Advances in the Theory of Nonlinear Analysis and its Applications 7 (2023) No. 2, 292–302.
https://doi.org/10.31197/atnaa.1281575
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications

ISSN: 2587-2648

Peer-Reviewed Scientific Journal

On the nonlinear Volterra equation with conformable derivative

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Abstract

In this paper, we are interested to study a nonlinear Volterra equation with conformable derivative. This kind of such equation has various applications, for example physics, mechanical engineering, heat conduction theory. First, we show that our problem have a mild soltution which exists locally in time. Then we prove that the convergence of the mild solution when the parameter tends to zero.

Keywords: conformable differential equation, memory term, local existence, Banach fixed point theorem.
2010 MSC: 35R11, 35B65, 26A33.

1. Introduction

In this paper, we consider the fractional Sobolev equation

$$\begin{cases} D_t^\alpha w - \Delta w - k\Delta D_t^\alpha w = F(w) + b\displaystyle\int_0^t (t-z)^{-\theta} w(z)dz, & (x,t) \in \Omega \times (0,T), \\ w = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ w(x,0) = f(x) \end{cases} \tag{1}$$

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Received October 15, 2022; Accepted: April 23, 2023; Online: April 25, 2023.

where F, f are the functions which are later mentioned. The constant k, b are two positive constant.

If $k = 0$, the main problem (30) is the classical heat equation[10, 11], is a special case of the non-classical diffusion equation and plays an important role in liquids mechanics, solid mechanics and heat conduction theory, such as [8, 9, 18]. Aifantis in the interesting paper [8] indicated that the classical reaction-diffusion equation does not include aspects of the reaction problem - diffusion. He has established mathematical models with many concrete examples that can contain elasticity and pressure according to the following equation:

$$w_t - \Delta w - k\Delta w_t = F(w) + g, \tag{2}$$

As we know, pseudoparabolic-type equations have applications in many areas of mathematics and physics to describe many physical phenomena. A significant point of interest in examining this equation is the presence of the quantity $-\Delta u_t$ in the first equation of (30) . Under the appearance of this term, we call Problem (30) pseudo-parabolic equation (see [17, 12]).

As we know, there are various papers on semilinear Volterra integrodifferential equations with integer order derivative. The paper [20] considered the following Volterra integro-differential equation. The paper [21] focused on the fractional Volterra integro-differential equation with ψ-Hilfer fractional derivative. In [2], the authors showed the existence of the mild solution of nonlinear fractional integro-differential equations in Banach spaces.

A new point of the main equation of (30) is the appearance of the conformable derivative D_t^α. In the following, we introduce the definition of conformable derivative. Let A be a Banach space and the function $\psi : [0, \infty) \to A$. If the following limitation

$$D_t^\alpha \psi := \lim_{\vartheta \to 0} \frac{\psi(t + \vartheta t^{1-\alpha}) - \psi(t)}{\vartheta} \tag{3}$$

for each $t > 0$, then we call it the conformable dertivative. More information about conformable, we can provide some papers [1, 3, 4, 5, 14, 15, 16].

Let us assume that the function f is a real function and $s > 0$, then f has a conformable fractional derivative of order α at s if and only if it is (classically) differentiable at s, and

$$D_t^\alpha f(s) = s^{1-\alpha} f'(s) \tag{4}$$

where $0 < \alpha \leq 1$.

To the best of our knowledge, there are not any results concerning to the (30). Motivated by this reason, in this paper, we study (30) with some various directions. The apperance of memory term on the right hand side of (30) making computation cumbersome. The principal contributions of our paper are described in detail as follows

- The main first result is to prove the existence of local solutions. The key of the proof is based on Banach fixed point theorem.
- The second major contribution is the proof that the solution of the problem (30) converges to the solution of the classical heat equation.

The analysis of our paper is learn from the ideas of the paper of Van [22]. However, we have a new few points different from [22] since the singularity of some proper integrals. In addition, we used some interesting techniques in the recent paper [15, 19]. The complexity of this problem also comes from the memory component.

The structure of the paper is shown as follows. In section 2, we introduce premilinaries. Section 3 shows that the local existence of mild solution. In section 4, we prove that the convergence of the mild solution when $k \to 0^+$.

2. Preliminaries

For each number $\theta \geq 0$, we define the following space

$$\mathbb{X}^{\theta}(\Omega) = \left\{ f = \sum_{n=1}^{\infty} f_n e_n \in L^2(\Omega) : \|f\|^2_{\mathbb{X}^{\theta}(\Omega)} = \sum_{n=1}^{\infty} f_n^2 \lambda_n^{2\theta} < \infty \right\}. \tag{5}$$

It is obvious to remind that $D_t^{\alpha} w(t) = t^{1-\alpha} w'(s)$. By a simple calculation, we get the following ordinary differential equation

$$t^{1-\alpha} \frac{dw_j}{dt} + \frac{\lambda_j}{1+k\lambda_j} w_j(t) = \frac{1}{1+k\lambda_j} F_j(w) + \frac{1}{1+k\lambda_j} G_j(w).$$

Then we get the following identity

$$\begin{aligned} w_n(t) &= \exp\Big(- \frac{\lambda_n}{1+k\lambda_n} \frac{t^{\alpha}}{\alpha} \Big) w_n^0 + \frac{1}{1+k\lambda_n} \int_0^t s^{\alpha-1} \exp\Big(- \frac{\lambda_n}{1+k\lambda_n} \frac{t^{\alpha}-s^{\alpha}}{\alpha} \Big) F_n(w)(s) ds \\ &\quad + \frac{b}{1+k\lambda_n} \int_0^t s^{\alpha-1} \exp\Big(- \frac{\lambda_n}{1+k\lambda_n} \frac{t^{\alpha}-s^{\alpha}}{\alpha} \Big) G_n(w)(s) ds, \end{aligned} \tag{6}$$

where

$$w_n(t) = \int_{\Omega} w(x) e_n(x), \quad F_n(w)(t) = \int_{\Omega} F(w(t)) e_n(x), \; G_n(w)(r) = \int_{\Omega} \int_0^r (r-z)^{-\theta} w(z) dz e_n(x)$$

Definition 2.1. Let us call w the mild solution of the problem (30) if it satisfies the following equality

$$\begin{aligned} w(t) &= \mathcal{B}\big(\frac{t^{\alpha}}{\alpha}\big) w^0 + \int_0^t s^{\alpha-1} \mathcal{B}\big(\frac{t^{\alpha}-s^{\alpha}}{\alpha}\big) F(w(s)) ds \\ &\quad + \int_0^t s^{\alpha-1} \mathcal{B}\big(\frac{t^{\alpha}-s^{\alpha}}{\alpha}\big) \int_0^s (s-z)^{-\beta} w(z) dz ds, \end{aligned} \tag{7}$$

where $\mathcal{B}(t)$ is defined by

$$\mathcal{B}(t)\varphi = \sum_{n=1}^{\infty} \exp\Big(- \frac{\lambda_n}{1+k\lambda_n} t \Big) \varphi_n e_n(x),$$

for any $f \in L^2(\Omega)$. Here β is chosen later.

Lemma 2.1. *Let φ be the function in $\mathbb{X}^m(\Omega)$. Let γ be a positive number such that $0 < \gamma < 1$. Then we get*

$$\Big\| \mathcal{B}(t)\varphi \Big\|_{\mathbb{X}^m(\Omega)} \leq \overline{C}_{\gamma} t^{-\gamma} \|\varphi\|_{\mathbb{X}^m(\Omega)}. \tag{8}$$

Proof. The proof of above Lemma can be found in [22]. □

3. Local existence

Theorem 3.1. *The function F satisfies the globally Lipschitz condition*

$$\big\| F(u) - F(v) \big\|_{\mathbb{X}^{\theta}(\Omega)} \leq K_f \big\| u - v \big\|_{\mathbb{X}^{\theta}(\Omega)}, \tag{9}$$

Let $f \in \mathbb{X}^{\theta}(\Omega)$. Then Problem (30) has a local existence $u \in L^{\infty}_m(0,T;\mathbb{X}^{\theta}(\Omega))$ where

$$\lambda\gamma \leq m < \frac{\alpha}{2}.$$

In addition, we also have

$$\Big\| u \Big\|_{L^{\infty}_m(0,T;\mathbb{X}^{\theta}(\Omega))} \leq C_1(\alpha,\gamma) T^{m-\alpha\gamma} \|f\|_{\mathbb{X}^{\theta}(\Omega)}. \tag{10}$$

Proof. Let us define the following function

$$\mathcal{Q}w(t) = \mathcal{B}\Big(\frac{t^\alpha}{\alpha}\Big)f + \int_0^t s^{\alpha-1}\mathcal{B}\Big(\frac{t^\alpha - s^\alpha}{\alpha}\Big)F(w(s))ds \\ + b\int_0^t s^{\alpha-1}\mathcal{B}\Big(\frac{t^\alpha - s^\alpha}{\alpha}\Big)\int_0^s (s-z)^{-\beta}w(z)dzds. \tag{11}$$

It is obvious to see that

$$\mathcal{Q}\left(w(t)=0\right) = \mathcal{B}\Big(\frac{t^\alpha}{\alpha}\Big)f.$$

Using Lemma (2.1), we have that

$$\Big\|\mathcal{B}\Big(\frac{t^\alpha}{\alpha}\Big)f\Big\|_{\mathbb{X}^\theta(\Omega)} \le \overline{C}_\gamma\Big(\frac{t^\alpha}{\alpha}\Big)^{-\gamma}\|f\|_{\mathbb{X}^\theta(\Omega)} \le C_1(\alpha,\gamma)t^{-\alpha\gamma}\|f\|_{\mathbb{X}^\theta(\Omega)}. \tag{12}$$

Multiplying both sides of the above expression by t^m, we immediately have

$$t^m\Big\|\mathcal{B}\Big(\frac{t^\alpha}{\alpha}\Big)f\Big\|_{\mathbb{X}^\theta(\Omega)} \le C_1(\alpha,\gamma)t^{m-\alpha\gamma}\|f\|_{\mathbb{X}^\theta(\Omega)}. \tag{13}$$

Since $m \ge \alpha\gamma$, we deduce that $\mathcal{Q}\left(w(t)=0\right) \in L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))$. Our next goal is to estimate the norm of the difference $\Big\|\mathcal{Q}u - \mathcal{Q}v\Big\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))}$ for any functions $u, v \in L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))$. Indeed, we have that

$$\mathcal{Q}u(t) - \mathcal{Q}v(t) = \int_0^t s^{\alpha-1}\mathcal{B}\Big(\frac{t^\alpha - s^\alpha}{\alpha}\Big)\Big(F(u(s)) - F(v(s))\Big)ds \\ + b\int_0^t s^{\alpha-1}\mathcal{B}\Big(\frac{t^\alpha - s^\alpha}{\alpha}\Big)\int_0^s (s-z)^{-\beta}\Big(u(z) - v(z)\Big)dzds = J_1(t) + J_2(t). \tag{14}$$

Let us consider the first term $J_1(t)$ on the right above. Using Lemma (2.1), one has

$$\Big\|\mathcal{B}\Big(\frac{t^\alpha - s^\alpha}{\alpha}\Big)f\Big\|_{\mathbb{X}^\theta(\Omega)} \le \overline{C}_\gamma\Big(\frac{t^\alpha - s^\alpha}{\alpha}\Big)^{-\gamma}\|f\|_{\mathbb{X}^\theta(\Omega)} \le C_1(\alpha,\gamma)\Big(t^\alpha - s^\alpha\Big)^{-\gamma}\|f\|_{\mathbb{X}^\theta(\Omega)}. \tag{15}$$

Here $\frac{1}{2} < \gamma < 1$. Since this equality and using the inequality, we bound J_1 as follows

$$\Big\|J_1(t)\Big\|_{\mathbb{X}^\theta(\Omega)} \le C_1(\alpha,\gamma)\int_0^t s^{\alpha-1}\Big(t^\alpha - s^\alpha\Big)^{-\gamma}\Big\|F(u(s)) - F(v(s))\Big\|_{\mathbb{X}^\theta(\Omega)}ds \\ \le K_f C_1(\alpha,\gamma)\int_0^t s^{\alpha-1}\Big(t^\alpha - s^\alpha\Big)^{-\gamma}\Big\|u(s) - v(s)\Big\|_{\mathbb{X}^\theta(\Omega)}ds. \tag{16}$$

Multiplying both sides of the above expression by t^m, we infer that

$$t^m\Big\|J_1(t)\Big\|_{\mathbb{X}^\theta(\Omega)} \le K_f C_1(\alpha,\gamma)t^m\int_0^t s^{\alpha-1}\Big(t^\alpha - s^\alpha\Big)^{-\gamma}\Big\|u(s) - v(s)\Big\|_{\mathbb{X}^\theta(\Omega)}ds \\ = K_f C_1(\alpha,\gamma)t^m\int_0^t s^{\alpha-1-m}\Big(t^\alpha - s^\alpha\Big)^{-\gamma}s^m\Big\|u(s) - v(s)\Big\|_{\mathbb{X}^\theta(\Omega)}ds \\ \le K_f C_1(\alpha,\gamma)t^m\Big[\int_0^t s^{\alpha-1-m}\Big(t^\alpha - s^\alpha\Big)^{-\gamma}ds\Big]\Big\|u - v\Big\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))}. \tag{17}$$

Let us construct the integral term on the right above. Using Hölder inequality, we find that

$$\Big(\int_0^t s^{\alpha-1-m}\Big(t^\alpha - s^\alpha\Big)^{-\gamma}ds\Big)^2 \le \Big(\int_0^t s^{\alpha-1-2m}ds\Big)\Big(\int_0^t s^{\alpha-1}\Big(t^\alpha - s^\alpha\Big)^{-2\gamma}ds\Big) \\ = \frac{t^{\alpha-2m}}{\alpha-2m}\Big(\int_0^t s^{\alpha-1}\Big(t^\alpha - s^\alpha\Big)^{-2\gamma}ds\Big). \tag{18}$$

It is not difficult to check that the following equality

$$\int_0^t s^{\alpha-1}\Big(t^\alpha - s^\alpha\Big)^{-2\gamma} ds = \frac{-1}{\alpha}\int_0^t \Big(t^\alpha - s^\alpha\Big)^{-2\gamma} d(t^\alpha - s^\alpha)$$
$$= \frac{1}{\alpha}\int_0^{t^\alpha} w^{-2\gamma} dw = \frac{t^{\alpha(1-2\gamma)}}{\alpha(1-2\gamma)}. \tag{19}$$

Combining (17), (18) and (19), we derive that

$$t^m \Big\|J_1(t)\Big\|_{\mathbb{X}^\theta(\Omega)} \le \frac{K_f C_1(\alpha,\gamma)}{\sqrt{\alpha(\alpha-2m)(1-2\gamma)}} t^{\alpha-\alpha\gamma}\Big\|u-v\Big\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))}. \tag{20}$$

Let us move to the considering of J_2. Indeed, by using similar claim of (16), we know that

$$\Big\|J_2(t)\Big\|_{\mathbb{X}^\theta(\Omega)} \le bC_1(\alpha,\gamma)\int_0^t s^{\alpha-1}\Big(t^\alpha - s^\alpha\Big)^{-\gamma}\Big\|\int_0^s (s-z)^{-\beta}\Big(u(z)-v(z)\Big)dz\Big\|_{\mathbb{X}^\theta(\Omega)} ds. \tag{21}$$

We have the following observation

$$\Big\|\int_0^s (s-z)^{-\beta}\Big(u(z)-v(z)\Big)dz\Big\|_{\mathbb{X}^\theta(\Omega)} \le \Big(\int_0^s (s-z)^{-\beta} z^{-m} dz\Big)\Big\|u-v\Big\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))}$$
$$= s^{1-\beta-m}B(1-\beta,1-m)\Big\|u-v\Big\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))}. \tag{22}$$

Combining (21) and (39), we derive that

$$t^m\Big\|J_2(t)\Big\|_{\mathbb{X}^\theta(\Omega)} \le bC_1(\alpha,\gamma)B(1-\beta,1-m)\Big\|u-v\Big\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))}$$
$$t^m\Big(\int_0^t s^{\alpha-\beta-m}\Big(t^\alpha - s^\alpha\Big)^{-\gamma} ds\Big). \tag{23}$$

Using Hölder inequality, we find that

$$\Big(\int_0^t s^{\alpha-\beta-m}\Big(t^\alpha - s^\alpha\Big)^{-\gamma} ds\Big)^2 \le \Big(\int_0^t s^{\alpha-\beta-2m} ds\Big)\Big(\int_0^t s^{\alpha-\beta}\Big(t^\alpha - s^\alpha\Big)^{-2\gamma} ds\Big)$$
$$= \frac{t^{\alpha-2m-\beta+1}}{\alpha-2m-\beta+1}\left(\int_0^t s^{\alpha-1}\Big(t^\alpha - s^\alpha\Big)^{-2\gamma} ds\right). \tag{24}$$

This estimate together with (19) allows us to get that

$$t^m\Big(\int_0^t s^{\alpha-\beta-m}\Big(t^\alpha - s^\alpha\Big)^{-\gamma} ds\Big)^2 \le \frac{t^{\frac{2\alpha-2m-2\alpha\gamma-\beta+1}{2}}}{\sqrt{\alpha(1-2\gamma)(\alpha-\beta-2m+1)}}. \tag{25}$$

Combibining (23) and (25), we find that

$$t^m\Big\|J_2(t)\Big\|_{\mathbb{X}^\theta(\Omega)} \le \frac{bC_1(\alpha,\gamma)B(1-\beta,1-m)}{\sqrt{\alpha(2\gamma-1)(\alpha-\beta-2m+1)}} t^{\frac{2\alpha-2m-2\alpha\gamma-\beta+1}{2}}\Big\|u-v\Big\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))}. \tag{26}$$

Combining (14), (17) and (26), we derive that

$$t^m\Big\|\mathcal{Q}u(t)-\mathcal{Q}v(t)\Big\|_{\mathbb{X}^\theta(\Omega)} \le t^m\Big\|J_1(t)\Big\|_{\mathbb{X}^\theta(\Omega)} + t^m\Big\|J_2(t)\Big\|_{\mathbb{X}^\theta(\Omega)}$$
$$\le \left(\frac{K_f C_1(\alpha,\gamma)}{\sqrt{\alpha(\alpha-2m)(1-2\gamma)}} t^{\alpha-\alpha\gamma} + \frac{bC_1(\alpha,\gamma)B(1-\beta,1-m)}{\sqrt{\alpha(1-2\gamma)(\alpha-\beta-2m+1)}} t^{\frac{2\alpha-2m-2\alpha\gamma-\beta+1}{2}}\right)\Big\|u-v\Big\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))} \tag{27}$$

Since $0 < \gamma < \frac{1}{2}$ and $\beta < 1$ and $m < \frac{\alpha}{2}$, we know that the following bound

$$\Big\|\mathcal{Q}u - \mathcal{Q}v\Big\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))} \leq \left(\frac{K_f C_1(\alpha,\gamma)}{\sqrt{\alpha(\alpha-2m)(1-2\gamma)}} t^{\alpha-\alpha\gamma} + \frac{bC_1(\alpha,\gamma)B(1-\beta,1-m)}{\sqrt{\alpha(1-2\gamma)(\alpha-\beta-2m+1)}} t^{\frac{2\alpha-2m-2\alpha\gamma-\beta+1}{2}} \right) \Big\|u-v\Big\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))} \tag{28}$$

By choosing the appropriate T, we will immediately have

$$\frac{K_f C_1(\alpha,\gamma)}{\sqrt{\alpha(\alpha-2m)(1-2\gamma)}} T^{\alpha-\alpha\gamma} + \frac{bC_1(\alpha,\gamma)B(1-\beta,1-m)}{\sqrt{\alpha(1-2\gamma)(\alpha-\beta-2m+1)}} T^{\frac{2\alpha-2m-2\alpha\gamma-\beta+1}{2}} \leq \frac{1}{2}.$$

This implies that $\mathcal{Q}$ is a contraction in the space $L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))$. By applying Banach fixed point theorem, we can deduce that $\mathcal{Q}$ has a fixed point $u \in L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))$. So, Problem (30) has a unique solution in the space $L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))$. In addition, using triangle inequality, we find that

$$\begin{aligned}
\Big\|u\Big\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))} &= \Big\|\mathcal{Q}u\Big\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))} \\
&\leq \Big\|\mathcal{Q}u - \mathcal{Q}(v=0)\Big\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))} + \sup_{0\leq t\leq T} t^m \Big\|\mathcal{B}\Big(\frac{t^\alpha}{\alpha}\Big)f\Big\|_{\mathbb{X}^\theta(\Omega)} \\
&\leq \frac{1}{2}\Big\|u\Big\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))} + C_1(\alpha,\gamma)T^{m-\alpha\gamma}\|f\|_{\mathbb{X}^\theta(\Omega)},
\end{aligned} \tag{29}$$

which claims that (10).

□

4. Convergence of the mild solution

In this section, we focus the convergence of the mild solution to problem (30) when $k \to 0^+$.

Theorem 4.1. *Let $w^{(k)}$ be the solution of followingProblem*

$$\begin{cases} D^\alpha_t w + (-\Delta)^\beta w - k\Delta D^\alpha_t w = F(w) + b\displaystyle\int_0^t (t-z)^{-\theta} w(z)dz, & (x,t) \in \Omega \times (0,T), \\ w = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ w(x,0) = f(x) \end{cases} \tag{30}$$

and w^* be the solution of the following classical heat problem

$$\begin{cases} D^\alpha_t w + (-\Delta)^\beta w = F(w) + b\displaystyle\int_0^t (t-z)^{-\theta} w(z)dz, & (x,t) \in \Omega \times (0,T), \\ w = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ w(x,0) = f(x) \end{cases} \tag{31}$$

Here $0 < \beta < 1$. Let us assume that $f \in \mathbb{X}^{-\beta p+\beta q+\theta+\frac{\varepsilon(p-q)}{2}}(\Omega) \cap \mathbb{X}^\theta(\Omega)$ for $q > p$. Then for T enough small then the following estimate holds

$$\begin{aligned}
\|w^{(k)}(t) - w^*(t)\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))} &\leq 2C(\alpha,p,q)k^{\frac{2q-\varepsilon q+\varepsilon p}{2}} T^{m+\alpha q-\alpha p}\Big\|f\Big\|_{\mathbb{X}^{-\beta p+\beta q+\theta+\frac{\varepsilon(p-q)}{2}}(\Omega)} \\
&\quad + 2C(m,\alpha,p,q)T^{\alpha q-\alpha p+\alpha} k^{\frac{2q-\varepsilon q+\varepsilon p}{2}} \Big\|f\Big\|_{\mathbb{X}^\theta(\Omega)} \\
&\quad + \frac{bC(\alpha,p,q,\beta,m)}{\alpha-\beta-m+1} k^{\frac{2q-\varepsilon q+\varepsilon p}{2}} \|f\|_{\mathbb{X}^\theta(\Omega)} T^{\alpha(q-p)+\alpha-\beta-m+1}
\end{aligned} \tag{32}$$

where m is defined in Theorem (3.1) *and* $\max(1,2\beta) < \varepsilon < 2$.

Proof. Let us define that

$$w^k(t) = \mathcal{M}_k\big(\frac{t^\alpha}{\alpha}\big)f + \int_0^t s^{\alpha-1}\mathcal{M}_k\big(\frac{t^\alpha - s^\alpha}{\alpha}\big)F(w^k(s))ds$$
$$+ b\int_0^t s^{\alpha-1}\mathcal{M}_k\big(\frac{t^\alpha - s^\alpha}{\alpha}\big)\int_0^s (s-z)^{-\beta}w^k(z)dzds, \tag{33}$$

where we remind that where $\mathcal{M}_k(t)$ is defined by

$$\mathcal{B}(t)f = \sum_{n=1}^{\infty}\exp\Big(-\frac{\lambda_n}{1+k\lambda_n}t\Big)f_n e_n(x),$$

for any $f \in L^2(\Omega)$. The function u^* is defined by

$$w^*(t) = \mathcal{M}_0\big(\frac{t^\alpha}{\alpha}\big)w^0 + \int_0^t s^{\alpha-1}\mathcal{M}_0\big(\frac{t^\alpha - s^\alpha}{\alpha}\big)F(w^*(s))ds$$
$$+ b\int_0^t s^{\alpha-1}\mathcal{M}_0\big(\frac{t^\alpha - s^\alpha}{\alpha}\big)\int_0^s (s-z)^{-\beta}w^*(z)dzds, \tag{34}$$

where $\mathcal{M}_0(t)f = \sum_{n=1}^{\infty}\exp\Big(-\lambda_n t\Big)f_n e_n(x)$ for any $f \in L^2(\Omega)$. Two equalities above provide us to get that

$$\begin{aligned}
w^{(k)}(t) - w^*(t) &= \mathcal{M}_k\big(\frac{t^\alpha}{\alpha}\big)f - \mathcal{M}_0\big(\frac{t^\alpha}{\alpha}\big)f + \int_0^t s^{\alpha-1}\mathcal{M}_k\big(\frac{t^\alpha - s^\alpha}{\alpha}\big)\Big(F(w^{(k)}(s)) - F(w^*(s))\Big)ds \\
&+ \int_0^t \Big(s^{\alpha-1}\mathcal{M}_k\big(\frac{t^\alpha - s^\alpha}{\alpha}\big) - s^{\alpha-1}\mathcal{M}_0\big(\frac{t^\alpha - s^\alpha}{\alpha}\big)\Big)F(w^*(s))ds \\
&+ b\int_0^t s^{\alpha-1}\mathcal{M}_k\big(\frac{t^\alpha - s^\alpha}{\alpha}\big)\int_0^s (s-z)^{-\beta}\Big(w^k(z) - w^0(z)\Big)dzds \\
&+ b\int_0^t s^{\alpha-1}\Big(\mathcal{M}_k\big(\frac{t^\alpha - s^\alpha}{\alpha}\big) - \mathcal{M}_0\big(\frac{t^\alpha - s^\alpha}{\alpha}\big)\Big)\int_0^s (s-z)^{-\beta}w^*(z)dzds \\
&= J_1(t) + J_2(t) + J_3(t) + J_4(t) + J_5(t).
\end{aligned} \tag{35}$$

In view of step 1 of Proof of Theorem 5.1 [15], we confirm the following result

$$\left|\exp\left(-\frac{\lambda_n^\beta}{1+k\lambda_n}\frac{t^\alpha}{\alpha}\right) - \exp\left(-\lambda_n^\beta\frac{t^\alpha}{\alpha}\right)\right| \le C(\alpha,p,q)k^{\frac{2q-\varepsilon q+\varepsilon p}{2}}t^{\alpha q-\alpha p}\lambda_n^{-\beta p+\beta q+\frac{\varepsilon(p-q)}{2}} \tag{36}$$

for any $q > p > 0$ and $1 < \varepsilon < 2$. Thus, we have

$$\begin{aligned}
\Big\|\mathcal{M}_k\big(\frac{t^\alpha}{\alpha}\big)f - \mathcal{M}_0\big(\frac{t^\alpha}{\alpha}\big)f\Big\|^2_{\mathbb{X}^\theta(\Omega)} &= \sum_{n=1}^{\infty}\lambda_n^{2\theta}\left(\exp\Big(-\frac{\lambda_n^\beta}{1+k\lambda_n}\frac{t^\alpha}{\alpha}\Big) - \exp\Big(-\lambda_n^\beta\frac{t^\alpha}{\alpha}\Big)\right)^2|f_n|^2 \\
&\le |C(\alpha,p,q)|^2 k^{2q-\varepsilon q+\varepsilon p}t^{2\alpha q-2\alpha p}\sum_{n=1}^{\infty}\lambda_n^{2\theta-2\beta p+2\beta q+\varepsilon(p-q)}|f_n|^2.
\end{aligned}$$

Hence, using Parseval's equality, we obtain that

$$\Big\|J_1(t)\Big\|_{\mathbb{X}^\theta(\Omega)} = \Big\|\mathcal{M}_k\big(\frac{t^\alpha}{\alpha}\big)f - \mathcal{M}_0\big(\frac{t^\alpha}{\alpha}\big)f\Big\|_{\mathbb{X}^\theta(\Omega)} \le C(\alpha,p,q)k^{\frac{2q-\varepsilon q+\varepsilon p}{2}}t^{\alpha q-\alpha p}\Big\|f\Big\|_{\mathbb{X}^{-\beta p+\beta q+\theta+\frac{\varepsilon(p-q)}{2}}(\Omega)}. \tag{37}$$

Multiplying bothsides to t^m, we deduce that

$$\|J_1\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))} \le C(\alpha,p,q)k^{\frac{2q-\varepsilon q+\varepsilon p}{2}} T^{m+\alpha q-\alpha p}\Big\|f\Big\|_{\mathbb{X}^{-\beta p+\beta q+\theta+\frac{\varepsilon(p-q)}{2}}(\Omega)}. \tag{38}$$

For the second term J_2, using the estimate (16), we obtain the following bound

$$\Big\|J_2(t)\Big\|_{\mathbb{X}^\theta(\Omega)} \le K_f C_1(\alpha,\gamma)\int_0^t s^{\alpha-1}\Big(t^\alpha - s^\alpha\Big)^{-\gamma}\Big\|w^k(s)-w^*(s)\Big\|_{\mathbb{X}^\theta(\Omega)}ds. \tag{39}$$

By a similar explanation as in (20), we obtain

$$t^m\Big\|J_2(t)\Big\|_{\mathbb{X}^\theta(\Omega)} \le \frac{K_f C_1(\alpha,\gamma)}{\sqrt{\alpha(\alpha-2m)(1-2\gamma)}} t^{\alpha-\alpha\gamma}\Big\|w^k-w^*\Big\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))}. \tag{40}$$

This implies that the following bound

$$\|J_2\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))} \le \frac{K_f C_1(\alpha,\gamma)}{\sqrt{\alpha(\alpha-2m)(1-2\gamma)}} T^{\alpha-\alpha\gamma}\Big\|w^k-w^*\Big\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))}. \tag{41}$$

Let us now return to the term J_3. Indeed, in view of Proof of Theorem 5.1 [15], we find that

$$\begin{aligned}&\Big\|\int_0^t \Big(s^{\alpha-1}\mathcal{M}_k\big(\frac{t^\alpha-s^\alpha}{\alpha}\big) - s^{\alpha-1}\mathcal{M}_0\big(\frac{t^\alpha-s^\alpha}{\alpha}\big)\Big) F(w^*(s))ds\Big\|_{\mathbb{X}^\theta(\Omega)}\\ &\le C(\alpha,p,q)k^{\frac{2q-\varepsilon q+\varepsilon p}{2}}\int_0^t s^{\alpha-1}\Big(t^\alpha-s^\alpha\Big)^{q-p}\Big\|F(w^*(s))\Big\|_{\mathbb{X}^{-\beta p+\beta q+\theta+\frac{\varepsilon(p-q)}{2}}(\Omega)}ds\end{aligned} \tag{42}$$

Let us choose ε such that

$$\max(1,2\beta) < \varepsilon < 2.$$

Then, it is easy to verify that the following condition

$$-\beta p+\beta q+\theta+\frac{\varepsilon(p-q)}{2} \le \theta.$$

Thus, using globally Lipschitz of F as in (9), we obtain the following estimate

$$\begin{aligned}\Big\|F(w^*(s))\Big\|_{\mathbb{X}^{-\beta p+\beta q+\theta+\frac{\varepsilon(p-q)}{2}}(\Omega)} &\le C(\beta,p,q,\theta)\Big\|F(w^*(s))\Big\|_{\mathbb{X}^\theta(\Omega)}\\ &\le C(\beta,p,q,\theta)K_f\Big\|w^*(s)\Big\|_{\mathbb{X}^\theta(\Omega)}\end{aligned} \tag{43}$$

It is easy to show that $w^* \in L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))$. By a similar proof of (10), we have the following result

$$\Big\|w^*(s)\Big\|_{\mathbb{X}^\theta(\Omega)} \le Cs^{-m}\Big\|f\Big\|_{\mathbb{X}^\theta(\Omega)}. \tag{44}$$

Combining (42) and (44), we derive that

$$\Big\|J_3(t)\Big\|_{\mathbb{X}^\theta(\Omega)} \le C(\alpha,p,q)k^{\frac{2q-\varepsilon q+\varepsilon p}{2}}\Big\|f\Big\|_{\mathbb{X}^\theta(\Omega)}\int_0^t s^{\alpha-1-m}\Big(t^\alpha-s^\alpha\Big)^{q-p}ds. \tag{45}$$

It is obvious to claim that

$$\int_0^t s^{\alpha-1-m}\Big(t^\alpha-s^\alpha\Big)^{q-p}ds \le t^{\alpha q-\alpha p}\int_0^t s^{\alpha-1-m}ds = \frac{t^{\alpha q-\alpha p+\alpha-m}}{\alpha-m} \tag{46}$$

From two latter obsevations, we have immediately that

$$t^m \Big\| J_3(t) \Big\|_{\mathbb{X}^\theta(\Omega)} \le C(m,\alpha,p,q) T^{\alpha q-\alpha p+\alpha} k^{\frac{2q-\varepsilon q+\varepsilon p}{2}} \Big\| f \Big\|_{\mathbb{X}^\theta(\Omega)}, \tag{47}$$

which allows us to obtain that

$$\|J_3\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))} \le C(m,\alpha,p,q) T^{\alpha q-\alpha p+\alpha} k^{\frac{2q-\varepsilon q+\varepsilon p}{2}} \Big\| f \Big\|_{\mathbb{X}^\theta(\Omega)}. \tag{48}$$

For the term J_4, we use the same techniques as in (26) in order to obtain that

$$\|J_4\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))} \le \frac{bC_1(\alpha,\gamma)B(1-\beta,1-m)}{\sqrt{\alpha(2\gamma-1)(\alpha-\beta-2m+1)}} T^{\frac{2\alpha-2m-2\alpha\gamma-\beta+1}{2}} \Big\| w^k - w^* \Big\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))}. \tag{49}$$

Let us now return to the term J_5. Indeed, in view of Proof of Theorem 5.1 [15], we find that

$$\begin{aligned} &\Big\| b\int_0^t s^{\alpha-1}\Big(\mathcal{M}_k\big(\frac{t^\alpha-s^\alpha}{\alpha}\big) - \mathcal{M}_0\big(\frac{t^\alpha-s^\alpha}{\alpha}\big)\Big)\int_0^s (s-z)^{-\beta} w^*(z)dzds \Big\|_{\mathbb{X}^\theta(\Omega)} \\ &\le bC(\alpha,p,q)k^{\frac{2q-\varepsilon q+\varepsilon p}{2}} \int_0^t s^{\alpha-1}\Big(t^\alpha-s^\alpha\Big)^{q-p} \Big\| \int_0^s (s-z)^{-\beta} w^*(z)dz \Big\|_{\mathbb{X}^{-\beta p+\beta q+\theta+\frac{\varepsilon(p-q)}{2}}(\Omega)} ds. \end{aligned} \tag{50}$$

It is easy to see that

$$\begin{aligned} &\int_0^s (s-z)^{-\beta} w^*(z)dz \Big\|_{\mathbb{X}^{-\beta p+\beta q+\theta+\frac{\varepsilon(p-q)}{2}}(\Omega)} \\ &\le \int_0^s (s-z)^{-\beta} z^{-m} z^m \Big\| w^*(z) \Big\|_{\mathbb{X}^\theta(\Omega)} dz \\ &\le \Big(\int_0^s (s-z)^{-\beta} z^{-m} dz\Big) \Big\| w^* \Big\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))} \\ &\le CB(1-\beta,1-m)s^{1-\beta-m} \Big\| f \Big\|_{\mathbb{X}^\theta(\Omega)}. \end{aligned} \tag{51}$$

Combining (50) and (51), we derive that

$$\begin{aligned} t^m \Big\| J_5(t) \Big\|_{\mathbb{X}^\theta(\Omega)} &\le bC(\alpha,p,q,\beta,m)k^{\frac{2q-\varepsilon q+\varepsilon p}{2}} \|f\|_{\mathbb{X}^\theta(\Omega)} \Big(\int_0^t s^{\alpha-\beta-m}\Big(t^\alpha-s^\alpha\Big)^{q-p} ds\Big) \\ &\le bC(\alpha,p,q,\beta,m)k^{\frac{2q-\varepsilon q+\varepsilon p}{2}} \|f\|_{\mathbb{X}^\theta(\Omega)} t^{\alpha(q-p)} \Big(\int_0^t s^{\alpha-\beta-m} ds\Big) \\ &= \frac{bC(\alpha,p,q,\beta,m)}{\alpha-\beta-m+1} k^{\frac{2q-\varepsilon q+\varepsilon p}{2}} \|f\|_{\mathbb{X}^\theta(\Omega)} t^{\alpha(q-p)+\alpha-\beta-m+1}. \end{aligned} \tag{52}$$

This inequality implies that

$$\|J_5\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))} \le \frac{bC(\alpha,p,q,\beta,m)}{\alpha-\beta-m+1} k^{\frac{2q-\varepsilon q+\varepsilon p}{2}} \|f\|_{\mathbb{X}^\theta(\Omega)} T^{\alpha(q-p)+\alpha-\beta-m+1}. \tag{53}$$

Combining (35), (38), (41), (48), (53), we derive that

$$\begin{aligned}
\|w^{(k)}(t)-w^*(t)\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))} &\le \|J_1\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))}+\|J_2\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))}+\|J_3\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))}\\
&\quad+\|J_4\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))}+\|J_5\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))}\\
&\le C(\alpha,p,q)k^{\frac{2q-\varepsilon q+\varepsilon p}{2}}T^{m+\alpha q-\alpha p}\Big\|f\Big\|_{\mathbb{X}^{-\beta p+\beta q+\theta+\frac{\varepsilon(p-q)}{2}}(\Omega)}\\
&\quad+\frac{K_f C_1(\alpha,\gamma)}{\sqrt{\alpha(\alpha-2m)(1-2\gamma)}}T^{\alpha-\alpha\gamma}\Big\|w^k-w^*\Big\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))}\\
&\quad+C(m,\alpha,p,q)T^{\alpha q-\alpha p+\alpha}k^{\frac{2q-\varepsilon q+\varepsilon p}{2}}\Big\|f\Big\|_{\mathbb{X}^\theta(\Omega)}\\
&\quad+\frac{bC_1(\alpha,\gamma)B(1-\beta,1-m)}{\sqrt{\alpha(2\gamma-1)(\alpha-\beta-2m+1)}}T^{\frac{2\alpha-2m-2\alpha\gamma-\beta+1}{2}}\Big\|w^k-w^*\Big\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))}\\
&\quad+\frac{bC(\alpha,p,q,\beta,m)}{\alpha-\beta-m+1}k^{\frac{2q-\varepsilon q+\varepsilon p}{2}}\|f\|_{\mathbb{X}^\theta(\Omega)}T^{\alpha(q-p)+\alpha-\beta-m+1}. \qquad (54)
\end{aligned}$$

Let us choose T such that

$$\frac{K_f C_1(\alpha,\gamma)}{\sqrt{\alpha(\alpha-2m)(1-2\gamma)}}T^{\alpha-\alpha\gamma}+\frac{bC_1(\alpha,\gamma)B(1-\beta,1-m)}{\sqrt{\alpha(2\gamma-1)(\alpha-\beta-2m+1)}}T^{\frac{2\alpha-2m-2\alpha\gamma-\beta+1}{2}}\le\frac{1}{2}.$$

Then we get

$$\begin{aligned}
\|w^{(k)}(t)-w^*(t)\|_{L^\infty_m(0,T;\mathbb{X}^\theta(\Omega))} &\le 2C(\alpha,p,q)k^{\frac{2q-\varepsilon q+\varepsilon p}{2}}T^{m+\alpha q-\alpha p}\Big\|f\Big\|_{\mathbb{X}^{-\beta p+\beta q+\theta+\frac{\varepsilon(p-q)}{2}}(\Omega)}\\
&\quad+2C(m,\alpha,p,q)T^{\alpha q-\alpha p+\alpha}k^{\frac{2q-\varepsilon q+\varepsilon p}{2}}\Big\|f\Big\|_{\mathbb{X}^\theta(\Omega)}\\
&\quad+\frac{bC(\alpha,p,q,\beta,m)}{\alpha-\beta-m+1}k^{\frac{2q-\varepsilon q+\varepsilon p}{2}}\|f\|_{\mathbb{X}^\theta(\Omega)}T^{\alpha(q-p)+\alpha-\beta-m+1} \qquad (55)
\end{aligned}$$

The proof is completed. □

5. Acknowledgement

This research is funded by Vietnam National University HoChiMinh City (VNU-HCM) under grant number C2022-44-12.

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