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Interval Estimation for the Difference of Two Independent Nonnormal Population Variances

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Abstract

In random experiments, most analyses are based on interpretation of the difference between the means of experiment and control groups. Therefore, studying the difference between the variances of the experiment and control groups may also be useful in interpreting the analysis results. This study focuses on interval estimation with sample variance estimators based on Winsorized Mean and Trimmed Mean for the difference of the variances of two nonnormal populations. In the simulation study, confidence intervals based on robust estimators for the difference of the variances of two non-normally distributed populations were compared in terms of coverage probabilities and average length widths. According to simulation study, it was determined that the coverage probabilities of confidence intervals based on robust estimators were very close to the nominal confidence level in any case. However, it was seen that the average length widths of confidence intervals obtained with sample variance estimator based on Trimmed Mean were narrower compared to the average length widths of confidence intervals obtained with sample variance estimator based on Winsorized Mean. In addition, it was determined that these results were the same when the Type I error is different. According to these results, it will be appropriate to prefer interval estimations obtained with sample variance estimator based on Trimmed Mean since it provides narrower confidence interval for the difference of the variances of two nonnormal populations.

1. INTRODUCTION

In randomized experiments or randomized trials with continuous outcomes, the focus of analysis is often on the difference in the mean outcomes of experimental and control groups. However, the difference in the variances of outcomes of experimental and control groups may also have a useful interpretation. In clinical experiments, the difference between variances equals to the variance of the effects of interventions [1]. Bell et al. [2] have used the difference between the variances of the treatment and control groups to test the quantitative change of treatment impacts on patients. Cojbasic and Tomovic [3] have obtained non-parametric confidence intervals based on Bootstrap method for the difference of the variances of two populations when the data is received from exponential family. Niwitpong [4] has studied coverage probabilities and average length widths of the generalized confidence interval and closed form confidence interval for the difference of two normal population variances. Later, Niwitpong [5] has presented an analytic definition of coverage probabilities and average length widths of closed form confidence interval and compared it to the confidence interval suggested in Niwitpong [4]. Herbert et al. [6] have obtained confidence intervals based on estimator of difference of sample variances for the difference of variances of two normal populations. Suwan and Niwitpong [7] have studied the interval estimation methods for a linear function of variances of nonnormal populations using the kurtosis coefficient.

In estimation of variances of normal populations, sample variance estimator S^2 is used. Sample variance estimator is the Maximum Likelihood Estimator of the population variance and the distribution of this

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estimator converges to asymptotically normal distribution [6]. In addition, it is known that $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$. Chi-square distribution is a special form of Gamma distribution with parameters $\alpha = (n-1)/2$ and $\beta = 2$. When the sample sizes and the variances of two normal populations are equal, it is known that the difference of two random Gamma variable with parameters (α, β) have McKay Type II distribution with parameters $a = (\alpha - 0.5)$, $b = \beta^2$ and $c = 0$. This distribution is close to normal distribution even if the sample size is too small [6]. Confidence interval for difference of variances of two normal populations based on estimator $\left(S_1^2 - S_2^2\right)$ is suggested by Herbert et al. [6] as follows:

$$P\left[\left(S_1^2 - S_2^2\right) - z_{\alpha/2}\sqrt{Var\left(S_1^2 - S_2^2\right)} \le \sigma_1^2 - \sigma_2^2 \le \left(S_1^2 - S_2^2\right) + z_{\alpha/2}\sqrt{Var\left(S_1^2 - S_2^2\right)}\right] = 1 - \alpha . \tag{1}$$

The variance of distribution of the estimator S^2 under normal distribution assumption is defined as [8];

$$Var(\mathrm{S}^2) = \frac{1}{n}\left\{\eta - \left(\frac{n-3}{n-1}\right)\sigma^4\right\} \tag{2}$$

where $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\mathrm{Y}_i - \bar{Y})^2$ and η is fourth central moment. With this information, provided that kurtosis coefficient is $\gamma = \frac{\eta}{\sigma^4}$, the variance of the difference between the sample variances of two independent normal populations is [6]:

$$\begin{aligned} Var(S_1^2 - S_2^2) &= Var(S_1^2) + Var(S_2^2) \\ &= \frac{1}{n_1}\left\{\gamma_1\sigma_1^4 - \left(\frac{n_1-3}{n_1-1}\right)\sigma_1^4\right\} + \frac{1}{n_2}\left\{\gamma_2\sigma_2^4 - \left(\frac{n_2-3}{n_2-1}\right)\sigma_2^4\right\}. \end{aligned} \tag{3}$$

Instead of unknown parameters, estimation values obtained from the sample are used here.

It is known that sample variance estimator S^2 does not display robust statistics features in the estimation of nonnormal population variance and the coverage probabilities of the confidence intervals obtained with this estimator have much lower values compared to the nominal confidence interval [9, 10]. In such cases, it is necessary to use robust scale estimators for estimation of population variance.

In this study, it was aimed that robust estimators are used to estimate the difference of variances of two nonnormal populations. These estimators are sample variance estimators based on Winsorized and Trimmed Mean which are robust estimators used instead of sample mean. Confidence intervals based on these estimators were obtained for the difference of variances of two nonnormal populations.

A simulation study was conducted in the following section for the distribution of these estimators.

2. ROBUST ESTIMATORS FOR VARIANCE

In this section, sample variance estimators based on Winsorized and Trimmed Mean and simulation results on distribution of these estimators are included.

2.1. Sample Variance Estimator Based on Winsorized Mean

When there are outliers in a data set, Winsorized Mean, which a robust estimator, may be used as the estimator of the population mean instead of sample mean. Winsorized Mean defines the centre of distribution for skew distributions better than the sample mean. This estimator was used for the first time in the field of sampling to reduce the effect of extreme values in the sample [11]. When Winsorized Mean is obtained, the lowest l_n number of observations is replaced with (l_n+1)th observations and the largest u_n number of observations is replaced with $(n-u_n)$th observations. If random sample of size n is $Y_1, Y_2, ..., Y_n$, ith order statistics is defined with $Y_{(i)}$. While replacement is made only on the high end of the consecutive data in operations performed with sample data produced from positively skewed distributions, replacement is made on both ends for the sample data produced from symmetric distributions [12]. When replacement is made only on the high end, Winsorized Mean is defined as follows [13]:

$$W_n(l_n,u_n)=\frac{1}{n}\left\{\sum_{i=1}^{n-u_n} Y_{(i)}+u_n\,Y_{(n-u_n)}\right\} \tag{4}$$

where u_n is the number of terms to be removed from the high end of the consecutive data. In that case, sample variance based on Winsorized Mean is expressed as;

$$S_W^2=\frac{1}{n-1}\left\{\sum_{i=1}^{n-u_n-1}\left(Y_i-W_n\right)^2+(u_n+1)\left(Y_{n-u_n}-W_n\right)^2\right\} \tag{5}$$

where $u_n=[\rho n+0.5]$, ρ is replacement percentage and $[.]$ expression indicates the largest integer function [14]. When replacement is made on both ends of the consecutive data, Winsorized Mean is obtained as [13];

$$W_n(l_n,u_n)=\frac{1}{n}\left\{\sum_{i=l_n+1}^{n-u_n} Y_{(i)}+l_nY_{(l_n+1)}+u_n\,Y_{(n-u_n)}\right\} \tag{6}$$

where l_n is the number of terms to be removed from the low end of the consecutive data and u_n is the number of terms to be removed from the high end of the consecutive data. In that case, sample variance based on Winsorized Mean is as follows;

$$S_W^2=\frac{1}{n-1}\left\{(l_n+1)\left(\left(Y_{l_n+1}-W_n\right)^2+\left(Y_{n-u_n}-W_n\right)^2\right)+\sum_{i=l_n+2}^{n-u_n-1}\left(Y_i-W_n\right)^2\right\}. \tag{7}$$

In order to determine the distribution of estimator S_W^2, a simulation study based on 10000 replications was realized for $\alpha=0.05$ with different sample sizes and distributions using the program written in Matlab R2009a. Although it is visually understood from the histograms obtained that the distribution of this estimator in various situations roughly resemble the normal distribution, it is necessary to use a Goodness of Fit test which is used for this fitting. In the simulation study, random samples are generated from Exponential, Gamma, Chi-square and t-distributions with the sample sizes of $n=10,20,50,100$ and Kolmogorov-Smirnov and Shapiro-Wilk Goodness of Fit tests are used to determine whether the distribution of this estimator complies with normal distribution. While replacement is made only on the high end of the consecutive data in operations conducted with sample data generated from Exponential, Gamma and Chi-square distributions, which are positively skewed distributions; replacement was made on both ends in the sample data generated from t-distribution. The average *p*-values for 10000 replications with 5% replacement proportion in these test operations are given in Table 1.

Table 1. *The average p-values of the Kolmogorov-Smirnov (KS) and Shapiro-Wilk (SW) Goodness of Fit tests for the* S_W^2 *estimator*

Sample sizes	Exponential (0.5)		Gamma (2, 0.5)		Chi-Square (1)		t (3)	
	KS	SW	KS	SW	KS	SW	KS	SW
10	0.2108	0.3210	0.2460	0.3502	0.1594	0.2099	0.2351	0.3369
20	0.2060	0.3206	0.2301	0.3500	0.1534	0.2045	0.2341	0.3310
50	0.2041	0.3099	0.2313	0.3278	0.1482	0.2000	0.2364	0.3299
100	0.2038	0.3060	0.2310	0.3214	0.1479	0.1999	0.2360	0.3269

When the average *p*-values of Kolmogorov Smirnov and Shapiro-Wilk Goodness of Fit tests in Table 1 are studied, it is seen that $p-value>\alpha$ in all cases for $\alpha=0.05$. Hence, the distribution of estimator S_W^2 complies with the normal distribution regardless of the sample size.

2.2. Sample Variance Estimator Based on Trimmed Mean

If there are outliers in a data set, another robust estimator for the estimation of population mean is Trimmed Mean. Trimmed Mean can be defined as the mean obtained after observation values at certain ratios are removed from the high/low end or both ends of the consecutive sample of size n If random sample of size n is $Y_1, Y_2, ..., Y_n$, ith order statistics is defined with $Y_{(i)}$. Trimming is made from the high end of the consecutive data in positively skewed distributions and both ends in symmetric distributions [12]. When trimming is made only on the high end, Trimmed Mean is defined as follows [13]:

$$T_n(l_n,u_n)=\frac{1}{n-u_n}\sum_{i=1}^{n-u_n}Y_{(i)} \tag{8}$$

where u_n is the number of terms to be removed from the high end of the consecutive data. In that case, sample variance based on Trimmed Mean is expressed as follows:

$$S_T^2=\frac{\sum_{i=1}^{n-u_n}\left(Y_i-T_n\right)^2}{n-u_n-1}. \tag{9}$$

When trimming is made on both ends, Trimmed Mean is expressed as;

$$T_n(l_n,u_n)=\frac{1}{n-2l_n}\sum_{i=l_n+1}^{n-u_n}Y_{(i)}. \tag{10}$$

In that case, sample variance is as follows;

$$S_T^2=\frac{\sum_{i=l_n+1}^{n-u_n}\left(Y_i-T_n\right)^2}{n-2l_n-1}. \tag{11}$$

It is understood from the histograms obtained as a result of simulation study that distribution of estimator S_T^2 resembles the normal distribution. Trimming is made from the high end of the consecutive data for random samples of size $n=10,20,50,100$ generated from Exponential, Gamma and Chi-square distributions and on both ends for random samples generated from t-distribution. When $\alpha=0.05$, the average *p*-values of the Kolmogorov-Smirnov and Shapiro- Wilk Goodness of Fit test for 10000 replications are given in Table 2.

Table 2. *The average p-values of the Kolmogorov-Smirnov (KS) and Shapiro-Wilk (SW) Goodness of Fit tests for the S_T^2 estimator*

Sample sizes	Exponential (0.5)		Gamma (2, 0.5)		Chi-Square (1)		t (3)	
	KS	SW	KS	SW	KS	SW	KS	SW
10	0.2247	0.3369	0.2636	0.3564	0.1702	0.2099	0.2265	0.3256
20	0.2074	0.3321	0.2440	0.3509	0.1660	0.2069	0.2250	0.3244
50	0.2027	0.3301	0.2427	0.3498	0.1681	0.2013	0.2253	0.3202
100	0.2023	0.3299	0.2423	0.3479	0.1671	0.2004	0.2251	0.3197

From the results in Table 2, it is understood that the distribution of estimator S_T^2 complies with normal distribution even in small sample sizes since $p-value>\alpha$ for both Goodness of Fit tests while $\alpha=0.05$

3. INTERVAL ESTIMATION METHODS FOR DIFFERENCE OF THE VARIANCES OF TWO NONNORMAL POPULATIONS

In this section, interval estimation methods for the difference of the variances of two nonnormal populations are included based on the difference of robust scale estimators. These estimators are discussed as $\left(S_{W_1}^2-S_{W_2}^2\right)$ and $\left(S_{T_1}^2-S_{T_2}^2\right)$ respectively.

3.1. Confidence Interval with Scale Estimator Based on Winsorized Mean

The result was obtained that the distribution of estimator S_W^2 with the sample data produced from distributions with different parameters and replacement 5% complied with the normal distribution even in small sample sizes. Given that the distribution of the difference of two estimators with normal distribution is also normal, the confidence interval based on estimator $\left(S_{W_1}^2-S_{W_2}^2\right)$ for the difference of two nonnormal population variances is obtained as follows:

$$P\left[(\mathrm{S}_{W_1}^2-\mathrm{S}_{W_2}^2)-z_{\alpha/2}\sqrt{Var(\mathrm{S}_{W_1}^2-\mathrm{S}_{W_2}^2)}\le\sigma_1^2-\sigma_2^2\le(\mathrm{S}_{W_1}^2-\mathrm{S}_{W_2}^2)+z_{\alpha/2}\sqrt{Var(\mathrm{S}_{W_1}^2-\mathrm{S}_{W_2}^2)}\right]=1-\alpha\,. \tag{12}$$

When the populations are nonnormally distributed, there is not a theoretical formula for $Var(S_{W_1}^2-S_{W_2}^2)$ in Equation (12). For that reason, for $Var(S_{W_1}^2-S_{W_2}^2)$, the variance estimation value obtained from the distribution of estimator $\left(S_{W_1}^2-S_{W_2}^2\right)$ with Monte Carlo Simulation Method or the variance estimation value obtained with Bootstrap Method can be used [15]. For $Var(S_{W_1}^2-S_{W_2}^2)$, Monte Carlo Simulation Method can be expressed as follows:

Provided that the value of the difference estimator obtained in the ith replication of the T repeated simulation study with sample data of size n is $D_{1i}=\left(S_{W_{1i}}^2-S_{W_{2i}}^2\right)$, $i=1,2,...,T$, the variance of the estimator $\left(S_{W_1}^2-S_{W_2}^2\right)$ is expressed as;

$$\mathrm{Var}\left(S_{W_1}^2 - S_{W_2}^2\right) = \frac{\sum_{i=1}^{T}(D_{1i} - \bar{D}_1)^2}{T-1} \tag{13}$$

where the $\bar{D}_1$ expressed in Equation (13) is the arithmetic mean of the differences.

In addition, variance estimation may also be determined with Bootstrap Method for $Var(S_{W_1}^2 - S_{W_2}^2)$. For the variance of estimator $\left(S_{W_1}^2 - S_{W_2}^2\right)$, bootstrap samples of size n are generated by simple random sampling with replacement. For each Bootstrap sample, Bootstrap estimation is obtained for estimator $\left(S_{W_1}^2 - S_{W_2}^2\right)$. This operation is repeated for B times. With the Bootstrap estimations obtained from B replications, the Bootstrap estimator for the variance of estimator $\left(S_{W_1}^2 - S_{W_2}^2\right)$ is given as [16];

$$\mathrm{Var}\left(S_{W_1}^2 - S_{W_2}^2\right) = \frac{\sum_{b=1}^{B}(D_{1b} - \bar{D}_1)^2}{B-1}. \tag{14}$$

Obtaining the $Var(S_{W_1}^2 - S_{W_2}^2)$ value with both methods yields quite similar results. In the case where the populations have Gamma distribution with parameters $\alpha = 3$ and $\beta = 1$ and t- distribution with parameter $\nu = 5$, variance estimation values based on 10000 replications which are obtained with Monte Carlo Simulation Method and Bootstrap Method are as follows.

Table 3. *Estimation values for* $Var(S_{W_1}^2 - S_{W_2}^2)$

	Gamma (3,1)		t(5)	
Sample sizes	MC*	Bootstrap	MC*	Bootstrap
10	3.5906	3.4099	0.3190	0.3867
20	2.3024	2.2387	0.2048	0.2057
50	1.7085	1.7012	0.1433	0.1435
100	0.9915	0.9899	0.0911	0.0974

*: Monte Carlo simulation method

According to the results obtained in Table 3, the values of variance estimations based on Monte Carlo Simulation and Bootstrap Methods have provided quite close results in all of the sample sizes.

3.2. Confidence Interval with Scale Estimator Based on Trimmed Mean

It was presented with the findings in Table 2 that the distribution of S_T^2 estimator complied with normal distribution in all sample sizes, even if the samples are not selected from normal populations. It is known that the distribution of the difference of two normally distributed estimator is normally distributed. For the difference of the variances of two nonnormal populations, confidence interval based on estimator $\left(S_{T_1}^2 - S_{T_2}^2\right)$ is obtained as follows using the normal distribution:

$$P\left[(\mathrm{S}_{T_1}^2 - \mathrm{S}_{T_2}^2) - z_{\alpha/2}\sqrt{Var(\mathrm{S}_{T_1}^2 - \mathrm{S}_{T_2}^2)} \le \sigma_1^2 - \sigma_2^2 \le (\mathrm{S}_{T_1}^2 - \mathrm{S}_{T_2}^2) + z_{\alpha/2}\sqrt{Var(\mathrm{S}_{T_1}^2 - \mathrm{S}_{T_2}^2)}\right] = 1 - \alpha. \tag{15}$$

The value of $Var(S_{T_1}^2 - S_{T_2}^2)$ in this confidence interval is found with the formula given in Section (3.1). In the case where the populations have Gamma distribution with parameters $\alpha = 3$ and $\beta = 1$ and t-distribution with parameter $\nu = 5$, variance estimation values based on 10000 replications for $Var(S_{T_1}^2 - S_{T_2}^2)$ which are obtained with Monte Carlo Simulation Method and Bootstrap Method are obtained as follows.

Table 4. *Estimation values for* $Var(S_{T_1}^2 - S_{T_2}^2)$

	Gamma (3,1)		t(5)	
Sample sizes	MC*	Bootstrap	MC*	Bootstrap
10	2.1390	2.1743	0.2035	0.2498
20	1.5794	1.4027	0.1253	0.1177
50	1.0791	1.0717	0.1030	0.1019
100	0.9000	0.8851	0.0599	0.0574

*: Monte Carlo simulation method

According to the results obtained in Table 4, the values of variance estimations based on Monte Carlo Simulation and Bootstrap Methods for $Var(S_{T_1}^2 - S_{T_2}^2)$ are quite similar to each other.

4. SIMULATION STUDY

A simulation study was conducted with the purpose of comparing the confidence intervals given in Equations (12) and (15). The confidence intervals for the difference of variances of two nonnormal populations were compared in terms of coverage probability and average length widths. In this simulation study, the data produced from Gamma and t-distributions with different parameters and it was used with the program written in Matlab R2009a. In obtaining confidence intervals based on robust estimator $\left(S_{W_1}^2 - S_{W_2}^2\right)$, replacement was made only on the high end of the consecutive data for the sample data produced from Gamma distribution and on both ends of the consecutive data for the sample data produced from t-distribution. Similarly, trimming was made only on the high end of the consecutive data for the sample data produced from Gamma distribution and on both ends of the consecutive data for the sample data produced from t-distribution for obtaining confidence intervals based on robust estimator $\left(S_{T_1}^2 - S_{T_2}^2\right)$ Simulation studies were conducted based on 10000 replications for $\alpha = 0.05$ and $\alpha = 0.10$ with different sample sizes, different replacement and trimming proportions.

With the simulation study, coverage probabilities and average length widths of confidence intervals based on robust estimators $\left(S_{W_1}^2 - S_{W_2}^2\right)$ and $\left(S_{T_1}^2 - S_{T_2}^2\right)$ for the difference of the variances of two nonnormal populations are summarized in Table 5-12. In this study, Monte Carlo Simulation Method was used for obtaining the coverage probabilities and average length widths of confidence intervals for $Var(S_{W_1}^2 - S_{W_2}^2)$ and $Var(S_{T_1}^2 - S_{T_2}^2)$.

Average length widths are obtained by dividing the total differences of the lower limit and upper limits of intervals found for each replication to the number of replications. Coverage probabilities are determined by dividing the number of cases where the difference between the variances of two populations were between the lower and upper interval limit values in the simulation study to the number of replications. ρ expresses the replacement proportion for Winsorized Mean and trimming proportion for Trimmed Mean.

Table 5. *Coverage probabilities and average length widths based on estimator* $\left(S_{W_1}^2 - S_{W_2}^2\right)$ *under Gamma distribution for* $\alpha = 0.05$

Sample sizes	Gamma (2,3)			Gamma (3,1)		
	ρ=5	ρ=10	ρ=20	ρ=5	ρ=10	ρ=20
10	0.9424 (46.4232)	0.9424 (46.4532)	0.9447 (31.7120)	0.9440 (7.1124)	0.9440 (7.1124)	0.9430 (5.1640)
20	0.9429 (38.6091)	0.9425 (31.6777)	0.9440 (21.5799)	0.9450 (6.2467)	0.9490 (5.0183)	0.9490 (3.6295)
50	0.9441 (22.5420)	0.9456 (19.1189)	0.9463 (13.0958)	0.9520 (3.3147)	0.9490 (2.9447)	0.9460 (2.1123)
100	0.9430 (16.1841)	0.9500 (12.9285)	0.9470 (9.2551)	0.9540 (2.4928)	0.9490 (2.1391)	0.9500 (1.5663)

*Values in the parenthesis are the average of the lengths of confidence interval.

In Table 5, it is seen that coverage probabilities of the confidence interval based on estimator $\left(S_{W_1}^2 - S_{W_2}^2\right)$ are quite close to the nominal confidence level in all sample sizes and all of the replacement proportions $\rho = 5,10,20$ when $\alpha = 0.05$. It is determined that the average length widths are reduced as the sample size increases. In each sample size, it is seen that average length widths increase as the replacement proportion ρ increases.

Table 6. *Coverage probabilities and average length widths based on estimator* $\left(S_{W_1}^2 - S_{W_2}^2\right)$ *under Gamma distribution for* $\alpha = 0.10$

Sample sizes	Gamma (2,3)			Gamma (3,1)		
	ρ =5	ρ=10	ρ=20	ρ=5	ρ=10	ρ=20
10	0.9009 (39.5071)	0.9009 (39.5071)	0.9019 (27.0066)	0.9020 (5.7853)	0.9020 (5.7853)	0.9000 (4.0574)
20	0.9055 (31.6530)	0.9018 (25.2729)	0.9024 (17.3476)	0.9010 (5.0392)	0.9014 (4.1962)	0.8990 (2.9792)
50	0.9020 (19.2433)	0.8970 (16.4238)	0.9000 (10.8138)	0.8990 (2.9900)	0.8990 (2.5457)	0.9020 (1.8080)
100	0.9090 (13.1040)	0.9010 (10.9875)	0.8990 (7.6307)	0.9090 (2.1737)	0.9070 (1.8423)	0.9020 (1.2522)

*Values in the parenthesis are the average of the lengths of confidence interval.

In Table 6, the case where $\alpha = 0.10$ is discussed. In this table, it is determined that coverage probabilities of the confidence interval based on estimator $\left(S_{W_1}^2 - S_{W_2}^2\right)$ are quite close to the nominal confidence level in all replacement proportions and sample sizes. In addition, it was seen that coverage probabilities increase and average length widths are reduced as the sample size increases.

Table 7. *Coverage probabilities and average length widths based on estimator* $\left(S_{W_1}^2 - S_{W_2}^2\right)$ *under Student-t distribution for* $\alpha = 0.05$

Sample sizes	t(5)			t(10)		
	ρ=5	ρ=10	ρ=20	ρ=5	ρ=10	ρ=20
10	0.9408 (3.4069)	0.9408 (3.4069)	0.9415 (2.0866)	0.9427 (2.5529)	0.9427 (2.5529)	0.9414 (1.7317)
20	0.9474 (3.0015)	0.9475 (2.2427)	0.9419 (1.3910)	0.9453 (2.1139)	0.9464 (1.7656)	0.9442 (1.2029)
50	0.9473 (1.6238)	0.9474 (1.3235)	0.9468 (0.8580)	0.9478 (1.2485)	0.9466 (1.0759)	0.9455 (0.7499)
100	0.9471 (1.2189)	0.9481 (0.9367)	0.9454 (0.5921)	0.9455 (0.9206)	0.9467 (0.7717)	0.9458 (0.5257)

*Values in the parenthesis are the average of the lengths of confidence interval.

In this part of the simulation study, replacement was made on both ends since t-distribution is symmetric in calculation of Winsorized Mean. In terms of coverage probabilities for $\alpha = 0.05$, it is determined that confidence interval based on estimator $\left(S_{W_1}^2 - S_{W_2}^2\right)$ are quite close to the nominal confidence level in all of the sample sizes and replacement proportions and narrower intervals are obtained in replacement proportion 20% .

Table 8. *Coverage probabilities and average length widths based on estimator* $\left(S_{W_1}^2 - S_{W_2}^2\right)$ *under Student-t distribution for* $\alpha = 0.10$

Sample sizes	t(5)			t(10)		
	ρ=5	ρ=10	ρ=20	ρ=5	ρ=10	ρ=20
10	0.9005 (2.7782)	0.9005 (2.7782)	0.9026 (1.7224)	0.9031 (2.1219)	0.9031 (2.1219)	0.9011 (1.4801)
20	0.9011 (2.4330)	0.9010 (1.8429)	0.9034 (1.1730)	0.9033 (1.7920)	0.9039 (1.5043)	0.9017 (1.0212)
50	0.9016 (1.3662)	0.9018 (1.1189)	0.9014 (0.7267)	0.9035 (1.0421)	0.9036 (0.9139)	0.9014 (0.6293)
100	0.9026 (1.0264)	0.9031 (0.7915)	0.9044 (0.5081)	0.9036 (0.7635)	0.9043 (0.6389)	0.9042 (0.4393)

*Values in the parenthesis are the average of the lengths of confidence interval.

In Table 8, it is seen that coverage probabilities of confidence interval based on estimator $\left(S_{W_1}^2 - S_{W_2}^2\right)$ for $\alpha = 0.10$ are quite close to the nominal confidence level even in small sample sizes. It is determined that the average length widths of confidence intervals are reduced as the sample size increases.

Table 9. *Coverage probabilities and average length widths based on estimator* $\left(S_{T_1}^2 - S_{T_2}^2\right)$ *under Gamma distribution for* $\alpha = 0.05$

Sample sizes	Gamma (2,3)			Gamma (3,1)		
	ρ=5	ρ=10	ρ=20	ρ=5	ρ=10	ρ=20
10	0.9440 (34.8914)	0.9440 (34.8914)	0.9450 (23.4142)	0.9450 (5.6808)	0.9450 (5.6808)	0.9430 (3.8510)
20	0.9480 (29.5252)	0.9470 (21.5476)	0.9460 (14.1125)	0.9460 (4.7855)	0.9450 (3.6900)	0.9440 (2.5371)
50	0.9440 (15.8907)	0.9460 (12.6574)	0.9470 (8.2883)	0.9490 (2.7197)	0.9490 (2.2636)	0.9480 (1.5759)
100	0.9500 (11.8213)	0.9490 (8.5550)	0.9510 (5.4862)	0.9490 (1.7942)	0.9500 (1.3996)	0.9490 (0.9947)

*Values in the parenthesis are the average of the lengths of confidence interval.

In Table 9, trimming operation for coverage probabilities and average length widths based on estimator $\left(S_{T_1}^2 - S_{T_2}^2\right)$ was made only on the high end of the consecutive data with the random samples produced from two populations with Gamma distribution for $n = 10, 20, 50, 100$. When $\alpha = 0.05$ in terms of coverage probabilities, it is determined that coverage probabilities of the confidence interval based on estimator $\left(S_{T_1}^2 - S_{T_2}^2\right)$ are close to the nominal confidence level in all sample sizes and trimming proportions. It is observed that the confidence interval widths are reduced as the sample size increases.

Table 10. *Coverage probabilities and average length widths based on estimator* $\left(S_{T_1}^2 - S_{T_2}^2\right)$ *under Gamma distribution for* $\alpha = 0.10$

Sample sizes	Gamma (2,3)			Gamma (3,1)		
	ρ=5	ρ=10	ρ=20	ρ=5	ρ=10	ρ=20
10	0.9010 (27.4044)	0.9010 (27.4044)	0.9030 (18.1078)	0.9010 (4.5887)	0.9010 (4.5887)	0.9020 (3.1551)
20	0.9020 (24.6182)	0.9080 (18.4344)	0.9080 (11.7004)	0.9020 (3.8028)	0.9030 (3.0102)	0.9030 (2.1635)
50	0.9030 (13.7027)	0.9080 (11.0327)	0.9060 (7.1019)	0.9030 (2.3241)	0.9040 (1.8664)	0.9050 (1.2589)
100	0.9040 (12.5890)	0.9080 (7.9106)	0.9090 (5.1991)	0.9040 (1.6655)	0.9050 (1.2847)	0.9060 (0.8783)

*Values in the parenthesis are the average of the lengths of confidence interval.

In Table 10, coverage probabilities and average length widths based on estimator $\left(S_{T_1}^2 - S_{T_2}^2\right)$ with the random samples produced from Gamma distribution for different sample sizes when $\alpha = 0.10$ are given. It is concluded that coverage probabilities of confidence intervals are quite close to the nominal confidence level in all cases. It is observed that average length widths are reduced as the trimming proportion increases.

Table 11. *Coverage probabilities and average length widths based on estimator* $\left(S_{T_1}^2 - S_{T_2}^2\right)$ *under Student-t distribution for* $\alpha = 0.05$

Sample sizes	t(5)			t(10)		
	ρ=5	ρ=10	ρ=20	ρ=5	ρ=10	ρ=20
10	0.9440 (2.4306)	0.9440 (2.4306)	0.9410 (1.5173)	0.9430 (1.9331)	0.9430 (1.9331)	0.9410 (1.2553)
20	0.9430 (2.1201)	0.9440 (1.5090)	0.9430 (0.9084)	0.9430 (1.6585)	0.9430 (1.2483)	0.9410 (0.8107)
50	0.9450 (1.1246)	0.9440 (0.8493)	0.9430 (0.4896)	0.9430 (0.9111)	0.9470 (0.7367)	0.9440 (0.4447)
100	0.9490 (0.9154)	0.9460 (0.6331)	0.9440 (0.3416)	0.9520 (0.6716)	0.9520 (0.5014)	0.9460 (0.2977)

*Values in the parenthesis are the average of the lengths of confidence interval.

In Table 11, coverage probabilities and average length widths of confidence intervals based on estimator $\left(S_{T_1}^2 - S_{T_2}^2\right)$ with the random samples produced from two populations Student-t distributed with parameters $\upsilon = 5$ and $\upsilon = 10$ for different sample sizes when $\alpha = 0.05$. It is seen that coverage probabilities of confidence interval based on estimator $\left(S_{T_1}^2 - S_{T_2}^2\right)$ are quite close to 95% confidence level regardless of the sample size and also it has quite narrow intervals for 20% trimming compared to other trimming proportions. It is determined that average length widths are reduced as the sample size increases.

Table 12. *Coverage probabilities and average length widths based on estimator* $\left(S_{T_1}^2 - S_{T_2}^2\right)$ *under Student-t distribution for* $\alpha = 0.10$

Sample sizes	t(5)			t(10)		
	ρ=5	ρ=10	ρ=20	ρ=5	ρ=10	ρ=20
10	0.9030 (2.3225)	0.9030 (2.3225)	0.9000 (1.4091)	0.8990 (1.6702)	0.8990 (1.6702)	0.8970 (1.0929)
20	0.9070 (1.7845)	0.9010 (1.2665)	0.8990 (0.7791)	0.9010 (1.3764)	0.9010 (1.0452)	0.9000 (0.6826)
50	0.9060 (1.0148)	0.9020 (0.7449)	0.9000 (0.4172)	0.9030 (0.7663)	0.9030 (0.6183)	0.8990 (0.3842)
100	0.9070 (0.7746)	0.9030 (0.5451)	0.9010 (0.2986)	0.9040 (0.5943)	0.9050 (0.4493)	0.9020 (0.2720)

*Values in the parenthesis are the average of the lengths of confidence interval.

In Table 12, random samples are produced from t-distribution and the case where $\alpha = 0.10$ is discussed. Coverage probabilities of the confidence interval based on estimator $\left(S_{T_1}^2 - S_{T_2}^2\right)$ are quite close to the nominal confidence level in all of the sample sizes and trimming proportions.

5. CONCLUSION

In the simulation study performed to determine the distribution of estimators S_W^2 and S_T^2, it is visually understood from the histograms that the distributions of this estimators in various situations roughly resemble the normal distribution. However, it is used the Goodness of Fit test for this fitting. Since $p - value > \alpha$ for $\alpha = 0.05$ and the sample size $n = 10, 20, 50, 100$, it is understood that the distributions of estimators S_W^2 and S_T^2 comply with the normal distribution regardless of the sample size. The distribution of the difference of two estimators which have normal distribution is also normal. With this information, the confidence intervals based on estimators $\left(S_{W_1}^2 - S_{W_2}^2\right)$ and $\left(S_{T_1}^2 - S_{T_2}^2\right)$ for the difference of two nonnormal population variances is obtained. For the difference of the variances of two nonnormal populations, confidence intervals based on robust estimators $\left(S_{W_1}^2 - S_{W_2}^2\right)$ and $\left(S_{T_1}^2 - S_{T_2}^2\right)$ are compared in terms of coverage probabilities. It is determined that these confidence intervals provide the results which are close to each other for the type I error is both 0.05 and 0.10. These probabilities are quite close to nominal confidence level even in small sample sizes. When these confidence intervals are compared in terms of average length widths, it is determined that confidence interval based on robust estimator $\left(S_{T_1}^2 - S_{T_2}^2\right)$ provided narrower intervals compared to the confidence interval based on robust estimator $\left(S_{W_1}^2 - S_{W_2}^2\right)$. In addition, it is determined that narrower confidence intervals are obtained as replacement and trimming proportions increase. According to this result, estimator $\left(S_{T_1}^2 - S_{T_2}^2\right)$ should be preferred if it is required to establish a narrower confidence interval for the difference of the variances of two nonnormal populations.

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CONFLICT OF INTEREST

No conflict of interest was declared by the author.

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