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Standardized Likelihood Ratio Test for Homogeneity of Variance under Normality

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Abstract

In this article, standardized likelihood ratio test is proposed for the homogeneity of variances under normality. The proposed method was compared with some of the existing methods via Monte Carlo simulation for various parameter combinations, different group sizes and sample sizes in terms of type I error rate and power of test. According to numeric results, the proposed method performs quite well according to its alternatives.

1. INTRODUCTION

Testing the homogeneity of variances is one of the main assumptions of some common statistical procedures and also it has been broadly used in many scientific application areas. The assumption of homogeneity of variances is required in many experimental design models such as analysis of variance (ANOVA). Testing the homogeneity of variances is also of interest as data from different sources is pooled to yield an improved estimated variance [12]. Besides, it is important to determine uniformity in the quality control of manufacturing processes, in biology, in agricultural production systems and in the development of educational methods [3]. Taguchi [28] demonstrates that reducing variability is an integral part of improving quality. In robust designs, experimentation is used to determine the factor levels so that the product or production processes is insensitive to potential variations in operating, environmental, and market conditions [25]. Besides, in quality control work, testing the homogeneity of variances is often a useful endpoint in analysis [12]. Biologists are interested in differences in the variability of populations for several reasons, for instance, as an indicator of generic diversity and in the investigation of mechanisms of adaptation [3]. Furthermore, testing the homogeneity of variances is used as a prelude to dose response modeling or discriminant analysis [6].

In the literature, there are many methods for testing the homogeneity of variances. Neyman and Pearson [26] suggested a statistic which is the ratio of a weighted geometric mean to a weighted arithmetic means of the mean squares. Bartlett [1] proposed an analogous test in which the sums of squares are weighted with their associated degrees of freedom instead of with the numbers of observations as in Neyman Pearson criterion [20]. Box and Andersen [4] modified the Bartlett test under permutation distribution for homogeneity of variances [18]. Cochran [11] proposed a test statistic in which the variance of only one group is greater than others. Hartley [19] suggested the F-max statistic which is the ratio of the maximum variance to the minimum variance of groups. Levene [22] gave a testing procedure based on ANOVA which uses the absolute difference of observations from their means. Brown and Forsythe [5] modified the Levene test using sample medians instead of sample means. Recently, Bhandary and Dai [2] proposed a test based on Bonferroni type adjustment procedure on the ordered p values to control the family wise type I error rate. Liu and Xu [23] proposed a test, using the generalized p value approach, and compared it with the Bartlett test. For homogeneity of variances, Gökpınar and Gökpınar [16] proposed a test statistic based on computational approach test (CAT) which is a special case of parametric bootstrap. The CAT method based

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on simulation and numerical computations uses the maximum likelihood estimates (MLEs), and does not require the knowledge of any sampling distribution. Some papers regarding the CAT are given as Chang and Pal [8], Chang et al. [7-9], Gökpınar and Gökpınar [13,16, 17], Gökpınar et al. [14], Mutlu et al. [24], etc. The likelihood ratio test (LRT) for homogeneity of variances performs poorly with respect to the type I error rate. Recently, Chang et al. [10] suggested a CAT version based on LRT for homogeneity of variances. Krishnamoorthy and Oral [25] proposed the standardized likelihood ratio test (SLRT) to improve the LRT for testing the equality of means of several log-normal distributions. In this study, by taking an idea from Krishnamoorthy and Oral [21], we proposed the SLRT for homogeneity of variances.

The rest of this study is organized as follows. For the homogeneity of variances, Bartlett test [1], Bhandary and Dai test [2], generalized p value approach given by Liu and Xu [23] and CAT given by Gökpınar and Gökpınar [16] were presented in Section 2. In Section 3, we developed the SLRT for homogeneity of variances. In Section 4, simulation results based on estimated Type I errors and powers of tests were presented. Concluding remarks were summarized in Section 5.

2. TEST STATISTICS

We consider the problem of testing the homogeneity of variances of k populations given random samples $\left\{X_{ij} : j=1,...,n_i, i=1,...,k\right\}$ from $N(\mu_i,\sigma_i^2)$ where μ_i and σ_i^2 are the mean and the variance of the ith population, respectively. The sample means and variances are

$$\bar{X}_i = \sum_{j=1}^{n_i} X_{ij} / n_i , \quad \text{and} \qquad S_i^2 = \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2 / (n_i - 1) , \qquad i=1,....k,$$

respectively and the pooled sample mean and the variance are

$$\bar{X} = \sum_{i=1}^{k} n_i \bar{X}_i / N \quad \text{and } S_p^2 = \sum_{i=1}^{k} (n-1) S_i^2 / (N-k) ,$$

respectively, where $N = \sum_{i=1}^{k} n_i$. The MLEs of the parameters μ_i and σ_i^2, $i=1,2,...,k$ are obtained as follows:

$$\hat{\mu}_{i(MLE)} = \sum_{j=1}^{n_i} X_{ij} \Big/ n_i = \bar{X}_i \quad \text{and} \quad \hat{\sigma}^2_{i(MLE)} = \sum_{j=1}^{n_i} \left(X_{ij} - \bar{X}_i \right)^2 \Big/ n_i = S_i^{2*}, \qquad i=1,2,...,k \tag{1}$$

The null hypothesis of interest is

$$H_0 : \sigma_1^2 = \sigma_2^2 = ... = \sigma_k^2 \text{ against } H_1 : \sigma_i^2 \neq \sigma_l^2 \text{ for at least one } i \neq l .$$

In the rest of this section, for testing Eq. (2), the Bartlett's test (BT), Bhandary and Dai's test (BDT), generalized p value approach (GPA) and CAT were given briefly.

2.1. Bartlett's Test

Bartlett [1] test statistic, B, is given by

$$B = \frac{(N-k)\log S_p^2 - \sum_{i=1}^{k} (n_i - 1) \log S_i^2}{1 + (1/3(k-1))\left(\sum_{i=1}^{k} 1/(n_i - 1) - 1/(N-k) \right)} .$$

For a given level α, and an observed value B_h of B, H_0 is rejected if $B_h > \chi^2_{k-1,\alpha}$ where χ^2_{k-1} is the Chi-square distribution with degrees of freedom $k-1$ under the null hypothesis.

2.2. Bhandary and Dai's test

In this section, simplified algorithm of Bhandary and Dai [2] test based on Bonferroni type adjustment procedure is given as follows:

1. Calculate the pooled sample variance on the merged data except *i*th group as follows:

$$S^2_{p:i} = ((N-k)S^2_p - (n_i - 1)S^2_i)/(N - n_i - (k-1)) \quad i = 1, 2, ..., k .$$

Determine F-test statistics to be $F_i = \dfrac{S^2_i}{S^2_p}$ and $F_i' = \dfrac{S^2_{p:i}}{S^2_i}$ $i = 1, 2, ..., k$.

2. Calculate $P_i = P(X > F_i)$ where $X \sim F_{n_i - 1, r_i}$ and $P_i' = P(X' > F_i')$ where $X' \sim F_{r_i, n_i - 1}$ and $r_i = \sum_{i \neq j}^{k} \left(n_j - 1\right)$.

3. Sort the P_i and P_i' values and denote them by $P_{(1)} < P_{(2)} < ... < P_{(2k)}$. Reject H_0 if $P_{(i)} < \dfrac{i}{2k}\alpha$ for some $i \in \{1, 2, ..., 2k\}$.

2.3. The generalized p value approach

In this section the algorithm of GPA obtained by Liu and Xu [23] is given as follows:

1) For a given data$\left(x_{i1}, x_{i2}, ..., x_{in_i}\right)$, $i = 1, ..., k,$ calculate observed statistics $\left(\bar{x}_1, \bar{x}_2, ..., \bar{x}_k\right)$ and $\left(s^2_1, ..., s^2_k\right)$.

2) Calculate the generalized pivotal quantity of $A\ln\sigma^2$, $R_{A\ln\sigma^2}$

$$R_{A\ln\sigma^2} = AR_{\ln\sigma^2} = A(R_{\ln\sigma^2_1}, R_{\ln\sigma^2_2}, ..., R_{\ln\sigma^2_k})',$$

where $R_{\ln\sigma^2_i} = \ln\dfrac{n_i s^2_i}{U_i}$, $U_i \sim \chi^2_{n_i - 1}$, $i = 1, ..., k$ and

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix}_{(k-1)xk} .$$

3) Calculate μ_R and Σ_R as follows:

$$\mu_R = A\left(E(R_{\ln\sigma^2_1}/(\bar{x}, s^2)), E(R_{\ln\sigma^2_2}/(\bar{x}, s^2)), ..., E(R_{\ln\sigma^2_k}/(\bar{x}, s^2))\right)' \quad \text{and}$$

$$\Sigma_R = Adiag\left(Var(R_{\ln\sigma^2_1}/(\bar{x}, s^2)), Var(R_{\ln\sigma^2_2}/(\bar{x}, s^2)), ..., Var(R_{\ln\sigma^2_k}/(\bar{x}, s^2))\right)A',$$

where

$$E\left(R_{\ln\sigma_i^2} / (\bar{x}, s^2)\right) = \ln(n_i s_i^2) - E(\ln U_i)$$

and

$$Var\left(R_{\ln\sigma_2^2} / (\bar{x}, s^2)\right) = E\left((\ln U_i)^2\right) - \left(E\left(\ln U_i\right)\right)^2.$$

4) Calculate $\|d\|^2 = \mu_R' \sum_R^{-1} \mu_R$.

5) Calculate $\|D\|^2 = \left(R_{A\ln\sigma^2} - \mu_R\right)' \sum_R^{-1} \left(R_{A\ln\sigma^2} - \mu_R\right)$.

6) Repeat step 2-5 a total m times and calculate p-value as $p = \#\left(\|D_j\|^2 > \|d_j\|^2\right) / m$, $j = 1,...,m$. In the case of $p < \alpha$, H_0 is rejected

2.4. The Computational Approach Test Approach

Gökpınar and Gökpınar [16] proposed a test based on the CAT for the homogeneity of variances under normality. The test statistic, η, is given as

$$\eta = \sum_{i=1}^{k} n_i \left(\log\sigma_i^2 - \log\overline{\sigma^2}\right)^2, \tag{3}$$

where $\overline{\sigma^2} = \sum_{i=1}^{k} n_i\sigma_i^2 \Big/ \sum_{i=1}^{k} n_i$. The algorithm of this approach is given as follows:

1) The MLE of σ_i^2 is given in Eq. (1), then the test statistic is calculated as

$$\hat{\eta}_{(\mathrm{ML})} = \sum_{i=1}^{k} n_i \left(\log S_i^{2*} - \log\overline{S^2}\right)^2, \text{ where } \overline{S^2} = \sum_{i=1}^{k} n_i S_i^{2*} \Big/ \sum_{i=1}^{k} n_i .$$

2. Under H_0, the restricted MLEs (RMLEs) of parameters are obtained as

$$\hat{\mu}_{i(RMLE)} = \frac{\sum_{j=1}^{n_i} X_{ij}}{n_i} = \bar{X}_i \quad \text{and} \quad \hat{\sigma}^2_{(RMLE)} = \frac{\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(X_{ij} - \bar{X}_i\right)^2}{\sum_{i=1}^{k} n_i} = \frac{\sum_{i=1}^{n} n_i S_i^{*2}}{\sum_{i=1}^{n} n_i} = \overline{S^2} \tag{4}$$

3. Generate artificial sample $X_{i1}, X_{i2}, ..., X_{in_i}$, $i = 1,...,k$ i.i.d from $N(\hat{\mu}_{i(RMLE)}, \hat{\sigma}^2_{RMLE})$ a large of number of times(say, m times). For each of these replicated samples, recalculate the values of $\hat{\eta}_{MLE}^{(j)}$ $(j = 1,...,m)$.

4. Calculate the p -value as $p = \#(\hat{\eta}_{MLE}^{(j)} > \hat{\eta}_{MLE}) / m$. In the case of $p < \alpha$, H_0 is rejected.

3. THE TESTS BASED ON LIKELIHOOD RATIO TEST FOR HOMOGENEITY OF VARIANCES

To obtain LRT statistic, we need to give the likelihood functions under H_1 and H_0 respectively. Under H_1 , log-likelihood function can be written as follows:

$$\ln L_1 = -\sum_{i=1}^{k} \frac{n_i}{2} \ln\left(2\pi\sigma_i^2\right) - \frac{1}{2}\left[\sum_{j=1}^{n_1} \frac{\left(X_{1j}-\mu_1\right)^2}{\sigma_1^2} + \ldots + \sum_{j=1}^{n_k} \frac{\left(X_{kj}-\mu_k\right)^2}{\sigma_k^2}\right].$$

Under H_0, log-likelihood function can be given as follows:

$$\ln L_0 = -\sum_{i=1}^{k} \frac{n_i}{2} \ln\left(2\pi\sigma^2\right) - \frac{1}{2\sigma^2}\left[\sum_{j=1}^{n_1}\left(X_{1j}-\mu_1\right)^2 + \ldots + \sum_{j=1}^{n_k}\left(X_{kj}-\mu_k\right)^2\right].$$

The LRT can be given as follows:

$$\Lambda = \sum_{i=1}^{k} n_i \left(\log S_i^{2*} - \log \overline{S^2}\right). \tag{5}$$

Λ has an approximate chi-square distribution with k-1 degrees of freedom under the null hypothesis.

3.1. The CAT Approach Based On The Likelihood Ratio Test

Chang et al. [10] suggested a CAT approach based on likelihood ratio test for homogeneity of variances. The test procedure is given as below:

1. Compute the LRT statistic Λ in Eq. (5)
2. Obtain the RMLEs of parameters μ and σ^2 in Eq. (4).
3. Generate artificial sample $X_{i1}, X_{i2}, \ldots, X_{in_i}$, $i=1,\ldots,k$ i.i.d from $N(\hat{\mu}_{i(RMLE)}, \hat{\sigma}^2_{RMLE})$ a large of number of times (say, m times), that is, $X_{ij} \sim N(\hat{\mu}_{i(RMLE)}, \hat{\sigma}^2_{RMLE})$. For each of these replicated samples, recalculate the values of $\Lambda^{(l)}$ $(l=1,\ldots,m)$.
4. Calculate the p -value as $p = \#(\Lambda^{(l)} > \Lambda)/m$. In the case of $p < \alpha$, H_0 is rejected.

3.2. The Proposed Test: The Standardized Likelihood Ratio Test

We proposed the standardized likelihood ratio test (SLRT) for testing the equality of variances of several normal distributions. The SLRT is defined as

$$\Lambda_S = \sqrt{2(\mathrm{k}-1)}\left(\frac{\Lambda - m(\Lambda)}{SD(\Lambda)}\right) + (k-1), \tag{6}$$

where $m(\Lambda)$ and $SD(\Lambda)$ are the mean and standard deviation of Λ, respectively. Λ_S has an approximate chi-square distribution with k-1 degrees of freedom under the null hypothesis. Krishnamoorthy and Oral [21] estimated the expressions of $m(\Lambda)$ ve $SD(\Lambda)$ through simulation because they are difficult to obtain. The proposed SLRT can be computed through the following steps.

1. Calculate the LRT statistic Λ in Eq. (5).
2. Generate artificial sample $X_{i1}, X_{i2}, \ldots, X_{in_i}$, $i=1,\ldots,k$ i.i.d from $N(\hat{\mu}_{i(RMLE)}, \hat{\sigma}^2_{RMLE})$ a large of number of times (say, m times), that is, $X_{ij} \sim N(\hat{\mu}_{i(RMLE)}, \hat{\sigma}^2_{RMLE})$. For each of these replicated samples, recalculate the values of the LRT statistic Λ in Eq. (5).
3. Calculate the mean $\left(m(\Lambda)\right)$ and standard deviation $\left(SD(\Lambda)\right)$ of these simulated LRT, and find the SLRT statistic Λ_S in Eq. (6).
4. If $\Lambda_S > \chi^2_{k-1;1-\alpha}$, then H_0 is rejected.

4. SIMULATION STUDY

In this section, the proposed SLRT was compared with GPA, BDT, BT, CAT test [17] and CATLR by means of simulation based on 5000 Monte Carlo runs. These tests were evaluated in terms of the type I error rate and power of test. The study includes different combination of sample sizes, number of groups and population variances. The samples were generated from the normal distribution. Without loss of generality, μ was taken to be equal to 0. For the specified nominal level of $\alpha = 0.05$, the estimated type I error rates of five tests were presented in from Table 1 to Table 4.

Table 1. *The estimated type I error rates for* $\alpha = 0.05$ *and k=3*

(n_1, n_2, n_3)	BT	GPA	BDT	CAT	CATLR	SLRT
3,5,7	0.050	0.050	0.042	0.049	0.050	0.052
3,6,9	0.046	0.047	0.040	0.049	0.047	0.051
3,8,13	0.055	0.058	0.045	0.054	0.058	0.055
10,15,20	0.048	0.044	0.044	0.045	0.047	0.052
20,25,30	0.050	0.051	0.047	0.051	0.051	0.052
10,20,30	0.053	0.054	0.050	0.053	0.055	0.046
15,25,35	0.046	0.046	0.044	0.044	0.046	0.054
20,30,40	0.047	0.047	0.043	0.049	0.050	0.046
3,3,3	0.046	0.050	0.048	0.049	0.049	0.047
5,5,5	0.051	0.053	0.048	0.053	0.052	0.050
7,7,7	0.048	0.050	0.046	0.051	0.050	0.047
9,9,9	0.047	0.049	0.043	0.049	0.048	0.055
15,15,15	0.048	0.047	0.045	0.045	0.047	0.051
20,20,20	0.050	0.051	0.045	0.050	0.050	0.050
25,25,25	0.048	0.047	0.045	0.048	0.048	0.046
30,30,30	0.047	0.046	0.043	0.046	0.047	0.051

Table 2*. The estimated type I error rates for* $\alpha = 0.05$ *and k=5*

$(n_1, n_2, n_3, n_4, n_5)$	BT	GPA	BDT	CAT	CATLR	SLRT
3,3,5,7,7	0.050	0.052	0.052	0.051	0.053	0.053
3,3,6,9,9	0.047	0.048	0.049	0.048	0.049	0.048
3, 3, 8,13,13	0.052	0.051	0.051	0.048	0.049	0.053
10,10,15,20,20	0.051	0.051	0.049	0.054	0.051	0.055
20,20,25,30,30	0.050	0.051	0.047	0.049	0.050	0.048
10,10, 20,30,30	0.047	0.045	0.048	0.045	0.046	0.050
15, 15, 25,35,35	0.050	0.052	0.053	0.051	0.050	0.047
20, 20, 30,40,40	0.048	0.051	0.054	0.050	0.049	0.054
3,3,3,3,3	0.045	0.051	0.051	0.051	0.050	0.047
5,5,5,5,5	0.045	0.044	0.048	0.043	0.045	0.049
7,7,7,7,7	0.049	0.049	0.046	0.048	0.049	0.051
9,9,9,9,9	0.051	0.049	0.052	0.052	0.051	0.047
15,15,15,15,15	0.056	0.054	0.052	0.054	0.055	0.048
20,20,20,20,20	0.051	0.051	0.046	0.051	0.051	0.051
25,25,25,25,25	0.049	0.046	0.046	0.045	0.049	0.051
30,30,30,30,30	0.050	0.052	0.051	0.053	0.050	0.050

Table 3. *The estimated type I error rates for* $\alpha = 0.05$ *and k=7*

$(n_1, n_2, n_3, n_4, n_5, n_6, n_7)$	BT	GPA	BDT	CAT	CATLR	SLRT
3, 3, 5, 5, 5, 7, 7	0.050	0.052	0.051	0.052	0.053	0.051
3, 3, 6, 6, 6, 9, 9	0.049	0.050	0.055	0.052	0.052	0.049
3, 3, 8, 8, 8, 13,13	0.050	0.050	0.047	0.053	0.051	0.047

10,10,15, 15, 15, 20,20	0.050	0.050	0.048	0.049	0.049	0.048
20,20,25,25,25, 30,30	0.053	0.055	0.053	0.053	0.053	0.054
10,10, 20, 20, 20, 30,30	0.057	0.056	0.051	0.053	0.055	0.051
15, 15, 25, 25, 25, 35, 35	0.046	0.045	0.052	0.049	0.046	0.049
20, 20, 30, 30, 30, 40,40	0.050	0.048	0.049	0.048	0.050	0.054
3, 3, 3, 3, 3, 3, 3	0.042	0.050	0.051	0.046	0.047	0.051
5, 5, 5, 5, 5, 5, 5	0.045	0.051	0.049	0.050	0.047	0.053
7, 7, 7, 7, 7, 7, 7	0.052	0.054	0.052	0.055	0.053	0.048
9, 9, 9, 9, 9, 9, 9	0.045	0.049	0.046	0.048	0.044	0.050
15,15,15,15,15,15,15	0.049	0.049	0.046	0.049	0.049	0.053
20,20,20,20,20,20,20	0.043	0.041	0.040	0.041	0.043	0.051
25,25,25,25,25,25,25	0.054	0.053	0.049	0.052	0.054	0.050
30,30,30,30,30,30,30	0.050	0.050	0.048	0.051	0.050	0.054

Table 4. *The estimated type I error rates for* $\alpha = 0.05$ *and k=9*

$(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9)$	**BT**	**GPA**	**BDT**	**CAT**	**CATLR**	**SLRT**
3, 3, 3, 5, 5, 5, 7, 7,7	0.047	0.050	0.047	0.051	0.049	0.050
3, 3, 3, 6, 6, 6, 9, 9,9	0.053	0.054	0.054	0.055	0.053	0.046
3, 3, 3, 8, 8, 8, 13,13,13	0.050	0.048	0.049	0.045	0.051	0.053
10,10,10, 15, 15,15,20,20,20	0.049	0.052	0.050	0.054	0.050	0.049
20,20,20,25,25,25, 30,30,30	0.048	0.047	0.049	0.046	0.048	0.051
10,10, 10, 20, 20,20, 30,30,30	0.048	0.047	0.054	0.048	0.048	0.053
15, 15, 15, 25, 25,25, 35, 35,35	0.051	0.049	0.050	0.048	0.051	0.048
20, 20, 20, 30, 30,30, 40,40,40	0.047	0.047	0.048	0.048	0.047	0.051
3, 3, 3, 3, 3, 3, 3,3,3	0.040	0.046	0.047	0.048	0.045	0.054
5, 5, 5, 5, 5, 5, 5, 5,5	0.055	0.055	0.051	0.054	0.055	0.054
7, 7, 7, 7, 7, 7, 7, 7,7	0.047	0.052	0.051	0.050	0.048	0.048
9, 9, 9, 9, 9, 9, 9, 9,9	0.052	0.052	0.050	0.052	0.051	0.052
15,15,15,15,15,15,15,15,15	0.052	0.051	0.044	0.052	0.052	0.049
20,20,20,20,20,20,20,20,20	0.050	0.050	0.049	0.050	0.049	0.055
25,25,25,25,25,25,25,25,25	0.049	0.050	0.048	0.050	0.050	0.049
30,30,30,30,30,30,30,30,30	0.048	0.048	0.051	0.048	0.050	0.048

From the numerical results in Table 1-Table 4, it appears that the estimated type I error rates of six tests are close to the nominal level for all cases.

For specified nominal level of $\alpha = 0.05$, Table 5-Table 8 present the estimated powers of the six tests.

Table 5. *The estimated powers of the tests for k=3*

$(\sigma_1^2, \sigma_2^2, \sigma_3^2)$	(n_1, n_2, n_3)	**BT**	**GPA**	**BDT**	**CAT**	**CATLR**	**SLRT**
(0.25, 0.5, 1)	3,5,7	0.129	0.126	0.131	0.156	0.163	0.157
	3,6,9	0.149	0.150	0.146	0.179	0.192	0.167
	3,8,13	0.177	0.169	0.177	0.185	0.213	0.213
	10,15,20	0.539	0.530	0.523	0.585	0.569	0.572
	20,25,30	0.847	0.848	0.835	0.862	0.857	0.848
	10,20,30	0.634	0.633	0.608	0.697	0.676	0.666
	15,25,35	0.788	0.792	0.774	0.830	0.812	0.822
	20,30,40	0.887	0.886	0.875	0.908	0.899	0.896
	3,3,3	0.083	0.078	0.085	0.081	0.091	0.094
	5,5,5	0.164	0.149	0.161	0.155	0.166	0.169
	7,7,7	0.269	0.253	0.259	0.260	0.270	0.243
	9,9,9	0.356	0.345	0.340	0.349	0.356	0.350
	15,15,15	0.595	0.587	0.578	0.588	0.594	0.601

	20,20,20	0.753	0.749	0.738	0.750	0.750	0.764
	25,25,25	0.849	0.849	0.836	0.848	0.849	0.854
	30,30,30	0.921	0.920	0.913	0.921	0.921	0.914
(0.25,0.75, 2.25)	3,5,7	0.270	0.252	0.263	0.316	0.332	0.332
	3,6,9	0.315	0.294	0.296	0.357	0.378	0.382
	3,8,13	0.404	0.380	0.373	0.401	0.465	0.461
	10,15,20	0.927	0.926	0.914	0.943	0.937	0.938
	20,25,30	0.997	0.998	0.996	0.998	0.998	0.999
	10,20,30	0.972	0.973	0.963	0.982	0.980	0.977
	15,25,35	0.995	0.995	0.993	0.997	0.996	0.997
	20,30,40	0.999	0.999	0.999	0.999	0.999	0.999
	3,3,3	0.149	0.132	0.162	0.143	0.162	0.157
	5,5,5	0.372	0.342	0.369	0.355	0.375	0.382
	7,7,7	0.599	0.581	0.587	0.591	0.601	0.601
	9,9,9	0.738	0.729	0.725	0.733	0.737	0.744
	15,15,15	0.950	0.949	0.943	0.949	0.949	0.954
	20,20,20	0.990	0.989	0.988	0.989	0.989	0.989
	25,25,25	0.998	0.998	0.997	0.998	0.998	0.997
	30,30,30	0.999	0.999	0.999	0.999	0.999	1.000

Table 6*. The estimated powers of the tests for k=5*

$(\sigma_1^2,\sigma_2^2,\sigma_3^2,\sigma_4^2,\sigma_5^2)$	(n_1,n_2,n_3,n_4,n_5)	**BT**	**GPA**	**BDT**	**CAT**	**CATLR**	**SLRT**
(0.25,0.25,0.5,1,1)	3,3,5,7,7	0.160	0.140	0.141	0.200	0.217	0.218
	3,3,6,9,9	0.182	0.158	0.143	0.219	0.249	0.262
	3, 3, 8,13,13	0.208	0.185	0.152	0.233	0.280	0.284
	10,10,15,20,20	0.753	0.738	0.584	0.802	0.792	0.792
	20,20,25,30,30	0.971	0.968	0.905	0.974	0.973	0.976
	10,10, 20,30,30	0.827	0.816	0.631	0.880	0.867	0.865
	15, 15, 25,35,35	0.945	0.944	0.850	0.964	0.957	0.956
	20, 20, 30,40,40	0.986	0.986	0.944	0.991	0.988	0.989
	3,3,3,3,3	0.112	0.085	0.119	0.097	0.125	0.119
	5,5,5,5,5	0.243	0.187	0.203	0.205	0.245	0.240
	7,7,7,7,7	0.391	0.329	0.306	0.347	0.391	0.408
	9,9,9,9,9	0.526	0.472	0.406	0.481	0.527	0.532
	15,15,15,15,15	0.817	0.798	0.693	0.800	0.817	0.820
	20,20,20,20,20	0.935	0.930	0.852	0.929	0.936	0.936
	25,25,25,25,25	0.981	0.979	0.934	0.980	0.980	0.982
	30,30,30,30,30	0.994	0.993	0.975	0.994	0.994	0.994
(0.25,0.25,0.75,2.25,2.25)	3,3,5,7,7	0.382	0.315	0.243	0.430	0.491	0.486
	3,3,6,9,9	0.441	0.371	0.258	0.485	0.552	0.542
	3, 3, 8,13,13	0.518	0.440	0.288	0.535	0.629	0.639
	10,10,15,20,20	0.997	0.996	0.965	0.998	0.998	0.998
	20,20,25,30,30	0.999	0.999	0.999	1.000	0.999	1.000
	10,10, 20,30,30	0.998	0.997	0.982	1.000	0.999	1.000
	15, 15, 25,35,35	1.000	1.000	0.999	1.000	1.000	1.000
	20, 20, 30,40,40	1.000	1.000	1.000	1.000	1.000	1.000
	3,3,3,3,3	0.233	0.153	0.218	0.187	0.249	0.257
	5,5,5,5,5	0.587	0.488	0.440	0.528	0.590	0.595
	7,7,7,7,7	0.816	0.769	0.664	0.784	0.816	0.819
	9,9,9,9,9	0.932	0.913	0.825	0.918	0.932	0.936
	15,15,15,15,15	0.997	0.997	0.987	0.997	0.997	0.999
	20,20,20,20,20	0.999	0.999	0.998	0.999	0.999	1.000
	25,25,25,25,25	1.000	1.000	1.000	1.000	1.000	1.000
	30,30,30,30,30	1.000	1.000	1.000	1.000	1.000	1.000

Table 7. *The estimated powers of the tests for k=7*

$(\sigma_1^2,\sigma_2^2,\sigma_3^2,\sigma_4^2,\sigma_5^2,\sigma_6^2,\sigma_7^2)$	$(n_1,n_2,n_3,n_4,n_5,n_6,n_7)$	BT	GPA	BDT	CAT	CATLR	SLRT
(0.25,0.25,0.5,0.5,0.5,1,1)	3, 3, 5, 5, 5, 7, 7	0.168	0.137	0.158	0.193	0.224	0.211
	3, 3, 6, 6, 6, 9, 9	0.192	0.160	0.172	0.217	0.255	0.254
	3, 3, 8, 8, 8, 13,13	0.256	0.208	0.202	0.248	0.318	0.315
	10,10,15, 15, 15, 20,20	0.714	0.672	0.549	0.746	0.749	0.752
	20,20,25,25,25, 30,30	0.959	0.955	0.883	0.965	0.963	0.965
	10,10, 20, 20, 20, 30,30	0.828	0.802	0.648	0.869	0.865	0.856
	15, 15, 25, 25, 25, 35, 35	0.936	0.927	0.821	0.951	0.947	0.950
	20, 20, 30, 30, 30, 40,40	0.975	0.973	0.918	0.982	0.979	0.981
	3, 3, 3, 3, 3, 3, 3	0.105	0.075	0.115	0.083	0.113	0.106
	5, 5, 5, 5, 5, 5, 5	0.203	0.142	0.187	0.162	0.206	0.220
	7, 7, 7, 7, 7, 7, 7	0.336	0.265	0.275	0.284	0.338	0.340
	9, 9, 9, 9, 9, 9, 9	0.468	0.407	0.366	0.421	0.470	0.468
	15,15,15,15,15,15,15	0.777	0.741	0.632	0.748	0.777	0.769
	20,20,20,20,20,20,20	0.902	0.887	0.786	0.892	0.902	0.895
	25,25,25,25,25,25,25	0.965	0.961	0.900	0.961	0.966	0.963
	30,30,30,30,30,30,30	0.983	0.982	0.951	0.982	0.982	0.988
(0.25,0.25,0.75,0.75,0.75,2.25,2.25)	3, 3, 5, 5, 5, 7, 7	0.403	0.285	0.287	0.415	0.495	0.491
	3, 3, 6, 6, 6, 9, 9	0.497	0.373	0.329	0.488	0.586	0.581
	3, 3, 8, 8, 8, 13,13	0.608	0.492	0.409	0.562	0.696	0.710
	10,10,15, 15, 15, 20,20	0.993	0.991	0.950	0.995	0.995	0.994
	20,20,25,25,25, 30,30	0.999	0.999	0.999	1.000	1.000	1.000
	10,10, 20, 20, 20, 30,30	0.997	0.997	0.978	0.998	0.998	0.999
	15, 15, 25, 25, 25, 35, 35	1.000	1.000	0.998	1.000	1.000	1.000
	20, 20, 30, 30, 30, 40,40	1.000	1.000	1.000	1.000	1.000	1.000
	3, 3, 3, 3, 3, 3, 3	0.215	0.115	0.217	0.154	0.228	0.227
	5, 5, 5, 5, 5, 5, 5	0.529	0.400	0.410	0.446	0.535	0.544
	7, 7, 7, 7, 7, 7, 7	0.772	0.692	0.618	0.717	0.773	0.767
	9, 9, 9, 9, 9, 9, 9	0.906	0.875	0.782	0.882	0.905	0.900
	15,15,15,15,15,15,15	0.996	0.996	0.979	0.996	0.996	0.994
	20,20,20,20,20,20,20	0.999	0.999	0.997	0.999	0.999	1.000
	25,25,25,25,25,25,25	1.000	1.000	0.999	1.000	1.000	1.000
	30,30,30,30,30,30,30	1.000	1.000	1.000	1.000	1.000	1.000

Table 8. *The estimated powers of the tests for k=9*

$(\sigma_1^2,\sigma_2^2,\sigma_3^2,\sigma_4^2,\sigma_5^2,\sigma_6^2,\sigma_7^2,\sigma_8^2,\sigma_9^2)$	$(n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8,n_9)$	**BT**	**GPA**	**BDT**	**CAT**	**CATLR**	**SLRT**
(0.25,0.25,0.25,0.5,0.5,0.5,1,1,1)	3, 3, 3, 5, 5, 5, 7, 7,7	0.209	0.154	0.154	0.225	0.279	0.267
	3, 3, 3, 6, 6, 6, 9, 9,9	0.240	0.176	0.165	0.262	0.329	0.322
	3, 3, 3, 8, 8, 8, 13,13,13	0.300	0.234	0.184	0.297	0.393	0.393
	10,10,10, 15, 15,15,20,20,20	0.854	0.823	0.591	0.884	0.881	0.892
	20,20,20,25,25,25, 30,30,30	0.992	0.990	0.933	0.994	0.994	0.995
	10,10, 10, 20, 20,20, 30,30,30	0.923	0.906	0.665	0.953	0.948	0.948
	15, 15, 15, 25, 25,25, 35, 35,35	0.986	0.983	0.871	0.991	0.990	0.989
	20, 20, 20, 30, 30,30, 40,40,40	0.998	0.998	0.961	0.999	0.998	0.997
	3, 3, 3, 3, 3, 3, 3,3,3	0.122	0.082	0.136	0.096	0.131	0.136
	5, 5, 5, 5, 5, 5, 5, 5,5	0.287	0.191	0.226	0.218	0.290	0.299
	7, 7, 7, 7, 7, 7, 7, 7,7	0.451	0.357	0.319	0.382	0.452	0.468
	9, 9, 9, 9, 9, 9, 9, 9,9	0.603	0.523	0.417	0.542	0.602	0.623
	15,15,15,15,15,15,15,15,15	0.904	0.884	0.715	0.888	0.904	0.902
	20,20,20,20,20,20,20,20,20	0.973	0.969	0.876	0.970	0.973	0.979
	25,25,25,25,25,25,25,25,25	0.996	0.995	0.952	0.995	0.996	0.995
	30,30,30,30,30,30,30,30,30	0.999	0.999	0.986	0.999	0.999	0.999
(0.25,0.25,0.25,0.75,0.75,0.75,2.25,2.25,2.25)	3, 3, 3, 5, 5, 5, 7, 7,7	0.521	0.368	0.286	0.539	0.639	0.634
	3, 3, 3, 6, 6, 6, 9, 9,9	0.604	0.458	0.310	0.612	0.720	0.708
	3, 3, 3, 8, 8, 8, 13,13,13	0.733	0.593	0.359	0.695	0.828	0.833
	10,10,10, 15, 15,15,20,20,20	0.999	0.999	0.976	0.999	0.999	1.000
	20,20,20,25,25,25, 30,30,30	1.000	1.000	1.000	1.000	1.000	1.000
	10,10, 10, 20, 20,20, 30,30,30	1.000	1.000	0.993	1.000	1.000	1.000
	15, 15, 15, 25, 25,25, 35, 35,35	1.000	1.000	1.000	1.000	1.000	1.000
	20, 20, 20, 30, 30,30, 40,40,40	1.000	1.000	1.000	1.000	1.000	1.000
	3, 3, 3, 3, 3, 3, 3,3,3	0.286	0.146	0.243	0.194	0.307	0.313
	5, 5, 5, 5, 5, 5, 5, 5,5	0.694	0.538	0.469	0.592	0.696	0.694
	7, 7, 7, 7, 7, 7, 7, 7,7	0.909	0.853	0.700	0.869	0.909	0.909
	9, 9, 9, 9, 9, 9, 9, 9,9	0.972	0.959	0.854	0.963	0.973	0.978
	15,15,15,15,15,15,15,15,15	1.000	1.000	0.994	1.000	1.000	1.000
	20,20,20,20,20,20,20,20,20	1.000	1.000	0.999	1.000	1.000	1.000
	25,25,25,25,25,25,25,25,25	1.000	1.000	1.000	1.000	1.000	1.000
	30,30,30,30,30,30,30,30,30	1.000	1.000	1.000	1.000	1.000	1.000

As seen from Table 5, when the number of groups is k=3 and the sample sizes are small and different, the power of the CATLR is higher than the other tests. For example, when (n_1, n_2, n_3) =(3,5,7) and $(\sigma_1^2, \sigma_2^2, \sigma_3^2)$ =(0.25,0.5,1), powers of tests were obtained as BT=0.129, GPA=0.126, BDT=0.131, CAT=0.156, CATLR=0.163 and SLRT=0.157. As the sample sizes are getting larger, the powers of the CATLR and SLRT approach each other. For example, while (n_1, n_2, n_3) =(3,8,13) and $(\sigma_1^2, \sigma_2^2, \sigma_3^2)$ =(0.25,0.5,1), the powers of the tests can be given as BT=0.177, GPA=0.169, BDT=0.178, CAT=0.186, CATLR=0.213 and SLRT=0.213. However, when all sample sizes are getting larger, the CAT is slightly more powerful than CATLR and SLRT. When sample sizes are equal, the powers of the CATLR and SLRT are higher than other tests, even the power of the SLRT is slightly higher than CATLR. However, when the equal sample sizes are getting larger, the powers of the CATLR, SLRT and BT approach each other.

When differences between population variances are getting higher and the sample sizes are small, the powers of the CATLR and SLRT are getting higher than the other tests, especially in the case of different sample sizes. For example, while (n_1, n_2, n_3) =(3,8,13) and $(\sigma_1^2, \sigma_2^2, \sigma_3^2)$ =(0.25,0.75,2.25), powers of tests can be given as BT=0.405, GPA=0.380, BDT=0.374, CAT=0.401, CATLR=0.465, SLRT=0.461. When the equal sample sizes are getting larger, the powers of the CATLR, SLRT and BT approach each other.

When the number of groups is k=5, for the small and different sample sizes, the powers of the CATLR and SLRT are close to each other and are higher than the other tests. However, when all sample sizes are getting larger, the CAT is slightly more powerful than the CATLR and SLRT. For the equal sample sizes, the powers of the CATLR and SLRT are close to each other and are higher than other tests, however, for the large and equal sample sizes, the powers of the CATLR, SLRT and BT approach each other. When the differences between population variances are getting higher, the powers of the CATLR and SLRT are getting higher than the other tests, especially in the case of different sample sizes.

As the number of group is getting larger, that is, while k=7 and the sample sizes are different, the CATLR and SLRT are more powerful than the other tests. For the equal sample sizes the powers of the CATLR and SLRT are better than the others. Besides, as the equal sample sizes increase, the CATLR, SLRT and BT approach each other. For k=9, the results are similar to the ones obtained for the number of group, k=7.

All in all, the CATLR and SLRT are better than the others, especially in the cases of small sample sizes. In addition to this, the powers of other tests are close to the CATLR and SLRT as the sample size increases.

5. CONCLUSION

In this study, we proposed a standardize likelihood ratio test (SLRT) for the homogeneity of variances. We have compared this approach with existing approaches. Simulation study indicates that regardless of the number of group and sample sizes, the powers of the CATLR and SLRT are close to each other and are higher than the other tests. When the differences between population variances are getting higher, in case of large equal sample sizes, the powers of the CATLR, SLRT and BT are close to each other and are higher than the other tests.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors

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