, e_{k,\,t-1}, e\}$ for $1 \le k \le n$. By the inclusion exclusion principle, there are $\binom{n}{2}\binom{b-(2t-1)}{j-(2t-1)+1}$ and $\binom{n}{1}[\binom{b-t}{j-t+1} - \binom{n-1}{1}\binom{b-(2t-1)}{j-(2t-1)+1}]$ such subsets respectively. There are $\binom{b}{j+1}$ subsets of E with cardinality $j+1$ in total. Hence, by the use of repetition of combinatorial formula $\sum\limits_{j=0}^{k}(-1)^j\binom{m}{j} = (-1)^k\binom{m-1}{k}$, we have that

$$\begin{aligned}
f_j &= \binom{b}{j+1} - \binom{n}{2}[\binom{b-(2t-1)}{j-(2t-2)+1} - \binom{n-2}{1}\binom{b-(3t-2)}{j-(3t-3)+1}] \\
&- \binom{n}{3}\binom{b-(3t-2)}{j-(3t-3)+1} - \binom{n}{1}[\binom{b-t}{j-t+1} - \binom{n-1}{1}\binom{b-(2t-1)}{j-(2t-1)+1}] \\
&- \binom{n}{2}\binom{b-(2t-1)}{j-(2t-1)+1} \\
&= \binom{b}{j+1} + \sum_{i=1}^{2}(-1)^i\binom{n}{i}\binom{b-[it-(i-1)]}{j-[it-(i-1)]+1} - \binom{n}{2}\binom{b-(2t-1)}{j-(2t-2)+1} \\
&+ [\binom{n}{2}\binom{n-2}{1} - \binom{n}{3}]\binom{b-(3t-2)}{j-(3t-3)+1} \\
&= \binom{b}{j+1} + \sum_{i=1}^{2}(-1)^i\binom{n}{i}\binom{b-[it-(i-1)]}{j-[it-(i-1)]+1} - \binom{n}{2}\binom{b-(2t-1)}{j-(2t-2)+1} \\
&- \binom{n}{3}[\binom{3}{0} - \binom{3}{1}]\binom{b-(3t-2)}{j-(3t-3)+1} \\
&= \binom{b}{j+1} + \sum_{i=1}^{2}(-1)^i\binom{n}{i}\binom{b-[it-(i-1)]}{j-[it-(i-1)]+1} - \binom{n}{2}\binom{b-(2t-1)}{j-(2t-2)+1} \\
&- \binom{n}{3}(-1)^1\binom{2}{1}\binom{b-(3t-2)}{j-(3t-3)+1} \\
&= \binom{b}{j+1} + \sum_{i=1}^{2}(-1)^i\binom{n}{i}\binom{b-[it-(i-1)]}{j-[it-(i-1)]+1} \\
&\qquad - \sum_{i=2}^{3}(-1)^i(i-1)\binom{n}{i}\binom{b-[it-(i-1)]}{j-(it-i)+1}.
\end{aligned}$$

For $\min\{3t-3, d+1\} \leq j \leq \min\{4t-5, d+1\}-1$, on the one hand, we not only need to count all subsets of E having the cardinality $j+1$ containing $\{e_{k1}, e_{k2}, \cdots, e_{k,t-1}, e_{l1}, e_{l2}, \cdots, e_{l,t-1}, e_{s1}, e_{s2}, \cdots, e_{s,t-1}, e\}$ for $1 \leq k < l < s \leq n$ of some three cyclic graphs G_{t_k}, G_{t_l} and G_{t_s} of n-cyclic graph $G_{t_1, t_2, \cdots, t_n}$, but also need to count all subsets of E having the cardinality $j+1$ containing $\{e_{i1}, e_{i2}, \cdots, e_{i,t-1}, e_{j1}, e_{j2}, \cdots, e_{j,t-1}, e\}$ of some two cyclic graphs G_{t_i} and G_{t_j}. By the inclusion exclusion principle, there are $\binom{n}{3}\binom{b-(3t-2)}{j-(3t-2)+1}$ and $\binom{n}{2}[\binom{b-(2t-1)}{j-(2t-1)+1} - \binom{n-2}{1}\binom{b-(3t-2)}{j-(3t-2)+1}]$ such subsets respectively. On the other hand, we need to compute all the subsets of E having the cardinality $j+1$ containing $\{e_{i1}, e_{i2}, \cdots, e_{i,t-1}, e\}$ of some cyclic graph G_{t_i}, we get that there are $\binom{n}{1}\{\binom{b-t}{j-t+1} - \binom{n-1}{1}[\binom{b-(2t-1)}{j-(2t-1)+1} - \binom{n-2}{1}\binom{b-(3t-2)}{j-(3t-2)+1}] - \binom{n-1}{2}\binom{b-(3t-2)}{j-(3t-2)+1}\}$ such subsets. Of course, we have to count all the subsets E with cardinality $j+1$ containing $\{e_{k1}, e_{k2}, \cdots, e_{k,t-1}, e_{l1}, e_{l2}, \cdots, e_{l,t-1}, e_{s1}, e_{s2}, \cdots, e_{s,t-1}\}$ for $1 \leq k < l < s \leq n$, and all subsets E with cardinality $j+1$ containing such edge set $\{e_{k1}, e_{k2}, \cdots, e_{k,t-1}, e_{l1}, e_{l2}, \cdots, e_{l,t-1}\}$ for $1 \leq k < l \leq n$. By the inclusion exclusion principle, there are $\binom{n}{3}[\binom{b-(3t-3)}{j-(3t-3)+1} - \binom{b-(3t-2)}{j-(3t-2)+1}] = \binom{n}{3}\binom{b-(3t-2)}{j-(3t-3)+1}$ and $\binom{n}{2}[\binom{b-(2t-1)}{j-(2t-2)+1} - \binom{n-2}{1}\binom{b-(3t-2)}{j-(3t-3)+1}]$. In total, there are $\binom{b}{j+1}$ subsets of E with cardinality $j+1$. By the use of repetition of the combinatorial formula $\sum\limits_{j=0}^{m-1}(-1)^j\binom{m}{j} = (-1)^{m-1}$, we have that

$$\begin{aligned}
f_j &= \binom{b}{j+1} - \binom{n}{3}\binom{b-(3t-2)}{j-(3t-2)+1} \\
&- \binom{n}{2}[\binom{b-(2t-1)}{j-(2t-1)+1} - \binom{n-2}{1}\binom{b-(3t-2)}{j-(3t-2)+1}] \\
&- \binom{n}{1}\{\binom{b-t}{j-t+1} - \binom{n-1}{1}[\binom{b-(2t-1)}{j-(2t-1)+1} - \binom{n-2}{1}\binom{b-(3t-2)}{j-(3t-2)+1}] \\
&- \binom{n-1}{2}\binom{b-(3t-2)}{j-(3t-2)+1}\} - \binom{n}{3}\binom{b-(3t-2)}{j-(3t-3)+1} \\
&- \binom{n}{2}[\binom{b-(2t-1)}{j-(2t-2)+1} - \binom{n-2}{1}\binom{b-(3t-2)}{j-(3t-3)+1}] \\
&= \binom{b}{j+1} - \binom{n}{1}\binom{b-t}{j-t+1} + [\binom{n}{1}\binom{n-1}{1} - \binom{n}{2}]\binom{b-(2t-1)}{j-(2t-1)+1} \\
&- [\binom{n}{3} - \binom{n}{2}\binom{n-2}{1} + \binom{n}{1}\binom{n-1}{1}\binom{n-2}{1} - \binom{n}{1}\binom{n-1}{2}]\binom{b-(3t-2)}{j-(3t-2)+1} \\
&- \binom{n}{2}\binom{b-(2t-1)}{j-(2t-2)+1} + [\binom{n}{2}\binom{n-2}{1} - \binom{n}{3}]\binom{b-(3t-2)}{j-(3t-3)+1}
\end{aligned}$$

$$
\begin{aligned}
&= \binom{b}{j+1} - \binom{n}{1}\binom{b-t}{j-t+1} - \binom{n}{2}[\binom{2}{0} - \binom{2}{1}]\binom{b-(2t-1)}{j-(2t-1)+1} \\
&- \{\binom{n}{3} - \binom{n}{2}\binom{n-2}{1} - \binom{n}{1}\binom{n-1}{2}[\binom{2}{0} - \binom{2}{1}]\}\binom{b-(3t-2)}{j-(3t-2)+1} \\
&- \sum_{i=2}^{3}(-1)^i(i-1)\binom{n}{i}\binom{b-[it-(i-1)]}{j-(it-i)+1} \\
&= \binom{b}{j+1} - \binom{n}{1}\binom{b-t}{j-t+1} + \binom{n}{2}\binom{b-(2t-1)}{j-(2t-1)+1} \\
&- \binom{n}{3}[\binom{3}{0} - \binom{3}{1} + \binom{3}{2}]\binom{b-(3t-2)}{j-(3t-2)+1} \\
&- \sum_{i=2}^{3}(-1)^i(i-1)\binom{n}{i}\binom{b-[it-(i-1)]}{j-(it-i)+1} \\
&= \binom{b}{j+1} - \binom{n}{1}\binom{b-t}{j-t+1} + \binom{n}{2}\binom{b-(2t-1)}{j-(2t-1)+1} - \binom{n}{3}\binom{b-(3t-2)}{j-(3t-2)+1} \\
&- \sum_{i=2}^{3}(-1)^i(i-1)\binom{n}{i}\binom{b-[it-(i-1)]}{j-(it-i)+1} \\
&= \binom{b}{j+1} + \sum_{i=1}^{3}(-1)^i\binom{n}{i}\binom{b-[it-(i-1)]}{j-[it-(i-1)]+1} \\
&- \sum_{i=2}^{3}(-1)^i(i-1)\binom{n}{i}\binom{b-[it-(i-1)]}{j-(it-i)+1}.
\end{aligned}
$$

Other cases can be shown in a similar way as the above. □

We can now give a formula for Hilbert series of $k[\Delta_s(G_{t_1,t_2,\cdots,t_n})]$ under the condition that the length of every cyclic graph G_{t_i} is t for $1 \le i \le n$.

Theorem 4.3. *Let $\Delta_s(G_{t_1,t_2,\cdots,t_n})$ be the spanning simplicial complex of n-cyclic graph $G_{t_1,t_2,\cdots,t_n}$, where $t_i = t$ for each $1 \le i \le n$. Then Hilbert series of the Stanley-Reisner ring $k[\Delta_s(G_{t_1,t_2,\cdots,t_n})]$ is given by*

$$
\begin{aligned}
H(k[\Delta_s(G_{t_1,t_2,\cdots,t_n})], z) &= 1 + \sum_{i=0}^{t-2}\frac{\binom{b}{j+1}z^{i+1}}{(1-z)^{i+1}} + \sum_{i=t-1}^{2t-4}\frac{[\binom{b}{j+1} - \binom{n}{1}\binom{b-t}{j-t+1}]z^{i+1}}{(1-z)^{i+1}} \\
&+ \frac{\{\binom{b}{j+1} + \sum_{i=1}^{2}(-1)^i\binom{n}{i}\binom{b-[it-(i-1)]}{j-[it-(i-1)]+1}\}z^{2t-2}}{(1-z)^{2t-2}}
\end{aligned}
$$

$$+\sum_{i=2t-2}^{3t-5}\frac{[\binom{b}{j+1}+\sum_{i=1}^{2}(-1)^i\binom{n}{i}\binom{b-[it-(i-1)]}{j-[it-(i-1)]+1}-\binom{n}{2}\binom{b-(2t-1)}{j-(2t-2)+1}]z^{i+1}}{(1-z)^{i+1}}$$

$$+\frac{\{\binom{b}{j+1}+\sum_{i=1}^{2}(-1)^i\binom{n}{i}\binom{b-[it-(i-1)]}{j-[it-(i-1)]+1}-\sum_{i=2}^{3}(-1)^i(i-1)\binom{n}{i}\binom{b-[it-(i-1)]}{j-(it-i)+1}\}z^{3t-3}}{(1-z)^{3t-3}}$$

$$+\ \cdots$$

$$+\frac{\{\binom{b}{j+1}+\sum_{i=1}^{n-2}(-1)^i\binom{n}{i}\binom{b-[it-(i-1)]}{j-[it-(i-1)]+1}-\sum_{i=2}^{n-1}(-1)^i(i-1)\binom{n}{i}\binom{b-[it-(i-1)]}{j-(it-i)+1}\}z^{(n-1)(t-1)}}{(1-z)^{(n-1)(t-1)}}$$

$$+\sum_{i=(n-1)(t-1)}^{n(t-1)-2}\frac{[\binom{b}{j+1}+\sum_{i=1}^{n-1}(-1)^i\binom{n}{i}\binom{b-[it-(i-1)]}{j-[it-(i-1)]+1}-\sum_{i=2}^{n-1}(-1)^i(i-1)\binom{n}{i}\binom{b-[it-(i-1)]}{j-(it-i)+1}]z^{i+1}}{(1-z)^{i+1}}.$$

Proof. From [5, Corollary 5.4.5], we have that if Δ is a simplicial complex and $f(\Delta)=(f_0,\ldots,f_d)$ is its f-vector, then the Hilbert series of Stanley-Reisner ring $k[\Delta]$ is given by

$$H(k[\Delta],z)=\sum_{i=-1}^{d}\frac{f_i z^{i+1}}{(1-z)^{i+1}},\qquad d=\dim(\Delta).$$

The desired formula follows from the above theorem at once. □

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