Konuralp Journal of Mathematics, 11 (2) (2023) 97-104

Konuralp Journal of Mathematics

Research Paper

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath

e-ISSN: 2147-625X

Spectrum and Fine Spectrum of the Triple Repetitive Double-Band Matrix Over the Sequence Space cs

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Abstract

The aim of our study is to obtain the spectrum, fine spectrum, approximate point spectrum, defect spectrum and compression spectrum of triple repetitive double-band matrix over the cs sequence space. In addition, the spectrum and fine spectrum of the n-repetitive form were investigated in the space of this matrix.

Keywords: *Band matrix, spectrum, fine spectrum, approximate point spectrum*
2010 Mathematics Subject Classification: *47A10, 47B37*

1. Introduction

The spectral theory is a branch of summability theory, which is an important branch of functional analysis. Some important studies regarding the recent developments in summability theory are [6], [8], [15], [20], [21] and [22]. The spectral theory extends the concept of eigenvalue and eigenvector of matrix theory to the spectrum of operators in several spaces. While examining the operators' spectrum, the fine separation defined by Goldberg and another separation that does not need to be disjoint, approximate, defect, compression spectrum were examined. Some studies on this topic are: The spectrum of the Cesàro operator is studied in [1]-[2] The spectrum of the $B(r,s)$ operator is studied in [5]-[11]. The spectrum of operator $B(r,s,t)$ [18]-[7] was studied. The spectrum of the Δ_{ab} is studied in [3]-[13]. The spectrum of the Δ^{uv} is studied in [16]-[14]. The spectrum of the $U(a,0,b)$is studied in [9], [10].
Let w, cs, bv denote the set of all sequences, convergent series, bounded variation sequences, respectively.

$$cs=\left\{x=(x_k)\in w:\lim_{n\to\infty}\sum_{k=1}^{n}x_k \text{ exists}\right\}$$

and

$$\|x\|_{cs}=\sup_n\left|\sum_{k}^{n}x_k\right|.$$

cs is isomorphic to $bv=\left\{x=(x_k)\in w:\sum\limits_{k=1}^{\infty}|x_k-x_{k+1}| \text{ exists}\right\}$ with the norm $\|x\|_{bv}=\sum\limits_{k=1}^{\infty}|x_k-x_{k+1}|$.
In [12], the spectrum and fine spectrum of the $U(a_0,a_1,a_2;b_0,b_1,b_2)$ matrix in the c_0 sequence space were examined. In this study, we will examine the spectrum, fine spectrum and spectral decomposition of this matrix on the cs sequence space. We will also give the spectrum and fine spectrum of the n-repetitive form of this matrix on the same space. For this purpose, we will first show that the matrix defines a limited linear operator on the cs sequence space, and then calculate the approximate point spectrum, defect spectrum and compression spectrum after performing the Golberg's classification of the spectrum.
Let X and Y be the Banach spaces, and $A:X\to Y$ be a bounded linear operator. We denominate the range of A by $R(A)=\{y\in Y: y=Ax,\ x\in X\}$, and the domain of A.by $D(A)=\{x\in X: y=Ax\}$. We show the set of all bounded linear operators on X into itself.by $B(X)$. Also let I be the identity operator.

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2. Fine Spectrum

In this section, the spectra and fine spectra of the $U(a_0,a_1,a_2;b_0,b_1,b_2)$ matrix will be calculated.
Let $A:D(A)\subset X\to X$ be a bounded linear operator where X is a complex normed space. Let $A_\zeta := \zeta I - A$ for $\zeta\in\mathbb{C}$. A_ζ^{-1} is known as the resolvent operator of A.
$\zeta\in\mathbb{C}$ is a regular value of A such that
a) A_ζ^{-1} exists,
b) A_ζ^{-1} is continuous
c) A_ζ^{-1} is defined on a set which is dense in X,
the set of all regular values is denoted by $\rho(A,X)$. The set $\sigma(A,X)=\mathbb{C}\backslash\rho(A,X)$ is called the spectrum of A.
The set $\sigma(A,X)$ is union of three disjoint sets as follows:
a) The point spectrum $\sigma_p(A,X)=\left\{\zeta\in\mathbb{C}: A_\zeta^{-1}\text{does not exist}\right\}$.
b) The continuous spectrum
$\sigma_c(A,X)=\left\{\zeta\in\mathbb{C}: A_\zeta^{-1}\text{is defined on a dense subspace of X and is discontinuous}\right\}$
c) The residual spectrum
$\sigma_r(A,X)=\left\{\zeta\in\mathbb{C}: A_\zeta^{-1}\text{exists and its domain of definition is not dense in } X\right\}$.
From the above definitions, the following table is obtained;

		1	2	3
		A_ζ^{-1} exists and is continuous	A_ζ^{-1} exists and is discontinuous	A_ζ^{-1} does not exists
I	$R(\zeta I-A)=X$	$\zeta\in\rho(A,X)$	–	$\zeta\in\sigma_p(A,X)$
II	$\overline{R(\zeta I-A)}=X$	$\zeta\in\rho(A,X)$	$\zeta\in\sigma_c(A,X)$	$\zeta\in\sigma_p(A,X)$
III	$\overline{R(\zeta I-A)}\neq X$	$\zeta\in\sigma_r(A,X)$	$\zeta\in\sigma_r(A,X)$	$\zeta\in\sigma_p(A,X)$

Table 1: Disjoint divisions of the spectra of a bounded linear operators.

The triangular double-band matrix $U(a_0,a_1,a_2;b_0,b_1,b_2)$ is an infinite matrix of form

$$U(a_0,a_1,a_2;b_0,b_1,b_2)=\begin{bmatrix} a_0 & b_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots\\ 0 & a_1 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots\\ 0 & 0 & a_2 & b_2 & 0 & 0 & 0 & 0 & 0 & \cdots\\ 0 & 0 & 0 & a_0 & b_0 & 0 & 0 & 0 & 0 & \cdots\\ 0 & 0 & 0 & 0 & a_1 & b_1 & 0 & 0 & 0 & \cdots\\ 0 & 0 & 0 & 0 & 0 & a_2 & b_2 & 0 & 0 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots \end{bmatrix} \quad (b_0,b_1,b_2\neq 0). \tag{2.1}$$

Lemma 2.1 ([23] Stieglitz, Tietz). *The matrix $A=(a_{nk})$ gives rise to a bounded linear operator $T\in(cs;cs)$ from cs to itself if and only if*
(i) $\sum\limits_{n=1}^{\infty} a_{nk}$ *converges, for each k,*
(ii) $\sup\limits_{N>0}\sum\limits_{k=1}^{\infty}\left|\sum\limits_{n=0}^{N}\left(a_{nk}-a_{n,k-1}\right)\right|<\infty$,

Theorem 2.2. *$U(a_0,a_1,a_2;b_0,b_1,b_2): cs\to cs$ is a bounded linear operator.*

Proof. (i) Since

$$\sum_n a_{nk}=\begin{cases} a_0, & k=0\\ b_0+a_1, & k=3i-2\\ b_1+a_2, & k=3i-1\\ b_2+a_0, & k=3i\end{cases} \quad i=1,2,3,\dots$$

it is clear.
(ii) Let $B_k^N=\sum\limits_{n=0}^{N}\left(a_{nk}-a_{n,k-1}\right)$ and if we calculate B_k^N then we have

	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$	$\cdots$
$k=1$	$b_0-a_0+a_1$	$b_0-a_0+a_1$	$b_0-a_0+a_1$	$b_0-a_0+a_1$	$b_0-a_0+a_1$	$\cdots$
$k=2$	$-b_0+b_1-a_1$	$-b_0+b_1-a_1+a_2$	$-b_0+b_1-a_1+a_2$	$-b_0+b_1-a_1+a_2$	$-b_0+b_1-a_1+a_2$	$\cdots$
$k=3$	$-b_1$	$-b_1+b_2-a_2$	$-b_1+b_2-a_2+a_0$	$-b_1+b_2-a_2+a_0$	$-b_1+b_2-a_2+a_0$	$\cdots$
$k=4$	0	$-b_2$	$-b_2+b_0-a_0$	$-b_2+b_0-a_0+a_1$	$-b_2+b_0-a_0+a_1$	$\cdots$
$k=5$	0	0	$-b_0$	$-b_0+b_1-a_1$	$-b_0+b_1-a_1+a_2$	$\cdots$.
$k=6$	0	0	0	$-b_1$	$-b_1+b_2-a_2$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\sum\limits_{k=1}^{\infty}\left\|B_k^N\right\|$	$\|a_0\|$	$\|a_0\|$	$\|a_0\|$	$\|a_0\|$	$\|a_0\|$	$\cdots$

Therefore we have

$$B_k^N = \begin{cases} 0, & N \le k-3 \\ -b_{k-2}, & N = k-2 \\ -b_{k-2} + b_{k-1} - a_{k-1}, & N = k-1 \\ -b_{k-2} + b_{k-1} - a_{k-1} + a_k, & N \ge k \end{cases}.$$

Hence, since $\sum_{k=1}^{\infty} \left|B_k^N\right| = |a_0|$ we get

$$\sup_{N>0} \sum_{k=1}^{\infty} \left| \sum_{n=0}^{N} \left(a_{nk} - a_{n,k-1}\right) \right| = \sup_{N>0} \sum_{k=1}^{\infty} \left|B_k^N\right| = |a_0| < \infty$$

Thus $U(a_0,a_1,a_2;b_0,b_1,b_2) : cs \to cs$ is a bounded linear operator. □

Lemma 2.3 (Golberg [19, p.60]). *T has a dense range if and only if T^* is 1-1.*

Lemma 2.4 (Golberg [19, p.60]). *T has a bounded inverse if and only if T^* is onto.*

Theorem 2.5. $\sigma_p(U(a_0,a_1,a_2;b_0,b_1,b_2), cs) = \{\zeta \in \mathbb{C} : |\zeta - a_0||\zeta - a_1||\zeta - a_2| < |b_0||b_1||b_2|\}$.

Proof. Let ζ be an eigenvalue of the operator $U(a_0,a_1,a_2;b_0,b_1,b_2)$. Then there exists $x \neq \theta = (0,0,0,...)$ in cs such that $U(a_0,a_1,a_2;b_0,b_1,b_2)x = \zeta x$. Then we obtain

$$\begin{cases} x_{3n} &=& d^n x_0, \\ x_{3n+1} &=& \dfrac{\zeta - a_0}{b_0} d^n x_0, \\ x_{3n+2} &=& \dfrac{(\zeta - a_0)(\zeta - a_1)}{b_0 b_1} d^n x_0, \end{cases}, \; n \ge 0$$

where $d = \dfrac{(\zeta - a_0)(\zeta - a_1)(\zeta - a_2)}{b_0 b_1 b_2}$. Thus we get

$$\begin{aligned} \sum_{n=0}^{\infty} x_n &= x_0 + x_1 + x_2 + \cdots + x_n + \cdots \\ &= \left(1 + \frac{\zeta - a_0}{b_0} + \frac{(\zeta - a_0)(\zeta - a_1)}{b_0 b_1}\right) x_0 \sum_{n=0}^{\infty} d^n \end{aligned}$$

$\sum_{n=0}^{\infty} d^n$ is absolutely convergent if and only if $|d| < 1$. Since absolutely convergent every series is convergent series, $(x_n) \in cs$. So $x = (x_n) \in cs$ if and only if $|\zeta - a_0||\zeta - a_1||\zeta - a_2| < |b_0||b_1||b_2|$. Therefore
$\sigma_p(U(a_0,a_1,a_2;b_0,b_1,b_2), cs) = \{\zeta \in \mathbb{C} : |\zeta - a_0||\zeta - a_1||\zeta - a_2| < |b_0||b_1||b_2|\}$. □

As it is known, if $T \in B(cs)$ is represented with a matrix A , then the adjoint operator $T^* \in B(cs^* \cong bv)$ and A^t is its matrix representation.

Theorem 2.6. $\sigma_p(U(a_0,a_1,a_2;b_0,b_1,b_2)^*, cs^* \cong bv) = \emptyset$.

Proof. Let ζ be an eigenvalue of the operator $U(a_0,a_1,a_2;b_0,b_1,b_2)^*$. Then there exists $x \neq \theta = (0,0,0,...)$ in bv such that $U(a_0,a_1,a_2;b_0,b_1,b_2)^* x = \zeta x$. Then, we obtain

$a_0 x_0 = \zeta x_0$

$b_0 x_0 + a_1 x_1 = \zeta x_1$

$b_1 x_1 + a_2 x_2 = \zeta x_2$

$b_2 x_2 + a_0 x_3 = \zeta x_3$

$\vdots$

$$n = 3k, \;\; b_0 x_n + a_1 x_{n+1} = \zeta x_{n+1} \tag{2.2}$$

$n = 3k+1, \;\; b_1 x_n + a_2 x_{n+1} = \zeta x_{n+1}$

$n = 3k+2, \;\; b_2 x_n + a_0 x_{n+1} = \zeta x_{n+1}$

Let x_k be the first non-zero of sequence (x_n). Let $n = 3k$. If $x_n = x_{k-1}$ satistify in (2.2) then from $b_0 x_{k-1} + a_1 x_k = \zeta x_k$, we have $a_1 = \zeta$. Again from (2.2), we get for $a_1 = \zeta$, $b_0 x_k = 0$ which implies $x_k = 0$ as $b_0 \neq 0$,a contradiction.Similarly, if $n = 3k+1$ and $n = 3k+2$ we obtain a contradiction. Hereby, $\sigma_p(\left(\Delta_3^{ab}\right)^*, cs^* \cong bv) = \emptyset$. □

Theorem 2.7. $\sigma_r(U(a_0,a_1,a_2;b_0,b_1,b_2), cs) = \emptyset$.

Proof. Owing to $\sigma_r(A,X) = \sigma_p(A^*,X^*) \backslash \sigma_p(A,X)$, we get required result from Theorems 2.5 and 2.6. □

We know that the inverse of an upper triangular matrix is also an upper triangular matrix. So if we calculate

$$\begin{bmatrix} a_0-\zeta & b_0 & 0 & 0 & 0 & \cdots \\ 0 & a_1-\zeta & b_1 & 0 & 0 & \cdots \\ 0 & 0 & a_2-\zeta & b_2 & 0 & \cdots \\ 0 & 0 & 0 & a_0-\zeta & b_0 & \cdots \\ 0 & 0 & 0 & 0 & a_1-\zeta & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} c_{00} & c_{01} & c_{02} & c_{03} & c_{04} & \cdots \\ 0 & c_{11} & c_{12} & c_{13} & c_{14} & \cdots \\ 0 & 0 & c_{22} & c_{23} & c_{24} & \cdots \\ 0 & 0 & 0 & c_{33} & c_{34} & \cdots \\ 0 & 0 & 0 & 0 & c_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

the inverse matrix of $(U(a_0,a_1,a_2;b_0,b_1,b_2)-\zeta I)$ will be

$$C=\begin{pmatrix} \frac{1}{a_0-\zeta} & -\frac{b_0}{(a_0-\zeta)(a_1-\zeta)} & \frac{b_0b_1}{(a_0-\zeta)(a_1-\zeta)(a_2-\zeta)} & -\frac{b_0b_1b_2}{(a_0-\zeta)^2(a_1-\zeta)(a_2-\zeta)} & \frac{b_0^2b_1b_2}{(a_0-\zeta)^2(a_1-\zeta)^2(a_2-\zeta)} & \cdots \\ 0 & \frac{1}{a_1-\zeta} & -\frac{b_1}{(a_1-\zeta)(a_2-\zeta)} & \frac{b_1b_2}{(a_0-\zeta)(a_1-\zeta)(a_2-\zeta)} & -\frac{b_0b_1b_2}{(a_0-\zeta)(a_1-\zeta)^2(a_2-\zeta)} & \cdots \\ 0 & 0 & \frac{1}{a_2-\zeta} & -\frac{b_2}{(a_0-\zeta)(a_2-\zeta)} & \frac{b_0b_2}{(a_0-\zeta)(a_1-\zeta)(a_2-\zeta)} & \cdots \\ 0 & 0 & 0 & \frac{1}{a_0-\zeta} & -\frac{b_0}{(a_0-\zeta)(a_1-\zeta)} & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{a_1-\zeta} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Theorem 2.8. $\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),cs)=\{\zeta\in\mathbb{C}:|\zeta-a_0||\zeta-a_1||\zeta-a_2|\le|b_0||b_1||b_2|\}.$

Proof. First, we prove that $(U(a_0,a_1,a_2;b_0,b_1,b_2)-\zeta I)^{-1}$ exists and is in (cs,cs) for $|b_0||b_1||b_2|<|\zeta-a_0||\zeta-a_1||\zeta-a_2|$ and then we show that the operator $(U(a_0,a_1,a_2;b_0,b_1,b_2)-\zeta I)$ is not invertible for $|\zeta-a_0||\zeta-a_1||\zeta-a_2|\le|b_0||b_1||b_2|$.
Let $\zeta\notin\{\zeta\in\mathbb{C}:|\zeta-a_0||\zeta-a_1||\zeta-a_2|\le|b_0||b_1||b_2|\}$. Since $b_n\neq 0$, $n=0,1,2$ we get $a_n\neq\zeta$, $n=0,1,2$.
Hence, since $(U(a_0,a_1,a_2;b_0,b_1,b_2)-\zeta I)$ is a upper triangle, $(U(a_0,a_1,a_2;b_0,b_1,b_2)-\zeta I)^{-1}$ is exists. Let $(U(a_0,a_1,a_2;b_0,b_1,b_2)-\zeta I)^{-1}=C$, then from Lemma 2.1, we get

$$c_{nk}=\begin{cases} 0 & , \quad n>k \\ \frac{1}{a_n-\zeta}\prod_{v=0}^{k-n-1}(-1)^{k-n}\frac{b_{k-1-v}}{a_{k-v}-\zeta} & , \quad n\le k \end{cases}$$

where we assume $\prod_{v=0}^{-1}\frac{b_{k-1-\iota}}{a_{k-\iota}-\zeta}=1$.

Now, we have to $(U(a_0,a_1,a_2;b_0,b_1,b_2)-\zeta I)^{-1}\in(cs,cs)$. Since the matrix is triangular, condition (i) of Lemma 2.1 is clear.

$$\begin{aligned} c_{nk}-c_{n,k-1} &= \frac{1}{a_n-\zeta}\left(\prod_{v=0}^{k-n-1}\frac{b_{k-1-v}}{a_{k-v}-\zeta}-\prod_{v=0}^{k-n-2}(-1)^{k-n-1}\frac{b_{k-2-v}}{a_{k-v-1}-\zeta}\right) \\ &= \frac{(-1)^{k-n}}{a_n-\zeta}\left(\left(\frac{b_n}{a_{n+1}-\zeta}+1\right)\prod_{v=0}^{k-n-2}\frac{b_{k-2-v}}{a_{k-v-1}-\zeta}\right) \\ &= \frac{A}{s^3}\frac{(-1)^{k-n}}{a_n-\zeta}\left(\frac{b_n}{a_{n+1}-\zeta}+1\right)s^{k-n} \end{aligned}$$

where $s=\left(\frac{b_0b_1b_2}{(a_0-\zeta)(a_1-\zeta)(a_2-\zeta)}\right)^{\frac{1}{3}}$ and

$$A=\left\{\begin{array}{c|ccc} & k=3i & k=3i-1 & k=3i-2 \\ \hline n=3j & \left|\frac{b_1b_0}{(a_2-\zeta)(a_1-\zeta)}\right| & \left|\frac{b_0}{a_1-\zeta}\right| & 1 \\ n=3j-1 & \left|\frac{b_1b_0b_2}{(a_2-\zeta)(a_1-\zeta)(a_0-\zeta)}\right| & \left|\frac{b_0b_2}{(a_1-\zeta)(a_0-\zeta)}\right| & \left|\frac{b_2}{a_0-\zeta}\right| \\ n=3i-2 & \left|\frac{b_1^2b_0b_2}{(a_2-\zeta)^2(a_1-\zeta)(a_0-\zeta)}\right| & \left|\frac{b_1b_0b_2}{(a_2-\zeta)(a_1-\zeta)(a_0-\zeta)}\right| & \left|\frac{b_2b_1}{(a_0-\zeta)(a_2-\zeta)}\right| \end{array}\right. . \tag{2.3}$$

So,

$$\begin{aligned}
\sum_{k=0}^{\infty}\left|\sum_{n=1}^{N}\frac{A}{s^3}\frac{1}{a_n-\zeta}\left(\frac{b_n}{a_{n+1}-\zeta}+1\right)(-s)^{k-n}\right| &\leq \frac{A}{s^3}\max_{m=0}^{2}\left|\frac{1}{a_m-\zeta}\right|\max_{m=0}^{2}\left|\frac{b_m}{a_{m+1}-\zeta}+1\right|\sum_{k=0}^{\infty}\frac{|s|^k\left(1-\frac{1}{|s|^{N+1}}\right)}{1-\frac{1}{|s|}}\\
&= \frac{\left(|s|^{N+1}-1\right)R}{|s|-1}\sum_{k=0}^{\infty}|s|^{k+1}\\
&= \frac{R\,|s|}{(|s|-1)^2}\left(\frac{1-|s|^{N+1}}{|s|^{N+1}}\right)
\end{aligned}$$

marked here as $\frac{A}{s^3}\max_{m=0}^{2}\left|\frac{1}{a_m-\zeta}\right|\max_{m=0}^{2}\left|\frac{b_m}{a_{m+1}-\zeta}+1\right|=R$. Thus we get

$$\begin{aligned}
\sup_{N>0}\sum_{k=1}^{\infty}\left|\sum_{n=0}^{N}\left(c_{nk}-c_{n,k-1}\right)\right| &\leq \frac{R\,|s|}{(|s|-1)^2}\sup_{N>0}\frac{1-|s|^{N+1}}{|s|^{N+1}}\\
&= \sup_{N>0}\left(\frac{1}{|s|^{N+1}}-1\right)
\end{aligned}$$

if $|s|>1$, then $\sup_{N>0}\left(\frac{1}{|s|^{N+1}}-1\right)<\infty$. So condition (ii) of Lemma 2.1 proven.

Consequently, if $|b_2||b_1||b_0|<|a_2-\zeta||a_1-\zeta||a_0-\zeta|$, then $(U(a_0,a_1,a_2;b_0,b_1,b_2)-\zeta I)^{-1}\in(cs,cs)$. Hereby, the operator $(U(a_0,a_1,a_2;b_0,b_1,b_2)-\zeta I)$ is not invertible for $|\zeta-a_0||\zeta-a_1||\zeta-a_2|\leq|b_0||b_1||b_2|$. So $\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),cs)=\{\zeta\in\mathbb{C}:|\zeta-a_0||\zeta-a_1||\zeta-a_2|\leq|b_0||b_1||b_2|\}$. □

Theorem 2.9. $\sigma_c(U(a_0,a_1,a_2;b_0,b_1,b_2),cs)=\{\zeta\in\mathbb{C}:|\zeta-a_0||\zeta-a_1||\zeta-a_2|=|b_0||b_1||b_2|\}$

Proof. Owing to $\sigma(A,X)$ is the disjoint union of $\sigma_p(A,X)$, $\sigma_r(A,X)$ and $\sigma_c(A,X)$, thence

$\sigma_c(A,X)=\sigma(A,X)\setminus\left(\sigma_p(A,X)\cup\sigma_r(A,X)\right)$.

By Theorem 2.5 and Theorem 2.7, we get required result. □

3. Subdivision of the Spectrum

In this section, another spectral decomposition of the $U(a_0,a_1,a_2;b_0,b_1,b_2)$ matrix will be calculated. The spectrum $\sigma(A,X)$ is divided into three sets that need not be discrete as follows:
a) The approximate point spectrum of A the set as

$\sigma_{ap}(A,X):=\{\zeta\in\mathbb{C}:\text{there exists a Weyl sequence for }\zeta I-A\}$

If there exists a sequence (x_n) in X such that $\|x_n\|=1$ and $\|Ax_n\|\to 0$ as $n\to\infty$ then (x_n) is called Weyl sequence for A.
b) The defect spectrum of A is the set as

$\sigma_\delta(A,X):=\{\zeta\in\sigma(A,X):\zeta I-A\text{ is not surjective}\}$

c) The compression spectrum of A the set as

$\sigma_{co}(A,X)=\{\zeta\in\mathbb{C}:\overline{R(\zeta I-A)}\neq X\}$

Proposition 1 ([4], Proposition 1.3)**.** *The spectra and subspectra of an operator $A\in B(X)$ and its adjoint $A^*\in B(X^*)$ are related by the following relations:*
(a) $\sigma(A^*,X^*)=\sigma(A,X)$,
(b) $\sigma_c(A^*,X^*)\subseteq\sigma_{ap}(A,X)$,
(c) $\sigma_{ap}(A^*,X^*)=\sigma_\delta(A,X)$,
(d) $\sigma_\delta(A^*,X^*)=\sigma_{ap}(A,X)$,
(e) $\sigma_p(A^*,X^*)=\sigma_{co}(A,X)$,
(f) $\sigma_{co}(A^*,X^*)\supseteq\sigma_p(A,X)$,
(g) $\sigma(A,X)=\sigma_{ap}(A,X)\cup\sigma_p(A^*,X^*)=\sigma_p(A,X)\cup\sigma_{ap}(A^*,X^*)$.

Goldberg's Classification of Spectrum
If $A\in B(X)$, then there are three cases for $R(A)$:
(I) $R(A)=X$, (II) $\overline{R(A)}=X$, but $R(A)\neq X$, (III) $\overline{R(A)}\neq X$
and three cases for A^{-1}:
(1) A^{-1} exists and continuous, (2) A^{-1} exists but discontinuous, (3) A^{-1} does not exist.
If these cases are combined in all possible ways, nine different states are created. These are labelled by: I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3 (see [19]).
$\sigma(A,X)$ can be divided into subdivisions $I_2\sigma(A,X)=\emptyset$, $I_3\sigma(A,X)$, $II_2\sigma(A,X)$, $II_3\sigma(A,X)$, $III_1\sigma(A,X)$, $III_2\sigma(A,X)$, $III_3\sigma(A,X)$.

<table>
<tr><td></td><td></td><td>1</td><td>2</td><td>3</td></tr>
<tr><td></td><td></td><td>A_ζ^{-1} exists
and is continuous</td><td>A_ζ^{-1} exists
and is continuous</td><td>A_ζ^{-1}
does not exists</td></tr>
<tr><td>I</td><td>$R(A_\zeta)=X$</td><td>$\zeta\in\rho(A,X)$</td><td>-</td><td>$\zeta\in\sigma_{ap}(A,X)$</td></tr>
<tr><td rowspan="2">II</td><td rowspan="2">$\overline{R(A_\zeta)}=X$</td><td rowspan="2">$\zeta\in\rho(A,X)$</td><td>$\zeta\in\sigma_{ap}(A,X)$</td><td>$\zeta\in\sigma_{ap}(A,X)$</td></tr>
<tr><td>$\zeta\in\sigma_\delta(A,X)$</td><td>$\zeta\in\sigma_\delta(A,X)$</td></tr>
<tr><td rowspan="3">III</td><td rowspan="3">$\overline{R(A_\zeta)}\neq X$</td><td rowspan="2">$\zeta\in\sigma_\delta(A,X)$</td><td>$\zeta\in\sigma_{ap}(A,X)$</td><td>$\zeta\in\sigma_{ap}(A,X)$</td></tr>
<tr><td>$\zeta\in\sigma_\delta(A,X)$</td><td>$\zeta\in\sigma_\delta(A,X)$</td></tr>
<tr><td>$\zeta\in\sigma_{co}(A,X)$</td><td>$\zeta\in\sigma_{co}(A,X)$</td><td>$\zeta\in\sigma_{co}(A,X)$</td></tr>
</table>

Table 2: Subdivisions of the spectra of a bounded linear operators.

Theorem 3.1. *$I\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),cs)=\emptyset$.*

Proof. For $\zeta\in I\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),cs)$, we should show that $U(a_0,a_1,a_2;b_0,b_1,b_2)-\zeta I$ is onto. Let $y=(y_n)\in cs$ be such that $(U(a_0,a_1,a_2;b_0,b_1,b_2)-\zeta I)x=y$ for $x=(x_n)$.Then

$$\begin{aligned}(a_0-\zeta)x_0+b_0x_1 &= y_0\\ (a_1-\zeta)x_1+b_1x_2 &= y_1\\ (a_2-\zeta)x_2+b_2x_3 &= y_2\\ &\vdots\\ (a_n-\zeta)x_n+b_nx_{n+1} &= y_n.\end{aligned}$$

Calculating x_k, we get

$$x_n=\frac{1}{b_{n-1}}\left(y_{n-1}+\sum_{k=1}^{n-2}y_k\prod_{u=1}^{n-k-1}\frac{\zeta-a_{n-u}}{b_{n-u-1}}\right)+x_0\prod_{u=1}^{n}\frac{\zeta-a_{n-u}}{b_{n-u}},\quad n=1,2,3,\dots$$

We have to show that $x=(x_k)\in cs$ and setting $d=\dfrac{(\zeta-a_0)(\zeta-a_1)(\zeta-a_2)}{b_0b_1b_2}$

since $\displaystyle\prod_{u=1}^{n-k-1}\frac{\zeta-a_{n-u}}{b_{n-u-1}}=Md^{\frac{k-n-1}{3}}$ and $\displaystyle\prod_{u=1}^{n}\frac{\zeta-a_{n-u}}{b_{n-u}}=Nd^{\frac{n}{3}}$ where M and N are constants similar to (2.3)

$$x_n=\frac{1}{b_{n-1}}y_{n-1}+\frac{1}{b_{n-1}}\sum_{k=1}^{n-2}y_kMd^{\frac{k-n-1}{3}}+x_0Nd^{\frac{n}{3}},\quad n=1,2,3,\dots$$

Now suppose $y=(e_{n-1})=(0,0,\dots,0,1,0,\dots)$ then we get

$$x_n=\frac{1}{b_{n-1}}\left(1+1.M.d^{\frac{0}{3}}\right)+x_0.N.d^{\frac{n}{3}}$$

$$x_n=\frac{1}{b_{n-1}}(1+M)+x_0.N.d^3\longrightarrow\frac{1}{\lim_{n\to\infty}b_{n-1}}(1+M)\neq 0.$$

Hence since $\sum x_n$ divergent, $(x_n)\notin cs$. Therefore $\zeta\in\mathbb{C}$ doesn't satisfies Golberg's condition I. So
$I\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),cs)=\emptyset$. □

Corollary 3.2. *$II_3\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),cs)=\{\zeta\in\mathbb{C}:|\zeta-a_0||\zeta-a_1||\zeta-a_2|<|b_0||b_1||b_2|\}$.*

Proof. $(U(a_0,a_1,a_2;b_0,b_1,b_2)-\zeta I)^*$ is injective from Theorem 2.6. So from Lemma 2.3 $U(a_0,a_1,a_2;b_0,b_1,b_2)-\zeta I$ has a dense range. Thus for $\zeta\in\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),cs)$, $\zeta\in I\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),\ell_p)$ or $\zeta\in II\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),cs)$.
$\zeta\in II\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),cs)$ is gotten from Theorem 2.9.
Also, if $|\zeta-a_0||\zeta-a_1||\zeta-a_2|<|b_0||b_1||b_2|$, then $\zeta\in 3\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),cs)$ from Theorem 2.9. Hence
$II_3\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),cs)=\{\zeta\in\mathbb{C}:|\zeta-a_0||\zeta-a_1||\zeta-a_2|<|b_0||b_1||b_2|\}$. □

Corollary 3.3. *$III_1\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),cs)=III_2\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),cs)=\emptyset$.*

Proof. Since $\sigma_r(A,X)=III_1\sigma(A,X)\cup III_2\sigma(A,X)$ from Table 1 and Table 2, the result is obtained by Theorem 2.8 and Theorem 2.7. □

Corollary 3.4. *$I_3\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),cs)=III_3\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),cs)=\emptyset$.*

Proof. Since $\sigma_p(A,X)=I_3\sigma(A,X)\cup II_3\sigma(A,X)\cup III_3\sigma(A,X)$ from Table 1 and Table 2, the result is obtained by Theorem 2.5 and Theorem 2.9. □

Theorem 3.5. *The following spectral decompositions are valid:*
(a) $\sigma_{ap}(U(a_0,a_1,a_2;b_0,b_1,b_2),cs)=\{\zeta\in\mathbb{C}:|\zeta-a_0||\zeta-a_1||\zeta-a_2|\leq|b_0||b_1||b_2|\}$,
(b) $\sigma_\delta(U(a_0,a_1,a_2;b_0,b_1,b_2),cs)=\{\zeta\in\mathbb{C}:|\zeta-a_0||\zeta-a_1||\zeta-a_2|\leq|b_0||b_1||b_2|\}$,
(c) $\sigma_{co}(U(a_0,a_1,a_2;b_0,b_1,b_2),cs)=\emptyset$.

Proof. (a) From Table 2, we obtain

$$\sigma_{ap}(A,X) = \sigma(A,X) \backslash III_1\sigma(A,X).$$

And so $\sigma_{ap}(U(a_0,a_1,a_2;b_0,b_1,b_2),cs) = \{\zeta \in \mathbb{C} : |\zeta - a| \leq |b|\}$ from Corollary 3.3.
(b) From Table 2, we obtain

$$\sigma_{\delta}(A,X) = \sigma(A,X) \backslash I_3\sigma(A,X).$$

So using Theorems 2.8 and 2.9, the result is gotten.
(c) By Proposition 1 (e), we obtain

$$\sigma_p(A^*,X^*) = \sigma_{co}(A,X).$$

Using Theorem 2.6, the result is gotten. □

Corollary 3.6. *The following spectral decompositions are valid:*
(a) $\sigma_{ap}(U(a_0,a_1,a_2;b_0,b_1,b_2)^*,cs^* \cong bs) = \{\zeta \in \mathbb{C} : |\zeta - a_0||\zeta - a_1||\zeta - a_2| \leq |b_0||b_1||b_2|\}$,
(b) $\sigma_{\delta}(U(a_0,a_1,a_2;b_0,b_1,b_2)^*,cs^* \cong bs) = \{\zeta \in \mathbb{C} : |\zeta - a_0||\zeta - a_1||\zeta - a_2| \leq |b_0||b_1||b_2|\}$.

Proof. By Proposition 1 (c) and (d), we obtain

$$\sigma_{ap}(U(a_0,a_1,a_2;b_0,b_1,b_2)^*,cs^* \cong bv) = \sigma_{\delta}(U(a_0,a_1,a_2;b_0,b_1,b_2),cs)$$

and

$$\sigma_{\delta}(U(a_0,a_1,a_2;b_0,b_1,b_2)^*,cs^* \cong bv) = \sigma_{ap}(U(a_0,a_1,a_2;b_0,b_1,b_2),cs).$$

from Theorem 3.5 (a) and (b), the results are gotten. □

4. Results

We can generalize our operator

$$U(a_0,a_1,\dots,a_{n-1};b_0,b_1,\dots,b_{n-1}) = \begin{bmatrix} a_0 & b_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_1 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \ddots & b_2 & \ddots & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & a_{n-1} & b_{n-1} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & a_0 & b_0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & a_1 & b_1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots \end{bmatrix}$$

where $b_0,b_1,\dots,b_{n-1} \neq 0$.

Theorem 4.1. *The following results are provided where* $D = \left\{\zeta \in \mathbb{C} : \prod_{k=0}^{n-1}\left|\frac{\zeta - a_k}{b_k}\right| \leq 1\right\}$*. Also* $\mathring{D}$ *be the interior of the set* D *and* ∂D *be the boundary of the set* D

1. $\sigma_p(U(a_0,a_1,\dots,a_{n-1};b_0,b_1,\dots,b_{n-1}),cs) = \mathring{D}$,
2. $\sigma_p(U(a_0,a_1,\dots,a_{n-1};b_0,b_1,\dots,b_{n-1})^*,cs^* \cong bv) = \emptyset$,
3. $\sigma_r(U(a_0,a_1,\dots,a_{n-1};b_0,b_1,\dots,b_{n-1}),cs) = \emptyset$,
4. $\sigma_c(U(a_0,a_1,\dots,a_{n-1};b_0,b_1,\dots,b_{n-1}),cs) = \partial D$,
5. $\sigma(U(a_0,a_1,\dots,a_{n-1};b_0,b_1,\dots,b_{n-1}),cs) = D$.

5. Conclusion

There is a large literature on the spectrum and fine spectrum of certain linear operators represented by particular limitation matrices over some sequence spaces. In this article, the spectrum, fine spectrum, and approximate point spectrum, defect spectrum, and compression spectrum of the triple repetitive double-band matrix on the *cs* sequence space are calculated as subdivisions of the spectrum. Additionally, the spectrum and fine spectrum of the *n*-repetitive form in the *cs* sequence space of this matrix are also given. This is the development of the spectrum of an infinite matrix over a sequence space in the usual sense.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

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