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SEMIREGULAR, SEMIPERFECT AND SEMIPOTENT MATRIX RINGS RELATIVE TO AN IDEAL

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Abstract. This paper investigates relative ring theoretical properties in the context of formal triangular matrix rings. The first aim is to study the semiregularity of formal triangular matrix rings relative to an ideal. We prove that the formal triangular matrix ring T is T'-semiregular if and only if A is I-semiregular, B is K-semiregular and $N = M$ for an ideal $T' = \left(\begin{smallmatrix} I & 0 \\ N & K \end{smallmatrix}\right)$ of $T = \left(\begin{smallmatrix} A & 0 \\ M & B \end{smallmatrix}\right)$. We also discuss the relative semiperfect formal triangular matrix rings in relation to the strong lifting property of ideals. Moreover, we have considered the behavior of relative semipotent and potent property of formal triangular matrix rings. Several examples are provided throughout the paper in order to highlight our results.

1. Introduction

The celebrated work of Wedderburn and Artin gave a key insight into the structure of a semisimple artinian ring, which makes it an attractive structure to study. Moreover, for a right artinian ring R, the Jacobson radical is nilpotent, and the ring $R/\mathrm{J}(R)$ is semisimple, so a main problem would be to "lift" the structure of the factor ring $R/\mathrm{J}(R)$ onto the ring R itself. As a consequence of this, we are led to the concept of lifting idempotents and, consequently, to the notion of a semiperfect ring.

Let I be an ideal in a ring R. Recall that an element $a \in R$ is an idempotent modulo I if $a + I \in R/I$ is an idempotent. In this case, we say that a can be lifted to an idempotent (modulo I) if there exists an idempotent $e \in R$ with $e - a \in I$. Note that the ideal I in R is called *idempotent lifting* if, whenever $a + I \in R/I$ is an idempotent, then there exists an idempotent $e \in R$ such that $e - a \in I$.

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Lifting of idempotents is a key method in transferring some structural properties of a factor ring of a ring R up to the ring itself. Several classes of rings are described in terms of the idempotent lifting property of ideals. For example, semiperfect rings are those rings R for which $R/\mathrm{J}(R)$ is semisimple and the Jacobson radical $\mathrm{J}(R)$ of R is idempotent lifting. Some nontrivial generalizations of semiperfect rings, such as semiregular rings and potent rings may be considered as further examples.

As it has been pointed out above, the idempotent lifting property of the Jacobson radical $J(R)$ of R is prominent in the study of semiregular and semiperfect rings. A stronger property than the idempotent lifting property, namely, strong lifting property of ideals, gives rise to a natural generalization of semiregular and semiperfect rings. Semiregular rings relative to an ideal first emerged in a paper [8] by Nicholson and Yousif. Then, Yousif and Zhou [10] studied further semiperfect and perfect rings relative to an ideal in connection with relative semiregular rings. Later, Nicholson and Zhou worked on a natural extension of this work together with strongly lifting ideals to characterize I-semiregular and I-semiperfect rings for an ideal I of a ring R in [9]. Recall that an ideal I of a ring R is called *strongly lifting* if, whenever $a + I \in R/I$ is an idempotent, then there exists an idempotent $e \in aR$ such that $a - e \in I$. In this work, Nicholson and Zhou further showed that the ring R is I-semiregular (semiperfect) if and only if R/I is regular (semisimple) and I is strongly lifting.

Recall that a ring R is called semipotent if each one-sided ideal of R that is not contained in its Jacobson radical $\mathrm{J}(R)$ contains a nonzero idempotent. A semipotent ring R is called potent if, in addition, $\mathrm{J}(R)$ is an idempotent lifting ideal of R. Semipotent rings has been generalized to semipotent rings relative to an ideal by Nicholson and Zhou in [9]. It is also important to consider the strong lifting properties of ideals in this setting, and relative potent rings are defined in relation to these ideals as well as relative semipotent rings.

One of important constructions in ring theory is the triangular ring construction. Let A, B be rings and M be a B-A bimodule. A *formal triangular matrix ring* is a ring of the form

$$\begin{pmatrix} A & 0 \\ M & B \end{pmatrix} = \left\{ \begin{pmatrix} a & 0 \\ m & b \end{pmatrix} \mid a \in A,\ b \in B,\ \text{and } m \in M \right\}$$

under the usual matrix operations. There are a number of important examples in this class, including lower (upper) triangular matrices over a known ring R. Moreover, many surprising examples and counterexamples have emerged via the triangular ring construction in literature by varying the choices of A, B and M. By using formal triangular matrix rings, Herstein in [5] provided a counterexample to the Jacobson conjecture, one of the oldest and most well-known conjectures in noncommutative ring theory. In [3], these rings were studied in detail, and in [4], various ring theoretic properties of formal triangular matrix rings were investigated.

This paper aims to unify all these relative properties in the framework of formal triangular matrix rings. In Section 2, we completely give a description of the semiregularity and semiperfectness of formal triangular matrix rings relative to an ideal, proving that T is T'-semiregular (resp. semiperfect) if and only if A is I-semiregular (resp. semiperfect), B is K-semiregular (resp. semiperfect) and $N = M$ for an ideal $T' = \left(\begin{smallmatrix} I & 0 \\ N & K \end{smallmatrix}\right)$ of $T = \left(\begin{smallmatrix} A & 0 \\ M & B \end{smallmatrix}\right)$ (Theorem 1 and Theorem 2). Then, we highlight our results by providing several examples, in particular we show that the "if" part of the above theorems are not in general true if we omit the condition $N = M$. We have further considered the behavior of relative semipotent property of formal triangular matrix rings. Since being a semipotent or a potent ring passes over to formal triangular matrix rings by a result due to Haghany and Varadarajan [4], it is natural to suspect that it may also pass over in the relative case.

Throughout this paper, all rings will be associative rings with an identity element $1 \neq 0$, not necessarily commutative. We will denote by $\mathrm{J}(R)$ the Jacobson radical of a ring R.

2. Results

Recall that a formal triangular matrix ring T is a ring of the form

$$T = \left\{ \begin{pmatrix} a & 0 \\ m & b \end{pmatrix} \mid a \in A,\ b \in B,\ m \in M. \right\}$$

under the usual matrix addition and multiplication where A, B are two rings and M is a left B right A bimodule. For simplicity, we write

$$T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$$

for the formal triangular matrix ring. The construction of examples and counterexamples for asymmetric ring-theoretic properties is among the major applications of such rings in noncommutative ring theory. In particular, [4] provides a comprehensive resource for various ring-theoretic properties of formal triangular matrix rings.

Moreover, Goodearl covered formal triangular matrix rings in his classic book "Ring Theory: Nonsingular rings and modules" [3]. In order to better understand the ideal structure of a ring of such a type, we must first recall the following fact.

Proposition 1. [3] *If I is a two-sided ideal of A, K a two-sided ideal of B, and N a B-A subbimodule of M which contains $MI + KM$, then $\left(\begin{smallmatrix} I & 0 \\ N & K \end{smallmatrix}\right)$ is a two-sided ideal of T. Conversely, every two-sided ideal of T has this form.*

We will begin by simplifying the following notation: T denotes the formal triangular matrix ring $\left(\begin{smallmatrix} A & 0 \\ M & B \end{smallmatrix}\right)$, while T' refers to an ideal of the form $\left(\begin{smallmatrix} I & 0 \\ N & K \end{smallmatrix}\right)$ with the additional properties outlined above.

Now we continue with a lemma that is implicit in [8] and proved in [10, Lemma 1.1] and leads us to a number of significant ring-theoretic properties relative to an ideal of a ring R.

Lemma 1. [10, Lemma 1.1] *Let I be an ideal of the ring R. The following conditions are equivalent for a right ideal I' of R:*

(1) *There exists $e^2 = e \in I'$ with $(1-e)I' \subseteq I$.*
(2) *There exists $e^2 = e \in I'$ with $I' \cap (1-e)R \subseteq I$.*
(3) *$I' = eR \oplus S$ where $e^2 = e$ and $S \subseteq I$.*

According to Nicholson and Zhou [9] an ideal I of the ring R *respects* a right ideal I' of R if the conditions in Lemma 1 are satisfied. Similarly, I *respects* a left ideal $L \subseteq R$ if $L = Re \oplus S$ where $e^2 = e$ and $S \subseteq I$. It is worth noting that this definition is left-right symmetric, i.e., if $I \triangleleft R$ and $a \in R$, then I respects aR if and only if I respects Ra.

Right (left) I-semiregular elements and rings first emerged in a paper [8] by Nicholson and Yousif and then were studied in [10] by Yousif and Zhou. Later, Nicholson and Zhou [9] dealt with these elements in terms of respecting a right (left) ideal as defined above and demonstrated that it is not necessary to distinguish between "right I-semiregular" and "left I-semiregular". Let I be an ideal of the ring R. Recall that an element $a \in R$ is called I-semiregular if I respects aR, i.e., if $e^2 = e \in aR$ exists with $(1-e)a \in I$, or alternatively, if $f^2 = f \in Ra$ exists with $a(1-f) \in I$. As expected, when all elements of the ring R are I-semiregular, the ring R is called a I-semiregular ring.

It is well known that the topic of lifting of idempotents is a crucial method for identifying the structure of semiregular and semiperfect rings. Nicholson and Zhou studied a natural extension of these notions in connection with strongly lifting ideals in [9]. Recall that an ideal I of a ring R is called *strongly lifting* if, for some $a \in R$, whenever $a^2 - a \in I$, then there exists an idempotent $e \in aR$ with $a - e \in I$. It is possible to replace the conclusion $e \in aR$ by $e \in Ra$ or $e \in aRa$ since this notion is left-right symmetric [9, Lemma 1]. In this work, Nicholson and Zhou further showed that the ring R is I-semiregular if and only if R/I is regular and I is strongly lifting. A recent work [1] has shed new light on the question: "What can be said about relative semiregular ideals of the the formal triangular matrix ring?". The author has provided a criterion to decide if a given ideal T' of the formal triangular matrix ring T is strongly lifting.

Our first Theorem is motivated by the above-mentioned results and characterizes the semiregularity of formal triangular matrix rings relative to an ideal.

Theorem 1. *Let T' be an ideal of T. Then T is T'-semiregular if and only if A is I-semiregular, B is K-semiregular and $N = M$.*

Proof. First recall the fact that T is T'-semiregular if and only if T/T' is regular and T' is strongly lifting. Now if T is T'-semiregular, then T/T' is regular, and so A/I and B/K regular. Further, $\mathrm{J}(T/T') = 0$ implies that $M/N = 0$, that is $N = M$. Moreover, Corollary 2.8 in [1] states that strongly lifting ideals T' of T are those ideals for which I and K are strongly lifting in A and B, respectively. Combining these two results with the above-mentioned fact, we get A is I-semiregular, B is K-semiregular, as desired.

For the converse, first note that $T/T' = \begin{pmatrix} A/I & 0 \\ 0 & B/K \end{pmatrix} \cong A/I \times B/K$. Hence, the regularity of A/I and B/K implies the regularity of T/T'. Now, the result is easily seen by again using Corollary 2.8 in [1]. □

As an application, we continue with an illustrative example.

Example 1. Let $A = \mathbb{Z}_{30}$, $B = \mathbb{Z}_9$ and $M = \mathbb{Z}_3 \oplus \mathbb{Z}_3$. Let us begin by considering the following ring:

$$T = \begin{pmatrix} \mathbb{Z}_{30} & 0 \\ \mathbb{Z}_3 \oplus \mathbb{Z}_3 & \mathbb{Z}_9 \end{pmatrix}.$$

It is our aim to determine all ideals T' of T with the property that T is T'-semiregular by using Theorem 1. To do this, we first need to specify the strongly lifting ideals I (K) of A (B) for which A/I (B/K) is von Neumann regular, respectively. Since A and B are exchange rings, all ideals of these two rings are strongly lifting and an easy computation shows that all factors of the form A/I and B/K are von Neumann regular except for the case $K = 0$ in B.

Taking into account the ideal structure of T described in Proposition 1 and letting $N = M$, the following ideals T' are those for which T is T'-semiregular:

- $\begin{pmatrix} 0 & 0 \\ M & \mathbb{Z}_9 \end{pmatrix}, \begin{pmatrix} 15\mathbb{Z}_{30} & 0 \\ M & \mathbb{Z}_9 \end{pmatrix}, \begin{pmatrix} 10\mathbb{Z}_{30} & 0 \\ M & \mathbb{Z}_9 \end{pmatrix}, \begin{pmatrix} 6\mathbb{Z}_{30} & 0 \\ M & \mathbb{Z}_9 \end{pmatrix};$
- $\begin{pmatrix} 5\mathbb{Z}_{30} & 0 \\ M & \mathbb{Z}_9 \end{pmatrix}, \begin{pmatrix} 3\mathbb{Z}_{30} & 0 \\ M & \mathbb{Z}_9 \end{pmatrix}, \begin{pmatrix} 2\mathbb{Z}_{30} & 0 \\ M & \mathbb{Z}_9 \end{pmatrix}, \begin{pmatrix} \mathbb{Z}_{30} & 0 \\ M & \mathbb{Z}_9 \end{pmatrix};$
- $\begin{pmatrix} 0 & 0 \\ M & 3\mathbb{Z}_9 \end{pmatrix}, \begin{pmatrix} 15\mathbb{Z}_{30} & 0 \\ M & 3\mathbb{Z}_9 \end{pmatrix}, \begin{pmatrix} 10\mathbb{Z}_{30} & 0 \\ M & 3\mathbb{Z}_9 \end{pmatrix}, \begin{pmatrix} 6\mathbb{Z}_{30} & 0 \\ M & 3\mathbb{Z}_9 \end{pmatrix};$
- $\begin{pmatrix} 5\mathbb{Z}_{30} & 0 \\ M & 3\mathbb{Z}_9 \end{pmatrix}, \begin{pmatrix} 3\mathbb{Z}_{30} & 0 \\ M & 3\mathbb{Z}_9 \end{pmatrix}, \begin{pmatrix} 2\mathbb{Z}_{30} & 0 \\ M & 3\mathbb{Z}_9 \end{pmatrix}, \begin{pmatrix} \mathbb{Z}_{30} & 0 \\ M & 3\mathbb{Z}_9 \end{pmatrix}.$

Remark 1. Note that the "if" part of the above theorem is not in general true if we omit the condition $N = M$ as shown in the following example.

Let $A = \mathbb{Z}_4$, $B = \mathbb{Z}_2$ and $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Consider the formal triangular matrix ring

$$T = \begin{pmatrix} \mathbb{Z}_4 & 0 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}.$$

We wish to find an ideal T' of T for which A is I-semiregular, B is K-semiregular, but T is not T'-semiregular due to the fact that $N \neq M$. For this, we first observe that the ideals $I = 2\mathbb{Z}_4$ of $\mathbb{Z}_4$ and $K = 0$ of $\mathbb{Z}_2$ are strongly lifting, respectively. Further, $A/I \cong B/K \cong \mathbb{Z}_2$ is clearly von Neumann regular. Hence, these two together imply that A is I-semiregular and B is K-semiregular.

Taking into account the ideal structure of T described in Proposition 1, we let $N = \mathbb{Z}_2 \oplus 0$ a (B, A)-subbimodule of M and

$$T' = \begin{pmatrix} 2\mathbb{Z}_4 & 0 \\ \mathbb{Z}_2 \oplus 0 & 0 \end{pmatrix}.$$

Then the ring T is not T'-semiregular since the ring

$$T/T' \cong \begin{pmatrix} \mathbb{Z}_2 & 0 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}$$

is not von Neumann regular.

It should be noted that the Jacobson radical $\mathrm{J}(R)$ of a ring R is not necessarily idempotent lifting. However, if it is idempotent lifting, it is also strongly lifting. In fact, as is well-known, a ring R is semiregular if and only if $R/\mathrm{J}(R)$ is regular and $\mathrm{J}(R)$ is idempotent lifting. Hence, the $\mathrm{J}(R)$-semiregular rings are just the semiregular rings and we get the following immediate corollary to the above result.

Corollary 1. *T is semiregular if and only if A and B are semiregular.*

A right I-semiperfect ring is one in which every right ideal M of R fulfills the equivalent conditions as stated in Lemma 1. Left I-semiperfect rings can be defined in a similar vein. A short proof of the right-left symmetry of this notion appears in [9] by showing the following equivalence

$$R \text{ is } I\text{-semiperfect} \Leftrightarrow R/I \text{ is semisimple and } I \text{ is strongly lifting.}$$

We now determine a necessary and sufficient condition for the triangular matrix ring T to be T'-semiperfect.

Theorem 2. *If T' is an ideal of T, then the following conditions are equivalent:*

(i) *T is T'-semiperfect;*
(ii) *A is I-semiperfect, B is K-semiperfect and $N = M$.*

Proof. To begin with, let us recall that T is T'-semiperfect if and only if T/T' is semisimple and T' is strongly lifting. Now if T is T'-semiperfect, then T/T' is semisimple, and so are A/I and B/K. Further, $\mathrm{J}(T/T') = 0$ implies that $M/N = 0$, that is $N = M$. Moreover, Corollary 2.8 in [1] states that strongly lifting ideals T' of T correspond to those ideals for which I and K are strongly lifting in A and B. Combining these two results with the above-mentioned fact, we get A is I-semiperfect, B is K-semiperfect, as desired.

For the converse, consider the ring isomorphism $T/T' = \left(\begin{smallmatrix} A/I & 0 \\ 0 & B/K \end{smallmatrix}\right) \cong A/I \times B/K$. Hence, the semisimplicity of A/I and B/K implies the semisimplicity of T/T'. We get our assertion by putting these and Corollary 2.8 in [1] together. $\square$

Example 2. Let $\mathbb{Z}_{(p)}$ be the localization of the ring of integers $\mathbb{Z}$ at a prime ideal $p\mathbb{Z}$, $\mathbb{Z}_{p^\infty}$ be the Prüfer group and let $\hat{\mathbb{Z}}_p$ be p-adic integers.

Consider the formal triangular matrix ring

$$T = \begin{pmatrix} \mathbb{Z}_{(p)} & 0 \\ \mathbb{Z}_{p^\infty} & \hat{\mathbb{Z}}_p \end{pmatrix}.$$

Our goal is to determine all ideals T' of T with the property that T is T'-semiperfect by using Theorem 2. For this, it is enough to identify all strongly lifting ideals of $\mathbb{Z}_{(p)}$ and $\hat{\mathbb{Z}}_p$ for which every factor ring of these two rings is semisimple Artinian. Since $\mathbb{Z}_{(p)}$ and $\hat{\mathbb{Z}}_p$ are exchange rings, all ideals of these two rings are strongly lifting. It is not difficult to see that factor rings are semisimple Artinian for the ideals $p\mathbb{Z}_{(p)}$ and $\mathbb{Z}_{(p)}$ and the ideals $p\hat{\mathbb{Z}}_p$ and $\hat{\mathbb{Z}}_p$ of the uniserial rings $\mathbb{Z}_{(p)}$ and $\hat{\mathbb{Z}}_p$, respectively.

Hence, the ring T is T'-semiperfect for the following list of ideals T':

$$\begin{pmatrix} p\mathbb{Z}_{(p)} & 0 \\ \mathbb{Z}_{p^\infty} & p\hat{\mathbb{Z}}_p \end{pmatrix}, \begin{pmatrix} p\mathbb{Z}_{(p)} & 0 \\ \mathbb{Z}_{p^\infty} & \hat{\mathbb{Z}}_p \end{pmatrix}, \begin{pmatrix} \mathbb{Z}_{(p)} & 0 \\ \mathbb{Z}_{p^\infty} & p\hat{\mathbb{Z}}_p \end{pmatrix}, \begin{pmatrix} \mathbb{Z}_{(p)} & 0 \\ \mathbb{Z}_{p^\infty} & \hat{\mathbb{Z}}_p \end{pmatrix}.$$

In the example below, it can be seen that the assumption "$N = M$" in the "if" part of Theorem 2 cannot simply be dropped.

Example 3. As an example of an ideal T' of T for which A is I-semiperfect, B is K-semiperfect, but T is not T'-semiperfect, we would like to recall an example of Berberian that was discussed in detail in [2, Example 1].

Let $\mathbb{C}$ be the complex field and $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ be the division ring of real quaternions.

Take $A = \mathbb{C}$, $B = \mathbb{H}$ and $M = \mathbb{H}$. Let us begin by considering the following ring

$$T = \begin{pmatrix} \mathbb{C} & 0 \\ \mathbb{H} & \mathbb{H} \end{pmatrix}.$$

The ring T is an exchange ring due to the fact that the rings $\mathbb{C}$ and $\mathbb{H}$ are all exchange rings [6, Proposition 2.1].

We further observe that the ideals $I = 0$ of $\mathbb{C}$ and $K = \mathbb{H}$ of $\mathbb{H}$ are strongly lifting as exchange rings are precisely the rings that every one-sided ideal is strongly lifting. Furthermore, A/I and B/K are clearly semisimple. Hence, we have A is I-semiperfect and B is K-semiperfect.

On the other hand, for the ideal

$$T' = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{H} \end{pmatrix}$$

of the ring T, T is not T'-semiperfect since the ring

$$T/T' \cong \begin{pmatrix} \mathbb{C} & 0 \\ \mathbb{H} & 0 \end{pmatrix}$$

is not semisimple Artinian.

A ring R is called semipotent if each right ideal of R that is not contained in its Jacobson radical $\mathrm{J}(R)$ contains a nonzero idempotent. Note that this notion is left-right symmetric. A semipotent ring R is called potent if, in addition, $\mathrm{J}(R)$ is an idempotent lifting ideal of R. Examples of these rings include exchange rings (see [7, Proposition 1.9]). It is well known that a formal triangular matrix ring T is semipotent (respectively, potent) if and only if A and B are semipotent (respectively, potent) [4, Theorem 6.4].

Semipotent rings has been generalized to semipotent rings relative to an ideal based on the following lemma proposed by Nicholson and Zhou [9].

Lemma 2. [9, Lemma 19] *The following are equivalent for $I \lhd R$:*

1. *If $I' \nsubseteq I$ is a right ideal, then there exists $e^2 = e \in I' - I$.*
2. *If $a \notin I$, then there exists $e^2 = e \in aR - I$.*
3. *If $a \notin I$ there exists $x \in R$ such that $xax = x \notin I$.*

Following Nicholson and Zhou, for an ideal I in a ring R, R is said to be I-semipotent if the above conditions in Lemma 2 are fulfilled, and is said to be I-potent if it is I-semipotent and I is strongly lifting in R. In other words, the semipotent (potent) rings are simply the $\mathrm{J}(R)$-semipotent ($\mathrm{J}(R)$-potent) rings. Since the property of being a semipotent or a potent ring transfers to formal triangular matrix rings by the above-mentioned result due to Haghany and Varadarajan [4, Theorem 6.4], it is natural to suspect that it may also transfer in the relative case.

We now interpret this notion in the language of formal triangular matrix rings. We mimic the proof of Haghany and Varadarajan [4, Theorem 6.4].

Theorem 3. *Let T' be an ideal of T. If T is T'-semipotent then A is I-semipotent and B is K-semipotent, respectively.*

Proof. Assume that T is T'-semipotent. We first claim that A is I-semipotent. Let $I' \nsubseteq I$ be a right ideal in A. Then $I'' = \begin{pmatrix} I' & 0 \\ 0 & 0 \end{pmatrix}$ is a right ideal not contained in T'. Hence there exists $e \in I'$ with $\left(\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right) \in I'' - T'$. This implies that $e^2 = e \in I' - I$, as desired. Secondly, we claim that B is K-semipotent. Let $K' \nsubseteq K$ be a right ideal in B. Then $K'' = \begin{pmatrix} 0 & 0 \\ K'M & K' \end{pmatrix}$ is a right ideal of T not contained in T'. Since T is T'-semipotent, there exists a nonzero element $\left(\begin{smallmatrix} 0 & 0 \\ m & f \end{smallmatrix}\right) \in K'' - T'$ with $\left(\begin{smallmatrix} 0 & 0 \\ m & f \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ m & f \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ m & f \end{smallmatrix}\right) \in K'' - T'$. This implies that $f^2 = f$ and $fm = m$. Since $\left(\begin{smallmatrix} 0 & 0 \\ m & f \end{smallmatrix}\right)$ is nonzero, we get $f \neq 0$ or $m \neq 0$. By considering $fm = m$, we get $f \neq 0$. Thus, $0 \neq f$ with $f^2 = f \in K' - K$, as desired. $\square$

The converse of Theorem 3 does not hold in general, as can be seen in the following example.

Example 4. There exist a formal triangular ring T and an ideal T' of T such that A is I-semipotent, B is K-semipotent, but T is not T'-semipotent.

Let $R = \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$ be the direct product of rings. Take into consideration the following subring of R:

$$A = \{(n, \bar{n}_2, \bar{n}_3, \cdots, \bar{n}_k, \bar{n}, \cdots) | n, n_i \in \mathbb{Z},\ k \geqslant 2\}.$$

Putting $I = \{(2m, \bar{0}, \bar{0}, \cdots) \mid m \in \mathbb{Z}\}$, it follows that A is I-semipotent by [9, Example 23].

Now, take $B = \mathbb{Z}_4$ and $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Consider the formal triangular matrix ring

$$T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}.$$

We further consider the ideal $K = 2\mathbb{Z}_4$ of $B = \mathbb{Z}_4$. Since B is K-semiregular by Remark 1, it is K-semipotent, too.

On the other hand, for the ideal

$$T' = \begin{pmatrix} I & 0 \\ \mathbb{Z}_2 \oplus 0 & K \end{pmatrix}$$

of the ring T, T is not T'-semipotent. To show the last statement, consider the following ideal of the ring T

$$\tilde{T} = \begin{pmatrix} I & 0 \\ 0 \oplus \mathbb{Z}_2 & K \end{pmatrix}.$$

Then $\tilde{T}$ is clearly not contained in the ideal T. On the other hand, an easy computation shows that the ideal $\tilde{T}$ of T doesn't contain any nonzero idempotent. Thus, there do not exist any idempotent in $\tilde{T}$ which is not in T'. By Lemma 2, T is not T'-semipotent, as desired.

As we mentioned above, relative potent rings is a proper subclass of the class of relative semipotent rings with the additional strongly lifting condition on the relative ideal. Due to the fact that strongly lifting ideals T' of T are those ideals for which I and K are strongly lifting in A and B, respectively [1, Corollary 2.8], Theorem 3 implies the the following immediate corollary.

Corollary 2. *Let T' be an ideal of T. If T is T'-potent, then A is I-potent and B is K-potent, respectively.*

The converse of Theorem 3 is not true in general, it is natural to ask the question what additional conditions are required for this to happen. We will show below that this question has an affirmative answer for ideals of the form $\left(\begin{smallmatrix} I & 0 \\ M & K \end{smallmatrix}\right)$ in the formal triangular matrix ring T.

Theorem 4. *Let T' be an ideal of the form $\left(\begin{smallmatrix} I & 0 \\ M & K \end{smallmatrix}\right)$ in T. If A is I-semipotent and B is K-semipotent, then T is T'-semipotent.*

Proof. Assume that A is I-semipotent and B is K-semipotent. We will show that T is T'-semipotent. Let $\tilde{T}$ be a right ideal of T with $\tilde{T} \nsubseteq T'$. Then there exists an element $t = \left(\begin{smallmatrix} a & 0 \\ m & b \end{smallmatrix}\right) \in \tilde{T} - T'$. This implies that either $a \notin I$ or $b \notin K$. First, assume that $a \notin I$. Then there exist a non-zero idempotent e in $aA - I$. Set $e = ar$ for some $r \in A$. Then $are = e^2 = e \neq 0$. Note that

$$\left(\begin{smallmatrix} e & 0 \\ mre & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} are & 0 \\ mre & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & 0 \\ m & b \end{smallmatrix}\right)\left(\begin{smallmatrix} re & 0 \\ 0 & 0 \end{smallmatrix}\right),$$

and so $\left(\begin{smallmatrix} e & 0 \\ mre & 0 \end{smallmatrix}\right) \in tT \subseteq \tilde{T}$. Also

$$\left(\begin{smallmatrix} e & 0 \\ mre & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} e & 0 \\ mre & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} e & 0 \\ mre & 0 \end{smallmatrix}\right).$$

Thus $\left(\begin{smallmatrix} e & 0 \\ mre & 0 \end{smallmatrix}\right)$ is a non-zero idempotent in $tT - T'$. If $b \notin K$, there exists a non-zero idempotent $f \in bB$ and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & f \end{smallmatrix}\right)$ is a non-zero idempotent in $tT - T'$. This proves that T is T'-semipotent. □

Considering the definition of a relative potent (resp. semipotent and potent) ring we end the paper with the following corollaries of Theorem 3 and Theorem 4.

Corollary 3. *Let T' be an ideal of of the form $\left(\begin{smallmatrix} I & 0 \\ M & K \end{smallmatrix}\right)$ in T. If A is I-potent and B is K-potent, then T is T'-potent.*

Corollary 4. [4, Theorem 6.4] *Let T be the formal triangular matrix ring of the form $\left(\begin{smallmatrix} A & 0 \\ M & B \end{smallmatrix}\right)$. Then T is semipotent (resp. potent) if and only if A and B are semipotent (resp. potent).*

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