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FRACTIONAL DIRAC SYSTEMS WITH MITTAG–LEFFLER KERNEL

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Abstract. In this paper, we study some fractional Dirac-type systems with the Mittag–Leffler kernel. We extend the basic spectral properties of the ordinary Dirac system to the Dirac-type systems with the Mittag–Leffler kernel. First, this problem was handled in a continuous form. The self-adjointness of the operator produced by this system, the reality of its eigenvalues, and the orthogonality of the eigenfunctions have been investigated. Later, similar results were obtained by considering the discrete state.

1. Introduction

In recent years, the subject of fractional differential equations has become very popular among mathematicians. The investigation of all kinds of problems in the theory of differential equations under the framework of fractional has revealed a very wide field of study. The Dirac equation, which is one of the important equations in the history of physics, should also be investigated. Although fractional Sturm–Liouville problems have been investigated a lot, research on fractional Dirac equivalents is less. Contributing to the gap in this area in the literature is the main motivation of this research.

There are many types of fractional derivatives. One of them is the one based on the Mittag–Leffler function. Atangana and Baleanu introduced a new fractional derivative with the Mittag–Leffler kernel [4]. In [1], Abdeljawad and Belanau defined integration with the part formula using the right fractional derivative and the right fractional integral corresponding to the Mittag–Leffler kernel. In [5], the

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authors studied the discrete versions of these fractional derivatives. In [12], Mert et al. studied fractional Sturm–Liouville operators with the Mittag–Leffler kernels. With the help of the Laplace transform, Ercan is obtained the representation of solutions for fractional Dirac system with the Mittag–Leffler kernel ([8]). Yalçınkaya handled some Dirac systems with exponential kernel in [13]. In [7], the authors studied a fractional Sturm–Liouville problem with exponential and Mittag-Leffler kernels.

In this study, we will investigate this type of fractional version of the Dirac system. Some basic features will be obtained for such systems. In the first chapter, the basic concepts and theorems that will be used in the study are given. In the following sections, the Dirac system with the Mittag–Leffler kernel in a continuous and discrete cases is discussed. This type of fractional Dirac system turns into the classical Dirac system by taking $\alpha \to 1$. It is transformed into a Riemann-Liouville type fractional Dirac system with a Laplace transform method. In this way, we examine these two systems under a single system. According to the knowledge of the authors, since there is no study on this subject in the literature, it will contribute to researchers working on this subject.

2. Preliminaries

This section covers the definitions and properties of fractional derivatives with the Mittag–Leffler kernel.

Definition 1. *([1]) Let $u \in H^1(a,b)$ (the usual Sobolev space), $a < b$, $\alpha \in [0,1]$. Then the definition of the left Caputo fractional derivative with the Mittag–Leffler kernel is given by*

$$ {}_{a}^{ABC}D^{\alpha}u\left(\xi\right) = \frac{B(\alpha)}{1-\alpha}\int_{a}^{\xi} E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(\xi-t)^{\alpha}\right) d\left(u\left(t\right)\right), \tag{1}$$

where $B(\alpha) > 0$ is a normalization function with $B(0) = B(1) = 1$;

$$E_{\alpha,\beta}\left(t\right) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma\left(\alpha k+\beta\right)}, \tag{2}$$

and $E_{\alpha}\left(t\right) = E_{\alpha,1}\left(t\right)$. The convergence condition of infinite series (2) is $\operatorname{Re}\alpha > 0$ and $\operatorname{Re}\beta > 0$ ([9]). Similarly, the left Riemann–Liouville fractional derivative with the Mittag–Leffler kernel has the following form

$$ {}_{a}^{ABR}D^{\alpha}u\left(\xi\right) = \frac{B(\alpha)}{1-\alpha}\frac{d}{d\xi}\int_{a}^{\xi} E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(\xi-t)^{\alpha}\right) u(t)dt. \tag{3}$$

The associated fractional integral is given by

$$ {}_{a}^{AB}I^{\alpha}u\left(\xi\right) = \frac{1-\alpha}{B(\alpha)}u(\xi) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{a}^{\xi}\left(\xi-t\right)^{\alpha-1}u(t)dt.$$

The right Caputo fractional derivative with the Mittag–Leffler kernel is given by

$$^{ABC}D_b^{\alpha}u\left(\xi\right)=-\frac{B(\alpha)}{1-\alpha}\int_{\xi}^{b}E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(t-\xi)^{\alpha}\right)u(t)dt, \tag{4}$$

and the right Riemann–Liouville derivative with the Mittag–Leffler kernel is defined by the formula

$$^{ABR}D_b^{\alpha}u\left(\xi\right)=-\frac{B(\alpha)}{1-\alpha}\frac{d}{d\xi}\int_{\xi}^{b}E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(t-\xi)^{\alpha}\right)u(t)dt.$$

Moreover, the corresponding fractional integral is given by

$$^{AB}I_b^{\alpha}u\left(\xi\right)=\frac{1-\alpha}{B(\alpha)}u(\xi)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{\xi}^{b}\left(t-\xi\right)^{\alpha-1}u(t)dt.$$

Proposition 1. *([1]) Let* $\alpha>0,\ p\geq 1,\ q\geq 1,$ *and* $\frac{1}{p}+\frac{1}{q}\leq 1+\alpha$ $(p\neq 1$ *and* $q\neq 1$ *when* $\frac{1}{p}+\frac{1}{q}=1+\alpha)$*.*

(1) If $u\in L_p(a.b)$ and $v\in L_q(a.b)$, then

$$\int_a^b u(\xi)_a^{AB}I^{\alpha}v(\xi)d\xi=\int_a^b v(\xi)^{AB}I_b^{\alpha}u(\xi)d\xi.$$

(2) If $u\in^{AB}I_b^{\alpha}(L_p)$ and $v\in_a^{AB}I^{\alpha}(L_q)$, then

$$\int_a^b u(\xi)_a^{ABR}D^{\alpha}v(\xi)d\xi=\int_a^b v(\xi)^{ABR}D_b^{\alpha}f(\xi)d\xi,$$

where

$$^{AB}I_b^{\alpha}(L_p)=\left\{u:u=^{AB}I_b^{\alpha}v,v\in L_p(a.b)\right\},$$

and

$$_a^{AB}I^{\alpha}(L_q)=\left\{u:u=_a^{AB}I^{\alpha}v,v\in L_q(a.b)\right\}.$$

Theorem 1. *([1]) Let* $u,v\in H^1(a,b),\ a<b$ *and* $\alpha\in(0,1)$*. Then we have*

(1)

$$\int_a^b u(\xi)_a^{ABC}D^{\alpha}v(\xi)d\xi=\int_a^b v(\xi)^{ABR}D_b^{\alpha}u(\xi)d\xi$$

$$+\frac{B(\alpha)}{1-\alpha}v(\xi)E_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}u(\xi)|_a^b,$$

where

$$E_{\alpha,\beta,w,b^-}u\left(\xi\right)=\int_{\xi}^{b}\left(t-\xi\right)^{\beta-1}E_{\alpha,\beta}\left(w\left(t-\xi\right)^{\alpha}\right)u\left(t\right)dt,\ \xi<b.$$

2.

$$\int_a^b u(\xi)^{ABC}D_b^\alpha v(\xi)d\xi = \int_a^b v(\xi)_a^{ABR}D^\alpha u(\xi)d\xi$$

$$-\frac{B(\alpha)}{1-\alpha}v(b)E_{\alpha,1,\frac{-\alpha}{1-\alpha},a^+}u(b)$$

$$+\frac{B(\alpha)}{1-\alpha}v(a)E_{\alpha,1,\frac{-\alpha}{1-\alpha},a^+}u(a),$$

where

$$E_{\alpha,\beta,w,a^+}u(\xi) = \int_a^\xi (\xi-t)^{\beta-1}E_{\alpha,\beta}\left(w(\xi-t)^\alpha\right)u(t)\,dt,\ \xi > a.$$

Let

$$\mathbb{N}_a = \{a, a+1, a+2, ...\},$$

$${}_b\mathbb{N} = \{..., b-2, b-1, b\},$$

$$\mathbb{N}_{a,b} = \{a, a+1, a+2, ..., b\},$$

where $a, b \in \mathbb{R}$ and $b-a$ is a positive integer.

Definition 2. *([5,6,10]) Let $u : \mathbb{N}_a \to \mathbb{R}$ and $\alpha \in (0, 1/2)$. Then the nabla discrete left Caputo difference with the Mittag–Leffler kernel is defined by*

$${}_a^{ABC}\nabla^\alpha u(\xi) = \frac{B(\alpha)}{1-\alpha}\sum_{i=a+1}^{\xi}\nabla_i u(i)\,E_{\overline{\alpha}}\left(\frac{-\alpha}{1-\alpha}, \xi-\rho(i)\right),\ \xi \in \mathbb{N}_{a+1},$$

and the left Riemannn–Liouville one by

$${}_a^{ABR}\nabla^\alpha u(\xi) = \frac{B(\alpha)}{1-\alpha}\nabla_\xi\sum_{i=a+1}^{\xi}u(i)\,E_{\overline{\alpha}}\left(\frac{-\alpha}{1-\alpha}, \xi-\rho(i)\right),\ \xi \in \mathbb{N}_{a+1},$$

where $\rho(i) = i-1$; and the discrete the Mittag–Leffler kernel is defined by the formula

$$E_{\overline{\alpha}}(\lambda, z) = \sum_{i=0}^{\infty}\lambda^i\frac{z^{\overline{i\alpha}}}{\Gamma(i\alpha+1)},$$

where $z^{\overline{i\alpha}} = \prod_{i=0}^{i\alpha-1}(t+i)$, $z^{\overline{0}} = 1$, $t \in \mathbb{R}$. Moreover, the associated fractional sum function

$${}_a^{AB}\nabla^{-\alpha}u(\xi) = \frac{1-\alpha}{B(\alpha)}u(\xi) + \frac{\alpha}{B(\alpha)}\nabla_a^{-\alpha}u(\xi),\ \xi \in \mathbb{N}_{a+1},$$

where

$$\nabla_a^{-\alpha}u(\xi)=\frac{1}{\Gamma(\alpha)}\sum_{i=a+1}^{\xi}(\xi-\rho(i))^{\overline{\alpha-1}}u(i),\ \xi\in\mathbb{N}_{a+1}\ \textit{(see [2, 3])}.$$

Definition 3. *([5]) Let $u :_b \mathbb{N}\to\mathbb{R}$ and $\alpha\in(0,1/2)$. Then the nabla discrete right Caputo difference with the Mittag–Leffler kernel is defined by*

$${}^{ABC}\nabla_b^{\alpha}u(\xi)=\frac{-B(\alpha)}{1-\alpha}\sum_{i=\xi}^{b-1}\Delta u\,(i)\,E_{\overline{\alpha}}(\frac{-\alpha}{1-\alpha},i-\rho(\xi)),\ \xi\in_{b-1}\mathbb{N},$$

and the right Reimann–Liouville one by

$${}^{ABR}\nabla_b^{\alpha}u(\xi)=\frac{-B(\alpha)}{1-\alpha}\Delta_\xi\sum_{i=\xi}^{b-1}u\,(i)\,E_{\overline{\alpha}}(\frac{-\alpha}{1-\alpha},i-\rho(\xi)),\ \xi\in_{b-1}\mathbb{N}.$$

Further, the associated fractional sum is defined by

$${}^{AB}\nabla_b^{-\alpha}u(\xi)=\frac{1-\alpha}{B(\alpha)}u(\xi)+\frac{\alpha}{B(\alpha)}\nabla_b^{-\alpha}u(\xi),\ \xi\in_{b-1}\mathbb{N},$$

where

$$\nabla_b^{-\alpha}u(\xi)=\frac{1}{\Gamma(\alpha)}\sum_{i=\xi}^{b-1}(i-\rho(\xi))^{\overline{\alpha-1}}u(i),\ \xi\in_{b-1}\mathbb{N}\ \textit{(see [2, 3])}.$$

Theorem 2. *([5]) Let $u,v:\mathbb{N}_{a,b}\to\mathbb{R}$ and $\alpha\in(0,1/2)$. Then we have*

$$\sum_{\xi=a+1}^{b-1}v(\xi){}_a^{AB}\nabla^{-\alpha}u(\xi)=\sum_{\xi=a+1}^{b-1}u(\xi){}^{AB}\nabla_b^{-\alpha}v(\xi),$$

$$\sum_{\xi=a+1}^{b-1}v(\xi){}_a^{ABR}\nabla^{\alpha}u(\xi)=\sum_{\xi=a+1}^{b-1}u(\xi){}^{ABR}\nabla_b^{\alpha}v(\xi),$$

and

$$\begin{aligned}\sum_{\xi=a+1}^{b-1}u(\xi){}_a^{ABC}\nabla^{\alpha}v(\xi)=&\sum_{\xi=a+1}^{b-1}v(\xi-1){}^{ABR}\nabla_b^{\alpha}u(\xi-1)\\&+v(b-1)\frac{B(\alpha)}{1-\alpha}E^{1}_{\overline{\alpha,1},\frac{-\alpha}{1-\alpha},b^-}u(b-1)\\&-v(a)\frac{B(\alpha)}{1-\alpha}E^{1}_{\overline{\alpha,1},\frac{-\alpha}{1-\alpha},b^-}u(a),\end{aligned}$$

where

$$E^{1}_{\overline{\rho,\mu},w,b^{-}}u(\xi)=\sum_{a=\xi}^{b-1}(a-\rho(\xi))^{\overline{\mu-1}}E_{\overline{\rho,\mu}}(w,a-\rho(\xi))\,u(\xi),\ \ \xi\in_{b}\mathbb{N}.$$

3. The Continuous Case

Let us consider the below continuous fractional Dirac system

$$Lu:=Bu+Qu=\lambda u,\ \ a\leq x\leq b<\infty, \tag{5}$$

where

$$Q=\begin{pmatrix} p & 0\\ 0 & r\end{pmatrix},\ u:=\begin{pmatrix} u_1\\ u_2\end{pmatrix},\ B=\begin{pmatrix} 0 & {}^{ABC}_{a}D^{\alpha}\\ {}^{ABR}D^{\alpha}_{b} & 0\end{pmatrix},$$

$\alpha\in(0,1)\,,\ \lambda\in\mathbb{C};\ p,r\in C\,[a,b]\,;\ p\,(x)>0,\ r\,(x)>0,\ \forall x\in[a,b]\,.$ We also consider the following boundary conditions

$$\varkappa_{11}E^{1}_{\overline{\alpha,1},\frac{-\alpha}{1-\alpha},b^{-}}u_{1}\,(a)+\varkappa_{12}u_{2}\,(a)=0, \tag{6}$$

$$\varkappa_{21}E^{1}_{\overline{\alpha,1},\frac{-\alpha}{1-\alpha},b^{-}}u_{1}\,(b)+\varkappa_{22}u_{2}\,(b)=0, \tag{7}$$

with $\varkappa_{11}^2+\varkappa_{12}^2\neq 0$ and $\varkappa_{21}^2+\varkappa_{22}^2\neq 0$.

Now let's define the inner product suitable for this system. Let $L^2((a,b);\mathbb{R}^2)$ denotes the Hilbert space with the following inner product

$$(u,v):=\int_a^b u_1v_1dx+\int_a^b u_2v_2dx, \tag{8}$$

where

$$u=\begin{pmatrix} u_1\\ u_2\end{pmatrix},\ v=\begin{pmatrix} v_1\\ v_2\end{pmatrix},$$

u_i and v_i $(i=1,2)$ are real-valued continuous functions defined on $[a,b]\,.$

Theorem 3. *The operator L defined by (5)-(7) is formally self-adjoint on $L^2\left((a,b)\,;\mathbb{R}^2\right).$*

Proof. Using (8), we get

$$\begin{aligned}(Lu,v)-(u,Lv)&=\int_a^b\left({}^{ABC}_{a}D^{\alpha}u_2+p\,(x)\,u_1\right)v_1dx\\ &+\int_a^b\left({}^{ABR}D^{\alpha}_{b}u_1+r\,(x)\,u_2\right)v_2dx\\ &-\int_a^b u_1\left({}^{ABC}_{a}D^{\alpha}v_2+p\,(x)\,v_1\right)dx\end{aligned}$$

$$-\int_a^b u_2 \left({}^{ABR}D_b^\alpha v_1 + r\left(x\right) v_2\right) dx$$

$$= \int_a^b \left({}_a^{ABC}D^\alpha u_2 v_1\right) dx + \int_a^b \ {}^{ABR}D_b^\alpha u_1 v_2 dx$$

$$-\int_a^b u_1 ({}_a^{ABC}D^\alpha v_2) dx - \int_a^b u_2 \left({}^{ABR}D_b^\alpha v_1\right) dx,$$

where $u, v \in L^2\left(\left(a,b\right);\mathbb{R}^2\right)$. From Proposition 1 and Theorem 1, we obtain

$$(Lu, v) - (u, Lv) = \left[u, v\right]_b - \left[u, v\right]_a \tag{9}$$

where

$$\left[u, v\right]_x = v_2(x)\frac{B(\alpha)}{1-\alpha}E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}u_1(x) - u_2(x)\frac{B(\alpha)}{1-\alpha}E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}v_1(x).$$

By conditions (6)-(7), we get the desired result. □

Corollary 1. *The eigenvalues of Eq. (5) subject to the boundary conditions (6)-(7) are real. The eigenfunctions corresponding to different eigenvalues of the system (5)-(7) are orthogonal.*

Let us define the Wronskian of u and v by

$$W\left(u, v\right)\left(x\right) = \left(\frac{B(\alpha)}{1-\alpha}E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}u_1\left(x\right)\right)v_2\left(x\right) - \left(\frac{B(\alpha)}{1-\alpha}E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}v_1\left(x\right)\right)u_2\left(x\right),$$

where

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},\ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in L^2((a,b);\mathbb{R}^2).$$

Theorem 4. *Let v_1 and v_2 be two solutions of Eq. (5). Then $W(v_1, v_2)$ is independent of x.*

Proof. By (9), we obtain

$$(\lambda v_1, v_2) - (v_1, \lambda v_2) = \left[v_1, v_2\right]_b - \left[v_1, v_2\right]_a,$$

since $Lv_1 = \lambda v_1$ and $Lv_2 = \lambda v_2$. Hence

$$\left[v_1, v_2\right]_b = \left[v_1, v_2\right]_a = W(v_1, v_2)(a).$$

□

Theorem 5. *Any two solutions of the Eq. (5) are linearly dependent if and only if their Wronskian is zero.*

Proof. Assume v_1 and v_2 be two linearly dependent solutions of Eq. (5). Then there exists a constant $\eta > 0$ such that $v_1 = \eta v_2$. Hence

$$W(v_1, v_2)(x) = \begin{vmatrix} v_{11}(x) & \frac{B(\alpha)}{1-\alpha} E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-} v_{12}(x) \\ v_{21}(x) & \frac{B(\alpha)}{1-\alpha} E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-} v_{22}(x) \end{vmatrix}$$

$$= \begin{vmatrix} \eta v_{21}(x) & \eta \frac{B(\alpha)}{1-\alpha} E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-} v_{22}(x) \\ v_{21}(x) & \frac{B(\alpha)}{1-\alpha} E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-} v_{22}(x) \end{vmatrix} = 0.$$

On the other hand, if the Wronskian $W(v_1, v_2)(x)$ is zero for some x in $[a, b]$, then we obtain

$$v_1 = \eta v_2$$

i.e., v_1 and v_2 are linearly dependent on $[a, b]$. □

Let us now give an example to illustrate our results.

Example 1. *If we take $\alpha \to 1^-$ in (5), we obtain the ordinary Dirac system ([11]) defined as*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{du}{dx} + Qu = \lambda u, \ a \le x \le b < \infty,$$

where

$$Q = \begin{pmatrix} p & 0 \\ 0 & r \end{pmatrix} \text{ and } u := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

In fact, for $\alpha \in (0, 1]$, the ABR and ABC fractional operators become well-defined due to the Mittag-Leffler kernel (2) doesn't have a convergence problem.

4. The Discrete Case

Let us consider the nabla discrete fractional Dirac systems

$$L_1 u = Cu + Qu = \lambda u, \ x \in \mathbb{N}_{a,b-1}, \tag{10}$$

$$Q = \begin{pmatrix} p & 0 \\ 0 & r \end{pmatrix}, \ u := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \ C = \begin{pmatrix} 0 & {}^{ABC}_{a}\nabla^{\alpha} \\ {}^{ABR}\nabla^{\alpha}_{b} & 0 \end{pmatrix},$$

where $\alpha \in (0, 1/2)$, $\lambda \in \mathbb{C}$; p and r are real-valued functions on $\mathbb{N}_{a,b-1}$; $p(x) > 0, r(x) > 0, \ \forall x \in \mathbb{N}_{a,b-1}$. We consider the following conditions

$$\varkappa_{11} \left({}^{ABR}\nabla^{\alpha}_{b} - \frac{B(\alpha)}{1-\alpha} E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-} \right) u_1(a) + \varkappa_{12} u_2(a) = 0, \tag{11}$$

$$\varkappa_{21} \left({}^{ABR}\nabla^{\alpha}_{b} - \frac{B(\alpha)}{1-\alpha} E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-} \right) u_1(b-1) + \varkappa_{22} u_2(b-1) = 0, \tag{12}$$

where $\varkappa_{11}^2 + \varkappa_{12}^2 \neq 0$ and $\varkappa_{21}^2 + \varkappa_{22}^2 \neq 0$.

Let $L^2_\nabla(\mathbb{N}_{a,b-1};\mathbb{R}^2)$ denotes the Hilbert space with the following inner product

$$\langle u,v\rangle := \sum_{x=a+1}^{b-1} u_1(x)v_1(x) + \sum_{x=a+1}^{b-1} u_2(x)v_2(x),$$

where

$$u=\begin{pmatrix} u_1 \\ u_2 \end{pmatrix},\ v=\begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

u_i and v_i $(i=1,2)$ are real-valued functions defined on $\mathbb{N}_{a,b-1}$.

Theorem 6. *The operator L_1 defined by (10)-(12) is formally self-adjoint on $L^2_\nabla(\mathbb{N}_{a,b-1};\mathbb{R}^2)$.*

Proof. Let $u,v\in L^2_\nabla(\mathbb{N}_{a,b-1};\mathbb{R}^2)$. Then we see that

$$\begin{aligned}
\langle L_1u,v\rangle - \langle u,L_1v\rangle &= \sum_{x=a+1}^{b-1}\left({}_{a}^{ABC}\nabla^{\alpha}u_2 + p\left(x\right)u_1\right)v_1 + \sum_{x=a+1}^{b-1}\left({}^{ABR}\nabla_b^{\alpha}u_1 + r\left(x\right)u_2\right)v_2 \\
&- \sum_{x=a+1}^{b-1} u_1\left({}_{a}^{ABC}\nabla^{\alpha}v_2 + p\left(x\right)v_1\right) - \sum_{x=a+1}^{b-1} u_2\left({}^{ABR}\nabla_b^{\alpha}\ v_1 + r\left(x\right)v_2\right) \\
&= \sum_{x=a+1}^{b-1} {}_{a}^{ABC}\nabla^{\alpha}u_2v_1 + \sum_{x=a+1}^{b-1} p(x)u_1(x)v_1(x) + \sum_{x=a+1}^{b-1} {}^{ABR}\nabla_b^{\alpha}\ u_1v_2 \\
&+ \sum_{x=a+1}^{b-1} r(x)u_2(x)v_2(x) - \sum_{x=a+1}^{b-1} u_1({}_{a}^{ABC}\nabla^{\alpha}v_2) - \sum_{x=a+1}^{b-1} p(x)u_1(x)v_1(x) \\
&- \sum_{x=a+1}^{b-1} u_2\left({}^{ABR}\nabla_b^{\alpha}\ v_1\right) - \sum_{x=a+1}^{b-1} r(x)u_2(x)v_2(x) \\
&= \sum_{x=a+1}^{b-1} {}_{a}^{ABC}\nabla^{\alpha}u_2\ v_1 + \sum_{x=a+1}^{b-1} {}^{ABR}\nabla_b^{\alpha}\ u_1v_2 \\
&- \sum_{x=a+1}^{b-1} u_1({}_{a}^{ABC}\nabla^{\alpha}v_2) - \sum_{x=a+1}^{b-1} u_2\left({}^{ABR}\nabla_b^{\alpha}\ v_1\right) \\
&= \sum_{x=a+1}^{b-1} u_2\left(x-1\right)^{ABR}\nabla_b^{\alpha}\ v_1\left(x-1\right) + \frac{B(\alpha)}{1-\alpha}u_2(x)E^{1}_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}v_1(x)\Big|_a^{b-1} \\
&+ \sum_{x=a+1}^{b-1}\left({}^{ABR}\nabla_b^{\alpha}u_1\right)v_2 - \sum_{x=a+1}^{b-1} v_2\left(x-1\right)^{CFR}\nabla_b^{\alpha}u_1\left(x-1\right) \\
&- \frac{B(\alpha)}{1-\alpha}v_2(x)E^{1}_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}u_1(x)\Big|_a^{b-1} - \sum_{x=a+1}^{b-1} u_2\left({}^{ABR}\nabla_b^{\alpha}\ v_1\right)
\end{aligned}$$

$$= u_2(a)^{ABR}\nabla_b^\alpha\; v_1(a) - u_2(b-1)^{ABR}\nabla_b^\alpha\; v_1(b-1)$$

$$+\frac{B(\alpha)}{1-\alpha}u_2(b-1)E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}v_1(b-1)-\frac{B(\alpha)}{1-\alpha}u_2(a)E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}v_1(a)$$

$$+ v_2(b-1)^{ABR}\nabla_b^\alpha\; u_1(b-1) - v_2(a)^{ABR}\nabla_b^\alpha\; u_1(a)$$

$$+\frac{B(\alpha)}{1-\alpha}v_2(a)E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}u_1(a)-\frac{B(\alpha)}{1-\alpha}v_2(b-1)E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}u_1(b-1)$$

$$= z_2(b-1)\left({}^{CFR}\nabla_b^\alpha-\frac{B(\alpha)}{1-\alpha}E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}\right)y_1(b-1)$$

$$- y_2(b-1)\left({}^{CFR}\nabla_b^\alpha-\frac{B(\alpha)}{1-\alpha}E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}\right)z_1(b-1)$$

$$-\begin{bmatrix} v_2(a)\left({}^{ABR}\nabla_b^\alpha-\frac{B(\alpha)}{1-\alpha}E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}\right)u_1(a)\\ -u_2(a)\left({}^{ABR}\nabla_b^\alpha-\frac{B(\alpha)}{1-\alpha}E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}\right)v_1\;\;(a)\end{bmatrix}.$$

It follows from (11) and (12) that

$$\langle L_1u,v\rangle-\langle u,L_1v\rangle=0.$$

□

Corollary 2. *All eigenvalues of the problem (10)-(12) are real. Eigenfunctions corresponding to different eigenvalues are orthogonal.*

Theorem 7. *Let*

$$W(u,v)(x)=\begin{vmatrix}\left({}^{ABR}\nabla_b^\alpha\;-\frac{B(\alpha)}{1-\alpha}E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}\right)u_1(x) & u_2(x)\\ \left({}^{ABR}\nabla_b^\alpha\;-\frac{B(\alpha)}{1-\alpha}E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}\right)v_1(x) & v_2(x)\end{vmatrix},$$

where

$$u=\begin{pmatrix}u_1\\u_2\end{pmatrix},\; v=\begin{pmatrix}v_1\\v_2\end{pmatrix}\in L^2_\nabla(\mathbb{N}_{a,b-1};\mathbb{R}^2)$$

and let θ_1 and θ_2 be two solutions of Eq. (5). Then $W(\theta_1,\theta_2)$ is independent of x. Moreover, any two linearly independent solutions φ_1,φ_2 of Eq. (10) are linearly dependent if and only if $W(\varphi_1,\varphi_2)=0$.

Proof. The proof is as in Theorem 4 and Theorem 5. □

5. Conclusion

In this work, we have considered some fractional Dirac systems with Mittag–Leffler kernel. Firstly, a continuous fractional Dirac system with Mittag–Leffler kernel is studied. Its spectral properties are investigated. Later, the nabla discrete fractional Dirac system with Mittag–Leffler kernel is constructed. Similar properties are studied. Since Dirac systems have an important place in quantum physics, the properties of such systems are studied intensively. In this context, investigating fractional Dirac systems with Mittag–Leffler kernel will contribute to researchers working in this field. In the future, Green's function can be created for this system and eigenfunction expansions can be investigated.

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