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On Analogues of Nilpotent and Total Graphs Associated Ring Z_n

Hatice Pınar Cantekin[1,a*], Sezer Sorgun[1,b]

[1]Department of Mathematics, Nevsehir Hacı Bekta¸s Veli University, Nevsehir 50300, Turkey.
*Corresponding author

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ABSTRACT

In the last 20 years, the graphs associated a ring have been introduced such as the zero-divisor graph, the nilpotent graph and the total graph, etc. The studies on these graphs are generally related to their graph invariants (diameter, girth etc.). In this paper, we define two novel analogue graphs over the integer rings and obtain some properties as well as the spectrum with respect to the adjacency matrix.

Key Words: *Graph, Nilpotent Graph, Total Graph, Zero Divisor, Spectrum*
2000 Mathematics Subject Classification: *05C50;05C75*

Z_n Halkası Üzerinde Tanımlı Nilpotent ve Total Çizgelerin Analogları Üzerine

Süreç

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ÖZ

Son 20 yıldır, sıfır-bölen çizge, nilpotent çizge ve total çizge gibi bir halka ile oluşturulmuş graflar tanıtılmıştır. Bu yapılar üzerine yapılan çalışmalar genellikle çizgenin çapı, en kısa döngü sayısı vb. çizge değişmezleri çalışmalarıdır. Bu çalışmada tam sayılar halkası üzerinde iki analog çizge tanıtılmış ve komşuluk matrisine göre bazı spektral özellikleri incelenmiştir.

Anahtar Kelimeler: *Çizge, Nilpotent Çizge, Total Çizge, Sıfır-Bölen, Spektrum.*
2000 Mathematik Konu Sınıflandırması: *05C50;05C75*

[a] pinarcantekin007@gmail.com https://orcid.org/0000-0002-9692-275X [b] srgnrzs@gmail.com https://orcid.org/0000-0001-8708-1226

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Introduction

Throughout the paper we consider only simple graphs. Let $G = (V, E)$ be a graph on vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G) = \{v_i v_j : v_i, v_j \in V\}$. Also let d_i be the degree of vertex v_i for $i = 1, 2, \ldots, n$. The minimum vertex degree is denoted by $\delta = \delta(G)$ and the maximum by $\Delta = \Delta(G)$. Let N_i be the neighbour set of the vertex $v_i \in V(G)$. If vertices v_i and v_j are adjacent, we denote that by $v_i v_j \in E(G)$ or $v_i \sim v_j$. The complete graph K_n is a graph such that all degrees of the vertices are $n - 1$. A complete bipartite graph $K_{m,n}$ is a graph whose vertices can be partitioned into two disjoint subsets U and W such that $V = U \cup W$ and for every $u \in U$ and $\in W$, uw is an edge in E. For the graphs G and H, $G + H$ is also a graph and called the union of G and H.

The adjacency matrix $A(G)$ of G is defined by its entries $a_{ij} = 1$ if $v_i v_j \in E(G)$ and 0 otherwise. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ denote the distinct eigenvalues of $A(G)$. The multiset of the eigenvalues is known as the spectrum and is shown by $Spec_A(G) = \{\lambda_1^{(m_1)}, \ldots, \lambda_k^{(m_k)}\}$ where m_i is the algebraic multiplicity of λ_i.

In recent years, the graphs associated the rings have become one of the interesting research topics. The main questions arising in the studies are as 'Can a graph be defined on different elements of the group or ring from commonly used algebraic structures?' and 'Which graph family does the obtained graph structure belong to?'. In view of these questions, various graphs associated a ring can be found in [Anderson and Livingston, 1999; Anderson, Levy and Shapiro, 2003; Anderson and Badawi, 2008; Anderson, Asir, Badawi and Chelvam, 2021; Li and Li, 2010; Nikmehr and Khojasteh, 2013; Singh and Bhat, 2020]. In addition to pure graph theory studies, there are also the studies in terms of graph spectra (see also [Bajaj and Panigrahi, 2022; Cantekin and Sorgun, 2017; Chattopadhyay, Patra and Sahoo, 2020; Pirzada, Rather, Aijaz and Chishti,2022; Pirzada, Rather, Sbahan and Chishti, 2023]). In this paper, we define analogues of total graph and nilpotent graph and obtain the graph structure. We also present the spectrum with respect to the adjacency matrix.

Main Results

Lemma 2.1. *[Brouwer and Haemers, 2012]*

1. For the complete graph K_m, we have $Spec_A(K_m) = \{(m-1)^1, (-1)^{m-1}\}$.
2. For the complete bipartite graph $K_{m,n}$, $Spec_A(K_{m,n}) = \left\{(\sqrt{mn})^{(1)}, (-\sqrt{mn})^{(1)}, 0^{(m+n-2)}\right\}$.
3. Let G and H be graphs. Then $Spec_A(G + H) = Spec_A(G) \cup Spec_A(H)$.

Lemma 2.2.

[Brouwer and Haemers, 2012] Let Q be a quotient matrix of any square matrix A corresponding to an equitable partition. Then the spectrum of A contains the spectrum of Q.

Definition 2.3.

Let R be a commutative ring with unity such that the set of zero- divisor elements and the set of the nilpotent elements of R are $Z(R)$ and $N(R)$, respectively.

1. The nilpotent-divisor graph $\Gamma_{ND}(R)$ is the graph such that for any two distinct vertices x and y in $Z(R) \setminus N(R)$ are adjacent if and only if xy is nilpotent.
2. Let R be a commutative ring with unity. The nilpotent-total graph $\Gamma_{NT}(R)$ is the graph such that for any two distinct vertices x and y in $Z(R) \setminus N(R)$ are adjacent if and only if $x + y$ is nilpotent.

Lemma 2.4.

Let $\mathbb{Z}_n$ be an integer ring such that $n = \sum_{i=1}^{3} {p_i}^{m_i}$ (p_i's prime numbers). Then the vertex of set $\Gamma_{ND}(\mathbb{Z}_n)$ has six pairwise disjoint zero divisor sets except the nilpotent elements of it and the cardinalities of the sets as the following:

$$c_1 = |\overline{p_2p_3} \setminus \overline{p_1p_2p_3}| = {p_2}^{m_2-1}{p_3}^{m_3-1}\Phi({p_1}^{m_1})$$
$$c_2 = |\overline{p_1p_2} \setminus \overline{p_1p_2p_3}| = {p_1}^{m_1-1}{p_2}^{m_2-1}\Phi({p_3}^{m_3})$$
$$c_3 = |\overline{p_1p_3} \setminus \overline{p_1p_2p_3}| = {p_1}^{m_1-1}{p_3}^{m_3-1}\Phi({p_2}^{m_2})$$
$$c_4 = |\overline{p_1} \setminus \overline{p_1p_2} \cup \overline{p_1p_3}| = {p_1}^{m_1-1}\Phi({p_2}^{m_2})\Phi({p_3}^{m_3})$$
$$c_5 = |\overline{p_3} \setminus \overline{p_1p_3} \cup \overline{p_2p_3}| = {p_3}^{m_3-1}\Phi({p_1}^{m_1})\Phi({p_2}^{m_2})$$
$$c_6 = |\overline{p_2} \setminus \overline{p_1p_2} \cup \overline{p_2p_3}| = {p_2}^{m_2-1}\Phi({p_1}^{m_1})\Phi({p_3}^{m_3}).$$

Proof. We have $N(\mathbb{Z}_n) = \{p_1p_2p_3, 2(p_1p_2p_3), \ldots, ({p_1}^{m_1-1}{p_2}^{m_2-1}{p_3}^{m_3-1})(p_1p_2p_3)\}$.

Also from the definition of nilpotent divisor graph the vertices of the graph are form of $\overline{p_i}, \overline{p_ip_j}$. For distinct i, j, k all sets which include the zero-divisor elements of $\mathbb{Z}_n$ are

$$\overline{p_i} = \{p_i, 2p_i, \ldots, ({p_i}^{m_i-1}{p_j}^{m_j}{p_k}^{m_k})p_i\} \quad (1)$$
$$\overline{p_ip_j} = \{p_ip_j, 2p_ip_j, \ldots, ({p_i}^{m_i-1}{p_j}^{m_j-1}{p_k}^{m_k})p_ip_j\} \quad (2)$$
$$\overline{p_ip_jp_k} = N(\mathbb{Z}_n) \quad (3)$$

From the definition of the graph, we get $p_i \sim p_jp_k$ and $p_ip_j \sim p_ip_k$ for all distinct i, j, k. Hence since we have $\overline{p_ip_jp_k} \subset \overline{p_ip_j} \subset \overline{p_i}$, the distinct set of the zero-divisors of the ring and hence the cardinalities are

$$|\overline{\mathrm{U}p_i}| = |\overline{p_i} \setminus (\overline{p_ip_j} \cup \overline{p_ip_k} \setminus \overline{\mathrm{U}p_ip_jp_k})| \quad (4)$$
$$|\overline{\mathrm{U}p_ip_j}| = |\overline{p_ip_j} \setminus \overline{p_ip_jp_k}| \quad (5)$$

Hence, from the sets in (1-3), we get

$$\begin{aligned}|\overline{\mathrm{U}p_i}| &= {p_i}^{m_i-1}{p_j}^{m_j}{p_k}^{m_k} - ({p_i}^{m_i-1}{p_j}^{m_j-1}{p_k}^{m_k} + {p_i}^{m_i-1}{p_j}^{m_j}{p_k}^{m_k-1} - {p_i}^{m_i-1}{p_j}^{m_j-1}{p_k}^{m_k-1}) \\ &= {p_i}^{m_i-1}({p_j}^{m_j}{p_k}^{m_k} - {p_j}^{m_j-1}{p_k}^{m_k} - {p_j}^{m_j}{p_k}^{m_k-1} + {p_j}^{m_j-1}{p_k}^{m_k-1}) \\ &= {p_i}^{m_i-1}\left({p_j}^{m_j}({p_k}^{m_k} - {p_k}^{m_k-1}) - {p_j}^{m_j-1}({p_k}^{m_k} - {p_k}^{m_k-1})\right) \\ &= {p_i}^{m_i-1}\Phi({p_j}^{m_j})\Phi({p_k}^{m_k}) \quad (6)\end{aligned}$$

and

$$\begin{aligned}|\overline{\mathrm{U}p_ip_j}| &= {p_i}^{m_i-1}{p_j}^{m_j}{p_k}^{m_k} - {p_i}^{m_i-1}{p_j}^{m_j-1}{p_k}^{m_k-1} \\ &= {p_i}^{m_i-1}{p_j}^{m_j-1}({p_k}^{m_k} - {p_k}^{m_k-1})\end{aligned}$$

$$= p_i^{m_i-1} p_j^{m_j-1} \Phi(p_k^{m_k}) \qquad (7)$$

Permuting i, j, k in the sets, the other cardinalities is provided similarly.

Theorem 2.5.

Let $\mathbb{Z}_n$ be an integer ring such that $n = \sum_{i=1}^{3} p_i^{m_i}$. Then

$$P_{A(\Gamma_{ND}(\mathbb{Z}_n))}(x) = x^{\sigma-6}(x^6 - u_1x^4 - 2u_2x^3 + u_3x^2 - u_4) \qquad (8)$$

where σ is the number of vertex of the graph and $u_1 = \rho\phi(n)(S+3)$, $u_2 = \rho^2\phi(n)$, $u_3 = \rho^2\phi(n)^2(S+3)$, $u_4 = \rho^3\phi(n)^3$ such that ρ is the number of nilpotent elements and

$$S = \frac{p_1^{m_1-1}}{\Phi(p_1^{m_1})} + \frac{p_2^{m_2-1}}{\Phi(p_2^{m_2})} + \frac{p_3^{m_3-1}}{\Phi(p_3^{m_3})}.$$

Proof. Let $n = \sum_{i=1}^{3} p_i^{m_i}$. From Lemma 2.4, we get

$$|V| = |\overline{p_2p_3} \setminus \overline{p_1p_2p_3}| + |\overline{p_1p_2} \setminus \overline{p_1p_2p_3}| + |\overline{p_1p_3} \setminus \overline{p_1p_2p_3}| + |\overline{p_1} \setminus \overline{p_1p_2} \cup \overline{p_1p_3}| + |\overline{p_3} \setminus \overline{p_1p_3} \cup \overline{p_2p_3}| + |\overline{p_2} \setminus \overline{p_1p_2} \cup \overline{p_2p_3}|$$

Let's say $|V| = \sigma$ and $|N(\mathbb{Z}_n)| = \rho$. For every $x \in \overline{p_i} \setminus (\overline{p_ip_j} \cup \overline{p_ip_k})$ and $y \in \overline{p_i}$ we get $x \sim y$. Also $x \sim y$ for all $x \in \overline{p_ip_j}$ and $y \in \overline{p_ip_k}$. Hence the graph has six partitions as seen Figure. The adjacency matrix of the graph can be blocked as

$$A(\Gamma_{ND}(\mathbb{Z}_n)) = \begin{bmatrix} 0_{c_1\times c_1} & J_{c_1\times c_2} & J_{c_1\times c_3} & J_{c_1\times c_4} & 0_{c_1\times c_5} & 0_{c_1\times c_6} \\ J_{c_2\times c_1} & 0_{c_2\times c_2} & J_{c_2\times c_3} & 0_{c_2\times c_4} & J_{c_2\times c_5} & 0_{c_2\times c_6} \\ J_{c_3\times c_1} & J_{c_3\times c_2} & 0_{c_3\times c_3} & 0_{c_3\times c_4} & 0_{c_3\times c_5} & J_{c_3\times c_6} \\ J_{c_4\times c_1} & 0_{c_4\times c_2} & 0_{c_4\times c_3} & 0_{c_4\times c_4} & 0_{c_4\times c_5} & 0_{c_4\times c_6} \\ 0_{c_5\times c_1} & J_{c_5\times c_2} & 0_{c_5\times c_3} & 0_{c_5\times c_4} & 0_{c_5\times c_5} & 0_{c_5\times c_6} \\ 0_{c_6\times c_1} & 0_{c_6\times c_2} & J_{c_6\times c_3} & 0_{c_6\times c_4} & 0_{c_6\times c_5} & 0_{c_6\times c_6} \end{bmatrix}$$

Since each ith block of the matrix has an identical rows, hence $x^{|V|-6}$ is a factor of

the characteristic polynomial of $A(\Gamma_{ND}(\mathbb{Z}_n))$.Also as seen the blocks, $A(\Gamma_{ND}(\mathbb{Z}_n))$ has an equitable partition 6 classes. Let Q be the quotient matrix of the partition as

$$Q = \begin{bmatrix} 0 & c_2 & c_3 & c_4 & 0 & 0 \\ c_1 & 0 & c_3 & 0 & c_5 & 0 \\ c_1 & c_2 & 0 & 0 & 0 & c_6 \\ c_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_3 & 0 & 0 & 0 \end{bmatrix}$$

From Lemma 2.2, the characteristic polynomial of Q is a factor of the characteristic polynomial of $A(\Gamma_{ND}(\mathbb{Z}_n))$. Hence we get

$$P_Q(x) = x^6 - (c_1c_2 + c_1c_3 + c_1c_4 + c_2c_3 + c_2c_5 + c_3c_6)x^4 + 2c_1c_2c_3x^3 + \left(\left(\left(\left((c_4 + c_5 + c_6)c_3 + c_4c_5\right)c_2 + c_3c_4c_6\right)c_1 + c_2c_3c_5c_6\right)x^2 - c_1c_2c_3c_4c_5c_6 \qquad (9)$$

by computation and hence subtituting the value of c_i, the desired result holds.

Corollary 2.6.

$\Gamma_{ND}(\mathbb{Z}_n) \; \tilde{=} \; K_{p^{\alpha-1}\Phi(q^\beta), q^{\beta-1}\Phi(p^\alpha)}$ for $n = p^\alpha q^\beta$.

Proof. It is easy to see the result from Lemma 2.1 and Theorem 2.5.

Theorem 2.7.

Let $\mathbb{Z}_n$ be an integer ring such that $n = \sum_{i=1}^{3} p_i^{m_i}$. Then $\Gamma_{NT}(\mathbb{Z}_n) \cong K_\rho + \frac{|V|-c_1}{2\rho} K_{\rho,\rho}$, when if $p_1 = 2$; $\Gamma_{NT}(\mathbb{Z}_n) \cong \frac{|V|}{2\rho} K_{\rho,\rho}$, when $p_i \neq 2$, where ρ is the number of nilpotent elements of $\mathbb{Z}_n$.

Proof. Assume that $p_1 = 2$. Recalling the disjoint zero divisor setss in Lemma 2.4. For every distinct $x, y \in \overline{p_2p_3} \setminus \overline{p_1p_2p_3}$ we get $x \sim y$ since $x + y \in N(\mathbb{Z}_n)$. In fact, if $x \in \overline{p_2p_3} \setminus \overline{p_1p_2p_3}$, then there is an odd t_1 integer such that $x = p_2p_3t_1$. Similarly if

$y \in \overline{p_2p_3} \setminus \overline{p_1p_2p_3}$, then there is an odd t_2 integer such that $x = p_2p_3t_2$. So we have $x + y \in p_2p_3(t_1 + t_2)$ and since $t_1 + t_2 \in 2\mathbb{Z}$, $x + y$ must be the nilpotent element. Hence $x + y \in N(\mathbb{Z}_n)$. Provided this, there is exactly c_1 -elements, hence this partition forms the complete graph K_{c_1} . Notice that $c_1 = \rho$ since $p_1 = 2$. Hence the partition forms K_ρ.

On the other hand, for every $x \in \overline{p_1p_j} \setminus \overline{p_1p_jp_k}$ $(j, k \neq 1)$, it is easy to say that $x + (p_1p_2p_3)t \in \overline{p_1p_j} \setminus \overline{p_1p_jp_k}$ since every elements have exactly one additive inverse. Also $-x + (p_1p_2p_3)t \in \overline{p_1p_j} \setminus \overline{p_1p_jp_k}$ because 0 is a nilpotent element. Let $U = \{x + (p_1p_2p_3)t : t \in \mathbb{Z}\}$ and $V = \{-x + (p_1p_2p_3)t : t \in \mathbb{Z}\}$. Then, we get $x \sim y$ for every $x \in U$ and $y \in V$. Hence it forms a bipartite graph whose partitions are the sets U and. By similar method, we get disjoint bipartite graph on the distinct sets, given in Lemma 2.4. Therefore $\Gamma_{NT}(\mathbb{Z}_n) \cong K_\rho + \frac{|V|-c_1}{2\rho} K_{\rho,\rho}$.

Let $p_i \neq 2$ for $i = 1, 2, 3$. In this case, the all partitions form the complete bipartite graph $K_{\rho,\rho}$ by the method which is similar to the proof of the first case.

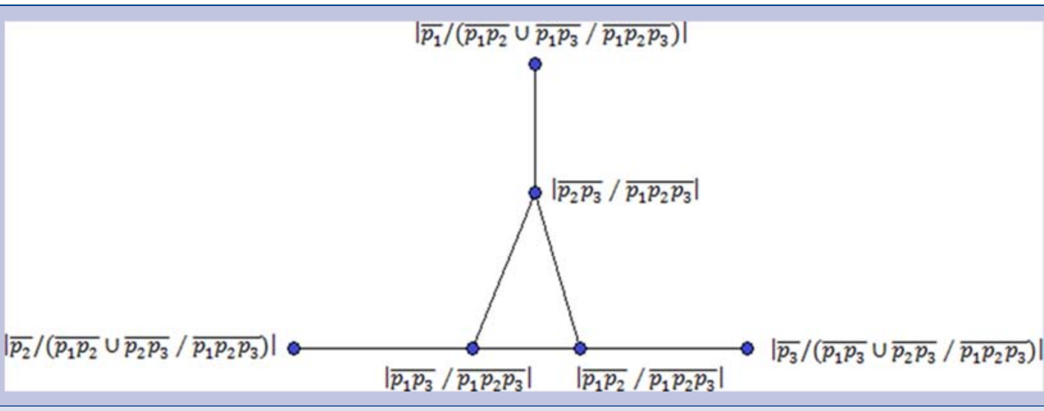

Figure 1: *The nilpotent divisor graph of* $\mathbb{Z}_n$

Corollary 2.8.

Let $\mathbb{Z}_n$ be an integer ring such that $n = \sum_{i=1}^{3} p_i^{m_i}$. Then

$$Spec_A(\,\Gamma_{NT}\,(\mathbb{Z}_n)) = \left\{(\rho - 1)^{(1)}, (-1)^{(\rho-1)}, (\rho)^{\left(\frac{|V|-c_1}{2\rho}\right)}, (-\rho)^{\left(\frac{|V|-c_1}{2\rho}\right)}, (0)^{\left(\frac{|V|-c_1}{\rho}(\rho-1)\right)}\right\}$$

when $p_1 = 2$;

$$Spec_A(\,\Gamma_{NT}\,(\mathbb{Z}_n)) = \left\{(\rho)^{\left(\frac{|V|}{2\rho}\right)}, (-\rho)^{\left(\frac{|V|}{2\rho}\right)}, (0)^{\left(\frac{|V|}{\rho}(\rho-1)\right)}\right\}$$

, otherwise.

Proof. It is obvious from Theorem 2.7 and Lemma 2.1.

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