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Eight-Dimensional Walker Locally Symmetric Manifolds

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Abstract

A pseudo-Riemannian manifold which admits a field of parallel null r-planes, with $r \leq \frac{m}{2}$ is a Walker m-manifold. The even-dimensional Walker manifolds $(m = 2r)$ with fields of parallel null planes of half dimension have some special interest. The main purpose of the present paper is to study a specifc Walker metric on a 8-dimensional manifold and to give a theorem for the metric to be locally symmetric.

Keywords: *Locally symmetric space; Walker manifold.*
2010 Mathematics Subject Classification: *53B30, 53B50, 53C35, 53C50.*

1. Introduction

Walker m-manifold is a pseudo-Riemannian manifold which admit a non-trivial parallel null r-plane field with $r \leq m$. Walker m-manifold is applicable in physics. Lorentzian Walker manifolds have been studied extensively in physics. They constitute the background metric of the pp-wave models. A pp-wave spacetime admits a covariantly constant null vector field U [1]. The theory of Walker manifolds is outlined in [2]. The authors treated hypersurfaces with nilpotent shape operators, locally conformally flat metrics with nilpotent Ricci operator, degenerate pseudo-Riemannian homogeneous structures, para-Kähler structures and 2-step nilpotent Lie groups with degenerate center. The curvature properties of a large class of 4-dimensional Walker metrics are treated in [3] and several interesting examples are given, (see [4, 6, 10] and references therein).

There are few studies in the class of 8-dimensional Walker manifolds. Some examples in this direction may be found in [5, 7, 8, 9, 11] and references therein. In [5], the geometric properties of some curvature tensors of an 8-dimensional Walker manifold are investigated, theorems for the metric to be Einstein, locally conformally flat and for the 8-dimensional manifold to admit a Kähler structure are given. Also, in [7], the authors study the curvature properties of a Walker 8-manifold (M,g) which admits a field of parallel null 4-planes. Mainly, they study the conditions for such a Walker metric to be Einstein, Osserman, or locally conformally flat. Recently, the author [8] study almost Norden structures on 8-dimensional Walker manifolds. He discuss the integrability and Kähler (holomorphic) conditions for these structures. The nonexistence of (non-Kähler) quasi-Kähler structures on almost Norden-Walker 8-manifolds is also proved.

We want to study a particular form of a Walker metric on an eight dimensional manifold. We derive the $(0,4)$-curvature tensor, and establish a theorem for the Walker metric to be locally symmetric. The paper is organized as follows. In Section 2, we give the canonical form of Walker metrics. A specific family of Walker metric on an 8-dimensional manifold is investigated in Section 3 and a necessary condition for the Walker metric to be locally symmetric is given.

2. The canonical form of Walker metrics

Let M be a pseudo-Riemannian manifold of signature (n,n). We suppose given a splitting of the tangent bundle in the form $TM = \mathscr{D}_1 \oplus \mathscr{D}_2$ where $\mathscr{D}_1$ and $\mathscr{D}_2$ are smooth subbundles which are called distributions. This define two complementary projection π_1 and π_2 of TM onto $\mathscr{D}_1$ and $\mathscr{D}_2$. We say that $\mathscr{D}_1$ is parallel distribution if $\nabla\pi_1 = 0$. Equivalently this means that if X_1 is any smooth vector field taking values in $\mathscr{D}_1$, then ∇X_1 again takes values in $\mathscr{D}_1$. If M is Riemannian, we can take $\mathscr{D}_2 = \mathscr{D}_1^{\perp}$ to be the orthogonal complement of $\mathscr{D}_1$ and in that case $\mathscr{D}_2$ is again parallel. In the pseudo-Riemannian setting, $\mathscr{D}_1 \cap \mathscr{D}_2$ need not be trivial. We say that $\mathscr{D}_1$ is a null parallel distribution if it is

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parallel and the metric restricted to $\mathscr{D}_1$ vanish identically.

Walker [12] studied pseudo-Riemannian manifolds (M,g) with a parallel field of null planes $\mathscr{D}$ and derived a canonical form. Motivated by this seminal work, one says that a pseudo-Riemannian manifold M which admits a null parallel (i.e., degenerate) distribution $\mathscr{D}$ is a Walker manifold.

It is known that Walker metrics have served as a powerful tool for constructing interesting indefinite metrics which exhibit various aspects of geometric properties not given by any positive definite metrics. Among these, the significant Walker manifolds are the examples of the non-symmetric and non-homogeneous Osserman manifolds [2].

Canonical forms were known previously for parallel non-degenerate distributions. In this case, the metric tensor, in matrix notation, expresses in canonical form as

$$(g_{ij}) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \tag{2.1}$$

where A is a symmetric $r\times r$ matrix whose coefficients are functions of $(u_1,\dots,u_r)$ and B is a symmetric $(m-r)\times(m-r)$ matrix whose coefficients are functions of $(u_{r+1},\dots,u_m)$. Here m is the dimension of M and r is the dimension of the distribution $\mathscr{D}$. We will refer [2] for the proof of the following theorems.

Theorem 2.1 ([2], Page 40)**.** *A canonical form for an m dimensional pseudo-Riemannian manifold (M,g) admitting a parallel field of null r dimensional planes $\mathscr{D}$ is given by the metric tensor in matrix form as*

$$(g_{ij}) = \begin{pmatrix} 0 & 0 & \mathrm{Id}_r \\ 0 & A & H \\ \mathrm{Id}_r & {}^tH & B \end{pmatrix}, \tag{2.2}$$

where Id_r is the $r\times r$ identity matrix and A,B,H are matrices whose coefficients are functions of the coordinates satisfying the following:

1. *A and B are symmetric matrices of order $(m-2r)\times(m-2r)$ and $r\times r$ respectively. H is a matrix of order $(m-2r)\times r$ and ${}^tH$ stands for the transpose of H.*
2. *A and H are independent of the coordinates $(u_1,\dots,u_r)$.*

Furthermore, the null parallel r-plane $\mathscr{D}$ is locally generated by the coordinate vector fields $\{\partial_{u_1},\dots,\partial_{u_r}\}$.

Theorem 2.2 ([2], Page 42)**.** *A canonical form for an m dimensional pseudo-Riemannian manifold (M,g) admitting a strictly parallel field of null r dimensional planes $\mathscr{D}$ is given by the metric tensor as in Theorem 2.1, where B is independent of the coordinates $(u_1,\dots,u_r)$.*

3. Locally Symmetry of the metric

A neutral g on an 8-manifold M is said to be a Walker metric if there exists a 4-dimensional null distribution $\mathscr{D}$ on M which is parallel with respect to g. From Walker theorem [12], there is a system of coordinates $(u_1,\dots,u_8)$ with respect to which g takes the local canonical form

$$(g_{ij}) = \begin{pmatrix} 0 & I_4 \\ I_4 & B \end{pmatrix}, \tag{3.1}$$

where I_4 is the 4×4 identity matrix and B is an 4×4 symmetric matrix whose coefficients are the functions of $(u_1,\dots,u_8)$. Note that g is of neutral signature $(4,4)$ and that the parallel null 4-plane $\mathscr{D}$ is spanned locally by $\{\partial_1,\dots,\partial_4\}$, where $\partial_i = \frac{\partial}{\partial_{u_i}}, i=1,2,3,4$.

In this paper, we consider the specific Walker metrics on 8-dimensional M with B of the form

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \tag{3.2}$$

where a,b are smooth functions of the coordinates $(u_1,\dots,u_4)$. We will denote $a_i = \frac{\partial a(u_1,\dots,u_4)}{\partial u_i}$ and $b_i = \frac{\partial b(u_1,\dots,u_4)}{\partial u_i}$. The non zero components of the Levi-Civita connection of the Walker metric (3.1) and (3.2) are given by:

$$\begin{aligned}
\nabla_{\partial_5}\partial_5 &= \frac{aa_1}{2}\partial_1 + \frac{aa_2}{2}\partial_2 + \frac{ba_3}{2}\partial_3 + \frac{ba_4}{2}\partial_4 - \frac{a_1}{2}\partial_5 - \frac{a_2}{2}\partial_6 - \frac{a_3}{2}\partial_7 - \frac{a_4}{2}\partial_8;\\
\nabla_{\partial_6}\partial_6 &= \frac{aa_1}{2}\partial_1 + \frac{aa_2}{2}\partial_2 + \frac{ba_3}{2}\partial_3 + \frac{ba_4}{2}\partial_4 - \frac{a_1}{2}\partial_5 - \frac{a_2}{2}\partial_6 - \frac{a_3}{2}\partial_7 - \frac{a_4}{2}\partial_8;\\
\nabla_{\partial_7}\partial_7 &= \frac{ab_1}{2}\partial_1 + \frac{ab_2}{2}\partial_2 + \frac{bb_3}{2}\partial_3 + \frac{bb_4}{2}\partial_4 - \frac{b_1}{2}\partial_5 - \frac{b_2}{2}\partial_6 - \frac{b_3}{2}\partial_7 - \frac{b_4}{2}\partial_8;\\
\nabla_{\partial_8}\partial_8 &= \frac{ab_1}{2}\partial_1 + \frac{ab_2}{2}\partial_2 + \frac{bb_3}{2}\partial_3 + \frac{bb_4}{2}\partial_4 - \frac{b_1}{2}\partial_5 - \frac{b_2}{2}\partial_6 - \frac{b_3}{2}\partial_7 - \frac{b_4}{2}\partial_8.
\end{aligned}$$

We observe that $\nabla_{\partial_5}\partial_5 = \nabla_{\partial_6}\partial_6$ and $\nabla_{\partial_7}\partial_7 = \nabla_{\partial_8}\partial_8$. From the above relations, after a long but straightforward calculation, the nonzero components of the $(1,3)$-curvature operator of any Walker metric (3.1) and (3.2) whith $a = a(u_1,\dots,u_4)$ and $b = b(u_1,\dots,u_4)$ are determined by:

$$\begin{aligned}
R(\partial_5,\partial_6)\partial_5 &= \frac{a_2}{2}\nabla_{\partial_5}\partial_5,\ R(\partial_5,\partial_6)\partial_6 = \frac{-a_1}{2}\nabla_{\partial_5}\partial_5,\ R(\partial_5,\partial_7)\partial_5 = R(\partial_6,\partial_7)\partial_6 = \frac{a_3}{2}\nabla_{\partial_7}\partial_7,\ R(\partial_5,\partial_7)\partial_7 = R(\partial_5,\partial_8)\partial_8 = \frac{-b_1}{2}\nabla_{\partial_5}\partial_5,\\
R(\partial_5,\partial_8)\partial_5 &= R(\partial_6,\partial_8)\partial_6 = \frac{a_4}{2}\nabla_{\partial_7}\partial_7,\ R(\partial_6,\partial_7)\partial_7 = R(\partial_6,\partial_8)\partial_8 = \frac{-b_2}{2}\nabla_{\partial_5}\partial_5,\ R(\partial_7,\partial_8)\partial_7 = \frac{b_4}{2}\nabla_{\partial_7}\partial_7,\ R(\partial_7,\partial_8)\partial_8 = \frac{-b_3}{2}\nabla_{\partial_7}\partial_7.
\end{aligned}$$

From the above relations, after a long but straightforward calculation, the nonzero components of the $(0,4)$-curvature tensor of any Walker metric (3.1) and (3.2) whith $a = a(u_1,\ldots,u_4)$ and $b = b(u_1,\ldots,u_4)$ are given by

$$\begin{aligned}
R_{1556} &= R_{5662} = \frac{a_1a_2}{4};\ R_{5772} = R_{5882} = \frac{a_2b_1}{4};\ R_{6771} = R_{6881} = \frac{a_1b_2}{4};\ R_{1557} = R_{5773} = R_{5883} = R_{1667} = \frac{a_3b_1}{4};\ R_{7881} = \frac{b_1b_3}{4};\\
R_{5663} &= \frac{a_1a_3}{4};\ R_{1558} = R_{5774} = R_{5884} = R_{1668} = \frac{a_4a_1}{4};\ R_{1778} = \frac{b_1b_4}{4};\ R_{5664} = \frac{a_1a_4}{4};\ R_{2557} = R_{6773} = R_{6883} = R_{2667} = \frac{a_3b_2}{4};\\
R_{7882} &= \frac{b_2b_3}{4};\ R_{3556} = \frac{a_2a_3}{4};\ R_{2558} = R_{6774} = R_{6884} = R_{2668} = \frac{a_4b_2}{4};\ R_{2778} = \frac{b_2b_4}{4};\ R_{4556} = \frac{a_2a_4}{4};\ R_{3558} = R_{3668} = \frac{a_4b_3}{4};\\
R_{3778} &= R_{7884} = \frac{b_3b_4}{4};\ R_{4557} = R_{4667} = \frac{a_3b_4}{4};\ R_{1665} = \frac{a_1^2}{4};\ R_{1775} = R_{1885} = \frac{a_1b_1}{4};\ R_{2556} = \frac{a_2^2}{4};\ R_{2776} = R_{2886} = \frac{a_2b_2}{4};\\
R_{3557} &= R_{3667} = \frac{a_3b_3}{4};\ R_{3887} = \frac{b_3^2}{4};\ R_{4558} = R_{4668} = \frac{a_4b_4}{4};\ R_{4778} = \frac{b_4^2}{4}.
\end{aligned}$$

Recall that a pseudo-Riemannian manifold is said to be locally symmetric if its Riemann curvature tensor R satisfy the condition $\nabla R = 0$, where ∇ is the Levi-Civita connection. Equivalently, we have:

$$\begin{aligned}
(\nabla_{\partial_i}R)(\partial_j,\partial_k)\partial_l &= \nabla_{\partial_i}R(\partial_j,\partial_k)\partial_l - R(\nabla_{\partial_i}\partial_j,\partial_k)\partial_l\\
&\quad -R(\partial_j,\nabla_{\partial_i}\partial_k)\partial_l - R(\partial_j,\partial_k)\nabla_{\partial_i}\partial_l
\end{aligned} \tag{3.3}$$

for $i,j,k,l = 1,\ldots,8$. The components of $(\nabla_{\partial_i}R)(\partial_j,\partial_k)\partial_l$ for $i,j,k,l = 1,\ldots,8$ of any Walker metric (3.1) and (3.2) whith $a = a(u_1,\ldots,u_4)$ and $b = b(u_1,\ldots,u_4)$ are given as;

$$\begin{aligned}
(\nabla_{\partial_5}R)(\partial_5,\partial_6)\partial_6 &= -\frac{a_3^2}{4}\nabla_{\partial_7}\partial_7 - \frac{a_4^2}{4}\nabla_{\partial_7}\partial_7;\ (\nabla_{\partial_5}R)(\partial_6,\partial_7)\partial_6 = -\frac{b_1a_3}{4}\nabla_{\partial_5}\partial_5;\ (\nabla_{\partial_5}R)(\partial_5,\partial_7)\partial_7 = -\frac{b_2a_2}{4}\nabla_{\partial_5}\partial_5 - \frac{b_4a_4}{4}\nabla_{\partial_7}\partial_7;\\
(\nabla_{\partial_5}R)(\partial_6,\partial_7)\partial_7 &= \frac{b_2a_1}{4}\nabla_{\partial_5}\partial_5;\ (\nabla_{\partial_5}R)(\partial_5,\partial_8)\partial_8 = -\frac{b_2a_2}{4}\nabla_{\partial_5}\partial_5 - \frac{b_3a_3}{4}\nabla_{\partial_7}\partial_7;\ (\nabla_{\partial_5}R)(\partial_6,\partial_8)\partial_6 = -\frac{b_1a_4}{4}\nabla_{\partial_5}\partial_5;\\
(\nabla_{\partial_5}R)(\partial_6,\partial_8)\partial_8 &= \frac{b_2a_1}{4}\nabla_{\partial_5}\partial_5;\ (\nabla_{\partial_5}R)(\partial_7,\partial_8)\partial_7 = -\frac{b_4b_1}{4}\nabla_{\partial_5}\partial_5;\ (\nabla_{\partial_5}R)(\partial_7,\partial_8)\partial_8 = \frac{b_1b_3}{4}\nabla_{\partial_5}\partial_5;\\
(\nabla_{\partial_6}R)(\partial_5,\partial_6)\partial_5 &= \frac{a_3^2}{4}\nabla_{\partial_7}\partial_7 - \frac{a_4^2}{4}\nabla_{\partial_7}\partial_7;\ (\nabla_{\partial_6}R)(\partial_5,\partial_7)\partial_5 = -\frac{a_3b_2}{4}\nabla_{\partial_5}\partial_5;\ (\nabla_{\partial_6}R)(\partial_5,\partial_7)\partial_7 = \frac{b_1a_2}{4}\nabla_{\partial_5}\partial_5;\\
(\nabla_{\partial_6}R)(\partial_5,\partial_8)\partial_5 &= -\frac{a_4b_2}{4}\nabla_{\partial_5}\partial_5;\ (\nabla_{\partial_6}R)(\partial_5,\partial_8)\partial_8 = \frac{a_2b_1}{4}\nabla_{\partial_5}\partial_5;\ (\nabla_{\partial_6}R)(\partial_6,\partial_7)\partial_7 = \frac{a_4b_4}{4}\nabla_{\partial_7}\partial_7;\\
(\nabla_{\partial_6}R)(\partial_6,\partial_7)\partial_6 &= -\frac{a_3b_2}{4}\nabla_{\partial_5}\partial_5 + \frac{a_2a_3}{4}\nabla_{\partial_7}\partial_7 + \frac{a_2a_3}{4}\nabla_{\partial_7}\partial_7 - \frac{a_3b_2}{4}\nabla_{\partial_5}\partial_5;\ (\nabla_{\partial_6}R)(\partial_6,\partial_8)\partial_6 = 2\frac{a_2a_4}{4}\nabla_{\partial_7}\partial_7;\\
(\nabla_{\partial_6}R)(\partial_6,\partial_8)\partial_8 &= -\frac{a_3b_3}{4}\nabla_{\partial_7}\partial_7;\ (\nabla_{\partial_6}R)(\partial_7,\partial_8)\partial_7 = -\frac{b_2b_4}{4}\nabla_{\partial_5}\partial_5;\ (\nabla_{\partial_6}R)(\partial_7,\partial_8)\partial_8 = \frac{b_3b_2}{4}\nabla_{\partial_5}\partial_5;\\
(\nabla_{\partial_7}R)(\partial_5,\partial_6)\partial_5 &= -\frac{a_2a_3}{4}\nabla_{\partial_7}\partial_7;\ (\nabla_{\partial_7}R)(\partial_5,\partial_6)\partial_6 = \frac{a_1a_3}{4}\nabla_{\partial_7}\partial_7;\ (\nabla_{\partial_7}R)(\partial_5,\partial_7)\partial_5 = \frac{a_2b_2}{4}\nabla_{\partial_5}\partial_5 + \frac{a_4b_4}{4}\nabla_{\partial_7}\partial_7;\\
(\nabla_{\partial_7}R)(\partial_5,\partial_7)\partial_7 &= 2\frac{a_3b_1}{4}\nabla_{\partial_7}\partial_7 - 2\frac{b_1b_3}{4}\nabla_{\partial_5}\partial_5;\ (\nabla_{\partial_7}R)(\partial_5,\partial_8)\partial_5 = \frac{a_4b_3}{4}\nabla_{\partial_7}\partial_7;\ (\nabla_{\partial_7}R)(\partial_5,\partial_8) = \frac{a_3b_1}{4}\nabla_{\partial_7}\partial_7;\\
(\nabla_{\partial_7}R)(\partial_6,\partial_7)\partial_6 &= \frac{a_1b_1}{4}\nabla_{\partial_5}\partial_5 - \frac{a_4^2}{4}\nabla_{\partial_7}\partial_7;\ (\nabla_{\partial_7}R)(\partial_6,\partial_7)\partial_7 = 2\frac{a_3b_2}{4}\nabla_{\partial_7}\partial_7 - \frac{b_3b_2}{4}\nabla_{\partial_5}\partial_5;\ (\nabla_{\partial_7}R)(\partial_6,\partial_8)\partial_6 = -\frac{a_4b_3}{4}\nabla_{\partial_7}\partial_7;\\
(\nabla_{\partial_7}R)(\partial_6,\partial_8)\partial_8 &= \frac{a_3b_2}{4}\nabla_{\partial_7}\partial_7;\ (\nabla_{\partial_7}R)(\partial_7,\partial_8)\partial_7 = 0;\ (\nabla_{\partial_7}R)(\partial_7,\partial_8)\partial_8 = -\frac{b_1^2}{4}\nabla_{\partial_5}\partial_5 - \frac{b_2^2}{4}\nabla_{\partial_5}\partial_5;\ (\nabla_{\partial_8}R)(\partial_5,\partial_6)\partial_5 = -\frac{a_2a_4}{4}\nabla_{\partial_7}\partial_7;\\
(\nabla_{\partial_8}R)(\partial_5,\partial_6)\partial_6 &= \frac{a_1a_4}{4}\nabla_{\partial_7}\partial_7;\ (\nabla_{\partial_8}R)(\partial_5,\partial_7)\partial_5 = -\frac{a_3b_4}{4}\nabla_{\partial_7}\partial_7;\ (\nabla_{\partial_8}R)(\partial_5,\partial_7)\partial_7 = \frac{b_1a_4}{4}\nabla_{\partial_7}\partial_7;\\
(\nabla_{\partial_8}R)(\partial_5,\partial_8)\partial_5 &= \frac{a_2b_2}{4}\nabla_{\partial_5}\partial_5 + \frac{a_3b_3}{4}\nabla_{\partial_7}\partial_7;\ (\nabla_{\partial_8}R)(\partial_5,\partial_8)\partial_8 = 2\frac{b_1a_4}{4}\nabla_{\partial_7}\partial_7 - 2\frac{b_1b_4}{4}\nabla_{\partial_5}\partial_5;\ (\nabla_{\partial_8}R)(\partial_6,\partial_7)\partial_6 = -\frac{a_3b_4}{4}\nabla_{\partial_7}\partial_7;\\
(\nabla_{\partial_8}R)(\partial_6,\partial_7)\partial_7 &= \frac{b_2a_4}{4}\nabla_{\partial_7}\partial_7;\ (\nabla_{\partial_8}R)(\partial_6,\partial_8)\partial_6 = \frac{a_1b_1}{4}\nabla_{\partial_5}\partial_5 + \frac{a_3b_3}{4}\nabla_{\partial_7}\partial_7;\ (\nabla_{\partial_8}R)(\partial_6,\partial_8)\partial_8 = 2\frac{b_2b_4}{4}\nabla_{\partial_7}\partial_7 - 2\frac{b_2b_4}{4}\nabla_{\partial_5}\partial_5;\\
(\nabla_{\partial_8}R)(\partial_7,\partial_8)\partial_7 &= \frac{b_1^2}{4}\nabla_{\partial_5}\partial_5 + \frac{b_2^2}{4}\nabla_{\partial_5}\partial_5;\ (\nabla_{\partial_8}R)(\partial_7,\partial_8)\partial_8 = 0.
\end{aligned}$$

From the above systems of equations, after a long but straightforward calculations, we obtain the results.

Theorem 3.1. *A Walker metric given by (3.1) and (3.2) is locally symmetric if the functions $a = a(u_1,u_2,u_3,u_4)$ and $b = b(u_1,u_2,u_3,u_4)$ are constants.*

Proof. Observe that if the functions $a = a(u_1,\ldots,u_4)$ and $b = b(u_1,\ldots,u_4)$ are constants, then $a_i, b_j = 0$ for all $i,j = 1,\ldots,8$, therefore, $(\nabla_{\partial_i}R)(\partial_j,\partial_k)\partial_l = 0$ for all $i,j,k,l = 1,\ldots,8$ and the result is obtained. □

4. Conclusion

Various geometric quantities are computed explicitly in terms of metrics coefficients, including the Levi-Civita connection, curvature operator, Riemann curvature and covariant derivative of curvature operator. Using these formulas, we have obtained a condition for a specific Walker metric to be locally symmetric.

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