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Research Article

A Note on the Laplacian Energy of the Power Graph of a Finite Cyclic Group

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ABSTRACT

In this study, the Laplacian matrix concept for the power graph of a finite cyclic group is redefined by considering the block matrix structure. Then, with the help of the eigenvalues of the Laplacian matrix in question, the concept of Laplacian energy for the power graphs of finite cyclic groups was defined and introduced into the literature. In addition, boundary studies were carried out for the Laplacian energy in question using the concepts the trace of a matrix, the Cauchy-Schwarz inequality, the relationship between the arithmetic mean and geometric mean, and determinant. Later, various results were obtained for the Laplacian energy in question for cases where the order of a cyclic group is the positive integer power of a prime.

1. Introduction

The power graph $P(G)$ of a finite group G, is the graph whose vertices are represented by the elements of G, and with adjacency relation between different two vertices v_x and v_y is defined as

$$v_x \sim v_y \Leftrightarrow \quad v_x = {v_y}^m \text{ and } v_y = {v_x}^m, m \in \mathbb{Z}^+.$$

Kelarev and Quinn introduced the concept of power graph to the mathematical literature in 2000 with their work on directed power graphs of finite semigroups [1]. Later, the concept of directed power graph for groups was defined, and various studies were carried out for directed power graphs on semigroups and groups [2-3]. Chakrabarty et al., inspired by these studies, introduced the concept of undirected power graph to the mathematical literature with their study in 2009 [4]. In addition, this study also revealed the relationship between the power graph being a complete graph and the structure and order of the group. Later, Cameron et al. abbreviated the concept of undirected power graph and named it as power graph [5-6] and this name passed into the mathematics literature and after that, studies for undirected power graphs were published under the name of power graph.

Another name that directs the study of power graph on the basis of spectral graph theory is Chattopadhyay. The study of Chattopadhyay et al. in 2018, the adjacency matrix concept was redefined on a power graph. Additionally, in this study, they obtained bounds for the largest eigenvalues of power graphs [7].

The energy of a graph was originated from the π-electron energy in the Hückel molecular orbital theory and motivated by this study, Gutman in 1978 defined the energy of a graph [8]. At the first time, the concept of graph energy did not receive much attention. However, in the past decade, the concept of graph energy has become popular with its widespread use in both theoretical and application areas and many different versions have been conceived. One of

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the most important of these versions is the Laplacian energy. The concept of Laplacian energy for graphs was defined as the sum of the absolute deviations of the eigenvalues of its Laplacian matrix by Gutman and Zhou in 2006 [9]. It is very important to do boundary studies in graph theory because it is not always easy to find the spectral structures of graphs with a large number of points. In this sense, boundary studies for the Laplacian energy of a graph has received much interest and has appeared frequently in many papers.

In this study, for the power graph ona finitecyclic group, the concept of Laplacian matrix is redefined by considering the block matrix structure, and then the concept of Laplacian energy is given with the help of Laplacian eigenvalues. Briefly mention the structures that we will use throughout the study.

Let C_n be a cyclic group with n elements, V_1 be the set of its the identity and generators and V_2 the set of its remaining elements. Thus $|V_1| = 1+\varphi(n) = t$ (say), where $\varphi(n)$ is Euler's φ function. In this case, the Laplacian matrix can be redefined as the block matrix structure below, considering the V_1 and V_2 structures.

$$L = \begin{pmatrix} (n\mathrm{I}-J)_{\mathrm{t\times t}} & -J_{\mathrm{t\times(n-t)}} \\ -J_{\mathrm{(n-t)\times t}} & L(P(V_2))_{\mathrm{(n-t)\times(n-t)}} \end{pmatrix}$$

where J is the matrix with all entires being 1 and I is the identity matrix. Also let $L(P(V_2)) = (l_{ij})$ is the Laplacian matrix formed by the elements of V_2, i.e.,

$$l_{ij} = \begin{cases} -1 & ; \quad i\sim j \\ d(i) & ; \quad i=j \\ 0 & ; \quad otherwise \end{cases},$$

where $d(i)$, is the degree of a vertex i. The Laplacian matrix of the power graph is a symmetric and real matrix. Therefore, all eigenvalues are real and are given in the following order.

$$\mu_n \geq \mu_{n-1} \geq \cdots \geq \mu_2 \geq \mu_1 = 0.$$

In the next section, in order to bring a different perspective to boundary studies, the Laplacian energy is defined by using the fact that the Laplacian matrix for a power graph is a block matrix and the structures of the Laplacian eigenvalues, and then the bounds on this concept are obtained.

2. Main Results

Let L be the Laplacian matrix of $P(C_n)$ and its eigenvalues are $\mu_n \geq \mu_{n-1} \geq \cdots \geq \mu_2 \geq \mu_1 = 0$. Using the concept of the Laplacian energy of a simple graph and the Laplacian matrix of a power graph being a block matrix, the Laplacian energy LE of the power graph $P(C_n)$ is defined as

$$LE = \sum_{i=1}^{n} |\gamma_i|,$$

where

$$\gamma_i = \mu_i - \frac{s}{n},$$

$$s = t(2n-t-1) - 2\sum_{t+1\leq i<j\leq n} l_{ij}.$$

Lemma 1. Let LE be the Laplacian energy of the power graph $P(C_n)$ and $n \geq 3$. Then

$$\sum_{i=1}^{n} \mu_i = t(2n-t-1) - 2\sum_{t+1\leq i<j\leq n} l_{ij}$$

$$= s,$$

$$\sum_{i=1}^{n} \mu_i^2 = s + t(n-1)^2 + \sum_{i=t+1}^{n} d^2(i),$$

where $d(i)$, is the degree of a vertex i.

Proof. Since the trace of a matrix is the sum of its eigenvalues, we have

$$\sum_{i=1}^{n} \mu_i = tr(L)$$

$$= t(n-1) + \sum_{i=t+1}^{n} d(i)$$

$$= t(n-1) + \sum_{i=t+1}^{n} \left(-\sum_{j=1,i\neq j}^{n} l_{ij}\right)$$

$$= t(n-1) - \sum_{i=t+1}^{n} \left(-t + \sum_{j=t+1,i\neq j}^{n} l_{ij}\right)$$

$$= t(2n-t-1) - 2\sum_{t+1\leq i<j\leq n} l_{ij}.$$

We now consider the matrix L^2.

$$\sum_{i=1}^{n} \mu_i^2 = tr(L^2)$$

$$= t((n-1)^2 + t - 1) + 2t(n-t)$$

$$+ tr\left[L\left(P(V_2)\right)^2\right].$$

The ii-th entry of $L^2\left(P(V_2)\right)$ is

$$\sum_{i=t+1}^{n} d^2(i) - 2\sum_{t+1 \le i < j \le n} l_{ij}.$$

Thus

$$\sum_{i=1}^{n} \mu_i^2 = t(2n-t-1) -$$

$$2\sum_{t+1 \le i < j \le n} l_{ij} + t(n-1)^2 + \sum_{i=t+1}^{n} d^2(i)$$

$$= s + t(n-1)^2 + \sum_{i=t+1}^{n} d^2(i)$$

This completes the proof.

Lemma 2. Let LE be the Laplacian energy of the power graph $P(C_n)$ and $n \ge 3$. Then

$$\sum_{i=1}^{n} \gamma_i = 0$$

and

$$\sum_{i=1}^{n} \gamma_i^2 = s + \sum_{i=1}^{n} \left(d(i) - \frac{s}{n}\right)^2.$$

Proof. Using the definition of γ_i and Lemma 1, we have

$$\sum_{i=1}^{n} \gamma_i = -s + \sum_{i=1}^{n} \mu_i = 0.$$

For the proof of the second equality, we have

$$\sum_{i=1}^{n} \gamma_i^2 = \sum_{i=1}^{n} \mu_i^2$$

$$-2\frac{t(2n-t-1) - 2\sum_{t+1 \le i < j \le n} l_{ij}}{n} \sum_{i=1}^{n} \mu_i$$

$$+ \frac{\left(t(2n-t-1) - 2\sum_{t+1 \le i < j \le n} l_{ij}\right)^2}{n}.$$

From Lemma 1, we have

$$= t(2n-t-1) - 2\sum_{t+1 \le i < j \le n} l_{ij}$$

$$+ t(n-1)^2 + \sum_{i=t+1}^{n} d^2(i)$$

$$- \frac{\left(t(2n-t-1) - 2\sum_{t+1 \le i < j \le n} l_{ij}\right)^2}{n}$$

$$= \sum_{i=t+1}^{n} d^2(i) + n^2 t - nt$$

$$+ \sum_{i=t+1}^{n} d^2(i) - \frac{\left(t(2n-t-1) - 2\sum_{t+1 \le i < j \le n} l_{ij}\right)^2}{n}$$

$$= t(2n-t-1) - 2\sum_{t+1 \le i < j \le n} l_{ij}$$

$$+ \sum_{i=1}^{n} \left(d(i) - \frac{t(2n-t-1) - 2\sum_{t+1 \le i < j \le n} l_{ij}}{n}\right)^2$$

$$= s + \sum_{i=1}^{n} \left(d(i) - \frac{s}{n}\right)^2$$

so the proof is complete.

Lemma 3. Let LE be the Laplacian energy of the power graph $P(C_n)$ and $n = q^k \ge 3$. Then

$$\sum_{i=1}^{n} \gamma_i^2 = n(n-1),$$

where q is a prime number and $k \in \mathbb{Z}^+$.

Proof. Since n is the positive integer power of a prime number then $P(C_n)$ is a complete graph. Using Lemma 1, we have

$$\sum_{i=1}^{n} \mu_i = tr(L) = n(n-1), \qquad (1)$$

$$\sum_{i=1}^{n} \mu_i^2 = tr(L^2) = n^2(n-1) \qquad (2)$$

and

$$\gamma_i = \mu_i - \frac{t(2n-t-1) - 2\sum_{t+1 \le i < j \le n} l_{ij}}{n},$$

$$= \mu_i - \frac{n(2n-n-1)}{n}$$

$$= \mu_i - (n-1). \qquad (3)$$

From (1), (2), (3), we have

$$\sum_{i=1}^{n} \gamma_i^2 = \sum_{i=1}^{n} \left(\mu_i - (n-1)\right)^2$$

$$= \sum_{i=1}^{n} \mu_i^2 + n(n-1)^2$$

$$-2(n-1)\sum_{i=1}^{n} \mu_i\left(P(C_n)\right)$$

$$= n(n-1).$$

This completes the proof.

Theorem 4. Let LE be the Laplacian energy of the power graph $P(C_n)$ and $n \geq 3$. Then

$$\sqrt{ns + n\sum_{i=1}^{n}\left(d(i) - \frac{s}{n}\right)^2} \geq LE$$

and

$$LE \geq \sqrt{s + \sum_{i=1}^{n}\left(d(i) - \frac{s}{n}\right)^2}.$$

Proof. Using the Cauchy-Schwarz inequality and Lemma 2, we have

$$LE^2 = \left(\sum_{i=1}^{n}|\gamma_i|\right)^2$$

$$\leq n\sum_{i=1}^{n}\gamma_i^2$$

$$= ns + n\sum_{i=1}^{n}\left(d(i) - \frac{s}{n}\right)^2$$

and thus

$$LE \leq \sqrt{ns + n\sum_{i=1}^{n}\left(d(i) - \frac{s}{n}\right)^2}.$$

For the second inequality of the theorem, we obtain

$$LE^2 = \left(\sum_{i=1}^{n}|\gamma_i|\right)^2$$

$$\geq s + \sum_{i=1}^{n}\left(d(i) - \frac{s}{n}\right)^2.$$

Therefore,

$$LE \geq \sqrt{s + \sum_{i=1}^{n}\left(d(i) - \frac{s}{n}\right)^2}.$$

Corollary 5. Let LE be the Laplacian energy of the power graph $P(C_n)$ and $n = q^k \geq 3$. Then

$$\sqrt{n^2 - n} \leq LE \leq \sqrt{n^3 - n^2},$$

where q is a prime number and $k \in \mathbb{Z}^+$.

Proof. Since n is the positive integer power of a prime numberthen $P(C_n)$ is a complete graph. Using definition of Laplacian energy and Lemma 3, we have

$$LE^2 \geq \sum_{i=1}^{n}\gamma_i^2 = n(n-1)$$

and then

$$LE \geq \sqrt{n^2 - n}.$$

For the proof of the second inequality, we have

$$LE^2 \leq n\sum_{i=1}^{n}\gamma_i^2 = n^2(n-1),$$

i.e.,

$$LE \leq \sqrt{n^3 - n^2}.$$

so the proof is completed.

Theorem 6. Let LE be the Laplacian energy of the power graph $P(C_n)$ and $n \geq 3$. Then

$$LE \leq \frac{s}{n}$$

$$+\sqrt{(n-1)\left[s + \sum_{i=1}^{n}\left(d(i) - \frac{s}{n}\right)^2 - \frac{s^2}{n^2}\right]}.$$

Proof. Since the definition of the Laplacian energy, Cauchy-Schwarz inequality and Lemma 2, we have

$$\left(LE - \frac{s}{n}\right)^2 = \left(\sum_{i=2}^{n}|\gamma_i|\right)^2$$

$$\leq (n-1)\left[\sum_{i=1}^{n}\gamma_i^2 - \frac{s^2}{n^2}\right]$$

$$= (n-1)\left[s + \sum_{i=1}^{n}\left(d(i) - \frac{s}{n}\right)^2 - \frac{s^2}{n^2}\right],$$

and thus

$$LE \leq \frac{s}{n}$$

$$+\sqrt{(n-1)\left[s + \sum_{i=1}^{n}\left(d(i) - \frac{s}{n}\right)^2 - \frac{s^2}{n^2}\right]},$$

The proof is complete.

Corollary 7. Let LE be the Laplacian energy of the power graph $P(C_n)$ and $n = q^k \geq 3$. Then

$$LE = 2(n-1),$$

where q is a prime number and $k \in \mathbb{Z}^+$.

Proof. Since n is the positive integer power of a prime number then $P(C_n)$ is a complete graph. Thus

$$s = t(2n - t - 1) - 2\sum_{t+1 \leq i < j \leq n} l_{ij}$$

$$= n(2n - n - 1)$$

$$= n(n-1).$$

From Lemma 3, the definition of the Laplacian energy, Cauchy-Schwarz inequality and the spectrum of a complete graph, we obtain

$$\left(LE - \frac{s}{n}\right)^2 = (n-1)\left[\sum_{i=1}^{n} \gamma_i^2 - \frac{s^2}{n^2}\right]$$

$$= \left(LE - (n-1)\right)^2 = (n-1)^2$$

and then

$$LE = 2(n-1)$$

so the proof is completed.

Theorem 8. Let LE be the Laplacian energy of the power graph $P(C_n)$ and $n \geq 3$.Then

$$LE \geq \sqrt{s + \sum_{i=1}^{n}\left(d(i) - \frac{s}{n}\right)^2 + n(n-1)det\left(L - \frac{s}{n}\mathrm{I}\right)^{\frac{2}{n}}}.$$

Proof. By Lemma 2, we have

$$LE^2 = \sum_{i=1}^{n} \gamma_i^2 + 2\sum_{1 \leq i,j \leq n} |\gamma_i||\gamma_j|$$

$$= s + \sum_{i=1}^{n}\left(d(i) - \frac{s}{n}\right)^2 + \sum_{i \neq j} |\gamma_i||\gamma_j|. \qquad (4)$$

Because the arithmetic mean of nonnegative numbers is greater than the geometric mean. So we have

$$\frac{1}{n(n-1)}\sum_{i \neq j} |\gamma_i||\gamma_j| \geq \left(\prod_{i \neq j} |\gamma_i||\gamma_j|\right)^{\frac{1}{n(n-1)}}$$

$$= \left(\prod_{i=1}^{n} |\gamma_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}}$$

$$= det\left(L - \frac{s}{n}\mathrm{I}\right)^{\frac{2}{n}}, \qquad (5)$$

where I is the identity matrix. By (4) and (5), we obtain

$$LE \geq \sqrt{\begin{array}{c} s + \sum_{i=1}^{n}\left(d(i) - \frac{s}{n}\right)^2 \\ + n(n-1)det\left(L - \frac{s}{n}\mathrm{I}\right)^{\frac{2}{n}} \end{array}}.$$

Hence the proof is completed.

Corollary 9. Let LE be the Laplacian energy of the power graph $P(C_n)$ and $n = q^k \geq 3$. Then

$$LE \geq \sqrt{n(n-1)\left[1 + (n-1)^{\frac{2}{n}}\right]},$$

where q is a prime number and $k \in \mathbb{Z}^+$.

Proof. Since n is the positive integer power of a prime number then $P(C_n)$ is a complete graph. Using Theorem 8 and Lemma 3, we have

$$LE^2 \geq \sum_{i=1}^{n} \gamma_i^2 + n(n-1)det\left(L - \frac{s}{n}I\right)^{\frac{2}{n}}$$

$$= n(n-1) + n(n-1)det\left(L - \frac{s}{n}I\right)^{\frac{2}{n}}$$

$$= n(n-1)\left[1 + det\left(L - \frac{s}{n}I\right)^{\frac{2}{n}}\right].$$

Since $\mathcal{P}(C_n)$ is a complete graph,

$$s = n(n-1)$$

and then

$$det\left(L - \frac{s}{n}I\right) = det(L - (n-1)I)$$

$$= det\left(-A\left(\mathcal{P}(C_n)\right)\right)$$

$$= (-1)^n det\left(A\left(\mathcal{P}(C_n)\right)\right) = 1 - n.$$

Thus,

$$LE^2 \geq n(n-1)\left[1+(1-n)^{\frac{2}{n}}\right]$$

and then

$$LE \geq \sqrt{n(n-1)\left[1+(n-1)^{\frac{2}{n}}\right]}.$$

The proof is complete.

Theorem 10. Let LE be the Laplacian energy of the power graph $P(C_n)$, $n \geq 3$ and $|\gamma_1| \geq |\gamma_2| \geq \ldots \geq |\gamma_n| \geq 0$. Then

$$LE \geq \frac{s+\sum_{i=1}^{n}\left(d(i)-\frac{s}{n}\right)^2+n|\gamma_1||\gamma_n|}{|\gamma_1|+|\gamma_n|}.$$

Proof. We note that, since C_n is a cyclic group with $n \geq 3$, $\mathcal{P}(C_n)$ has at least two edges. Thus, L has at least one non-zero eigenvalue. Now for every $i = 1,2,\ldots,n$, $|\gamma_1| \geq |\gamma_i| \geq |\gamma_n|$. Thus

$$(|\gamma_1|-|\gamma_i|)(|\gamma_i|-|\gamma_n|) \geq 0. \qquad (6)$$

On the other hand

$$(|\gamma_1|-|\gamma_i|)(|\gamma_i|-|\gamma_n|) = |\gamma_i|(|\gamma_1|+|\gamma_n|)$$

$$-(\gamma_i^2+|\gamma_1||\gamma_n|). \qquad (7)$$

From (6) and (7), we obtain

$$|\gamma_i|(|\gamma_1|+|\gamma_n|) \geq \gamma_i^2+|\gamma_1||\gamma_n|. \qquad (8)$$

By summing the sides of the (8) for every $1 \leq i \leq n$, we have

$$(|\gamma_1|+\cdots+|\gamma_n|)(|\gamma_1|+|\gamma_n|) \geq$$

$$(\gamma_1^2+\ldots+\gamma_n^2)+n|\gamma_1||\gamma_n|$$

and thus

$$LE \geq \frac{\sum_{i=1}^{n}\gamma_i^2+n|\gamma_1||\gamma_n|}{|\gamma_1|+|\gamma_n|}.$$

From Lemma 2, we obtain

$$LE \geq \frac{s+\sum_{i=1}^{n}\left(d(i)-\frac{s}{n}\right)^2+n|\gamma_1||\gamma_n|}{|\gamma_1|+|\gamma_n|}.$$

Hence the proof is completed.

Corollary 11. Let LE be the Laplacian energy of the power graph $P(C_n)$ and $n = q^k \geq 3$. Assume that $|\gamma_1| \geq |\gamma_i| \geq |\gamma_n|$. Then

$$LE = 2(n-1),$$

where q is a prime number and $k \in \mathbb{Z}^+$.

Proof. Since n is the positive integer power of a prime number then $P(C_n)$ is a complete graph and its Laplacian spectrum is $\left\{0,\underbrace{n,n,\ldots,n}_{n-1}\right\}$. Also

$$s = n(n-1)$$

and for every $i = 2,3,\ldots,n$, $\gamma_i = \mu_i - \frac{s}{n}$, we obtain

$$\gamma_1 = n-1, \gamma_2 = \ldots = \gamma_n = 1.$$

Thus, using by Teorem 10 and Lemma 3, we have

$$LE = \frac{\sum_{i=1}^{n}\gamma_i^2+n|\gamma_1||\gamma_n|}{|\gamma_1|+|\gamma_n|}$$

$$= \frac{n(n-1)+n(n-1)}{n-1+1}$$

$$= 2(n-1).$$

The proof is complete.

3. Conclusion

In this study, Laplacian matrix concept is defined for power graphs of finite cyclic groups, inspired by the concepts of Laplacian matrix defined on simple connected graphs and adjecency matrix defined on power graphs. Then, using Laplacian eigenvalues, the concept of Laplacian energy for the power graph of a cyclic group is given and boundary studies are done on it. Although some bounds give results very close to the Laplacian energy, it is not always possible to make comparison between the bounds as the boundary results will change as the graph structure changes.

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