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q-Leonardo Bicomplex Numbers

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Abstract

In the paper, we define the q-Leonardo bicomplex numbers by using the q-integers. Also, we give some algebraic properties of q-Leonardo bicomplex numbers such as recurrence relation, generating function, Binet's formula, D'Ocagne's identity, Cassini's identity, Catalan's identity and Honsberger identity.

Keywords: *; Bicomplex Fibonacci number; Bicomplex number; Fibonacci number Leonardo number; q-Leonardo number; q-Leonardo bicomplex number;q-integer.*

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1. Introduction

Bicomplex Fibonacci and bicomplex Lucas numbers were first described by Nurkan and Güven in 2015 [16] respectively, as follows

$$\begin{aligned}\mathscr{BF}_n &= F_n + i\,F_{n+1} + j\,F_{n+2} + i\,j\,F_{n+3} \\ &= (F_n + i\,F_{n+1}) + j\,(F_{n+2} + i\,F_{n+3}),\end{aligned} \tag{1.1}$$

and

$$\mathscr{BL}_n = L_n + i\,L_{n+1} + j\,L_{n+2} + i\,j\,L_{n+3}, \tag{1.2}$$

where i, j and $i\,j$ satisfy the conditions

$$i^2 = -1,\ \ j^2 = -1,\ \ i\,j = j\,i. \tag{1.3}$$

The multiplication is done using bicomplex units (table 1), this product is commutative.

Table 1: Multiplication scheme of bicomplex units

x	1	i	j	$i\,j$
1	1	i	j	$i\,j$
i	i	-1	$i\,j$	$-j$
j	j	$i\,j$	-1	$-i$
$i\,j$	$i\,j$	$-j$	$-i$	1

For more details about the bicomplex numbers and bicomplex Fibonacci quaternions, the readers can refer to [6, 7, 8, 11, 14, 15, 16, 17, 18]. The theory of the q-calculus has been extensively studied in many branches of mathematics as well as in other areas in biology, physics, electrochemistry, economics, probability theory and statistics [1, 4, 12] for $n \in \mathbb{N}_0$, q-integer $[n]_q$ is defined as follows

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots + q^{n-1},\ q \neq 1. \tag{1.4}$$

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By (1.3) for all $m,n\in\mathbb{Z}$ can be easily obtained $[m+n]_q=[m]_q+q^m[n]_q$. For more details related to the quantum q-calculus, we refer to [4, 12].
In 2019, q-Fibonacci hybrid and q-Lucas hybrid numbers defined by Kızılateş [13], respectively as follows

$$\mathbb{HF}_n(\alpha;q)=\alpha^{n-1}[n]_q+\alpha^n[n+1]_q\mathbf{i}+\alpha^{n+1}[n+2]_q\,\varepsilon \\ +\alpha^{n+2}[n+3]_q\mathbf{h}, \tag{1.5}$$

and

$$\mathbb{HL}_n(\alpha;q)=\alpha^n\frac{[2n]_q}{[n]_q}+\alpha^{n+1}\frac{[2n+2]_q}{[n+1]_q}\mathbf{i}+\alpha^{n+2}\frac{[2n+4]_q}{[n+2]_q}\,\varepsilon+\alpha^{n+3}\frac{[2n+6]_q}{[n+3]_q}\mathbf{h}. \tag{1.6}$$

where i, ε and h satisfy the conditions

$$\mathbf{i^2}=-1,\ \varepsilon^2=0,\ \mathbf{h^2}=1,\ \mathbf{i}\mathbf{h}=\mathbf{h}\mathbf{i}=\varepsilon+\mathbf{i}. \tag{1.7}$$

Also, Kızılateş [13] derived several interesting properties of these numbers such as Binet's formula, exponential generating functions, summation formulas, Cassini's, Catalan's and d'Ocagne's identities.

In 2019, the Leonardo sequence denoted with Le_n is defined by Catarino and Borges in [9] the following recurrence relation for $n\geq 2$

$$Le_n=Le_{n-1}+Le_{n-2}+1, \tag{1.8}$$

with the initial conditions $Le_0=Le_1=1$.
This sequence is also expressed as:

$$Le_{n+1}=2Le_n-Le_{n-2},\ n\geq 2. \tag{1.9}$$

Also, Catarino and Borges [9] derived some properties for example Cassini's identity, Catalan's identity and d'Ocagne's identity with Leonardo numbers. There are many identities between Leonardo numbers, Fibonacci and Lucas numbers. Two of these are the most used in this paper as follows

$$Le_n=2F_{n+1}-1,\ n\geq 0, \tag{1.10}$$

$$Le_n=L_{n+2}-F_{n+2}-1,\ n\geq 0. \tag{1.11}$$

In [10], the authors introduced incomplete Leonardo numbers provied some properties and generating functions of this sequence of integers. In [19], Shannon has defined generalized Leonardo numbers. In [20], the authors investigated the two dimensional recurrences relations of Leonardo numbers from its one-dimensional model. In [2], authors have given some properties of Leonardo numbers.
In [7], the q-Fibonacci bicomplex numbers and the q-Lucas bicomplex numbers were defined by Aydın Torunbalcı as follows

$$\begin{aligned}\mathscr{BF}_n(\alpha;q)&=\alpha^{n-1}[n]_q+\alpha^n[n+1]_q\,i+\alpha^{n+1}[n+2]_q\,j+\alpha^{n+2}[n+3]_q\,i\,j,\\ &=\alpha^n\left(\tfrac{1-q^n}{\alpha-\alpha q}\right)+\alpha^{n+1}\left(\tfrac{1-q^{n+1}}{\alpha-\alpha q}\right)i+\alpha^{n+2}\left(\tfrac{1-q^{n+2}}{\alpha-\alpha q}\right)j\quad+\alpha^{n+3}\left(\tfrac{1-q^{n+3}}{\alpha-\alpha q}\right)i\,j,\end{aligned} \tag{1.12}$$

or

$$\mathscr{BF}_n(\alpha;q)=\frac{\alpha^n\,\widehat{\gamma}-(\alpha q)^n\,\widehat{\delta}}{\alpha-\alpha q}$$

and

$$\begin{aligned}\mathscr{BL}_n(\alpha;q)&=\alpha^n\tfrac{[2n]_q}{[n]_q}+\alpha^{n+1}\tfrac{[2n+2]_q}{[n+1]_q}\,i+\alpha^{n+2}\tfrac{[2n+4]_q}{[n+2]_q}\,j+\alpha^{n+3}\tfrac{[2n+6]_q}{[n+3]_q}\,i\,j,\\ &=\alpha^{2n}\left(\tfrac{1-q^{2n}}{\alpha^n-(\alpha q)^n}\right)+\alpha^{2n+2}\left(\tfrac{1-q^{2n+2}}{\alpha^{n+1}-(\alpha q)^{n+1}}\right)i+\alpha^{2n+4}\left(\tfrac{1-q^{2n+4}}{\alpha^{n+2}-(\alpha q)^{n+2}}\right)j+\alpha^{2n+6}\left(\tfrac{1-q^{2n+6}}{\alpha^{n+3}-(\alpha q)^{n+3}}\right)i\,j,\\ &=\left(\tfrac{\alpha^{2n}-(\alpha q)^{2n}}{\alpha^n-(\alpha q)^n}\right)+\left(\tfrac{\alpha^{2n+2}-(\alpha q)^{2n+2}}{\alpha^{n+1}-(\alpha q)^{n+1}}\right)i+\left(\tfrac{\alpha^{2n+4}-(\alpha q)^{2n+4}}{\alpha^{n+2}-(\alpha q)^{n+2}}\right)j+\left(\tfrac{\alpha^{2n+6}-(\alpha q)^{2n+6}}{\alpha^{n+3}-(\alpha q)^{n+3}}\right)i\,j,\\ &=(\alpha^n+(\alpha q)^n)+(\alpha^{n+1}+(\alpha q)^{n+1})\,i+(\alpha^{n+2}+(\alpha q)^{n+2})\,j+(\alpha^{n+3}+(\alpha q)^{n+3})\,i\,j.\end{aligned} \tag{1.13}$$

or

$$\mathscr{BL}_n(\alpha;q)=\alpha^n\,\widehat{\gamma}+(\alpha q)^n\,\widehat{\delta}.$$

where $\widehat{\gamma}=1+i\,\alpha+j\,(\alpha)^2+i\,j\,(\alpha)^3$ *and* $\widehat{\delta}=1+i\,(\alpha q)+j\,(\alpha q)^2+i\,j\,(\alpha q)^3$.
The Binet's formula of the $\mathscr{BF}_n$ sequence

$$\mathscr{BF}_n(\alpha;q)=\frac{\alpha^n\,\widehat{\gamma}-(\alpha q)^n\,\widehat{\delta}}{\alpha-\alpha q}, \tag{1.14}$$

and

$$\mathscr{BL}_n(\alpha;q) = \alpha^n \widehat{\gamma} + (\alpha q)^n \widehat{\delta}. \tag{1.15}$$

In [8], for $n \geq 1$, the n-th bicomplex Leonardo numbers $\mathscr{BL}e_n$ are defined by using the Leonardo numbers as follows

$$\mathscr{BL}e_n = Le_n + i Le_{n+1} + j Le_{n+2} + i j Le_{n+3}, \tag{1.16}$$

Also, initial values are $\mathscr{BL}e_0 = 1 + i + 3j + 5ij$, $\mathscr{BL}e_1 = 1 + 3i + 5j + 9ij$.

$$\begin{aligned} \mathscr{BL}e_n = & \; Le_n + i Le_{n+1} + j Le_{n+2} + i j Le_{n+3}, \\ = & \; (2F_{n+1} - 1) + i(2F_{n+2} - 1) + j(2F_{n+3} - 1) + ij(2F_{n+4} - 1), \\ = & \; 2(F_{n+1} + iF_{n+2} + jF_{n+3} + ijF_{n+4}) - A \\ = & \; 2\mathscr{BF}_{n+1} - A. \end{aligned} \tag{1.17}$$

$$\begin{aligned} \mathscr{BL}e_n = & \; (Le_{n-1} + Le_{n-2} + 1) + i(Le_n + Le_{n-1} + 1), + j(Le_{n+1} + Le_n + 1) + ij(Le_{n+2} + Le_{n+1} + 1), \\ & = \mathscr{BL}e_{n-1} + \mathscr{BL}e_{n-2} + A. \end{aligned} \tag{1.18}$$

where $A = 1 + i + j + ij$.
The characteristic equation of recurence $\mathscr{BL}e_{n+1} = 2\mathscr{BL}e_n - \mathscr{BL}e_{n-2}$ [8]

$$\lambda^3 - 2\lambda^2 + 1 = 0. \tag{1.19}$$

The Binet's formula of the $\mathscr{BL}e_n$ sequence is

$$\mathscr{BL}e_n = \frac{2\alpha^{n+1}\widehat{\gamma} - 2\beta^{n+1}\widehat{\delta}}{\alpha - \beta} - A, \tag{1.20}$$

where α and β are roots of characteristic equation and $\hat{\gamma}$, $\hat{\delta}$ respectively, as follows

$$\hat{\gamma} = 1 + i\alpha + j\alpha^2 + ij\alpha^3,$$

$$\hat{\delta} = 1 + i\beta + j\beta^2 + ij\beta^3.$$

In this study, we used the identities (eq:1.8) and (eq:1.10) which are among the most important identities between Leonardo numbers and Fibonacci numbers [8, 9].

2. q-Leonardo bicomplex numbers

In this section, we define q-Leonardo bicomplex numbers by using the basis $\{1, i, j, ij\}$, where i, j and ij satisfy the conditions (eq:1.2) and identities (eq:1.4) and (eq:1.9) as follows

$$\begin{aligned} \mathscr{BL}e_n(\alpha;q) = & \; (2\alpha^n[n+1]_q - 1) + (2\alpha^{n+1}[n+2]_q - 1)i + (2\alpha^{n+2}[n+3]_q - 1)j + (2\alpha^{n+3}[n+4]_q - 1)ij, \\ = & \; 2[(\alpha^{n+1}(\tfrac{1-q^{n+1}}{\alpha-\alpha q}) + \alpha^{n+2}(\tfrac{1-q^{n+2}}{\alpha-\alpha q})i, + \alpha^{n+3}(\tfrac{1-q^{n+3}}{\alpha-\alpha q})j + \alpha^{n+4}(\tfrac{1-q^{n+4}}{\alpha-\alpha q})ij] - (1 + i + j + ij), \\ = & \; 2[\tfrac{\alpha^{n+1}}{\alpha-(\alpha q)}(1 + \alpha i + \alpha^2 j + \alpha^3 ij) - \tfrac{(\alpha q)^{n+1}}{\alpha-(\alpha q)}(1 + (\alpha q)i + (\alpha q)^2 j + (\alpha q)^3 ij)] - A, \\ = & \; \tfrac{2\alpha^{n+1}\widehat{\gamma} - 2(\alpha q)^{n+1}\widehat{\delta}}{\alpha - \alpha q} - A. \end{aligned} \tag{2.1}$$

where

$$A = 1 + i + j + ij,$$

$$\widehat{\gamma} = 1 + \alpha i + \alpha^2 j + \alpha^3 ij,$$

$$\widehat{\delta} = 1 + (\alpha q)i + (\alpha q)^2 j + (\alpha q)^3 ij.$$

If we take $\alpha = \frac{1+\sqrt{5}}{2}$ and $q = \frac{-1}{\alpha^2}$ in (2.1), then we have $\mathscr{BL}e_n(\alpha;q) = \mathscr{BL}e_n$. So, the q-Leonardo bicomplex numbers are a generalization of the known Leonardo bicomplex numbers.
Now we give the recurrence relation corresponding to expressions Eq.(1.8)and (1.10).

Theorem 2.1. *For $n \in \mathbb{N}$, we have*

$$\mathscr{BL}e_n(\alpha;q) = 2\mathscr{BF}_{n+1}(\alpha;q) - A. \tag{2.2}$$

Proof. Using relations (1.8) and (1.10) we obtain that,

$$\begin{aligned}\mathscr{BL}e_n(\alpha;q) = & \ (2\,\alpha^n[n+1]_q-1)+(2\,\alpha^{n+1}[n+2]_q-1)\,i+(2\,\alpha^{n+2}[n+3]_q-1)\,j+(2\,\alpha^{n+3}[n+4]_q-1)\,i\,j,\\ = & \ 2\,(\tfrac{\alpha^{n+1}}{\alpha-(\alpha q)}\,(1+\alpha\,i+\alpha^2\,j+\alpha^3\,i\,j))-2\,(\tfrac{(\alpha q)^{n+1}}{\alpha-(\alpha q)}\,(1+(\alpha\,q)\,i+(\alpha\,q)^2\,j+(\alpha\,q)^3\,i\,j)-(1+i+j+i\,j)),\\ = & \ \tfrac{2\alpha^{n+1}\,\widehat{\gamma}-2(\alpha\,q)^{n+1}\,\widehat{\delta}}{\alpha-\alpha\,q}-A\,.\end{aligned}$$

□

Now, we give Binet's formula, exponential generating functions and some other identities for the q-Fibonacci bicomplex number.

Theorem 2.2. ***Binet's formula***
Let $\mathscr{BL}e_n(\alpha;q)$ be the q-Leonardo bicomplex number. For $n\geq 1$, Binet's formula for these numbers respectively, is as follows:

$$\mathscr{BL}e_n(\alpha;q)=(\frac{2\alpha^{n+1}\,\widehat{\gamma}-2(\alpha\,q)^{n+1}\,\widehat{\delta}}{\alpha-\alpha\,q})-A\,, \tag{2.3}$$

where

$\widehat{\gamma}=1+\alpha\,i+\alpha^2\,j+\alpha^3\,i\,j,\quad \alpha=\frac{1+\sqrt{5}}{2}$

$\widehat{\delta}=1+(\alpha\,q)\,i+(\alpha\,q)^2\,j+(\alpha\,q)^3\,i\,j,\quad \alpha\,q=\frac{-1}{\alpha}$

and $A=1+i+j+i\,j$.

Proof. (2.3): Using (1.4) and (2.1), we find that

$$\begin{aligned}\mathscr{BL}e_n(\alpha;q) = & \ (2\,\alpha^n[n+1]_q-1)+i\,(2\,\alpha^{n+1}[n+2]_q-1)+j\,(2\,\alpha^{n+2}[n+3]_q-1)+i\,j\,(2\,\alpha^{n+3}[n+4]_q-1)\\ = & \ (\tfrac{2\alpha^{n+1}-2(\alpha\,q)^{n+1}}{\alpha-\alpha\,q}-1)+i\,(\tfrac{2\alpha^{n+2}-2(\alpha\,q)^{n+2}}{\alpha-\alpha\,q}-1)+j\,(\tfrac{2\alpha^{n+3}-2(\alpha\,q)^{n+3}}{\alpha-\alpha\,q}-1)+i\,j\,(\tfrac{2\alpha^{n+4}-2(\alpha\,q)^{n+4}}{\alpha-\alpha\,q}-1)\\ = & \ \tfrac{2\,\alpha^{n+1}\,[1+\alpha i+\alpha^2\,j+\alpha^3\,i\,j]-(\alpha\,q)^n\,[1+(\alpha\,q)\,i+(\alpha\,q)^2\,j+(\alpha\,q)^3\,i\,j]}{\alpha-(\alpha\,q)}-(1+i+j+i\,j)\\ = & \ (\tfrac{2\alpha^{n+1}\,\widehat{\gamma}-2(\alpha\,q)^{n+1}\,\widehat{\delta}}{\alpha-\alpha\,q})-A\,.\end{aligned}$$

where $\widehat{\gamma}=1+\alpha\,i+\alpha^2\,j+\alpha^3\,i\,j$, $\widehat{\delta}=1+(\alpha\,q)\,i+(\alpha\,q)^2\,j+(\alpha\,q)^3\,i\,j$ and $\widehat{\gamma}\widehat{\delta}=\widehat{\delta}\,\widehat{\gamma}$.
Thus, the proof is completed. □

Theorem 2.3. ***(Exponential generating function)***
Let $\mathscr{BL}e_n(\alpha;q)$ be the q-Leonardo bicomplex number. For the exponential generating functions for these numbers are as follows:

$$g_{\mathscr{BL}e_n(\alpha;q)}\,(\frac{t^n}{n!})=\sum_{n=0}^{\infty}\mathscr{BL}e_n(\alpha;q)\,\frac{t^n}{n!}=2\,(\frac{\widehat{\gamma}e^{\alpha t}\ -q\,\widehat{\delta}\,e^{(\alpha\,q)\,t}}{1-q})-A\,e^t\,. \tag{2.4}$$

Proof. (2.4): Using Binet's formula for q-Leonardo bicomplex numbers, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\mathscr{BL}e_n(\alpha;q)\,\tfrac{t^n}{n!} & =\sum_{n=0}^{\infty}[2\,(\tfrac{\alpha^{n+1}\,\widehat{\gamma}-(\alpha\,q)^{n+1}\,\widehat{\delta}}{\alpha-\alpha\,q})\,\tfrac{t^n}{n!}\,,\\ & =2\,\tfrac{\widehat{\gamma}}{\alpha-\alpha\,q}\sum_{n=0}^{\infty}\tfrac{(\alpha\,t)^n}{n!}-2\,\tfrac{(\alpha\,q)\widehat{\delta}}{\alpha-\alpha\,q}\sum_{n=0}^{\infty}\tfrac{(\alpha\,q\,t)^n}{n!}-A\sum_{n=0}^{\infty}\tfrac{t^n}{n!}\,,\\ & =2\,(\tfrac{\widehat{\gamma}e^{\alpha t}\ -q\,\widehat{\delta}\,e^{(\alpha q)t}}{1-q})-A\,e^t\,.\end{aligned}$$

Thus, the proof is completed. □

Theorem 2.4. *Let $(\mathscr{BL}e_n(\alpha;q))$ be q-Leonardo bicomplex number. In this case, for non-negative integer numbers n and k, we can give the following relations:*

$$\sum_{i=0}^{n}\binom{n}{i}\,(-\alpha^2\,q)^{n-i}\,\mathscr{BL}e_{2i+k}(\alpha;q)$$
$$=\left\{\begin{array}{ll}2\Delta^{\frac{n}{2}}\,\mathscr{BF}_{n+k+1}(\alpha;q)-A\,(1-\alpha^2\,q)^n & \text{if } n \text{ is even,}\\ 2\Delta^{\frac{n-1}{2}}\,\mathscr{BL}_{n+k+1}(\alpha;q)-A\,(1-\alpha^2\,q)^n & \text{if } n \text{ is odd.}\end{array}\right\} \tag{2.5}$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}(-\alpha^{2}q)^{n-i}\,\mathscr{BL}e_{2i+k}(\alpha;q)=2(-\alpha[2]_{q})^{n}\,\mathscr{BF}_{n+k+1}(\alpha;q)+A(-1)^{n+1}(1+\alpha^{2}q)^{n}. \tag{2.6}$$

here $\Delta=(\alpha-\alpha q)^{2}$ *and* $(-\alpha[2]_{q})^{n}=(-\alpha(1+q))^{n}$.

Proof. (2.5): Using the binomial coefficients, we have

$$\begin{aligned}
\sum_{i=0}^{n}\tbinom{n}{i}(-\alpha^{2}q)^{n-i}\,\mathscr{BL}e_{2i+k}(\alpha;q) &= \sum_{i=0}^{n}\tbinom{n}{i}(-\alpha^{2}q)^{n-i}[(\tfrac{2\alpha^{2i+k+1}\widehat{\gamma}-2(\alpha q)^{2i+k+1}\widehat{\delta}}{\alpha-\alpha q})-A],\\
&= \tfrac{2}{\alpha-\alpha q}[\sum_{i=0}^{n}\tbinom{n}{i}(-\alpha^{2}q)^{n-i}(\alpha^{2})^{i}\alpha^{k+1}\widehat{\gamma}]+\tfrac{2}{\alpha-\alpha q}[\sum_{i=0}^{n}\tbinom{n}{i}(-\alpha^{2}q)^{n-i}((\alpha q)^{2})^{i}(\alpha q)^{k+1}\widehat{\delta}]\\
&\quad -\sum_{i=0}^{n}\tbinom{n}{i}(-\alpha^{2}q)^{n-i}A,\\
&= 2[\tfrac{(\alpha^{2}-\alpha^{2}q)^{n}\alpha^{k+1}\widehat{\gamma}-((\alpha q)^{2}-\alpha^{2}q)^{n}(\alpha q)^{k+1}\widehat{\delta}}{\alpha-\alpha q}]-(1-\alpha^{2}q)^{n}A,\\
&= 2(\alpha\sqrt{\Delta})^{n}\ \tfrac{\alpha^{n+k+1}\widehat{\gamma}+(-1)^{n+1}(\alpha q)^{n+k+1}\widehat{\delta}}{\alpha-\alpha q}]-(1-\alpha^{2}q)^{n}.A,\\
\text{if n is even}\quad & \\
&= 2\sqrt{\Delta}^{n}[\tfrac{\alpha^{n+k+1}\widehat{\gamma}-(\alpha q)^{n+k+1}\widehat{\delta}}{\alpha-\alpha q}]-(1-\alpha^{2}q)^{n}A\\
&= 2\Delta^{\frac{n}{2}}\,\mathscr{BF}_{n+k+1}(\alpha;q)-(1-\alpha^{2}q)^{n}A.\\
\text{if n is odd}\quad & \\
&= 2\sqrt{\Delta}^{n}[\tfrac{\alpha^{n+k+1}\widehat{\gamma}+(\alpha q)^{n+k+1}\widehat{\delta}}{\alpha-\alpha q})]-(1-\alpha^{2}q)^{n}A,\\
&= 2\Delta^{\frac{n-1}{2}}\,\mathscr{BL}_{n+k+1}(\alpha;q)-(1-\alpha^{2}q)^{n}A.
\end{aligned}$$

(2.6): Using (2.3), (1.14) and binomial coefficients, we have

$$\begin{aligned}
\sum_{i=0}^{n}\tbinom{n}{i}(-1)^{i}(-\alpha^{2}q)^{n-i}\,\mathscr{BL}e_{2i+k}(\alpha;q) &= \sum_{i=0}^{n}\tbinom{n}{i}(-1)^{i}(-\alpha^{2}q)^{n-i}[\tfrac{2\alpha^{2i+k+1}\widehat{\gamma}-(\alpha q)^{2i+k+1}\widehat{\delta}}{\alpha-\alpha q}]-A,\\
&= \tfrac{2}{\alpha-\alpha q}[\sum_{i=0}^{n}\tbinom{n}{i}(-\alpha^{2}q)^{n-i}(-\alpha^{2})^{i}\alpha^{k+1}\widehat{\gamma}]+\tfrac{2}{\alpha-\alpha q}[-\sum_{i=0}^{n}\tbinom{n}{i}(-\alpha^{2}q)^{n-i}(-(\alpha q)^{2})^{i}(\alpha q)^{k+1}\widehat{\delta}]\\
&\quad -\sum_{i=0}^{n}\tbinom{n}{i}(-\alpha^{2}q)^{n-i}(-1)^{i}.A\\
&= 2[\tfrac{(-\alpha^{2}-\alpha^{2}q)^{n}\alpha^{k+1}\widehat{\gamma}-(-(\alpha q)^{2}-\alpha^{2}q)^{n}(\alpha q)^{k+1}\alpha^{k+1}\widehat{\delta}}{\alpha-\alpha q}]-(-1-\alpha^{2}q)^{n}A,\\
&= (-\alpha)^{n}(1+q)^{n}2[\tfrac{\alpha^{n+k+1}\widehat{\gamma}-(\alpha q)^{n+k+1}\widehat{\delta}}{\alpha-\alpha q}-(-1-\alpha^{2}q)^{n}A,\\
&= 2(-\alpha[2]_{q})^{n}\,\mathscr{BF}_{n+k+1}(\alpha;q)+(-1)^{n+1}(1+\alpha^{2}q)^{n}A.
\end{aligned}$$

□

Theorem 2.5. *For* $n\in\mathbb{N}$*, we have*

$$\sum_{i=0}^{n}\binom{n}{i}(\alpha(1+q))^{i}(-\alpha^{2}q)^{n-i}\,\mathscr{BL}e_{i}(\alpha;q)=\mathscr{BL}e_{2n}(\alpha;q)+A-(\alpha(1+q)-\alpha^{2}q)^{n}A. \tag{2.7}$$

Proof. (2.7): Using the binomial coefficients, we have

$$\begin{aligned}
&\sum_{i=0}^{n}\tbinom{n}{i}(\alpha(1+q))^{i}(-\alpha^{2}q)^{n-i}\,\mathscr{BL}e_{i}(\alpha;q)\\
&= \sum_{i=0}^{n}\tbinom{n}{i}(\alpha(1+q))^{i}(-\alpha^{2}q)^{n-i}[\tfrac{2\alpha^{i+1}\widehat{\gamma}-2(\alpha q)^{i+1}\widehat{\delta}}{\alpha-\alpha q}-A],\\
&= \tfrac{2}{\alpha-\alpha q}[\sum_{i=0}^{n}\tbinom{n}{i}(\alpha^{2}(1+q))^{i}(-\alpha^{2}q)^{n-i}\alpha\widehat{\gamma}]\\
&\quad +\tfrac{2}{\alpha-\alpha q}[-\sum_{i=0}^{n}\tbinom{n}{i}(\alpha^{2}q(1+q))^{i}(-\alpha^{2}q)^{n-i}(\alpha q)\widehat{\delta}]-\sum_{i=0}^{n}\tbinom{n}{i}(\alpha(1+q))^{i}(-\alpha^{2}q)^{n-i}A,\\
&= \tfrac{2}{\alpha-\alpha q}(\alpha^{2n+1}\widehat{\gamma}-(\alpha q)^{2n+1})-(\alpha(1+q)-\alpha^{2}q)^{n}A,\\
&= \mathscr{BL}e_{2n}(\alpha;q)+A-(\alpha(1+q)-\alpha^{2}q)^{n}A.
\end{aligned}$$

Thus, the proof is completed. □

Theorem 2.6. ***Honsberger identity***
For $n,m \geq 0$ the Honsberger identity for the q-Leonardo bicomplex numbers $\mathscr{BF}_n(\alpha;q)$ and $\mathscr{BF}_m(\alpha;q)$ are as follows:

$$\begin{aligned} \mathscr{BL}e_n(\alpha;q)\,\mathscr{BL}e_m(\alpha;q)+\mathscr{BL}e_{n+1}(\alpha;q)\,\mathscr{BL}e_{m+1}(\alpha;q) = \; & 4\,\tfrac{\alpha^{n+m+2}}{(\alpha-\alpha q)^2}\,[\,(1+\alpha^2)\,\widehat{\gamma}^2+(1+(\alpha q)^2)\,(q^{n+m+2})\,\widehat{\delta}^2 \\ & -\widehat{\gamma}\widehat{\delta}\,(1+\alpha^2 q)\,(q^{n+1}+q^{m+1})\,]-[\mathscr{BL}e_n(\alpha;q)+\mathscr{BL}e_{n+1}(\alpha;q)]A \\ & -[\mathscr{BL}e_m(\alpha;q)+\mathscr{BL}e_{m+1}(\alpha;q)]A\,,-2A^2\,. \end{aligned} \tag{2.8}$$

Proof. (2.8): By using (2.1) and (2.3) we get,

$$\begin{aligned} \mathscr{BL}e_n(\alpha;q)\,\mathscr{BL}e_m(\alpha;q)+\mathscr{BL}e_{n+1}(\alpha;q)\,\mathscr{BL}e_{m+1}(\alpha;q) = \; & (2\mathscr{BF}_{n+1}(\alpha;q)-A)\,(2\mathscr{BF}_{m+1}(\alpha;q)-A) \\ + \; & (2\mathscr{BF}_{n+2}(\alpha;q)-A)\,(2\mathscr{BF}_{m+2}(\alpha;q)-A)\,, \\ = \; & 4\,[\,(\tfrac{\alpha^{n+1}\,\widehat{\gamma}-(\alpha q)^{n+1}\,\widehat{\delta}}{\alpha-\alpha q})\,(\tfrac{\alpha^{m+1}\,\widehat{\gamma}-(\alpha q)^{m+1}\,\widehat{\delta}}{\alpha-\alpha q}) \\ + \; & (\tfrac{\alpha^{n+2}\,\widehat{\gamma}-(\alpha q)^{n+2}\,\widehat{\delta}}{\alpha-\alpha q})\,(\tfrac{\alpha^{m+2}\,\widehat{\gamma}-(\alpha q)^{m+2}\,\widehat{\delta}}{\alpha-\alpha q})\,] \\ - \; & A\,[\,(\tfrac{2\,\alpha^{n+1}\,\widehat{\gamma}-2\,(\alpha q)^{n+1}\,\widehat{\delta}}{\alpha-\alpha q})+(\tfrac{2\,\alpha^{m+1}\,\widehat{\gamma}-2\,(\alpha q)^{m+1}\,\widehat{\delta}}{\alpha-\alpha q})\,] \\ - \; & A\,[\,(\tfrac{2\,\alpha^{n+2}\,\widehat{\gamma}-2\,(\alpha q)^{n+2}\,\widehat{\delta}}{\alpha-\alpha q})+(\tfrac{2\,\alpha^{m+2}\,\widehat{\gamma}-2\,(\alpha q)^{m+2}\,\widehat{\delta}}{\alpha-\alpha q})\,]\,, \\ = \; & 4\,\tfrac{\alpha^{n+m+2}}{(\alpha-\alpha q)^2}\,[\,(1+\alpha^2)\,\widehat{\gamma}^2+(1+(\alpha q)^2)\,(q^{n+m+2})\,\widehat{\delta}^2 \\ & -\widehat{\gamma}\widehat{\delta}\,(1+\alpha^2 q)\,(q^{n+1}+q^{m+1})\,] \\ & -[\mathscr{BL}e_n(\alpha;q)+\mathscr{BL}e_{n+1}(\alpha;q)]A-[\mathscr{BL}e_m(\alpha;q)+\mathscr{BL}e_{m+1}(\alpha;q)]A \\ & -2A^2\,. \end{aligned}$$

Here, $\widehat{\gamma}\widehat{\delta}=\widehat{\delta}\,\widehat{\gamma}$ is used. Thus, the proof is completed. □

Theorem 2.7. ***D'Ocagne's identity***
For $n,m \geq 0$, the d'Ocagne's identity for the q-Leonardo bicomplex numbers $\mathscr{BL}e_n(\alpha;q)$ are as follows:

$$\begin{aligned} \mathscr{BL}e_m(\alpha;q)\,\mathscr{BL}e_{n+1}(\alpha;q)-\mathbb{BL}e_{m+1}(\alpha;q)\,\mathscr{BL}e_n(\alpha;q) = \; & 4\,[\tfrac{\alpha^{n+m+1}(q^{n+1}-q^{m+1})\,\widehat{\gamma}\widehat{\delta}}{(1-q)}\,]+A\,(\mathscr{BL}e_{m+1}(\alpha;q)-\mathscr{BL}e_m(\alpha;q)) \\ + \; & A\,(\mathscr{BL}e_n(\alpha;q)-\mathscr{BL}e_{n+1}(\alpha;q))\,. \end{aligned} \tag{2.9}$$

Proof. (2.9): By using (2.1) and (2.10) we get,

$$\begin{aligned} \mathscr{BL}e_m(\alpha;q)\,\mathscr{BL}e_{n+1}(\alpha;q)-\mathscr{BL}e_{m+1}(\alpha;q)\,\mathscr{BL}e_n(\alpha;q) = \; & (2\mathscr{BF}_{m+1}(\alpha;q)-A)\,(2\mathscr{BF}_{n+2}(\alpha;q)-A) \\ - \; & (2\mathscr{BF}_{m+2}(\alpha;q)-A)\,(2\mathscr{BF}_{n+1}(\alpha;q)-A)\,, \\ = \; & 4\,[\,(\tfrac{\alpha^{m+1}\,\widehat{\gamma}-(\alpha q)^{m+1}\,\widehat{\delta}}{\alpha-\alpha q})\,(\tfrac{\alpha^{n+2}\,\widehat{\gamma}-(\alpha q)^{n+2}\,\widehat{\delta}}{\alpha-\alpha q}) \\ - \; & (\tfrac{\alpha^{m+2}\,\widehat{\gamma}-(\alpha q)^{m+2}\,\widehat{\delta}}{\alpha-\alpha q})\,(\tfrac{\alpha^{n+1}\,\widehat{\gamma}-(\alpha q)^{n+1}\,\widehat{\delta}}{\alpha-\alpha q})\,] \\ + \; & A\,[\tfrac{2\,\alpha^{m+2}\,\widehat{\gamma}-2(\alpha q)^{m+2}\,\widehat{\delta}}{\alpha-\alpha q}-\tfrac{2\,\alpha^{m+1}\,\widehat{\gamma}-2(\alpha q)^{m+1}\,\widehat{\delta}}{\alpha-\alpha q}\,] \\ + \; & A\,[\tfrac{2\,\alpha^{n+1}\,\widehat{\gamma}-2(\alpha q)^{n+1}\,\widehat{\delta}}{\alpha-\alpha q}-\tfrac{2\,\alpha^{n+2}\,\widehat{\gamma}-2(\alpha q)^{n+2}\,\widehat{\delta}}{\alpha-\alpha q}\,]\,, \\ = \; & 4\,[\tfrac{\alpha^{n+m+1}(q^{n+1}-q^{m+1})\,\widehat{\gamma}\widehat{\delta}}{(1-q)}\,] \\ + \; & A\,(\mathscr{BL}e_{m+1}(\alpha;q)-\mathscr{BL}e_m(\alpha;q)) \\ + \; & A\,(\mathscr{BL}e_n(\alpha;q)-\mathscr{BL}e_{n+1}(\alpha;q))\,. \end{aligned}$$

Here, $\widehat{\gamma}\widehat{\delta}=\widehat{\delta}\,\widehat{\gamma}$ is used. Thus, the proof is completed. □

Theorem 2.8. ***Cassini's identity***
For $n \geq 1$, Cassini's identity for the q-Leonardo bicomplex numbers $\mathscr{BL}e_n(\alpha;q)$ are as follows:

$$\begin{aligned} \mathscr{BL}e_{n+1}(\alpha;q)\,\mathbb{BL}e_{n-1}(\alpha;q)-\mathscr{BL}e_n^2(\alpha;q) = 4\,(\,\frac{\alpha^{2n}\,q^{n+1}\,(1-q^{-1})\,\widehat{\gamma}\widehat{\delta}}{(1-q)}\,)+A\,(\mathscr{BL}e_n(\alpha;q)-\mathscr{BL}e_{n+1}(\alpha;q)) \\ +A\,(\mathscr{BL}e_n(\alpha;q)-\mathscr{BL}e_{n-1}(\alpha;q))\,. \end{aligned} \tag{2.10}$$

Proof. (2.10): By using (2.1) and (2.10) we get

$$\begin{aligned}\mathscr{BL}e_{n+1}(\alpha;q)\,\mathscr{BL}e_{n-1}(\alpha;q)-\mathscr{BL}e_n^2(\alpha;q) &= (2\mathscr{BF}_{n+2}(\alpha;q)-A)\,(2\mathscr{BF}_{n}(\alpha;q)-A)-(2\mathscr{BF}_{n+1}(\alpha;q)-A)\,(2\mathscr{BF}_{n+1}(\alpha;q)-A)\\
&=4\,[\,(\tfrac{\alpha^{n+2}\,\widehat{\gamma}-(\alpha\,q)^{n+2}\,\widehat{\delta}}{\alpha-\alpha\,q})\,(\tfrac{\alpha^{n}\,\widehat{\gamma}-(\alpha\,q)^{n}\,\widehat{\delta}}{\alpha-\alpha\,q})-(\tfrac{\alpha^{n+1}\,\widehat{\gamma}-(\alpha\,q)^{n+1}\,\widehat{\delta}}{(\alpha-\alpha\,q)})^2\,]\\
&+A\,[\tfrac{2\,\alpha^{n+1}\,\widehat{\gamma}-2(\alpha\,q)^{n+1}\,\widehat{\delta}}{(\alpha-\alpha\,q)})-\tfrac{2\,\alpha^{n+2}\,\widehat{\gamma}-2\,(\alpha\,q)^{n+2}\,\widehat{\delta}}{(\alpha-\alpha\,q)}\,]+A\,[\tfrac{2\,\alpha^{n+1}\,\widehat{\gamma}-2(\alpha\,q)^{n+1}\,\widehat{\delta}}{(\alpha-\alpha\,q)}-\tfrac{2\,\alpha^{n}\,\widehat{\gamma}-2\,(\alpha\,q)^{n}\,\widehat{\delta}}{(\alpha-\alpha\,q)}\,]\\
&=4\,[\,\tfrac{\alpha^{2n}\,q^{n+1}\,(1-q^{-1})\,\widehat{\gamma}\widehat{\delta}}{(1-q)}\,]+A\,(\mathscr{BL}e_n(\alpha;q)-\mathscr{BL}e_{n+1}(\alpha;q))\\
&+A\,(\mathscr{BL}e_n(\alpha;q)-\mathscr{BL}e_{n-1}(\alpha;q))\,.\end{aligned}$$

Here, $\widehat{\gamma}\widehat{\delta}=\widehat{\delta}\,\widehat{\gamma}$ is used. Thus, the proof is completed. □

Theorem 2.9. ***Catalan's identity***
For $n\geq r$, Catalan's identity for the q-Leonardo bicomplex numbers $\mathscr{BL}e_n(\alpha;q)$ are as follows:

$$\begin{aligned}\mathscr{BL}e_{n+r}(\alpha;q)\,\mathscr{BL}e_{n-r}(\alpha;q)-\mathscr{BL}e_n^2(\alpha;q) =& 4\,[\,\frac{\alpha^{2n}\,q^{n+1}\,(1-q^{r})(1-q^{-r})\,\widehat{\gamma}\widehat{\delta}}{(1-q)^2}\,]+A\,(\mathscr{BL}e_{n+1}(\alpha;q)-\mathscr{BL}e_{n+r+1}(\alpha;q))\\
&+A\,(\mathscr{BL}e_{n+1}(\alpha;q)-\mathscr{BL}e_{n-r+1}(\alpha;q))\,.\end{aligned}\tag{2.11}$$

Proof. (2.11): By using (2.1) and (2.10) we get

$$\begin{aligned}\mathscr{BL}e_{n+r}(\alpha;q)\,\mathscr{BL}e_{n-r}(\alpha;q)-\mathscr{BL}e_n^2(\alpha;q) &= (2\mathscr{BF}_{n+r+1}(\alpha;q)-A)\,(2\mathscr{BF}_{n-r+1}(\alpha;q)-A)\\
&-(2\mathscr{BF}_{n+1}(\alpha;q)-A)\,(2\mathscr{BF}_{n+1}(\alpha;q)-A)\,,\\
&=4\,[\,(\tfrac{\alpha^{n+r+1}\,\widehat{\gamma}-(\alpha\,q)^{n+r+1}\,\widehat{\delta}}{\alpha-\alpha\,q})\,(\tfrac{\alpha^{n-r+1}\,\widehat{\gamma}-(\alpha\,q)^{n-r+1}\,\widehat{\delta}}{\alpha-\alpha\,q})-(\tfrac{\alpha^{n+1}\,\widehat{\gamma}-(\alpha\,q)^{n+1}\,\widehat{\delta}}{(\alpha-\alpha\,q)})^2\,]\\
&+A\,[\,(\tfrac{2\,\alpha^{n+1}\widehat{\gamma}-2(\alpha\,q)^{n+1}\widehat{\delta}}{\alpha-\alpha\,q})-(\tfrac{2\,\alpha^{n+r+1}\widehat{\gamma}-2(\alpha\,q)^{n+r+1}\widehat{\delta}}{\alpha-\alpha\,q})\,]\\
&+A\,[\,(\tfrac{2\,\alpha^{n+1}\widehat{\gamma}-2(\alpha\,q)^{n+1}\widehat{\delta}}{\alpha-\alpha\,q})-(\tfrac{2\,\alpha^{n-r+1}\widehat{\gamma}-2(\alpha\,q)^{n-r+1}\widehat{\delta}}{\alpha-\alpha\,q})\,]\,,\\
&=\tfrac{4}{(1-q)^2}\,\widehat{\gamma}\widehat{\delta}\,[\alpha^{2n}\,q^{n+1}\,(1-q^{-r})(1-q^{r})]+A\,(\mathscr{BL}e_{n+1}(\alpha;q)-\mathscr{BL}e_{n+r+1}(\alpha;q))\\
&+A\,(\mathscr{BL}e_{n+1}(\alpha;q)-\mathscr{BL}e_{n-r+1}(\alpha;q))\,.\end{aligned}$$

Here, $\widehat{\gamma}\widehat{\delta}=\widehat{\delta}\,\widehat{\gamma}$ is used. Thus, the proof is completed. □

3. Conclusion

In this paper, algebraic and analytic properties of the q-Leonardo bicomplex numbers are investigated. Thanks to the q-calculus, many mathematical concepts are generalized. In this study, we generalized the q-Leonardo bicomplex numbers using the q-calculus.

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References

[1] Adler S. L., and others. *Quaternionic quantum mechanics and quantum fields*, Oxford University Press on Demand, 88, (1995).
[2] Y. Alp and E.G. Koçer, *Some properties of Leonardo numbers.*, Konuralp Journal of Mathematics (KJM), 9(1), (2021), 183-189.
[3] I. Akkus, G. Kizilaslan, *Quaternions: Quantum calculus approach with applications.*, Kuwait Journal of Science, 46 (4), (2019).
[4] Andrews G. E., Askey R., and Roy R. *Special functions*, Cambridge University Press, 71, (1999).
[5] Arfken G. B., and Weber H. J. *Mathematical methods for physicists*, American Association of Physics Teachers, (1999).

[6] F. Torunbalcı Aydın, *Bicomplex fibonacci quaternions.*, Chaos Solitons Fractals, Vol. 106, (2018), 147-153.
[7] F. Torunbalcı Aydın, *q-Fibonacci bicomplex and q-Lucas bicomplex numbers.*, Notes on Number Theory and Discrete Mathematics, 28(2),(2022), 261-275.
[8] M. Turan, S. Özkaldı Karakuş and S. Nurkan, *A New Perspective on Bicomplex Numbers: Leonardo Component Extension.*, preprint, (2022).
[9] P. Catarino, and Anabela Borges, *On leonardo numbers.*, Acta Mathematica Universitatis Comenianae, 89(1), (2019), 75-76.
[10] P. Catarino and A. Borges, *A note on incomplete Leonardo numbers.*, Integers, Vol. 20(7), (2020).
[11] S. Halıcı, *On bicomplex Fibonacci numbers and their generalization.*, Models and Theories in Social Systems, (2019), 509-524.
[12] Kac V. G., and Cheung P. *Quantum calculus*, Springer 113, (2002).
[13] C. Kızılateş, *A new generalization of Fibonacci hybrid and Lucas hybrid numbers.*, Chaos, Solitons and Fractals, 130, 109449, (2020).
[14] M. Luna-Elizarraras, M. Shapiro, D. C. Struppa, A. Vajiac, *Bicomplex numbers and their elementary functions.*, Cubo (Temuco) 14 (2), (2012), 61-80.
[15] M. Luna-Elizarraras, M. Shapiro, D. C. Struppa, A. Vajiac, *Bicomplex holomorphic functions: the algebra, geometry and analysis of bicomplex numbers.*, Birkhauser, (2015)
[16] S. Kaya Nurkan and A. I. Güven, *A note on bicomplex Fibonacci and Lucas numbers.*, arXiv preprint arXiv:1508.03972, (2015).
[17] D. Rochon and M. Shapiro, *On algebraic properties of bicomplex and hyperbolic numbers.*, Anal. Univ. Oradea, fasc. math, 11, (71), (2004), 110.
[18] Segre C. Le rappresentazioni reali delle forme complesse e gli ente iper-algebrici, *Mathematische Annalen*, Vol. 40(3), 413-467 (1892).
[19] A. G. Shannon, *A note on generalized Leonardo numbers.*, Notes on Number Theory and Discrete Mathematics, 25(3), (2019), 97-101.
[20] R. P. M. Vieira, F. R. V. Alves, and P. M. M. C. Catarino, *Relações bidimensionais e identidades da sequência de Leonardo.*, Revista Sergipana de Matemática e Educação Matemática, 4(2), (2019), 156-173 .