http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 222–234 (2024)
DOI:10.31801/cfsuasmas.1225525
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received:December 28, 2022; Accepted: November 2, 2023

MULTIGRID METHODS FOR NON COERCIVE VARIATIONAL INEQUALITIES

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Abstract. In this study, our examination centers around the numerical resolution of non-coercive issues using a multi-grid approach. Our particular emphasis is directed towards employing multi-grid methodologies to tackle non-linear variational inequalities. Our primary goal involves confirming the consistent convergence of the multi-grid algorithm. To attain this objective, we make use of fundamental sub-differential calculus and glean insights from the convergence principles of non-linear multi-grid techniques.

1. Introduction

Contemporary literature showcases a diverse array of computational technique that are harnessed to address intricate real-world challenges spanning various scientific and engineering domains. These methodologies have been crafted and utilized to confront demanding problems, yielding efficient resolutions within their respective fields. Many researchers have explored these computational strategies to tackle a number of applied problems, propelling comprehension and advance understanding and progress in many scientific fields.
Commonly used numerical methods for solving boundary problems generally lead, after discretisation, to the solution of systems of algebraic equations. These numerical techniques, encompassing iterative methods like Jacobi, Gauss-Seidel iteration, and relaxation methods, are frequently chosen due to their conventional nature.However, they may show a slow convergence of fine mesh sizes and complexity when applied to general ellipticity problems. In contrast, multi-grid methods offer

2020 *Mathematics Subject Classification.* 65M55, 65N30, 49N05, 65K15, 35J86.
Keywords. Multigrid method, variational inequality, finite element, iterative method, HJB equation, non coercive operator.

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a clear advantage. These algorithms exhibit linear expenses based on the number of discretization points. These algorithms exhibit linear expenses based on the number of discretization points, regardless of the problem's dimensions. Particularly, these methods are adept at resolving linear and non-linear partial differential equations (PDEs) as well as linear V.Is (Variational inequalities)[12, 10, 7]. Their linear complexity makes them powerful tools for large problems, greatly reducing computational requirements while ensuring accurate solutions. Multi-grid techniques are widely praised as a fast approach to tackling various forms of variational equations and inequalities [11], particularly in the area the discretized elliptic problems that leads to an M-matrix [6].
Through a conforming finite element method P_1 [4], we will be providing an overview of non-linear variational inequalities (N.V.I) problems and their discretization in the following section. Additionally, The Hoppe multi-grid method [14, 9] served as an inspiration for our algorithm, which views the V.I as stationary Hamilton-Jacobi-Bellman(H.J.B) equations. The iteration matrices are provided for an algorithm known as the M.G.H.J.B, or multi-grid Hierarchy Jacobi.
First, we present original results on the approximation and smoothness properties within the L^∞ norm. We then demonstrate the consistent convergence of the M.G.H.J.B algorithm. Finally, we apply the numerical method to a specific scenario where the operator is linear and unconstrained, and the second element is independent of the solution. In this context, we implemented the Gauss-Seidel method and the multigrid method V and W cycles. Numerical experiments are performed to evaluate the efficiency and performance of these methods in solving the proposed problem.

2. Multigrid Method

2.1. **Assumptions and Notations.** Suppose that Ω is an open in $\mathbb{R}^N$ with a sufficiently regular border $\partial\Omega$.
We define second order operators with $u, v \in H^1(\Omega)$,

$$\mathfrak{A} = \sum_{1\leq j,k\leq N} \mathfrak{d}_{jk}(x)\frac{\partial^2}{\partial x_j \partial x_k} + \sum_{k=1}^{N} \mathfrak{b}_k(x)\frac{\partial}{\partial x_k} + \mathfrak{b}_0(x),$$

where $\mathfrak{d}_{jk}(x), \mathfrak{b}_k(x), \mathfrak{b}_0(x)$ are sufficiently regular coefficients such that:

$$\mathfrak{d}_{kj}(x) = \mathfrak{d}_{jk}(x), \quad \mathfrak{b}_0(x) \geq \beta > 0; \quad (x \in \Omega).$$

Also, we define the associated bilinear non-coercive forms

$$\mathfrak{a}(u,v) = \int_\Omega \left(\sum_{1\leq j,k\leq N} \mathfrak{d}_{jk}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_k} + \sum_{k=1}^{N} \mathfrak{b}_k \frac{\partial u}{\partial x_k} v + \mathfrak{b}_0(x)uv \right) dx,$$

and the operators

$$\mathcal{B}=\sum_{1\leq j,k\leq N}\eth_{jk}(x)\frac{\partial^2}{\partial x_j\partial x_k}+\sum_{k=1}^{N}\mathfrak{b}_k(x)\frac{\partial}{\partial x_k}+(\mathfrak{b}_0(x)+\lambda)\,, \tag{1}$$

we choose $\lambda>0$ is sufficiently large so that $\mathcal{B}=\mathfrak{A}+\lambda I$ are strongly elliptic on $H^1(\Omega)$ and

$$\mathfrak{b}(u,v)=\mathfrak{a}(u,v)+\lambda(u,v). \tag{2}$$

Additionally, we consider f a second member as following:

$$f\in L^\infty(\Omega);\quad f\geq 0$$

and obstacle $\psi\in W^{2,\infty}$, where $\psi>0$.

2.2. **Problem Continuous.** The aim is to find u the solution of the problem presented by the following V.Is:
Find u solution of:

$$\begin{cases}\mathfrak{b}(u,v-u)\geq(f+\lambda u,v-u), & \forall v\in H^1(\Omega),\\ u\leq\psi;\quad v\leq\psi.\end{cases} \tag{3}$$

It has been confirmed that this issue has a singular solution, as demonstrated by the theorem of fixed point and from the aforementioned assumptions (see [1]).

2.3. **Discretization.** In order to build a multi-grid loop, we create a sequence of discretization steps referred to as $0<\mathfrak{h}_{k+1}<\mathfrak{h}_k<1$ such that the grids are nested $\mathfrak{h}_{k+1}=\frac{\mathfrak{h}_k}{2}$.
Subsequently, we delineate $\Omega_k=\Omega_{\mathfrak{h}_k},V_k=V_{\mathfrak{h}_k},\mathfrak{A}_k=\mathfrak{A}_{\mathfrak{h}_k}$ and we establish a series of uniform regular triangulations referred to as $\{T_k,k\in\mathbb{N}_0\}$. For all T_k,we have

$$\begin{gathered}\Omega_k\subset\Omega_{k+1}\subset\Omega,\\ \operatorname{dist}(\partial\Omega_k,\partial\Omega)\leq c_0\mathfrak{h}_k^2,\\ \mathfrak{h}_k\mathfrak{h}_{k+1}\leq c_1.\end{gathered}$$

We introduce $V_{\mathfrak{h}_k}=\left\{v_{\mathfrak{h}_k}\in C(\Omega)\cap H^1\ ;v_{\mathfrak{h}_k}/_T\in P_1\right\}$, for simplicity we write:

$$V_k=\left\{v_k\in C(\Omega)\cap H^1;\quad v_k/r\in P_1\right\}.$$

The shape function $\varphi_k^i,i\in(1,\ldots,m(\mathfrak{h}_k))$ of the usual basis is defined as: $\varphi_k^i\left(x_k^j\right)=\delta_{ij}$, where x_k^j be a node of the T_k triangulation .
So, the ordinary restriction operator r_k is defined like:

$$r_kv(x)=\sum_{i=1}^{m(\mathfrak{h}_k)}v\left(M_k^i\right)\varphi_k^i(x). \tag{4}$$

If we suppose $U_k = \mathbb{R}^{m_k}$. Then, $r_k : U_k \to V_k$ is a bijection.
U_k is equipped with the scalar product

$$< u, v >= \mathfrak{h}_k^2 \sum_{i=1}^{m(\mathfrak{h}_k)} u_i v_i, \quad \|u\|_k =< u, v >_k^{1/2} .$$

The maximum norms in U_k and V_k are equivalent, we denote them $\|\cdot\|_\infty$. We have the following lemma (see [2]).

Lemma 1. *There exists C_1, C_2 independent of k such that*

$$\begin{array}{ll} \|r_k(u)\|_\infty & = \|u\|_\infty, \quad \forall u \in U_k. \\ C_1\|v\|_\infty & \leq \|r_k^*(v)\|_\infty \leq C_2\| v\|_\infty, \quad \forall v \in V_k. \end{array} \tag{5}$$

2.4. **Problem Discrete.** Continuing in a logical sequence, we present the discretization matrices $\mathcal{B}_k$ and the bilinear form $b\left(\varphi_k^1, \varphi_k^s\right)$, where φ_s the shape functions. With these descriptions established. Now, we are positioned to formulate the discrete problem in the subsequent manner:
Find $u_k \in V_k$ solution of:

$$\left\{ \begin{array}{l} < \mathcal{B}_k u_k, v_k - u_k >\geq< f_k + \lambda u_k, v_k - u_k >, \quad \forall v_k \in V_k \\ u_k \leq r_k\psi, \quad v_k \leq r_k\psi \end{array} \right. \tag{6}$$

We make the assumption that the matrices $\mathcal{B}_k$ are M-matrices.(see [3]).

2.5. **H.J.B form.** The correspondence between the finite-dimensional V.I (3) and a representation in Hamilton-Jacobi-Bellman (H.J.B) form is easily discernible (see [10]). We detail the selected numerical technique for resolving the stationary H.J.B equations.
In the traditional framework, we recollect certain convergence outcomes that will play a crucial role in affirming the M.G.H.J.B algorithm's convergence expounded in the following:
Iterative diagram:
Step 1: Choose $u_k^0 \in \mathbb{R}^{n_k}$ as initial vector.
Step 2 : Calculate the solution $u_k^{\nu+1} \in \mathbb{R}^{n_k}$ of the following recurrence equation

$$\mathcal{B}_k^\nu u_k^{\nu+1} - Z_k^\nu = 0, \tag{7}$$

such that

$$Z_k^\nu = F_k^\nu + \lambda u_k^\nu$$

where

$$\mathcal{B}_{k,i}^\nu = \left\{ \begin{array}{ccc} \mathcal{B}_{k,i}\left(u_k\right) & \text{if} & \mathcal{B}_{k,i}u_{k,i}^\nu - Z_{k,i} > u_{k,i}^\nu - \psi_{k,i}, \\ u_{k,i} & \text{if} & 1 \leq i \leq N, \end{array} \right. \tag{8}$$

$$Z_{k,i}^\nu = \left\{ \begin{array}{ccc} Z_{k,i} & \text{if} & \mathcal{B}_{k,i}u_{k,i}^\nu - Z_{k,i} > u_{k,i}^\nu - \psi_{k,i}, \\ u_{k,i} & \text{if} & 1 \leq i \leq N. \end{array} \right. \tag{9}$$

Let the discrete H.J.B equation where u_k^* be the unique solution

$$\max_{1\leq i\leq N}\left(\mathcal{B}_{k,i}u_k^* - Z_{k,i}, u_{k,i}^* - \psi_{k,i}\right) = 0. \tag{10}$$

We will formulate the subsequent theorem and introduce our problem derived from the (H.J.B) equation, drawing inspiration from Hoppe's [10].

Theorem 1. *Let u_k^ν be the solution in the iteration defined and it satisfies the H.J.B equation. Furthermore, We make that $\mathcal{B}_k$ is continuously differentiable then the sequence $(u_k^\nu)_{\nu\geq 0}$ converges and approaches u_k^*.*

Previously moving forward with presenting the findings, it is relevant to revisit the subsequent theorem:

Theorem 2. *(see* [1] , [5]*) If the previous notations and assumptions are satisfies. So , we have:*

$$\|u - u_k^*\|_\infty \leq C\mathfrak{h}_k^2 \left|\log \mathfrak{h}_k\right|^2 \|g(u)\|_\infty. \tag{11}$$

2.6. **Multi-grid (M.G.H.J.B) algorithm for V.Is.** For the multi-grid method we choose an iteration $u_k^\nu, \nu > 0$.So, we obtain $\bar{u}_k^\nu$, by using an iterative method to solve the system (7) by α

$$\bar{u}_k^\nu = S_k^\alpha\left(u_k^\nu\right) \tag{12}$$

where S_k is the smoothing operator and α is the number performed of iterations. The solution of (7) is denoted by u_k^*. The error setting $e_k^\nu = \bar{u}_k^\nu - u_k^*$, and the residual $d_k^{(\nu)} = Z_k^\nu - \mathcal{B}_k^\nu\bar{u}_k^\nu$, the equation (7) can be write as

$$\mathcal{B}_k^\nu\left(\bar{u}_k^\nu + e_k^\nu\right) = Z_k^\nu.$$

This leads to the residual equation

$$\mathcal{B}_k^\nu e_k^\nu = Z_k^\nu - \mathcal{B}_k^\nu\bar{u}_k^\nu = d_k^\nu.$$

After the relaxation on $\mathcal{B}_k^\nu\bar{u}_k^\nu = Z_k^\nu$ on the fine grid, the error will display a continuous nature. However, the error on the coarse grid appears to be more oscillatory, leading to the relaxation. At the $(k-1)$ level, we need to compute e_{k-1}^ν for determine e_k^ν, where e_{k-1}^ν is the solution of the coarse grid system

$$\mathcal{B}_{k-1}^\nu e_{k-1}^\nu = d_{k-1}^\nu. \tag{13}$$

We can interpret $e_{k-1}^\nu\left(\text{resp}\,\mathcal{B}_{k-1}^\nu, d_{k-1}^\nu\right)$ and $e_k^\nu\left(\text{resp}\,\mathcal{B}_k^\nu, d_k^\nu\right)$ as approximation operator at level $(k-1)$ and (k) respectively. Additionally, we have $\mathcal{R}_k$ the restriction operator and $\mathcal{P}_k$ its reverse .
consecquently, at the (k) level we identify an improved iteration

$$u_k^{\nu+1} = \bar{u}_k^\nu + \mathcal{P}_k\left(e_{k-1}^\nu\right). \tag{14}$$

Because of the nested structure, we employ the well-defined identity operator

$$\pi : V_{k-1} \longrightarrow V_k; \quad \pi v = v,$$

the operators of extension and restriction define like

$$\mathcal{P}_k = r_k^{-1}r_{k-1}, \quad \mathcal{R}_k = \mathcal{P}_k^t. \tag{15}$$

2.7. **Matrix of the M.G.H.J.B Algorithm.** For each iteration, The matrix of the two-grid method with α_1 pre-smoothing and α_2 post-smoothing iterations at the (k) level is given by

$$TG_k\left(\alpha_1,\alpha_2\right) = S_k^{\alpha_2}\left(\left(\mathcal{B}_k^{\nu}\right)^{-1} - \mathcal{P}_k\left(\mathcal{B}_{k-1}^{\nu}\right)^{-1}\mathcal{R}_k\right)\left(\mathcal{B}_k^{\nu}\right)S_k^{\alpha_1}. \tag{16}$$

Theorem 3. *(see* [13] *)The multi-grid technique embodies a linear iterative approach, with the iteration matrix referred to as MG_k*

$$\begin{aligned} MG_0 &= 0, \\ MG_k &= S_k^{\alpha_2}\left(I_k - \mathcal{P}_k\left(I_k - MG_{k-1}\right)\left(\mathcal{B}_{k-1}^{\nu}\right)^{-1}\mathcal{R}_k\right)\left(\mathcal{B}_k^{\nu}\right)S_k^{\alpha_1}, \\ &= TG_k + S_k^{\alpha_2}\mathcal{P}_k MG_{k-1}\left(\mathcal{B}_{k-1}^{\nu}\right)^{-1}\mathcal{R}_k\left(\mathcal{B}_k^{\nu}\right)S_k^{\alpha_1}, \quad k = 1,2,.. \end{aligned} \tag{17}$$

3. Convergence of the Multi-grid algorithm in L^∞-norm

This section is devoted to presenting a unified convergence analysis of multi-grid algorithm. To prove the convergence, we need the following proprieties

3.1. Approximation property.

Theorem 4. *(see* [8] *) The matrix $\Upsilon_k = \left[\left(\mathcal{B}_k^{\nu}\right)^{-1} - \mathcal{P}_k\left(\mathcal{B}_{k-1}^{\nu}\right)^{-1}\mathcal{R}_k\right]$ has the approximation property*

$$\left\|\Upsilon_k\right\|_\infty \leq Ch_k^2\left|\ln h_k\right|^2. \tag{18}$$

Proof. The proof was proposed by Arnold in [14] on Theorem 1. □

3.2. **Property of Smoothing.** To prove the smoothness property, we consider the decomposition $\mathcal{B}_k^{\nu} = E_k - N_k$ and using the following assumptions:
for all k

$$E_k \text{ is regular and } \left\|E_k^{-1}N_k\right\|_\infty \leq 1,. \tag{19}$$

$$\left\|E_k\right\|_\infty \leq Ch_k^{-2}, \text{ with C independent of k.} \tag{20}$$

In the process of smoothing, we utilize a relaxation method with an iterative matrix

$$S_k = I_k - \omega E_k^{-1}N_k, \quad \omega \in (0,1).$$

For the following theorem, the concept of Arnold Reusken [14] is relevant to our work.

Theorem 5. *Under the previous assumptions, there exists a constant C, which is independent of both k and α. Such that:*

$$\left\|\left(\mathcal{B}_k^{\nu}\right)S_k^{\alpha}\right\|_\infty \leq C\frac{1}{\sqrt{\alpha}}h_k^{-2}. \tag{21}$$

(smoothness properties)

By switching to the norm in (14), from (18) and (21) we can proving the following estimation:

$$\exists C_s : \|S_k^{\alpha}\|_{\infty} \leq C_s, \text{ for all } k \text{ and } \alpha. \tag{22}$$

From the equation (16) with two lattices iterate (two-grid) and $\alpha_2 = 0$, we have the following estimate:

$$\begin{aligned}\|TG_k(\alpha_1, 0)\|_{\infty} &= \left\|\left((\mathcal{B}_k^{\nu})^{-1} - \mathcal{P}_k \left(\mathcal{B}_{k-1}^{\nu}\right)^{-1} \mathcal{R}_k\right) (\mathcal{B}_k^{\nu}) S_k^{\alpha_1}\right\|_{\infty} \\ &\leq \left\|\left((\mathcal{B}_k^{\nu})^{-1} - \mathcal{P}_k \left(\mathcal{B}_{k-1}^{\nu}\right)^{-1} \mathcal{R}_k\right)\right\|_{\infty} \|(\mathcal{B}_k^{\nu}) S_k^{\alpha_1}\|_{\infty}.\end{aligned}$$

Typically, we choose a hierarchy of more than two-grids. in this case, we can define the iterative matrices (17) by the recurrence of (16) for all (k) levels.

Theorem 6. *([13]) Consider a multi-grid method for a given iterative matrix (17). Then under the previous assumption, for the parameter value $\alpha_2 = 0, \alpha_1 = \alpha > 0, \tau \geq 2$. For each $\zeta \in (0,1)$ there is aα^*such that for all $\alpha \geq \alpha^*$*

$$\|MG_k\|_{\infty} \leq \zeta, k = 0, 1, ... \tag{23}$$

hold.

Proof. If the previous properties are related with (22), then we can stratify the same steps as in [[13] , Theorem 7.20]. □

The main result of our study was in the following theorem.

Theorem 7. *For two meshes (k) and $(k-1)$ and the previous given the iterated $u_k^{\nu}, \nu \geq 0$ satisfy:*

$$\left\|u_k^{\nu+1} - u_k^*\right\|_{\infty} \leq \left(\frac{C}{\sqrt{\alpha}} \left|\mathrm{Logh}_k\right|^2\right) \|u_k^{\nu} - u_k^*\|_{\infty}. \tag{24}$$

Proof. We have

$$\begin{aligned}\left\|u_k^{\nu+1} - u_k^*\right\|_{\infty} = &\quad \left\|\left(\left(I_k - \mathcal{P}_k \left(I_k - MG_{k-1}\right) \left(\mathcal{B}_{k-1}^{\nu}\right)^{-1} \mathcal{R}_k\right) (\mathcal{B}_k^{\nu}) S_k^{\alpha_1}\right) (u_k^{\nu} - u_k^*)\right\|_{\infty} \\ \leq &\quad \left\|\left(I_k - \mathcal{P}_k \left(I_k - MG_{k-1}\right) \left(\mathcal{B}_{k-1}^{\nu}\right)^{-1} \mathcal{R}_k\right)\right\|_{\infty} \|\mathcal{B}_k^{\nu} S_k^{\alpha_1}\|_{\infty} \|u_k^{\nu} - u_k^*\|_{\infty} \\ \leq &\quad \left(\frac{C_2}{\sqrt{\alpha}} h_k^{-2}\right) \left(C_1 h_k^2 \left|\log h_k\right|^2\right) \|(u_k^{\nu} - u_k^*)\|_{\infty} \\ \leq &\quad \left(\frac{C_1 C_2}{\sqrt{\alpha}}\right) \left|\log h_k\right|^2 \|u_k^{\nu} - u_k^*\|_{\infty}\end{aligned}$$

□

4. Numerical Simulation

In this part, we applied this method to the numerical example of a non-linear variational inequality.
We suppose that the problem to be sufficiently smooth data and we apply the dynamic programming principle of Bellman, then we solve (3) as we discussed before, using the following datas:

- **Mixed operator**

$$\begin{cases} \mathcal{B}u \geq f, \quad \text{in } \Omega = [0,1]^2 \\ \langle \mathcal{B}u - f, u - \psi \rangle = 0, \\ u \leq \psi, \\ u = 0, \quad \text{in } \partial\Omega. \end{cases} \tag{25}$$

Where

$$\begin{aligned} \mathcal{B}u \quad &= -\Delta u - 0.02 \frac{\partial^2 u}{\partial x \partial y} + 0.15 \tfrac{\partial u}{\partial x} + 0.1 \tfrac{\partial u}{\partial y} + (1+\lambda)u, \\ f \quad &= \sin(\pi x)\sin(2\pi y) sin(\pi(x+y)) + \lambda u, \\ \lambda \quad &= 2, \\ \psi \quad &= 0. \end{aligned}$$

We are constrain ourselves to the discretization of finite element method with a uniform triangulation and P_1 shape functions. For the domain, we have decretized by Matlab PDE toolbox (Matlab R2017b) for mesh generation. We solve the equation (25) by the M.G with 64 triangle and 41 nodes in the domain. This numerical illustration is performed to showcase the high efficiency of the M.G method. For the pre/post-smoothing of the M.G, we choose the Gauss-Seidel (G.S) method. The degrees of freedom chooses lower than 5 (recursion number of M.G method). Figure 1 illustrates the convergence behaviour of the M.G solver (green and red curves of M.G (V and W cycle)) with respect to the number of iterations performed. For comparison, the convergence behavior of Gauss-Seidel (blue curves) are included.

Norm of residual obtained after 100 iterations :

by Gauss Seidel method	by multi-grid V-cycle	by multi-grid W-cycle
$4.058087199609872\mathrm{e}^{-12}$	$4.440892098500626\mathrm{e}^{-16}$	$4.440892098500626\mathrm{e}^{-16}$

We have applied the Matlab-backslash-operator(M.B.O), G.S and the M.G (V and W-cycle) are carried out on the finest grid (41 grids) and on the coarsest one (4 nodes) then we get the solutions in figures 2.

Norm of residual obtained after 20 iterations :

by Gauss Seidel method	by multi-grid V-cycle	by multi-grid W-cycle
0.001165086612534	$4.440892098500626\mathrm{e}^{-16}$	$4.440892098500626\mathrm{e}^{-16}$

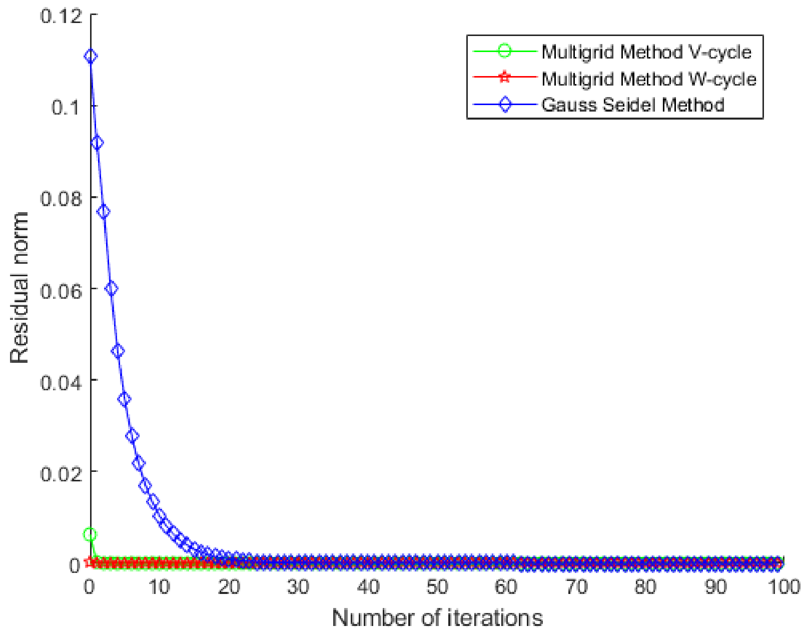

FIGURE 1. Comparison between the convergence of maximum residual norm by M.G and G.S.

• **Simple operator**

$$\begin{cases} \mathcal{B}u \geq f, \quad \text{in } \Omega = [0,1]^2 \\ \langle \mathcal{B}u - f, u - \psi \rangle = 0, \\ u \leq \psi, \\ u = 0, \quad \text{in } \partial\Omega. \end{cases} \tag{26}$$

Where

$$\begin{array}{ll} \mathcal{B}u & = -\Delta u + 0.5x\dfrac{\partial u}{\partial x} + 0.5y\frac{\partial u}{\partial y} + (0.045 + \lambda)u, \\ f & = \sin(2\pi x)\sin(2\pi y) + \lambda u, \\ \lambda & = 1, \\ \psi & = 0. \end{array}$$

With the same steps,we have:

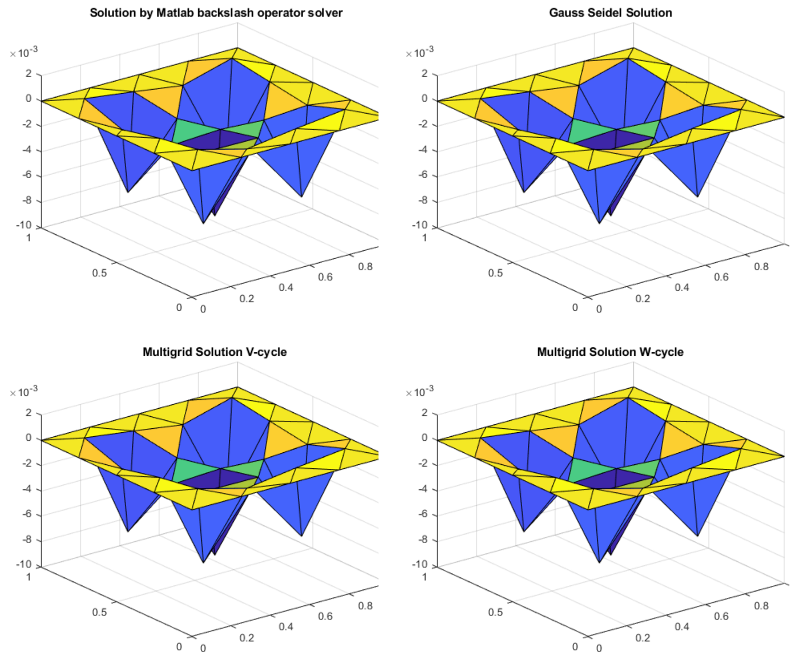

FIGURE 2. Solution of (25) on fine grid using M.B.O, G.S, M.G V-cycle and W-cycle after 100 iterations.

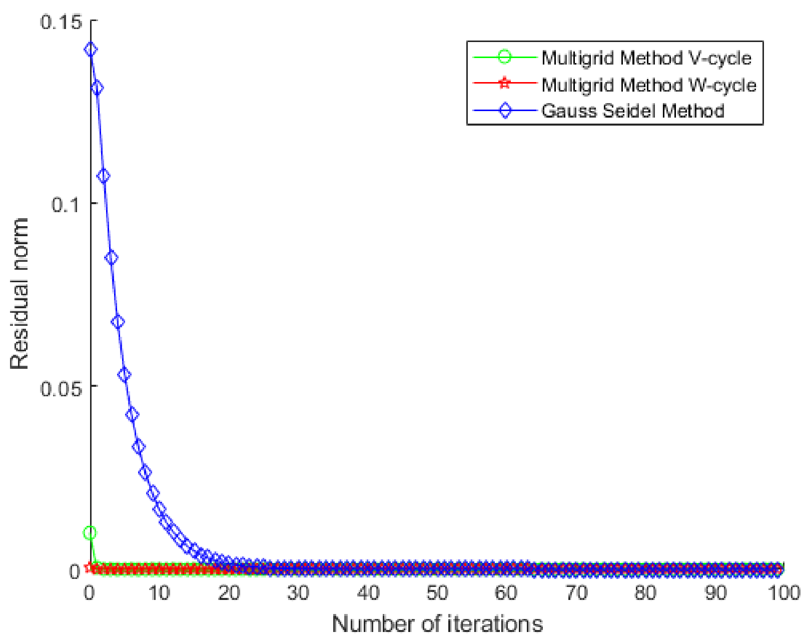

FIGURE 3. Comparison between the convergence of maximum residual norm by M.G and G.S.

Norm of residual obtained after 100 iterations :

by Gauss Seidel method	by multi-grid V-cycle	by multi-grid W-cycle
$1.076361222374089e^{-11}$	$2.220446049250313e^{-16}$	$2.220446049250313e^{-16}$

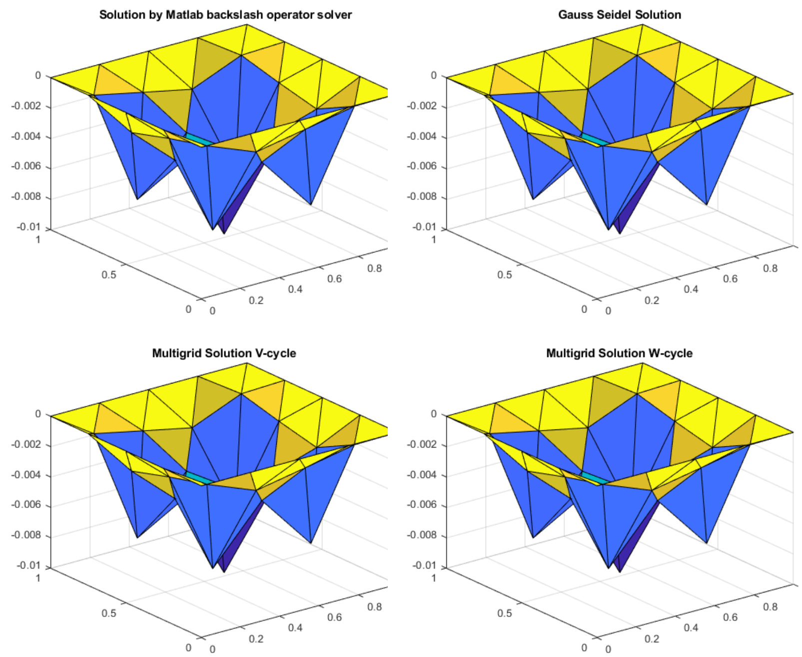

FIGURE 4. Solution of (26) on fine grid using M.B.O, G.S, M.G V-cycle and W-cycle after 100 iterations.

Norm of residual obtained after 10 iterations :

by Gauss Seidel method	by multi-grid V-cycle	by multi-grid W-cycle
0.020709274936256	$4.884981308350689e^{-15}$	$2.220446049250313e^{-16}$

Remark 1. *Should we conduct more than 10 iterations, the M.G approach emerges as the optimal method.*

4.1. **Conclusion.** Discretizing elliptic V.I. via efficient iterative solutions is the main focus of our study, employing algebraic M.G. The goal is to tackle loop domains' discretization using adaptive finite element approximation. Once discretization is complete, we successfully apply M.G to address the discrete problems at hand. Our main objective is to establish uniform convergence through our approach, and our research demonstrates the M.G's significant reduction in iteration count compared to the maximum norm method.

By means of numerical experimentation, we have constructed an example of a variational inequality. Our results indicate that the G.S. method, despite a substantial number of iterations, is unsuccessful in producing satisfactory outcomes. On the other hand, through the use of an error-damping mechanism that reduces high-frequency errors and transfers low-frequency errors to a coarser grid for alleviation, M.G. significantly enhances convergence and achieves it within a limited number of iterations. Our team recognizes the exceptional potential for further development using these methodologies.
Our numerical solution could be even more efficient and scalable if we explore the prospect of applying a parallel full M.G to surmount unconstrained elliptical inequalities. This avenue presents an interesting opportunity to cater to a broader range of problem domains.

Author Contribution Statements All authors contributed equally in the writing of this paper. All authors read and approved the final manuscript.

Declaration of Competing Interests The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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