http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 72, Number 3, Pages 650–662 (2023)
DOI:10.31801/cfsuasmas.1127781
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: June 8, 2022; Accepted: December 10, 2022

DEVELOPABLE NORMAL SURFACE PENCIL

Mustafa DEDE

Department of Mathematics, Kilis 7 Aralık University, 79000 Kilis, TÜRKİYE

Abstract. In this paper, we introduce a new class of surfaces, called as normal surface pencil. We parameterize a normal surface pencil by using the principal normal vector **n** and the binormal vector **b** of the Frenet frame of a space curve $\alpha(s)$ as follows $\varphi(s,t)=\alpha(s)+y(s,t)\mathbf{n}+z(s,t)\mathbf{b}$. A well known example of normal surface pencil is a canal surface. Finally, we propose the sufficient conditions of a normal surface pencil being a developable surface. Then several new examples of developable normal surface pencil are constructed from these conditions.

1. Introduction

Let $\varphi=\varphi(u,v)$ be a local parametrization of a surface parameterized by

$$\varphi(u,v)=(x(u,v),y(u,v),z(u,v)).$$

A well known Gauss curvature K of a surface is given by

$$K=\frac{LN-M^2}{EG-F^2}, \tag{1}$$

where E,F,G and L,M,N are the coefficients of the first and the second fundamental forms of a surface, respectively [11].

An important topic in differential geometry is the study of curvature conditions of a surface [12, 13]. For instance, the surfaces with constant Gauss curvature are investigated in many papers [8]. Recently, Lopez and Moruz investigated the constant Gauss curvature of translation and homothetical surfaces [14]. A special case of constant Gauss curvature surface is flat ones. A surface with vanishing Gaussian curvature is called a flat surface ($K=0$) [9, 18]. The geometric meaning of a flat surface is that if we flattened a developable surface (flat ruled surface) into

2020 *Mathematics Subject Classification.* 53A04, 65D17.
Keywords. Flat surfaces, Gauss curvature, surface pencil, developable.
mustafadede@kilis.edu.tr; 0000-0003-2652-637X.

a planar figure (with no distortion), any geodesic on it will be mapped to a straight line in the planar figure [4].

Let $\alpha(t)$ be a regular space curve [10], then the Frenet frame is defined as follows

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \mathbf{b} = \frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|}, \mathbf{n} = \mathbf{b} \wedge \mathbf{t}.$$

The well-known Frenet formulas are given by

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{bmatrix} = \|\alpha'(t)\| \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix},$$

where the curvature κ and the torsion τ of the curve are given by

$$\kappa = \frac{\|\alpha' \wedge \alpha''\|}{\|\alpha'\|^3}, \tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \wedge \alpha''\|^2}.$$

Let us consider the Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ along a unit speed space curve $\alpha(s)$. By using the Frenet frame, we can define lots of special class of surfaces. For instance, we define the ruled surfaces $F(s,u) = \alpha(s) + u\mathbf{n}(s)$ or $F(s,u) = \alpha(s) + u\mathbf{b}(s)$ which are called the principal normal surface or principal binormal surface of the curve $\alpha(s)$, respectively [15]. The other example is that a canal surface is defined by $F(s,u) = \alpha(s) + r(s)\cos(u)\mathbf{n} + r(s)\sin(u)\mathbf{b}$ where $r(s)$ is radii function [3]. Moreover a pipe surface (tube) is a canal surface with a constant radii [2].

Theorem 1. *The principal normal or principal binormal surfaces are flat (developable) if and only if the corresponding curve is a planar [1].*

Theorem 2. *The regular canal surface is developable if and only if the canal surface is a cylinder or cone. That is, the curvature $\kappa(s) = 0$; the spine curve is a line and radii function $r(s)$ is a constant or linear function of s [3].*

A surface pencil can be parameterized by using the Frenet frame as follows

$$\varphi(s,t) = \alpha(s) + u(s,t)\mathbf{t}(s) + v(s,t)\mathbf{n}(s) + w(s,t)\mathbf{b}(s),$$

where $u(s,t)$,$v(s,t)$ and $w(s,t)$ are functions of s and t [4, 6]. Wang et al. used the surface pencil to answer the problem "assume we are given a space curve, how to characterize those surfaces that possess this curve as a common geodesic". The generalized solution of this problem is studied in [5]. The study of surface pencil has been extended Minkowski and Galilean spaces [7, 17]. Recently, Zhao and Wang derived the necessary and sufficient conditions to construct a developable surface through a given curve [16]. However they studied this problem with some constraints such as the curve is isoparametric on surface. In this paper we have studied this problem without any constraints. Moreover we derived possible parameterizations of flat normal surface pencil. Surprisingly we obtained new class of space curve which we call it the helical extension of a space curve. In [20] we give the characterization of this class of curve.

Rotation Minimizing Frame(RMF) or sometimes called as Bishop frame which is well defined even when the curve has vanishing second derivative in 3-dimensional Euclidean space [21]. Because Bishop frame is formed with the tangent vector and any convenient arbitrary basis for the remainder of the frame [22, 23].

2. Flat Normal Surface Pencils

In this section, we introduce the normal surface pencil to generalize two special classes of surfaces, namely, the principal normal surfaces and the canal surfaces. Then we derive the necessary and sufficient conditions for a normal surface pencil to be flat.

Definition 1. *Let $\alpha(s)$ be a unit speed space curve with the Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$, then a normal surface pencil is parameterized by*

$$\varphi(s,t) = \alpha(s) + y(s,t)\mathbf{n}(s) + z(s,t)\mathbf{b}(s), \tag{2}$$

where $y(s,t)$ and $z(s,t)$ are functions of s and t. For simplicity, we take the functions $y(s,t)$ and $z(s,t)$ that can be decomposed into two factors that allows us instead of solving partial differential equations we deal with ordinary differential equations

$$y(s,t) = y(s)w(t), z(s,t) = z(s)l(t).$$

Here $y(s), w(t), z(s)$ and $l(t)$ are all functions of s and t. Then a normal surface pencil is parameterized by

$$\varphi(s,t) = \alpha(s) + y(s)w(t)\mathbf{n} + z(s)l(t)\mathbf{b}. \tag{3}$$

Now, we can give the following theorems for classification of flat normal surface pencil.

Theorem 3. *Let $\alpha(s)$ be a unit speed space curve ($\tau \neq 0, \kappa \neq 0$). A normal surface pencil is flat, if and only if it is either*

(1) *a surface parameterized by $\varphi(s,t) = \alpha(s) + \dfrac{1}{\kappa(s)}\mathbf{n} + z(s)l(t)\mathbf{b}$,*

(2) *a surface parameterized by $\varphi(s,t) = \alpha(s) + y(s)w(t)\mathbf{n} - \int \dfrac{\tau}{\kappa} ds\mathbf{b}$,*

(3) *a surface parameterized by $\varphi(s,t) = \alpha(s) + z(s)(\tan \int \tau + c_{11})l(t)\mathbf{n} + z(s)l(t)\mathbf{b}$.*

If the curve $\alpha(s)$ is a unit speed plane curve ($\tau = 0$). Then a normal surface pencil is flat, if and only if it is either

(1) *a surface parameterized by $\varphi(s,t) = \alpha(s) + y(s)c_2\mathbf{n} + z(s)l(t)\mathbf{b}$*

(2) *a surface parameterized by $\varphi(s,t) = \alpha(s) + y(s)w(t)\mathbf{n} + c_7c_8\mathbf{b}$*

If the curve $\alpha(s)$ is a unit speed line ($\kappa = \tau = 0$). We use Rotation Minimizing Frame basis ($\mathbf{e}_1$ and $\mathbf{e}_2$), then a normal surface pencil is flat, if and only if it is either

(1) *a surface parameterized by* $\varphi(s,t) = \alpha(s) + c_3(c_4 s + c_5)w(t)\mathbf{e}_1 + (c_4 s + c_5)l(t)\mathbf{e}_2$.

(2) *a surface parameterized by* $\varphi(s,t) = \alpha(s) + y(s)w(t)\mathbf{e}_1 + z(s)c_6\mathbf{e}_2$.

where $c_i(i = 1..11) \in \mathbb{R}$.

The rest of the section is devoted to the proof of the above theorem.

Proof. From (1) and (3), it is easy to see that a surface is flat if and only if it satisfies the following equation:

$$\langle \varphi_s \wedge \varphi_t, \varphi_{ss} \rangle \langle \varphi_s \wedge \varphi_t, \varphi_{tt} \rangle - \langle \varphi_s \wedge \varphi_t, \varphi_{st} \rangle^2 = 0. \tag{4}$$

By using the well known Frenet formulas, we obtain the partial derivatives of the normal surface pencil as follows

$$\varphi_s = (1-\kappa(s)y(s)w(t))\mathbf{t} + (y_s(s)w(t) - \tau(s)z(s)l(t))\mathbf{n} + (\tau(s)y(s)w(t) + z_s(s)l(t))\mathbf{b}. \tag{5}$$

and

$$\varphi_t = y(s)w_t(t)\mathbf{n} + z(t)l_t(t)\mathbf{b}. \tag{6}$$

where φ_s and φ_t denote the partial derivatives of the surface with respect to s and t.

It follows that the cross product of φ_s and φ_t is obtained as

$$\varphi_s \wedge \varphi_t = ((y_s w - \tau z l) z l_t - (\tau y w + z_s l) y w_t)\mathbf{t} - (1-\kappa y w) z l_t \mathbf{n} + (1-\kappa y w) y w_t \mathbf{b}. \tag{7}$$

Combining the equations (4), (5), (6) and (7) and higher order partial derivatives of normal surface pencil, we have a non-linear partial differential equation (PDE) in the following form

$$\begin{gathered} \tau^2 z^4(-y^2 w^2 \kappa^2 + 2w\kappa y - 1)l_t^4 + (...)l_t^3 w_t + ... + (...)l_t - \\ y^4(z_s^2 c^2 \kappa^2 + \tau^2 + 2z_s c\kappa\tau)w_t^4 + (...)w_t^3 l_t + ... + (...)w_t = 0. \end{gathered} \tag{8}$$

We can rearrange the equation (8) as follows

$$\sum_{i=1}^{4} A_i(s,t) l_t^i(t) + B_i(s,t) w_t^i(t) + \sum_{i=1}^{3} C_i(s,t) l_t^i(t) w_t^{4-i}(t) = 0, \tag{9}$$

where upper "i" indicates the degree of function and the coefficients $A_i(s,t)$ $B_i(s,t)$ $(i = 1..4)$ and $C_i(s,t)$ $(i = 1..3)$ are smooth functions on s and t.

We will solve the equation (9) whether the set of functions $\{l, w\}$ is linearly independent or linearly dependent.

- If l and w functions are not linearly dependent. Then a normal surface pencil is flat if the coefficients $A_i(s,t), B_i(s,t)$ and $C_i(s,t)$ vanishes.

From (8) and (9) the coefficient $A_4(s,t)$ of $l_t^4(t)$ can be computed as follows

$$A_4 = -\tau^2 z^4 (y w \kappa - 1)^2. \tag{10}$$

It follows that the coefficient $A_4(s,t)$ vanishes if and only if $yw\kappa - 1 = 0, \tau(s) = 0, z(s) = 0$ or $l_t = 0$, respectively.

Now, we discuss these four cases:
Case 1) If $yw\kappa - 1 = 0$ then we have

$$y(s)w(t) = \frac{1}{\kappa(s)}. \tag{11}$$

Thus, observe that $w(t)$ is a constant function.

Combining (11) with (9) we obtain that all the coefficients A_i, B_i and C_i in (9) vanishes. Therefore, substituting (11) into (3) allow us to parameterize a flat normal surface pencil as follows

$$\varphi(s,t) = \alpha(s) + \frac{1}{\kappa(s)}\mathbf{n} + z(s)l(t)\mathbf{b}. \tag{12}$$

Conversely, a simply calculation implies that the Gauss curvature of the surface is zero.

Now, let us construct an example belonging to case 1.

Example 1. *Let us consider a space curve parameterized by*

$$\alpha(s) = (\cos(s), \sin(s), s).$$

It is easy to see that the curvature $\kappa = 1/2$. Hence, from (11) we have $y(s)w(t) = 2$. Then a flat normal surface pencil is illustrated in Figure 1, in which $z(s)l(t) = t\cosh(t)$, if we set $z(s)l(t) = t\cos(s)$, we obtain another member of flat normal surface pencil shown in Figure 1.

Case 2) If $l_t(t) = 0 (l(t) = c_1, c_1 \in \mathbb{R})$ in equation (10), then we have the coefficient $B_4(s,t)$ of $w_t^4(t)$ as follows

$$B_4 = -y^4(z_s^2c_1^2\kappa^2 + \tau^2 + 2z_sc_1\kappa\tau).$$

It follows that for $y(s) \neq 0, w_t(t) \neq 0$ (in these cases it is not a surface) the coefficient $B_4(s,t)$ vanishes if and only if the following equation is satisfied:

$$z_s^2c_1^2\kappa^2 + 2z_sc_1\kappa\tau + \tau^2 = 0.$$

The solution of the above differential equation can be obtained as

$$z(s) = -\int \frac{\tau(s)}{c_1\kappa(s)}ds. \tag{13}$$

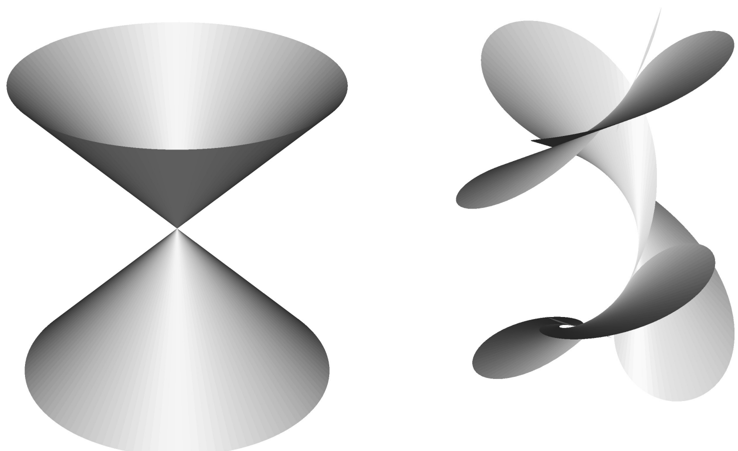
It follows that substituting $l(t) = c_1$ and (13) into (9) gives a flat normal surface pencil parameterized by

$$\varphi(s,t) = \alpha(s) + y(s)w(t)\mathbf{n} - \int \frac{\tau}{\kappa}ds\mathbf{b}.$$

Note that if the curve $\alpha(s)$ is an arbitrary speed curve, then one calculates the function $z(s)$ as follows

$$z(s) = -\int \frac{\tau}{c_1\kappa}\left\|\alpha'\right\| ds. \tag{14}$$

Now, let us construct an example about case 2.



(a) A member of flat normal surface pencil, with $zl = tcosh(t)$.

(b) A member of flat normal surface pencil, with $zl = tcos(s)$.

FIGURE 1. Flat normal surface pencil.

Example 2. *Assume that a space curve is given by*

$$\alpha(s) = (\frac{3s^2-1}{3s^2+3}, \frac{s(s^2-3)}{3s^2+3}, \frac{2\sqrt{2}\sqrt{s^2+1}}{3}).$$

When $c_1 = 1$, from (14) we have

$$z(s) = -\frac{1}{2}s^2.$$

In addition, if we set $y(s)w(t) = 5\cos(t)\sin(t)$ or $y(s)w(t) = st$ then the flat normal surface pencils are illustrated in Figure 2.

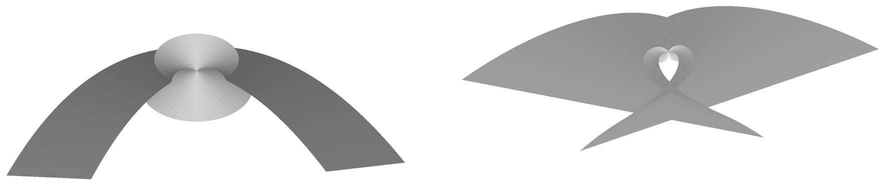
Case 3) If the curve $\alpha(s)$ is a plane curve ($\tau = 0$) in equation (10) then the coefficient $B_4(s,t)$ of $w_t^4(t)$ is calculated as

$$B_4 = -y^4 z_s^2 l^2 \kappa^2. \tag{15}$$

Now, we distinguish the following five cases:

Subcase 3.1) If $w_t(t) = 0(w(t) = c_2, c_2 \in \mathbb{R})$ in equation (15) then the Gauss curvature vanishes ($K = 0$), which implies that the normal surface pencil is a flat surface parameterized by

$$\varphi(s,t) = \alpha(s) + y(s)c_2\mathbf{n} + z(s)l(t)\mathbf{b}.$$



(a) A member of flat normal surface pencil, with $yw = 5cos(t)sin(t)$.

(b) A member of flat normal surface pencil, with $yw = st$.

FIGURE 2. Flat normal surface pencil.

Now, lets construct an example about subcase 3.1.

Example 3. *Assume that a plane curve is given by*

$$\alpha(s) = (\cos(s), \sin(s), 0).$$

A straightforward computation shows that $\tau = 0$, *therefore for* $w(t) = 3$, *we can set* $l(t) = \cos(t), y(s) = 2s/3$ *and* $z(s) = \cosh(s/5)$ *or* $l(t) = \cos(t), y(s) = \cos(s)$ *and* $z(s) = \cos(5s)$ *to construct members of flat normal surface pencil, shown in Figure 3.*

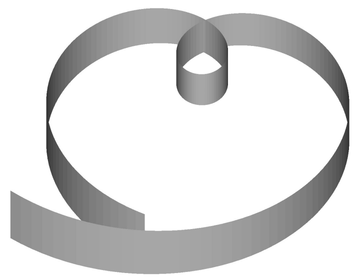
Subcase 3.2) If $y(s) = 0$ in equation (15) then the Gauss curvature vanishes, thus a flat normal surface pencil is parameterized by

$$\varphi(s,t) = \alpha(s) + z(s)l(t)\mathbf{b}.$$

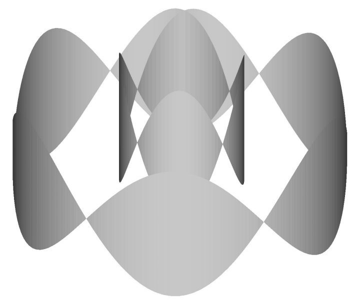
It is easy to see that if we set $z(s) = 1$ and $l(t) = t$ in the above equation, then we have a principal binormal surface, therefore the above result coincides with Theorem 1.

Subcase 3.3) If $l(t) = 0$ in equation (15), then we have $K = 0$, thus a flat normal surface pencil is parameterized by

$$\varphi(s,t) = \alpha(s) + y(s)w(t)\mathbf{n}.$$



(a) A member of flat normal surface pencil, with $l = cos(t)$, $y = 2s/3$ and $z = cosh(s/5)$.



(b) A member of normal flat surface pencil, with $l = cos(t)$, $y = cos(s)$ and $z = cos(5s)$.

FIGURE 3. Flat normal surface pencil.

Note that when we set $y(s) = 1$ and $w(t) = t$ in the above equation, then we have a principal normal surface, therefore the above result coincides with Theorem 1.

Subcase 3.4) If the curve is a line ($\kappa(s) = 0$) then the coefficient $C_2(s,t)$ of $l_t^2(t)w_t^2(t)$ is calculated as

$$C_2 = 2y_s zyz_s - z^2 y_s^2 - y^2 z_s^2.$$

The condition $w_t = 0$ is investigated in subcase 3.1. Therefore, two subcases must be considered.

Subsubcase 3.4.1) If $2y_s zyz_s - z^2 y_s^2 - y^2 z_s^2 = 0$ then we have

$$y(s) = c_3 z(s).$$

where $c_3 \in \mathbb{R}$.

With these conditions (9) becomes

$$c_3^2 z^3 z_{ss}(l_t w - w_t l)(l_{tt} w_t - l_t w_{tt}) = 0.$$

It is easy to see that $z(s) = 0$ and $l = kw$,$k \in \mathbb{R}$ contradiction. therefore, there is just one subcase:

If $z_{ss}(s) = 0(z(s) = c_4 s + c_5)$ $c_4, c_5 \in \mathbb{R}$, then the normal surface pencil is a flat surface parameterized by

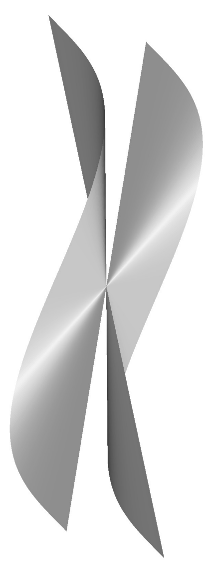
$$\varphi(s,t) = \alpha(s) + c_3(c_4 s + c_5)w(t)\mathbf{e}_1 + (c_4 s + c_5)l(t)\mathbf{e}_2. \tag{16}$$

Observe that if we set $c_3 = 1, w(t) = \cos(t)$ and $l(t) = \sin(t)$ in the equation (16), then the surface is a flat canal surface, therefore this result coincides with the Theorem 2. Now, let us construct an example about this case.

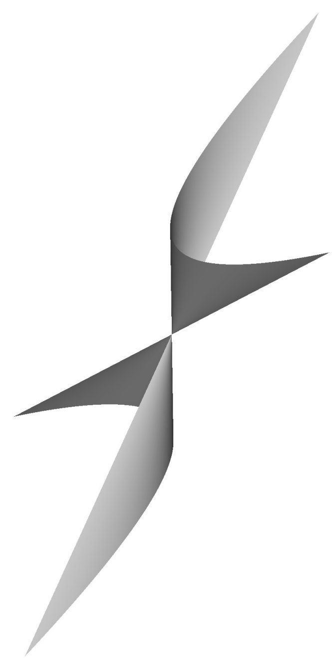
Example 4. *Assume that a line is given by*

$$\alpha(s) = (s, 3, 0).$$

In this case the Frenet frame is undefined, thus we can choose an arbitrary basis such as $\mathbf{n} = (0, 1, 0)$ *and* $\mathbf{b} = (0, 0, 1)$. *For* $c_3 = 3, c_5 = 0$ *and* $c_4 = 1$ *in (16), if we set* $w(t) = \cos(t/3)\sin(t/3)$ *and* $l(t) = t$ *or* $w(t) = \cosh(t/3)/3$ *and* $l(t) = \sinh(t/5)$ *then the flat normal surface pencils are illustrated in Figure 4.*



(a) A member of flat normal surface pencil, with $w = \sin(t)\cos(t)$ and $l = t$.



(b) A member of flat normal surface pencil, with $w = t - 5$ and $l = t^3$.

FIGURE 4. Flat normal surface pencil.

Subsubcase 3.4.2) If $l_t(t) = 0(l(t) = c_6, c_6 \in \mathbb{R})$ then the normal surface pencil is a flat surface parameterized by

$$\varphi(s,t) = \alpha(s) + y(s)w(t)\mathbf{e}_1 + z(s)c_6\mathbf{e}_2.$$

Subcase 3.5) If $z_s(s) = 0(z(s) = c_7)$ in equation (15) then the coefficient $C_2(s,t)$ of $l_t^2(t)w_t^2(t)$ is obtained as

$$C_2 = -y_s^2 z^2.$$

Since the case $w_t(t) = 0$ is investigated in subcase 3.1, there are three cases:

Subsubcase 3.5.1) If $z(s) = 0$ or $l_t(t) = 0(l(t) = c_8)$ then the normal surface pencil is a flat surface parameterized by

$$\varphi(s,t) = \alpha(s) + y(s)w(t)\mathbf{n}.$$

and

$$\varphi(s,t) = \alpha(s) + y(s)w(t)\mathbf{n} + c_7c_8\mathbf{b},$$

respectively.

Subsubcase 3.5.2) If $y_s(s) = 0(y(s) = c_9)$, then the equation (9) becomes

$$-l_t y\kappa z^2(yw\kappa - 1)^3(l_{tt}w_t - l_t w_{tt}) = 0.$$

Since we investigated all the cases, we omit all of these cases.

Case 4) If $z(s) = 0$ in equation (10) then we have the coefficient $B_4(s,t)$ of $w_t^4(t)$ as follows

$$B_4 = -y^4\tau^2.$$

For $y(s) \neq 0$ and $w_t(t) \neq 0$ (in these cases it is not a surface) when $\tau = 0$ this case is investigated in subsubcase 3.5.1.

- If l and w are linearly dependent. Then we have $w = c_{10}l$ and the equation (9) becomes

  $$-(c_{10}(-y_s z + yz_s) + \tau c_{10}^2 y^2 + \tau z^2)^2 l_t^4 = 0$$

  where $c_{10} \in \mathbb{R}$. The solution of the above differential equation can be obtained as

  $$y = z\frac{\tan\int\tau + c_{11}}{c_{10}}$$

  where c_{11} is a integration constant. The equation (3) is written as

  $$\varphi(s,t) = \alpha(s) + z(s)(\tan\int\tau + c_{11})l(t)\mathbf{n} + z(s)l(t)\mathbf{b}.$$

Note that if the curve $\alpha(s)$ is an arbitrary speed curve, then one calculates the function $y(s)$ as follows

$$y = z\frac{\tan\int\tau\,\|\alpha'\| + c_{11}}{c_{10}}. \tag{17}$$

Example 5. *Let us consider a space curve parameterized by*

$$\alpha(s) = (2s, s^2, \frac{s^3}{3}).$$

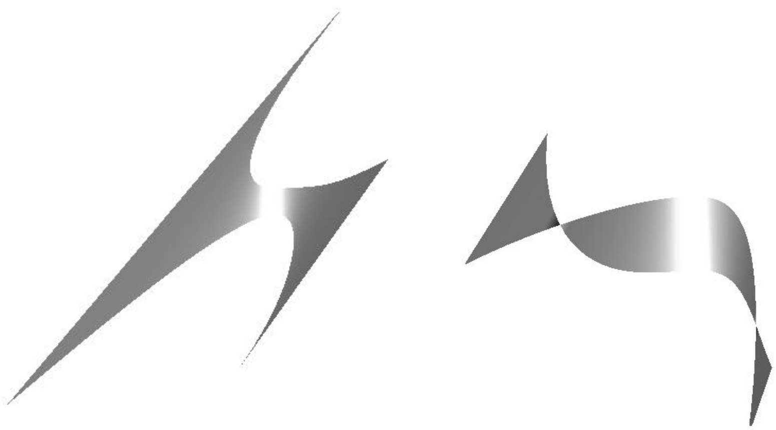
It is easy to see that the curvature and torsion are

$$\kappa = \frac{2}{(s^2+2)^2} = \tau$$

For $z(s) = \cosh(s)$, $w(t) = 6t$ *and* $l(t) = t$, *by using (17) we have*

$$y = \frac{\cosh(s)\tan(\sqrt{2}\arctan(\frac{s\sqrt{2}}{2}))}{6}$$

Moreover if we set $z(s) = (s)$, $w(t) = 6t$ *and* $l(t) = t$, *then we obtain another member of flat normal surface pencil shown in Figure 5.*



(a) A member of flat normal surface pencil, with $zl = tcosh(s)$ and $w = 6t$.

(b) A member of flat normal surface pencil, with $zl = tcos(s)$ and $w = 6t$.

FIGURE 5. Flat normal surface pencil.

□

3. Conclusion

The surface pencil and flat surfaces are both important subjects in computer aided design (CAD), and in this paper we describe somewhat novel analysis on special cases of flat normal surface pencils. We recommend this new approach for the following reasons:

- We can construct lots of flat normal surface pencil by using this method.
- The designer can select different sets of functions $y(s), w(t), z(s)$ and $l(t)$ to adjust the shape of the surface.
- We have studied this problem without any constraints such as curves that have isoparametric properties.

Declaration of Competing Interests The author declare that he has no competing interest.

References

[1] Kreyszig, E., Differential Geometry, Dover Publications Inc. Courier Corporation, New York, 2013.

[2] Maekawa, T., Patrikalakis, N. M., Sakkalis, T., Yu, G., Analysis and applications of pipe surfaces. *Computer Aided Geometric Design*, 15 (1988) 437-458.

[3] Xu, Z., Feng, R., Sun, J., Analytic and algebraic properties of canal surfaces, *Journal of Computational and Applied Mathematics*, 195 (2006), 220–228. https://doi.org/10.1016/j.cam.2005.08.002

[4] Wang, G. J., Tang, K., Tai, C. L., Parametric representation of a surface pencil with a common spatial geodesic, *Computer Aided Geometric Design*, 36 (5) (2004), 447–459.

[5] Kasap, E., Akyıldız, F. T., Orbay, K., A generalization of surfaces family with common spatial geodesic, *Appl. Math. Comput.*, 201 (2008), 781-789. https://doi.org/10.1016/j.amc.2008.01.016

[6] Li, C. Y., Wang, R. H., Zhu, C. G., Parametric representation of a surface pencil with a common line of curvature, *Computer Aided Design*, 43 (9) (2011), 1110–1117.

[7] Küçükarslan, Z. Y., On a family of surfaces with common asymptotic curve in the Galilean space G_3, *J. Nonlinear Sci. Appl.*, 9 (2016) 518–523.

[8] Lopez, R., Cyclic surfaces of constant Gauss curvature, *Houston J. Math.*, 27 (2001), 799–805.

[9] Lopez, R., Surfaces of constant Gauss Curvature in Lorentz-Minkowski Three-Space, *Rocky Mountain J. Math.*, 33 (2003) 971-993.

[10] Do Carmo, M. P., Differential Geometry of Curves and Surfaces, Prentice Hall, NJ, 1976.

[11] Gray, A., Modern Differential Geometry of Curves and Surfaces with Mathematica, CRC Press. Boca Raton, 1998.

[12] Goemans, W., Van de Woestyne, I., Twisted surfaces in Euclidean and Minkowski 3-space, *In: Pure and Applied Differential Geometry Padge, Shaker Verlag Aachen, Germany*, (2012), 143-151.

[13] Goemans, W., Van de Woestyne, I., Twisted surfaces with null rotation axis in Minkowski 3-space, *Results Math.*, 70(1) (2016), 81-93. https://doi.org/10.1007/s00025-015-0462-2

[14] Lopez, R., Moruz, M., Translation and homothetical surfaces in Euclidean space with constant curvature, *J. Korean Math. Soc.*, 52(3) (2015), 523-535.

[15] Izumiya, S., Takeuchi, N., New special curves and developable surfaces, *Turkish J. Math.*, 28(2) (2004), 153-163.

[16] Zhao, H. Y., Wang, G. J., A new method for designing a developable surface utilizing the surface pencil through a given curve, *Progress in Nature Science*, 18 (2008), 105–110.

[17] Ergün, E., Bayram, E., Kasap, E., Surface pencil with a common line of curvature in Minkowski 3-space, *Acta Mathematica Sinica-English Series*, 30(12) (2014), 2103-2118.

[18] Alegre, P., Arslan, K., Carriazo, A., Murathan C., Öztürk, G., Some special types of developable ruled surface, *Hacettepe Journal of Mathematics and Statistics*, 39(3) (2010), 319–325.

[19] Hanson, A. J., Ma, H., Parallel transport approach to curve framing, *Tech. Report*, 425 (1995).
[20] Dede, M., Helical extension curve of a space curve, *Mediterranean Journal of Mathematics*, 18 (2021), 1-10.
[21] Ates, F., Gok, I., Ekmekci, F., N., Yaylı, Y., Characterizations of inclined curves according to parallel transport frame in $\mathbb{E}^4$ and Bishop frame in $\mathbb{E}^3$, *Konuralp Journal of Mathematics*, 7(1) (2019), 16-24.
[22] Bishop, R. L., There is more than one way to frame a curve, *Amer. Math. Monthly* 82, (1975) 246–251.
[23] Bloomenthal, J., Calculation of Reference Frames Along a Space Curve, Graphics Gems, Academic Press Professional, Inc., San Diego, CA, 1990.