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TRAJECTORY CURVES AND SURFACES: A NEW PERSPECTIVE VIA PROJECTIVE GEOMETRIC ALGEBRA

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Abstract. The aim of this work is to define quaternion curves and surfaces and their conjugates via operators in Euclidean projective geometric algebra (EPGA). In this space, quaternions were obtained by the geometric product of vector fields. New vector fields, which we call trajectory curves and surfaces, were obtained by using this new quaternion operator. Moreover, dual quaternion curves are determined by a similar method and then their generated motion is studied. Illustrative examples are given.

1. Introduction

Understanding what complex numbers indicate geometrically has always been a matter of curiosity. Since the problem of finding the roots of a quadratic equation, we use a combination of a complex unit and real numbers, or their ordered binary representation, to show complex numbers. So what does this imaginary part show geometrically? Common usage is an axis orthogonal to the real axis. Thus, it shows the 2-dimensional real space in binary terms. However, complex numbers seem to contain more than that.

While working on the algebra of complex numbers of the form $a + bi$, Argand noticed that when the complex number is multiplied by the imaginary unit i, i.e. $i(a + bi) = -b + ai$ represents the rotation of this point, a geometric indicator of the complex number, about the origin in the plane by $90°$ [1,2]. Hamilton thought that this rotational property of complex numbers might also have a counterpart in 3-dimensional space. So he predicted that an ordered 3 with two complex units could show the rotation in 3-dimensional space. However, it was not that easy to establish the algebra. An undesirable complex expression was coming from the product of two triplets. He used a combination of three imaginary numbers and

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a real number to overcome this problem, and thus multiplication must be closed. Therefore, he invented the quaternions [3]:

A (real) quaternion is represented by $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ (general assumption) where $a, b, c, d \in \mathbb{R}$ and the conditions for units

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

held.

Thanks to quaternion algebra, one can obtain information about the rotation of any rigid body in space. Especially, in computer graphics, there are two options to rotate an object: use a rotation matrix (3x3 matrix) or a quaternion. Besides, there are some advantages to using quaternion like memory space, speed, and not showing gimbal locks as matrices do.

One can assume coefficients of the quaternion as dual numbers and then get the dual quaternion, i.e. $A, B, C, D \in \mathbb{D}$ [4–6].
Dual quaternions store both rotation and translation information about a rigid body in space. A similar operation can be done with 4x4 matrices. It is used quite often in the theory of mechanisms. This type of quaternion pair (called a dual quaternion) stores the same type of information but in two different quaternions. While the dual part of the dual quaternion represents translation information, the real one represents information about rotation. When mentioned together with these features, it is understood that it is a tool for software in robotics. Rather than its usual general definitions, it will be given here in accordance with projective geometric algebra.

Geometric algebra has very useful content to determine objects, especially complex ones geometrically, and transform one to another in space. Generally, geometric algebra over n-dimensional vector space is represented by the set $\mathcal{G}(n, 0)$ where n is the dimension of vector space and the second part is the grade of the inner product. Since the vector space studied here is real, the grade of the inner product is 0. Basically, geometric algebra allows us to symbolize scalars, vectors, areas, and volumes using a simple and consistent notation. Such items are closed under the algebraic operation. It is not difficult to see that many variables of this type are closed under addition. However, the product is somewhat unusual. So these expressions can be tough to visualize. Furthermore, the orientation of an object in space is important in physics research, for instance, spinors occupy an important place in quantum mechanics. Geometric algebra tools also provide the orientation of objects.

There are two expansions of this algebra: Conformal geometric algebra (CGA) and projective geometric algebra (PGA).

EPGA is based on duality: that is, we can represent work (wedge) done in one (exterior algebra) to be equivalent (join- $\vee$) in the other (dual exterior algebra). Similar to this situation, the meet ($\wedge$) operator is also defined. The reason we work in dual space is because all of the Euclidean operations can be represented. Working in a projective dual space also prevents special cases from occurring.

In this paper, we use Euclidean curves to generate quaternion curves via geometric product. Thus, we describe what a quaternion operator looks like visually. Bearing this motivation, we wonder about the motion that will occur around a moving (dual) quaternion rather than around a fixed (dual) quaternion in the EPGA language.

2. Fundamentals

In this section, we try to explain the properties of GA and PGA. There are some operators. We explore how these operators lead to rotations just as complex numbers do.

One of the most remarkable works on quaternion algebra in the literature is Aslan and Yaylı, [22], where they define quaternion operators on curves and surfaces in Euclidean 3-space using geometric algebra. These operators generate motions that have orbits along the generated curve or surface and can be expressed as 1-parameter or 2-parameter homothetic motions. Besides, Shoemake presents a new kind of spline curve suitable for smoothly interpolating sequences of arbitrary rotations. The motion generated is smooth and natural, without quirks found in earlier methods, [23].

Since geometric algebra is a very broad topic from kinematics [7–9] to robot dynamics [10], from neuroscience [17] to modeling [11], our aim here is not to explain all the basic topics of geometric algebra. For more details, see [12, 15–21]. Only definitions required in the article will be given.

2.1. **Geometric algebra.** Let $\mathbb{R}^2$, 2D-real space, be spanned by two linear independent orthonormal vectors: $\{\mathbf{e}_1, \mathbf{e}_2\}$. The inner and outer product of these elements produces new bases for geometric algebra:

$$< \mathbf{e}_1, \mathbf{e}_1 > = < \mathbf{e}_2, \mathbf{e}_2 > = 1 \text{ and } \mathbf{e}_1 \wedge \mathbf{e}_2$$

Bearing with these elements we have the following bases for geometric algebra $\mathcal{G}(2,0)$:

$$\{1(\text{scalar}), \mathbf{e}_1, \mathbf{e}_2(\text{vectors}), \mathbf{e}_1 \wedge \mathbf{e}_2(\text{pseudo-scalar})\}.$$

The algebra has also general elements called multivectors like that: $a1 + b\mathbf{e}_1 + c\mathbf{e}_2 + d\mathbf{e}_1 \wedge \mathbf{e}_2$, where $a, b, c, d \in \mathbb{R}$. Partially, $x1 + y\mathbf{e}_1 \wedge \mathbf{e}_2 = x + iy$, represent a complex number.

$\mathbb{R}^3$, 3D-real space, is spanned by three independent vectors: $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. New basis for geometric algebra over $\mathbb{R}^3$ are:

$$< \mathbf{e}_1, \mathbf{e}_1 > = < \mathbf{e}_2, \mathbf{e}_2 > = < \mathbf{e}_3, \mathbf{e}_3 > = 1$$

and

$$\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_3, \mathbf{e}_2 \wedge \mathbf{e}_3.$$

So vector spaces of the geometric algebra $\mathcal{G}(3,0)$ are

$$\{\underbrace{1}_{\text{scalars}},\underbrace{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3}_{\text{vector space}},\underbrace{\mathbf{e}_1\wedge\mathbf{e}_2,\mathbf{e}_2\wedge\mathbf{e}_3,\mathbf{e}_3\wedge\mathbf{e}_1}_{\text{bivector space}},\underbrace{\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3}_{\text{trivector space.}}\}$$

All geometric algebra has $\sum_{k=0}^{3}\begin{pmatrix}3\\k\end{pmatrix}=2^3=8$ elements where the number of k-blades of geometric algebra over $\mathbb{R}^3$ is computed by $\begin{pmatrix}3\\k\end{pmatrix}$ combination. So any multi-vector is of the form: $a1+b\mathbf{e}_1+c\mathbf{e}_2+d\mathbf{e}_3+e\mathbf{e}_1\wedge\mathbf{e}_2+f\mathbf{e}_1\wedge\mathbf{e}_3+g\mathbf{e}_2\wedge\mathbf{e}_3+h\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3$, where $a,b,c,d,e,f,g,h\in\mathbb{R}$.

Operations are important for the derivation of elements in a mathematical structure such as vector space or algebra. These processes should always show the feature of closure. The next definition gives the fundamental operator for geometric algebra.

Definition 1. *Clifford defined the geometric product of two vectors,* $\mathbf{u}$ *and* $\mathbf{v}$*, as follows [9]: Let* $\mathbf{u}=\sum_{i=1}^{n}u_i\mathbf{e}_i$ *and* $\mathbf{v}=\sum_{i=1}^{n}v_i\mathbf{e}_i$ *then*

$$\mathbf{uv}=<\mathbf{u},\mathbf{v}>+\mathbf{u}\wedge\mathbf{v}=\sum_{i=1}^{n}u_iv_i+\sum_{i,j=1}^{n}u_iv_j\mathbf{e}_i\mathbf{e}_j. \tag{1}$$

where $\mathbf{e}_i$ *are unit bases of* $\mathbb{R}^n$*. Here we use* $\mathbf{e}_{ij}=\mathbf{e}_i\mathbf{e}_j=\mathbf{e}_i\wedge\mathbf{e}_j$ *for simplicity.*

Proposition 1. *Let* $\mathbf{u}$*,* $\mathbf{v}$ *and* $\mathbf{w}$ *are vectors in* $\mathbb{R}^n$ *then following rules are provided*

(i) Associativity:

$$\mathbf{u}(\mathbf{vw})=\mathbf{u}(\mathbf{v})\mathbf{w}=\mathbf{uvw}$$

(ii) Distributivity:

$$\mathbf{u}(\mathbf{v}+\mathbf{w})=\mathbf{uv}+\mathbf{uw}$$

and

$$(\mathbf{v}+\mathbf{w})\mathbf{u}=\mathbf{vu}+\mathbf{wu}$$

(iii) Modulus:

$$\|\mathbf{u}\|^2=\mathbf{uu}=<\mathbf{u},\mathbf{u}>.$$

Products of fundamental elements of geometric algebra are given as follows,

$$\mathbf{e}_i\mathbf{e}_i=1,\mathbf{e}_i\mathbf{e}_j=\mathbf{e}_i\wedge\mathbf{e}_j=-\mathbf{e}_j\wedge\mathbf{e}_i=-\mathbf{e}_j\mathbf{e}_i$$
$$(\mathbf{e}_i\wedge\mathbf{e}_j)(\mathbf{e}_i\wedge\mathbf{e}_j)=\mathbf{e}_i\mathbf{e}_j\mathbf{e}_i\mathbf{e}_j=-\mathbf{e}_i\mathbf{e}_i=-1$$
$$(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)=-\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1\mathbf{e}_3=-1$$

where $i,j=1,2,3$ (for $i\neq j$).

Definition 2. *In geometric algebra, there is a Hodge duality for elements and defined as follows [18]: Let $I = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ be the pseudo-scalar then,*

$$1^* = -1I = -I,$$

So, scalar and pseudo-scalar are Hodge dual of each other (Hodge duality generally represented by *). Similarly, vectors and blades are Hodge dual of each other:

$$\begin{aligned}
\mathbf{e}_1^* &= -\mathbf{e}_1 I = -\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = -\mathbf{e}_2\mathbf{e}_3,\\
\mathbf{e}_2^* &= -\mathbf{e}_2 I = -\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_3,\\
\mathbf{e}_3^* &= -\mathbf{e}_3 I = -\mathbf{e}_3\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = -\mathbf{e}_1\mathbf{e}_2.
\end{aligned}$$

Definition 3. *Let $\mathbf{w}$ be a vector and $B = \mathbf{u} \wedge \mathbf{v}$ be a 2-vector, then*

$$\mathbf{w}B = \langle \mathbf{w}, B \rangle + \mathbf{w} \wedge B = (\mathbf{w}B)_1 + (\mathbf{w}B)_3.$$

where $(.)_k$ represent the grade of element.

2.2. **Projective geometric algebra.** Although affine transformations of geometric objects can be achieved with vector algebra, this can cause some difficulties in 3D computer graphics, such as the algebraic separation of point and vector. For this, it is the algebraic structure that we call homogeneous coordinates and allows us to represent n-dimensional real space in (n+1)-dimensional space. Let $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ be a point in Euclidean 3-space then $X = \mathbf{x} + \mathbf{e}_0$ be a homogeneous point in 4D geometric algebra.

This space can also be defined in geometric algebra by adding an explicit extra basis: $\mathbf{e}_0$, satisfying $\mathbf{e}_0^2 = 0$, which corresponds to a null vector providing only linear terms expansion of an exponential function. So the metric structure would be $\mathbf{e}_i.\mathbf{e}_j = \delta_{ij}$ for $i, j = 1, 2$ and $\mathbf{e}_i.\mathbf{e}_0 = 0$. Since this metric structure is the same as the Euclidean metric, it also preserves isometry. There are also other bases like $\mathbf{e}^2 = -1$ or 1 and generate higher dimensional projective geometric algebra. In general, these unusual bases are called geometric numbers. Summing up, its general notation in this point-based structure is $\mathbf{P}(\mathbb{R}_{p,n,z})$ where p, n, z stand for positive, negative and zero, respectively. Besides, plane-based model, for instance, the algebra $\mathbf{P}\left(\mathbb{R}^*_{2,0,1}\right)$ represents the proper 2D Euclidean space. As far as we know from linear algebra, if V is a vector space then there is a dual vector space V^*. So each geometric object in the exterior algebras in $\mathbf{P}(V)$ and $\mathbf{P}(V^*)$ has a representation in both. This is the Poincaré duality [13, 14].

$\mathbf{P}\left(\mathbb{R}^*_{3,0,1}\right)$ provides coordinate-free, uniform representation for Euclidean elements: points, lines, and planes.

$\wedge$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_0$	0	$\mathbf{e}_{01}$	$\mathbf{e}_{02}$	$\mathbf{e}_{03}$
$\mathbf{e}_1$	$\mathbf{e}_{10}$	0	$\mathbf{e}_{12}$	$\mathbf{e}_{13}$
$\mathbf{e}_2$	$\mathbf{e}_{20}$	$\mathbf{e}_{21}$	0	$\mathbf{e}_{23}$
$\mathbf{e}_3$	$\mathbf{e}_{30}$	$\mathbf{e}_{31}$	$\mathbf{e}_{32}$	0

Although it has degenerate metric we can explain the reason why this algebra shows Euclidean isometries as follows: The basic elements in geometric algebra do not actually have Euclidean representations. Therefore, we can understand what they are geometrically by looking at their dual structure. However, dual PGA performs its operations directly with Euclidean elements. So we can call it EPGA for short. The basic linear elements of this algebra are planes (1-vector), and they are defined as follows,

$$\mathbf{a} = \sum_{i=0}^{3} a_i \mathbf{e}_i.$$

It also includes meet and join operators. These operators decrease and increase of grades of elements of algebra, respectively. Thus, the union and intersection operations of points, lines, and planes in PGA can be done with wedge and progressive product, respectively.

Quaternions are also even subalgebra (zero and two graded) of projective geometric algebra $\mathbf{P}\left(\mathbb{R}^*_{3,0,1}\right)$ and a quaternion is represented by

$$q = a + b\mathbf{e}_{12} + c\mathbf{e}_{31} + d\mathbf{e}_{23}$$

where $a, b, c, d \in \mathbb{R}$.

3. Quaternion Curves and Surfaces

Grounded in the elegant framework of geometric algebra, quaternion curves and surfaces stand as a cornerstone in the field of 3D geometry and computer graphics. In this chapter, we enter a world where advanced mathematics meets practical applications, exploring the profound implications of quaternions in representing and manipulating curves and surfaces.

As a hyper-complex extension of complex numbers, quaternions offer a concise and efficient way to handle 3D rotations and orientations, finding extensive applications in fields ranging from computer graphics and robotics to physics simulations. Represented by geometric algebra, these quaternions are not just abstract mathematical constructs, but powerful tools that allow us to describe the complex motion of objects in space with remarkable precision and flexibility.

Definition 4. *Let* $\mathbf{a}(t), \mathbf{b}(t)$ *be vector fields then*

$$q(t) = \mathbf{a}(t)\mathbf{b}(t) =< \mathbf{a}(t), \mathbf{b}(t) > + \mathbf{a}(t) \wedge \mathbf{b}(t)$$

is a quaternion curve and its conjugate is given by reverse order product

$$\widetilde{q}(t) = \mathbf{b}(t)\mathbf{a}(t) =< \mathbf{b}(t), \mathbf{a}(t) > + \mathbf{b}(t) \wedge \mathbf{a}(t).$$

Definition 5. *Let* $\mathbf{a}(t), \mathbf{b}(s)$ *be vector fields then*

$$q(t,s) = \mathbf{a}(t)\mathbf{b}(s) =< \mathbf{a}(t), \mathbf{b}(s) > + \mathbf{a}(t) \wedge \mathbf{b}(s)$$

is a quaternion surface.

To see the behavior of these quaternion curves in 3-dimensional space, it is necessary to apply them to a point. Let us now formulate here the rotation of a point through a quaternion operator generated by the meeting of two unit vectors in space:

Let $\|\mathbf{u}\| = \|\mathbf{v}\| = 1, q = \mathbf{uv} = \cos(\theta) + \sin(\theta)\mathrm{B}$ and its conjugate $\widetilde{q} = \mathbf{vu} = \cos(\theta) - \sin(\theta)\mathrm{B}$. Take a point in EPGA as $P = \mathbf{e}_{123} + P_E$ then the rotated point P_r is as follows;

$$P_r = qP\widetilde{q} = \mathbf{e}_{123} + qP_E\widetilde{q} \tag{2}$$

Thus, applying real quaternion operators to EPGA elements gives the same result as in EGA.

3.1. **Trajectory curves.** The concept of trajectories generated by quaternion curves involves representing rotations in 3D space using quaternions and understanding how these rotations affect the orientation of objects over time.

In the context of trajectories, quaternions are used to smoothly interpolate between different orientations of an object, creating a continuous curve that describes the object's rotation over time. Quaternions have certain advantages over other rotation representations (such as Euler angles) because they do not suffer from gimbal lock and provide smooth interpolation without singularities.

Corollary 1. *Let* $q(t)$ *be a quaternion curve and* P *be a point in* $\mathbf{P}\left(\mathbb{R}^*_{3,0,1}\right)$ *then*

$$\alpha(t) = q(t)P\widetilde{q}(t) \tag{3}$$

is a trajectory curve.

Example 1. *Let* $\mathbf{u}(t) = \cos(t)\mathbf{e}_1 + \sin(t)\mathbf{e}_2$*,* $\mathbf{v}(t) = \cos(t)\mathbf{e}_1 + \sin(t)\mathbf{e}_3$*,* $P = \mathbf{e}_{123} + \mathbf{e}_{032}$ *be unit vector fields and a point in* $\in \mathbf{P}\left(\mathbb{R}^*_{3,0,1}\right)$*, respectively. Then*

$$\begin{aligned} q(t) &= \mathbf{u}(t)\mathbf{v}(t) \\ &=< \cos(t)\mathbf{e}_1 + \sin(t)\mathbf{e}_2, \cos(t)\mathbf{e}_1 + \sin(t)\mathbf{e}_3 > \\ &+ (\cos(t)\mathbf{e}_1 + \sin(t)\mathbf{e}_2) \wedge \cos(t)\mathbf{e}_1 + \sin(t)\mathbf{e}_3 \\ &= \cos(t)^2 + \cos(t)\sin(t)\mathbf{e}_1\mathbf{e}_3 - \cos(t)\sin(t)\mathbf{e}_1\mathbf{e}_2 \\ &+ \sin^2(t)\mathbf{e}_2\mathbf{e}_3. \end{aligned}$$

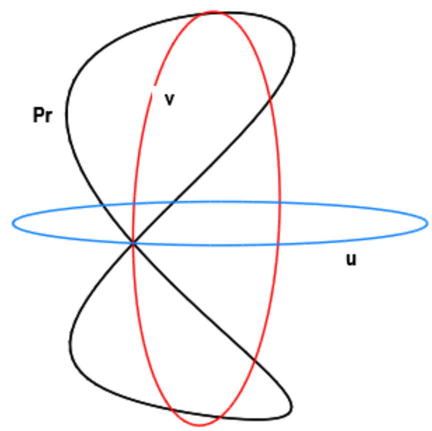

FIGURE 1. The curve that is generated by the quaternion curve.

So the curve of quaternion rotation is (see Fig.1)

$$\begin{aligned}\alpha(t) =& q(t)P\widetilde{q}(t)\\ =& \mathbf{e}_{123} + \cos(2t)^2\mathbf{e}_{230} + \frac{\sin(4t)}{2}\mathbf{e}_{301} - \sin(2t)\mathbf{e}_{012}.\end{aligned}$$

This curve is also called Viviani's curve.

3.2. **Trajectory surfaces.** Similar situations to the trajectory curves can also be done for surfaces. The only difference here is that the parameters of the vector fields forming the quaternion are different from each other.

Corollary 2. *Let $q(t,s)$ be a quaternion surface and P be a point in $\mathbf{P}\left(\mathbb{R}^*_{3,0,1}\right)$ then*

$$X(t,s) = q(t,s)P\widetilde{q}(t,s) \tag{4}$$

is a trajectory surface.

Corollary 3. *Let q_θ be a quaternion, where θ is the angle between vectors that are constructing the quaternion, and $\alpha(t)$ be a regular curve in $\mathbf{P}\left(\mathbb{R}^*_{3,0,1}\right)$ then*

$$\beta(t,\theta) = q_\theta\alpha(t)\widetilde{q_\theta} \tag{5}$$

is a rotational surface. This is also a special case of trajectory curves.

Corollary 4. *Let $q(t)$ be a quaternion curve $\alpha(s)$ be a regular curve in $\mathbf{P}\left(\mathbb{R}^*_{3,0,1}\right)$ then*

$$\beta(t,s) = q(t)\alpha(s)\widetilde{q}(t) \tag{6}$$

is a trajectory surface.

Let's define some surfaces as trajectory surfaces with the tool we have generated.

3.2.1. *Sphere as a trajectory surface.* This example can be used for surfaces (one can take a different parameter for the second vector field as we defined in Def 3.2). The resulting trajectory surface is this time, a 2-sphere. This is just like the product of two curves in the topological sense: $\mathbb{S}^2 \simeq \mathbb{S}^1 \times \mathbb{S}^1$.

3.2.2. *Cone as a trajectory surface.* It is easy to obtain a cone surface using this tool. As is well known, a cone is formed by a line passing through two points in space, one of the points being fixed and the other orbiting on a circle. We can make this imaginary geometric idea as it is with the help of geometric algebra. The flexibility in the choice of the vector fields that make up the quaternion and the object to be rotated in this tool offers us different ways and offers us the opportunity to create the geometric object to be created in different ways. Everything depends on your imagination. For example, for this cone surface, we can choose one of the vector fields forming the quaternion as a unit circle and the other as a point perpendicular to the plane in which this circle is located. The geometric object to be rotated is then a line.

3.3. **Dual quaternions and rigid motions.** The projective 4D analog of a quaternion is called a dual quaternion. This is where the real difference of EPGA comes into play. So all Euclidean motions can be described in this space and we can assume the dual number unit ϵ as a pseudo-scalar $\mathbf{e}_{0123}$, i.e. provide the mystique properties of its: $\mathbf{e}_{0123}^2 = 0$. Thus, we define the dual element as an algebraic basis.

Let $q = x_0 + x_1e_2e_3 + x_2e_3e_1 + x_3e_1e_2, r = y_0 + y_1e_2e_3 + y_2e_3e_1 + y_3e_1e_2$ be two quaternions. Then we can construct the dual quaternions in geometric algebra way as follows [10]:

$$\begin{aligned} Q &= q + r\mathbf{e}_{0123} = q - r^* \\ &= x_0 + x_1e_2e_3 + x_2e_3e_1 + x_3e_1e_2 \\ &+ y_0e_0e_1e_2e_3 + y_1e_1e_0 + y_2e_2e_0 + y_3e_3e_0 \end{aligned}$$

Therefore, $P_m = QP\overline{Q}$ represents a rigid transformation of a point in the space.

Definition 6. *Let $q(t), r(t)$ be quaternion curves, then*

$$Q(t) = q(t) + r(t)\mathbf{e}_{0123}$$

is a dual quaternion curve.

FIGURE 2. Trajectory curves that are generated by the dual quaternion.

Corollary 5. *Let $Q(t)$ be dual quaternion curve and P be a point in $\mathbf{P}\left(\mathbb{R}^{*}_{3,0,1}\right)$ then*

$$X(t) = Q(t)P\overline{Q}(t)$$

is a trajectory curve.

Example 2. *For the most basic situation, let $Q(t) = \cos(t)\mathbf{e}_{12} + \sin(t)\mathbf{e}_{23} + \mathbf{e}_{01} + \mathbf{e}_{0123}$ be a dual quaternion curve, then for points $P_x = \mathbf{e}_{123} + \mathbf{e}_{032}$, $P_y = \mathbf{e}_{123} + \mathbf{e}_{013}$, $P_z = \mathbf{e}_{123} + \mathbf{e}_{102}$ there are three trajectory curves, see Fig.2.*

An argument similar to the one above can be constructed with the basis (1-vectors) of EPGA. At this time, one can get a line that does not pass through the origin by meeting the two 1-vectors: Let $\mathbf{a}, \mathbf{b}$ be two 1-vectors and at least one of them is not passing through the origin, then their geometric product is

$$\mathbf{ab} = q_0 + \mathcal{L}.$$

where q_0 and $\mathcal{L}$ are scalar and plücker coordinates of the line, respectively. This operator generates a trajectory (orbit) of a point around a moving line in space.

It is decided whether a quaternion operator formed by the geometric multiplication of two 1-vectors is real or dual, by looking at whether its first terms (zeroth index) are zero. In other words, the line obtained from the intersection (i.e., the meet operator) of two planes passing through the origin represents the quaternion operator that represents a rotation around this line passing through the origin. On the other hand, if one of the first terms is nonzero, the resulting quaternion will represent a screw motion. Therefore, this quaternion acts like a dual quaternion operator.

4. Conclusion

Geometric algebra has lately garnered significant attention due to its profound applications for both imaging and performing fine operations on geometric objects. The field has seen remarkable progress, particularly in enhancing our capability to fantasize about complex geometric generalities. In this study, we embarked on a disquisition that extended the operation of geometric algebra from traditional vectorground representations of curves and surfaces in classical figures to the realm of quaternions. Our purpose is to demonstrate the unique capabilities of quaternions as an important fine driver, shedding light on their part in generating topological structures when applied to classical 3D geometric objects.

The implications of our findings extend far beyond the realm of classical Euclidean geometry. While our study primarily focused on classical 3D geometric objects, the inherent flexibility of quaternions suggests that similar investigations can be carried out in the domain of non-Euclidean geometries. This exciting prospect hints at a wealth of fascinating results waiting to be uncovered, as the interplay between quaternions and non-Euclidean geometries promises to yield profound insights and applications in various scientific and engineering disciplines.

In summary, our research represents a pivotal contribution to the field of geometric algebra by showcasing the remarkable utility of quaternions as operators in transforming classical geometric objects and elucidating the emergence of topological structures. This work not only deepens our understanding of the relationship between algebra and geometry but also opens up a tantalizing avenue for future research, where quaternions can be harnessed to explore the rich landscapes of non-Euclidean geometries, potentially revolutionizing how we perceive and interact with the mathematical underpinnings of the physical world.

Declaration of Competing Interests The author has no competing interests to declare.

References

[1] Vince, J., Imaginary Mathematics for Computer Science, Springer, ISBN-10: 3319946366, 2018. https://doi.org/10.1007/978-3-319-94637-5

[2] Argand, J. R., Essai Sur Une Manière de Représenter des Quantités Imaginaires Dans les Constructions Géométriques, 2nd edn. Gauthier-Villars, Paris, 1874.

[3] Hamilton, W. R., On quaternions: or a new system of imaginaries in algebra, *Phil. Mag.3rd ser.*, 25(163) (1844), 10-13. doi.org/10.1080/14786444408644923

[4] Clifford, W. K., Preliminary sketch of bi-quaternions, *Proceedings of the London Mathematical Society*, s1–4(1) (1873), 381–395. doi.org/10.1112/plms/s1-4.1.381

[5] Study, E., Geometrie der Dynamen, Teubner, Leipzig, 1901.
[6] Bottema, O., Roth, B., Theoretical Kinematics (Vol. 24), Courier Corporation, ISBN 10: 0486663469, ISBN 13: 9780486663463, 1990.
[7] Hestenes, D., New Foundations for Classical Mechanics (Vol. 15), Springer Science & Business Media, 2012. doi.org/10.1007/0-306-47122-1
[8] Hestenes, D., Sobczyk, G., Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics (Vol. 5), Springer Science & Business Media, 2012. doi.org/10.1007/978-94-009-6292-7
[9] Hestenes, D., Space-Time Algebra (Vol. 67), Birkhäuser, Basel, 2015. doi.org/10.1007/978-3-319-18413-5
[10] Selig, J. M., Geometric Fundamentals of Robotics, Springer Science & Business Media, 2004. doi.org/10.1007/b138859
[11] Hildenbrand, D., Geometric computing in computer graphics using conformal geometric algebra *Computers & Graphics*, 29(5) (2005), 795-803. doi.org/10.1016/j.cag.2005.08.028
[12] Dorst, L., Fontijne, D., Mann, S., Geometric Algebra for Computer Science: An Object-Oriented Approach to Geometry, Elsevier, ISBN: 9780123749420, 2010.
[13] Gunn, C. G., Doing Euclidean plane geometry using projective geometric algebra, *Adv. Appl. Clifford Algebras*, 27 (2017), 1203–1232. https://doi.org/10.1007/s00006-016-0731-5
[14] Gunn, C., Geometric algebras for Euclidean geometry, *Adv. Appl. Clifford Algebras*, 27 (2017), 185–208. https://doi.org/10.1007/s00006-016-0647-0
[15] Vince, J. A., Geometric Algebra for Computer Graphics, Springer, 2008. doi.org/10.1007/978-1-84628-997-2
[16] Perwass, C., Edelsbrunner, H., Kobbelt, L., Polthier, K., Geometric Algebra with Applications in Engineering (Vol. 4), Springer, Berlin, 2009. doi.org/10.1007/978-3-540-89068-3
[17] Bayro-Corrochano, E., Daniilidis, K., Sommer, G., Motor algebra for 3D kinematics: the case of the hand-eye calibration. *Journal of Mathematical Imaging and Vision*, 13 (2000), 79–100. https://doi.org/10.1023/A:1026567812984
[18] Kanatani, K., Understanding Geometric Algebra: Hamilton, Grassmann, and Clifford for Computer Vision and Graphics, CRC Press, 2015. https://doi.org/10.1201/b18273
[19] Vaz, J., Da Rocha, J., An Introduction to Clifford Algebras and Spinors, Oxford University Press, Oxford, 2016. https://doi.org/10.1093/acprof:oso/9780198782926.001.0001
[20] Josipović, M., Geometric Multiplication of Vectors: An Introduction to Geometric Algebra in Physics, Birkhäuser, Basel, 2019. https://doi.org/10.1007/978-3-030-01756-9
[21] Lasenby, J. (Ed.)., Guide to Geometric Algebra in Practice (pp. 371-389), Springer, New York, 2011. https://doi.org/10.1007/978-0-85729-811-9
[22] Aslan, S., Yaylı, Y., Motions on curves and surfaces using geometric algebra, *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, 71(1) (2022), 39-50. doi.org/10.31801/cfsuasmas.878766
[23] Shoemake, K., Animating rotation with quaternion curves, *In Proceedings of the 12th annual conference on Computer graphics and interactive techniques*, (1985), 245-254. https://doi.org/10.1145/325334.325242
[24] https://enkimute.github.io/ganja.js/