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On Some Properties of the Space $L_w^p(\mathbb{R}^n) \cap L_\vartheta^{q(.)}(\mathbb{R}^n)$

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Abstract: In this paper, we define $A_{w,\vartheta}^{p,q(.)}(\mathbb{R}^n)$ to be space of the intersection of the spaces $L_w^p(\mathbb{R}^n)$ and $L_\vartheta^{q(.)}(\mathbb{R}^n)$. Also, we investigate some inclusions and embedding properties of the space. Moreover, we discuss other basic properties of $A_{w,\vartheta}^{p,q(.)}(\mathbb{R}^n)$.

Keywords: Variable exponent, embedding, maximal operator.

1. Introduction

Kovacik and Rakosnik introduced the variable exponent Lebesgue space $L^{p(.)}(\mathbb{R}^n)$ and the Sobolev space $W^{k,p(.)}(\mathbb{R}^n)$ as an alternative approach for coping with non-linear Dirichlet boundary value problems in [18]. The study of these spaces has been triggered by the problems of elasticity, fluid dynamics, calculus of variations, see [25], [32], [33]. Also, Diening [6] proved for the first time the boundedness of the maximal operator in variable exponent Lebesgue spaces over bounded domains if $p(.)$ satisfies the locally log-Hölder continuous condition, that is,

$$|p(x)-p(y)| \leq \frac{C}{-\ln|x-y|},\ x,y \in \Omega,\ \ |x-y| \leq \frac{1}{2},$$

where Ω is a bounded domain. Diening later extended the result to unbounded domains by assuming, in addition, that the exponent $p(.) = p$ is a constant function outside a large ball. After this paper, many exciting and important papers appeared in non-weighted and weighted variable exponent spaces, see [7], [11], [18] and [27]. For $1 \leq p < \infty$, the space $B^p(G) = L^1(G) \cap L^p(G)$ is a Banach algebra with the norm $\|.\|_{B^p(G)}$ such that $\|f\|_{B^p(G)} := \|f\|_1 + \|f\|_p$ and the usual convolution product. Warner and Yap have studied the Banach algebras $B^p(G)$, for details see [29], [30] and [31]. Moreover, Sağir and Gurkanlı investigated some properties of $B_{w,\vartheta}^{p,q}(G) = L_w^p(G) \cap L_\vartheta^q(G)$ and endowed it with the sum norm $\|f\|_{w,\vartheta}^{p,q} := \|f\|_{p,w} + \|f\|_{q,\vartheta}$ in [26]. The aim of this paper is to generalize some of the results in [26] to the space $A_{w,\vartheta}^{p,q(.)}(\mathbb{R}^n) = L_w^p(\mathbb{R}^n) \cap L_\vartheta^{q(.)}(\mathbb{R}^n)$. Also, we obtain several inclusions and embedding properties in this space. Using these results, we obtain some applications for the intersection space $X \cap Y$ and the sum space $X+Y$, where X and Y are two normed spaces. We refer to [19] and [23] for a detailed historical background. The boundedness of the maximal operator in variable exponent spaces is very effective such that there are consequential papers e.g. [5], [6], [21], [22], [27]. By using this result under the conditions $w \in A_p$ and $\vartheta \in A_{q(.)}$, we will present Theorem 10, Theorem 11 and Corollary 2. These consequences can be used for some function spaces such as Sobolev spaces, Lorentz spaces, Amalgam spaces and Morrey spaces.

In this paper, we will work on $\mathbb{R}^n$ with Lebesgue measure dx. We write $(L^p(\mathbb{R}^n), \|.\|_p)$ for the Lebesgue spaces for $1 \leq p < \infty$. We denote by $C_c(\mathbb{R}^n)$ the space of all continuous, complex-valued functions with compact support in $\mathbb{R}^n$. We denote the family of all measurable functions $p(.) : \mathbb{R}^n \to [1,\infty)$ (called the variable exponent on $\mathbb{R}^n$) by the symbol $\mathscr{P}(\mathbb{R}^n)$. In this paper, the function $p(.)$ always denotes a variable exponent. For $p(.) \in \mathscr{P}(\mathbb{R}^n)$, define

$$p^- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \qquad p^+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x).$$

For every measurable function f on $\mathbb{R}^n$ we define the function

$$\rho_{p(.)}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx.$$

The function $\rho_{p(.)}$ is convex modular. $L^{p(.)}(\mathbb{R}^n)$ is denoted as the set of all (equivalence classes) measurable functions f on $\mathbb{R}^n$ such that $\rho_{p(.)}(\lambda f) < \infty$ for some $\lambda > 0$, equipped with the Luxemburg norm

$$\|f\|_{p(.)} = \inf\left\{\lambda > 0 : \rho_{p(.)}\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

If $p^+ < \infty$, then $f \in L^{p(.)}(\mathbb{R}^n)$ iff $\rho_{p(.)}(f) < \infty$. The set $L^{p(.)}(\mathbb{R}^n)$ is a Banach space with the norm $\|.\|_p$. If $p(.) = p$ is a constant function, then the Luxemburg coincides with the usual L^p-norm. A positive, measurable and locally integrable function $\vartheta : \mathbb{R}^n \to (0,\infty)$ is called a weight function. A Beurling weight ϑ on $\mathbb{R}^n$ is a measurable and locally bounded function on $\mathbb{R}^n$ and ϑ satisfies $1 \leq \vartheta(x)$ and $\vartheta(x+y) \leq \vartheta(x)\vartheta(y)$ for all $x,y \in \mathbb{R}^n$. We say that $\vartheta_1 \prec \vartheta_2$ if and only if there exists $c > 0$ such that $\vartheta_1(x) \leq c\vartheta_2(x)$ for all $x \in \mathbb{R}^n$. Two weight functions are called equivalent, written as $\vartheta_1 \approx \vartheta_2$, if $\vartheta_1 \prec \vartheta_2$ and $\vartheta_2 \prec \vartheta_1$. We set $L^p_w(\mathbb{R}^n) = \left\{ f : f w^{\frac{1}{p}} \in L^p(\mathbb{R}^n)\right\}$ for $1 \leq p < \infty$. It is a Banach space under the natural norm $\|f\|_{p,w} = \left\| f w^{\frac{1}{p}} \right\|_p$. Recall that $L^p_{w_1}(\mathbb{R}^n) \hookrightarrow L^p_{w_2}(\mathbb{R}^n)$ if and only if $w_2 \prec w_1$ [12]. The weighted modular is defined by

$$\rho_{p(.),\vartheta}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, \vartheta(x)\, dx.$$

The weighted variable exponent Lebesgue spaces $L^{p(.)}_{\vartheta}(\mathbb{R}^n)$ consist of all measurable functions f on $\mathbb{R}^n$ where $\|f\|_{p(.),\vartheta} = \left\| f \vartheta^{\frac{1}{p(.)}} \right\|_{p(.)} < \infty$. The space $L^{p(.)}_{\vartheta}(\mathbb{R}^n)$ is a Banach space with respect to

$\|.\|_{p(.),\vartheta}$. Also, some basic properties of this space were investigated in [2], [3], [4], [13], [14], [15], [16], [17]. We say that $p_1(.)$ is non-weaker than $p_2(.)$ if and only if there exist positive constants K_1, K_2 and $h \in L^1(\mathbb{R}^n), h \geq 0$ such that

$$t^{p_1(.)}\vartheta_1(x) \leq K_1 (K_2 t)^{p_2(.)}\vartheta_2(x) + h(x),$$

for a.e. $x \in \mathbb{R}^n$ and all $t \geq 0$. We say $p_1(.) \preceq p_2(.)$. Moreover, the embedding $L^{p_2(.)}_{\vartheta_2}(\mathbb{R}^n) \hookrightarrow L^{p_1(.)}_{\vartheta_1}(\mathbb{R}^n)$ holds if and only if $p_1(.) \preceq p_2(.)$, [10]. The space $L^1_{loc}(\mathbb{R}^n)$ is the space of all measurable functions f on $\mathbb{R}^n$ such that $f.\chi_K \in L^1(\mathbb{R}^n)$ for any compact subset $K \subset \mathbb{R}^n$. A Banach function space (shortly BF-space) on $\mathbb{R}^n$ is a Banach space $(B, \|.\|_B)$ of measurable functions which is continously embedded into $L^1_{loc}(\mathbb{R}^n)$, i.e. for any compact subset $K \subset \mathbb{R}^n$ there exists some constant $C_K > 0$ such that $\|f.\chi_K\|_{L^1} \leq C_K.\|f\|_B$ for all $f \in B$. A BF-space $(B, \|.\|_B)$ is called solid if $g \in L^1_{loc}(\mathbb{R}^n)$, $f \in B$ and $|g(x)| \leq |f(x)|$ almost everywhere (shortly a.e.) implies that $g \in B$ and $\|g\|_{L^1} \leq \|f\|_B$. A BF- space $(B, \|.\|_B)$ is solid iff it is a $L^\infty(\mathbb{R}^n)$-module. Let f be a measurable function on $\mathbb{R}^n$. The translation and character operators L_y and Λ_t are defined by $L_y f(x) = f(x-y)$ and $\Lambda_t f(y) = \langle y, t\rangle . f(y)$ respectively for $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}^n$. Also $(B, \|.\|_B)$ is strongly translation invariant if one has $L_y B \subseteq B$ and $\|L_y f\|_B = \|f\|_B$ and strongly character invariant if $\Lambda_t B \subseteq B$ and $\|\Lambda_t f\|_B = \|f\|_B$ for all $f \in B$, $y \in \mathbb{R}^n$ and $t \in \mathbb{R}^n$.

2. The Space $A^{p,q(.)}_{w,\vartheta}(\mathbb{R}^n)$

Throughout this paper, we assume that w, ϑ are Beurling weight functions.

Let $q(.) \in \mathscr{P}(\mathbb{R}^n)$. We set

$$A^{p,q(.)}_{w,\vartheta}(\mathbb{R}^n) = \left\{ f : f \in L^p_w(\mathbb{R}^n) \cap L^{q(.)}_\vartheta(\mathbb{R}^n) \right\}$$

and equip this vector space with the norm

$$\|f\|^{p,q(.)}_{w,\vartheta} = \|f\|_{p,w} + \|f\|_{q(.),\vartheta}$$

for any $f \in A^{p,q(.)}_{w,\vartheta}(\mathbb{R}^n)$.

Theorem 1. If $q^+ < \infty$, then $\left(A^{p,q(.)}_{w,\vartheta}(\mathbb{R}^n), \|.\|^{p,q(.)}_{w,\vartheta}\right)$ is a Banach space.

Proof. Let $\{f_n\}_{n\in IN}$ be a Cauchy sequence in $A^{p,q(.)}_{w,\vartheta}(\mathbb{R}^n)$. Thus given $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n, m \geqslant n_0$ implies

$$\|f_n - f_m\|^{p,q(.)}_{w,\vartheta} = \|f_n - f_m\|_{p,w} + \|f_n - f_m\|_{q(.),\vartheta} < \varepsilon.$$

Therefore, $\{f_n\}_{n\in IN} \subset L^p_w(\mathbb{R}^n)$ and $\{f_n\}_{n\in IN} \subset L^{q(.)}_\vartheta(\mathbb{R}^n)$ are Cauchy sequences with respect to $\|.\|_{p,w}$ and $\|.\|_{q(.),\vartheta}$ norms, respectively. Since the spaces $\left(L^p_w(\mathbb{R}^n), \|.\|_{p,w}\right)$ and $\left(L^{q(.)}_\vartheta(\mathbb{R}^n), \|.\|_{q(.),\vartheta}\right)$

are two Banach spaces, then given $\varepsilon > 0$, there are $n_1, n_2 \in \mathbb{N}$ such that for all $n \geqslant n_1, n \geqslant n_2$ imply

$$\begin{aligned} \|f_n - f\|_{p,w} &< \frac{\varepsilon}{2} \\ \|f_n - g\|_{q(.),\vartheta} &< \frac{\varepsilon}{2}. \end{aligned} \tag{1}$$

Thus $f_n \longrightarrow g$ in $L^{q(.)}(\mathbb{R}^n)$. Since $q^+ < \infty$, convergence in $L^{q(.)}$is necessity with convergence in measure [18] and then there is a sequence $\{f_{n_k}\}_{k\in IN} \subset \{f_n\}_{n\in IN}$ such that $f_{n_k} \longrightarrow g$, $a.e.$ [24]. Also, it is easy to see that $f_{n_k} \longrightarrow f$. Therefore given $\varepsilon > 0$, there exists $n_3 \in \mathbb{N}$ such that for all $n_k \geqslant n_3$ and $x \notin K$ implies

$$|f(x) - g(x)| \leqslant |f(x) - f_{n_k}(x)| + |f_{n_k}(x) - g(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

where K is the set of $\{f_{n_k}\}_{k\in IN}$ are not convergent. Then we write $f = g$, $a.e.$ and since elements of $L^{q(.)}_{\vartheta}(\mathbb{R}^n)$ are equivalence classes, we yield $f = g$. Since $f \in L^p_w(\mathbb{R}^n)$ and $f \in L^{q(.)}_{\vartheta}(\mathbb{R}^n)$, then $f \in A^{p,q(.)}_{w,\vartheta}(\mathbb{R}^n)$. Let $n_0 = \max\{n_1, n_2\}$. If we use (1), then given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ it holds that

$$\|f_n - f\|^{p,q(.)}_{w,\vartheta} = \|f_n - f\|_{p,w} + \|f_n - f\|_{q(.),\vartheta} < \varepsilon.$$

This completes the proof. ∎

Proposition 1. If $q^+ < \infty$, then

(i) $C_c(\mathbb{R}^n) \subset A^{p,q(.)}_{w,\vartheta}(\mathbb{R}^n)$

(ii) The space $A^{p,q(.)}_{w,\vartheta}(\mathbb{R}^n)$ is dense in $L^p_w(\mathbb{R}^n)$ and $L^{q(.)}_{\vartheta}(\mathbb{R}^n)$.

Proof. *(i)* Let $f \in C_c(\mathbb{R}^n)$ be any function such that $\sup p f = K$ compact. It is known that $C_c(\mathbb{R}^n) \subset L^p_w(\mathbb{R}^n)$. Also, the inclusion $C_c(\mathbb{R}^n) \subset L^{q(.)}_{\vartheta}(\mathbb{R}^n)$ is satisfied by [28]. Then, we get $f \in L^p_w(\mathbb{R}^n)$ and $f \in L^{q(.)}_{\vartheta}(\mathbb{R}^n)$. Therefore, $f \in A^{p,q(.)}_{w,\vartheta}(\mathbb{R}^n)$ which is the desired.

(ii) It is clear that the space $C_c(\mathbb{R}^n)$ is dense in $L^p_w(\mathbb{R}^n)$. Using the inclusion

$$C_c(\mathbb{R}^n) \subset A^{p,q(.)}_{w,\vartheta}(\mathbb{R}^n) \subset L^p_w(\mathbb{R}^n),$$

we get that $A^{p,q(.)}_{w,\vartheta}(\mathbb{R}^n)$ is dense in $L^p_w(\mathbb{R}^n)$. Similarly, we can conclude that $A^{p,q(.)}_{w,\vartheta}(\mathbb{R}^n)$ is dense in $L^{q(.)}_{\vartheta}(\mathbb{R}^n)$.

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Theorem 2. *(i)* The space $A^{p,q(.)}_{w,\vartheta}(\mathbb{R}^n)$ is strongly character invariant.

(ii) If $q^+ < \infty$, then the function $t \longrightarrow \Lambda_t f$ is continuous from $\mathbb{R}^n$ into $A^{p,q(.)}_{w,\vartheta}(\mathbb{R}^n)$.

(iii) The function $f \longrightarrow \Lambda_t f$ is continuous from $A^{p,q(.)}_{w,\vartheta}(\mathbb{R}^n)$ into $A^{p,q(.)}_{w,\vartheta}(\mathbb{R}^n)$.

Proof. *(i)* Take any $f \in A^{p,q(.)}_{w,\vartheta}(\mathbb{R}^n)$. Also, we have

$$|\Lambda_t f(x)| = |\langle x,t\rangle f(x)| = |f(x)|.$$

Using the definitions of norms of the spaces $L_w^p(\mathbb{R}^n)$ and $L_\vartheta^{q(.)}(\mathbb{R}^n)$, we get

$$\begin{aligned}\|\Lambda_t f_n-\Lambda_t f\|_{w,\vartheta}^{p,q(.)} &= \|\Lambda_t f_n-\Lambda_t f\|_{p,w}+\|\Lambda_t f_n-\Lambda_t f\|_{q(.),\vartheta}\\ &= \|f_n-f\|_{p,w}+\|f_n-f\|_{q(.),\vartheta}=\|f_n-f\|_{w,\vartheta}^{p,q(.)}.\end{aligned}$$

(ii) If we use same method in Proposition 2.4 by [2], then we can conclude

$$\|\Lambda_t f-f\|_{w,\vartheta}^{p,q(.)}=\|\Lambda_t f-f\|_{p,w}+\|\Lambda_t f-f\|_{q(.),\vartheta}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

for any $f\in A_{w,\vartheta}^{p,q(.)}(\mathbb{R}^n)$ and $t\in\mathbb{R}^n$.

(iii) Let $(f_n)_{n\in\mathbb{N}}\subset A_{w,\vartheta}^{p,q(.)}(\mathbb{R}^n)$ and $f\in A_{w,\vartheta}^{p,q(.)}(\mathbb{R}^n)$. Also, $f_n\longrightarrow f$ in $A_{w,\vartheta}^{p,q(.)}(\mathbb{R}^n)$. Then, we get

$$\begin{aligned}\|\Lambda_t f_n-\Lambda_t f\|_{w,\vartheta}^{p,q(.)} &= \|\Lambda_t f_n-\Lambda_t f\|_{p,w}+\|\Lambda_t f_n-\Lambda_t f\|_{q(.),\vartheta}\\ &= \|\Lambda_t(f_n-f)\|_{p,w}+\|\Lambda_t(f_n-f)\|_{q(.),\vartheta}\\ &= \|f_n-f\|_{w,\vartheta}^{p,q(.)}<\varepsilon.\end{aligned}$$

That is the desired.

■

Lemma 1. *(i)* The space $A_{w,\vartheta}^{p,q(.)}(\mathbb{R}^n)$ is a BF-space on $\mathbb{R}^n$.

(ii) If $q^+<\infty$, then $A_{w,\vartheta}^{p,q(.)}(\mathbb{R}^n)$ is a solid BF-space on $\mathbb{R}^n$.

Proof. It is known that $\left(A_{w,\vartheta}^{p,q(.)}(\mathbb{R}^n),\|.\|_{p(.),\vartheta}\right)$ is a Banach space. Let $K\subset\mathbb{R}^n$ be a compact subset and $\frac{1}{p(.)}+\frac{1}{q(.)}=1$. If we use the generalized Hölder inequality for variable exponent Lebesgue space, we obtain a $C>0$ such that

$$\begin{aligned}\int_K |f(x)|\,dx &\le C\,\|\chi_K\|_{t(.)}\,\|f\|_{q(.)}\le C\,\|\chi_K\|_{t(.),\vartheta}\,\|f\|_{q(.),\vartheta}\\ &\le C\,\|\chi_K\|_{t(.),\vartheta}\,\|f\|_{w,\vartheta}^{p,q(.)}\end{aligned}$$

for all $f\in A_{w,\vartheta}^{p,q(.)}(\mathbb{R}^n)$,where χ_K is the characteristic function of K. It is clear that $\|\chi_K\|_{t(.),\vartheta}<\infty$ iff $\rho_{t(.),\vartheta}(\chi_K)<\infty$ for $t^+<\infty$. Since $\vartheta(x)\ge 1$ for $x\in\mathbb{R}^n$, we get

$$\rho_{t(.),\vartheta}(\chi_K)=\int_K \vartheta(x)^{q(x)}dx=\left(\sup_{x\in K}\vartheta(x)^{q^+}\right)\mu(K)<\infty$$

Therefore, $A_{w,\vartheta}^{p,q(.)}(\mathbb{R}^n)\hookrightarrow L_{loc}^1(\mathbb{R}^n)$. This completes (i). If we use (i), then the proof of (ii) is clear.

■

Theorem 3. Let $q^+<\infty$. Then $A_{w_1,\vartheta_1}^{p,q(.)}(\mathbb{R}^n)\subset A_{w_2,\vartheta_2}^{p,q(.)}(\mathbb{R}^n)$ if and only if $A_{w_1,\vartheta_1}^{p,q(.)}(\mathbb{R}^n)\hookrightarrow A_{w_2,\vartheta_2}^{p,q(.)}(\mathbb{R}^n)$.

Proof. The sufficient condition of the theorem is clear by the definition of embedding. Now, let $A_{w_1,\vartheta_1}^{p,q(.)}(\mathbb{R}^n)\subset A_{w_2,\vartheta_2}^{p,q(.)}(\mathbb{R}^n)$. Put the sum norm $|||f|||=\|f\|_{w_1,\vartheta_1}^{p,q(.)}+\|f\|_{w_2,\vartheta_2}^{p,q(.)}$ on $A_{w_1,\vartheta_1}^{p,q(.)}(\mathbb{R}^n)$. Hence it is easy to see that $\left(A^{p,q(.)}_{w_1,\vartheta_1}(\mathbb{R}^n), |||.|||\right)$ is a Banach space. Now, let us define the unit function I from $\left(A^{p,q(.)}_{w_1,\vartheta_1}(\mathbb{R}^n), |||.|||\right)$ into $\left(A^{p,q(.)}_{w_1,\vartheta_1}(\mathbb{R}^n), \|.\|^{p,q(.)}_{w_1,\vartheta_1}\right)$. Then I is continuous. Because, we can obtain the inequality $\|I(f)\|^{p,q(.)}_{w_1,\vartheta_1} = \|f\|^{p,q(.)}_{w_1,\vartheta_1} \leq |||f|||$. Using the closed graph mapping theorem, it is clear that I is a homeomorphism. That means the norms $|||.|||$ and $\|.\|^{p,q(.)}_{w_1,\vartheta_1}$ are equivalent. Thus, for every $f \in A^{p,q(.)}_{w_1,\vartheta_1}(\mathbb{R}^n)$ there exists $k > 0$ such that

$$|||f||| \leq k\|f\|^{p,q(.)}_{w_1,\vartheta_1}. \tag{2}$$

Therefore, by using (2) and the definition of norm $|||.|||$, we write

$$\|f\|^{p,q(.)}_{w_2,\vartheta_2} \leq |||f||| \leq k\|f\|^{p,q(.)}_{w_1,\vartheta_1}.$$

That is the desired. ■

Theorem 4. If $w_2 \prec w_1$, then $A^{p,q(.)}_{w_1,\vartheta}(\mathbb{R}^n) \hookrightarrow A^{p,q(.)}_{w_2,\vartheta}(\mathbb{R}^n)$.

Proof. The proof is straightforward. ■

Theorem 5. If $\vartheta_2 \prec \vartheta_1$, then $A^{p,q(.)}_{w,\vartheta_1}(\mathbb{R}^n) \hookrightarrow A^{p,q(.)}_{w,\vartheta_2}(\mathbb{R}^n)$.

Proof. It is clear that $\vartheta_2 \prec \vartheta_1$ implies and $L^{p(.)}_{\vartheta_1}(\mathbb{R}^n) \hookrightarrow L^{p(.)}_{\vartheta_2}(\mathbb{R}^n)$ by [2]. Hence, we write $A^{p,q(.)}_{w,\vartheta_1}(\mathbb{R}^n) \hookrightarrow A^{p,q(.)}_{w,\vartheta_2}(\mathbb{R}^n)$. ■

Theorem 6. If $q_1(.) \preceq q_2(.)$, then $A^{p,q_2(.)}_{w,\vartheta_2}(\mathbb{R}^n) \hookrightarrow A^{p,q_1(.)}_{w,\vartheta_2}(\mathbb{R}^n)$.

Proof. Let $q_1(.) \preceq q_2(.)$. Then, we write $L^{q_2(.)}_{\vartheta_2}(\mathbb{R}^n) \hookrightarrow L^{q_1(.)}_{\vartheta_1}(\mathbb{R}^n)$. Using this embedding, we conclude that $A^{p,q_2(.)}_{w,\vartheta_2}(\mathbb{R}^n) \hookrightarrow A^{p,q_1(.)}_{w,\vartheta_1}(\mathbb{R}^n)$. ■

Theorem 7. If $q_1^+ < \infty$ and $\left\|\frac{\vartheta_2}{\vartheta_1}\right\|_{\frac{q_1(.)}{q_1(.)-q_2(.)},\vartheta_1} < \infty$, then $A^{p,q_1(.)}_{w,\vartheta_1}(\mathbb{R}^n) \hookrightarrow A^{p,q_2(.)}_{w,\vartheta_2}(\mathbb{R}^n)$.

Proof. Let us assume that $\left\|\frac{\vartheta_2}{\vartheta_1}\right\|_{\frac{q_1(.)}{q_1(.)-q_2(.)},\vartheta_1} < \infty$. It is known by Theorem 5.1 in [10] that $L^{q_1(.)}_{\vartheta_1}(\mathbb{R}^n) \hookrightarrow L^{q_2(.)}_{\vartheta_2}(\mathbb{R}^n)$. Thus there exists a constant $c > 0$ such that

$$\|f\|_{q_2(.),\vartheta_2} \leq c\|f\|_{q_1(.),\vartheta_1}.$$

for any $f \in L^{p(.)}_{\vartheta_1}(\mathbb{R}^n)$.

Therefore, we can write as follows

$$\begin{aligned}\|f\|^{p,q_2(.)}_{w,\vartheta_2} &\leq \|f\|_{p,w} + \|f\|_{q_2(.),\vartheta_2} \leq \|f\|_{p,w} + c\|f\|_{q_1(.),\vartheta_1} \\ &\leq \max\{1,c\}\left\{\|f\|_{p,w} + \|f\|_{q_1(.),\vartheta_1}\right\} = \max\{1,c\}\|f\|^{p,q_1(.)}_{w,\vartheta_1}.\end{aligned}$$

for any $f \in A^{p,q_1(.)}_{w,\vartheta_1}(\mathbb{R}^n)$. That is $A^{p,q_1(.)}_{w,\vartheta_1}(\mathbb{R}^n) \hookrightarrow A^{p,q_2(.)}_{w,\vartheta_2}(\mathbb{R}^n)$. ■

The following corollary can be easily proved using Theorem 3, Theorem 4, Theorem 5, Theorem 6 and Theorem 7.

Corollary 1. *(i)* The equality $A^{p,q(.)}_{w_1,\vartheta}(\mathbb{R}^n) = A^{p,q(.)}_{w_2,\vartheta}(\mathbb{R}^n)$ holds if and only if $w_2 \approx w_1$.

(ii) If $\vartheta_2 \approx \vartheta_1$, then $A^{p,q(.)}_{w,\vartheta_1}(\mathbb{R}^n) = A^{p,q(.)}_{w,\vartheta_2}(\mathbb{R}^n)$.

(iii) If $w_2 \prec w_1$ and $\vartheta_2 \prec \vartheta_1$, then $A^{p,q(.)}_{w_1,\vartheta_1}(\mathbb{R}^n) \hookrightarrow A^{p,q(.)}_{w_2,\vartheta_2}(\mathbb{R}^n)$.

(iv) If $w_2 \approx w_1$ and $\vartheta_2 \approx \vartheta_1$, then $A^{p,q(.)}_{w_1,\vartheta_1}(\mathbb{R}^n) = A^{p,q(.)}_{w_2,\vartheta_2}(\mathbb{R}^n)$.

(v) The embedding $A^{p,q_1(.)}_{w_1,\vartheta_1}(\mathbb{R}^n) \hookrightarrow A^{p,q_2(.)}_{w_2,\vartheta_2}(\mathbb{R}^n)$ holds if $w_2 \prec w_1$ and $q_2(.) \preceq q_1(.)$.

(vi) The embedding $A^{p,q_1(.)}_{w_1,\vartheta_1}(\mathbb{R}^n) \hookrightarrow A^{p,q_2(.)}_{w_2,\vartheta_2}(\mathbb{R}^n)$ holds if $w_2 \prec w_1$ and $\left\| \frac{\vartheta_2}{\vartheta_1} \right\|_{\frac{q_1(.)}{q_1(.)-q_2(.)},\vartheta_1} < \infty$.

Now, we characterize the embeddings of the sum and intersection of variable exponent Lebesgue spaces. For two normed spaces X and Y (which are both embedded into a Hausdorff topological vector space Z) we equip the intersection $X \cap Y := \{f : f \in X,\ f \in Y\}$ and the sum $X+Y := \{g+h : g \in X, h \in Y\}$ with the norms

$$\begin{aligned} \|f\|_{X\cap Y} &: = \max\{\|f\|_X, \|f\|_Y\}, \\ \|f\|_{X+Y} &: = \inf_{f=g+h, g\in X, h\in Y} \{\|g\|_X + \|h\|_Y\}. \end{aligned}$$

The following theorem is well known by [8].

Theorem 8. Let $p(.)\, q(.), r(.) \in \mathscr{P}(\mathbb{R}^n)$ with $p(.) \leq q(.) \leq r(.)$, almost everywhere in $\mathbb{R}^n$. Then

$$L^{p(.)}(\mathbb{R}^n) \cap L^{r(.)}(\mathbb{R}^n) \hookrightarrow L^{q(.)}(\mathbb{R}^n) \hookrightarrow L^{p(.)}(\mathbb{R}^n) + L^{r(.)}(\mathbb{R}^n).$$

Theorem 9. Let $1 \leq p < \infty$, $q(.), r(.) \in \mathscr{P}(\mathbb{R}^n)$ with $p \leq r(.) \leq q(.)$, almost everywhere in $\mathbb{R}^n$. Then

$$A^{p,q(.)}_{w,\vartheta}(\mathbb{R}^n) \hookrightarrow L^{r(.)}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n) + L^{q(.)}(\mathbb{R}^n).$$

Proof. Since w, ϑ are Beurling's weight functions, we can write embeddings $L^p_w(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ and $L^{q(.)}_\vartheta(\mathbb{R}^n) \hookrightarrow L^{q(.)}(\mathbb{R}^n)$. Using the last two embeddings and Theorem 8, we get

$$A^{p,q(.)}_{w,\vartheta}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n) \cap L^{q(.)}(\mathbb{R}^n) \hookrightarrow L^{r(.)}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n) + L^{q(.)}(\mathbb{R}^n).$$

■

The exponent $q(.)$ satisfies the locally log-Hölder continuous if

$$|q(x) - q(y)| \leq \frac{C}{-\ln|x-y|},\ x, y \in \Omega, |x-y| \leq \frac{1}{2}.$$

where $\Omega \subset \mathbb{R}^n$. We denote the family of all locally log-Hölder continous functions by the symbol $P^{\log}(\mathbb{R}^n)$.

For $f \in L^1_{loc}(\mathbb{R}^n)$, we denote the (centered) Hardy-Littlewood moximal operator Mf of f by

$$Mf(x) = \sup_{r>0} \frac{1}{B(x,r)} \int\limits_{B(x,r)} |f(y)|\, dy.$$

where the supremum is taken over all balls $B(x,r)$.

Let $1 \leq p < \infty$. A weight w satisfies Muckenhoupt's $A_p(\mathbb{R}^n) = A_p$ condition, or $w \in A_p$, if there exist positive constants C and c such that, for every ball $B \subset \mathbb{R}^n$,

$$\left(\frac{1}{|B|} \int\limits_B w dx \right) \left(\frac{1}{|B|} \int\limits_B w^{-\frac{1}{p-1}} dx \right)^{p-1} \leq C,\ 1 < p < \infty$$

or

$$\left(\frac{1}{|B|} \int\limits_B w dx \right) \left(ess \sup_B \frac{1}{w} \right) \leq c,\ \ p = 1.$$

Let $1 < p < \infty$. Then Muckenhoupt proved that $w \in A_p$ if and only if the Hardy-Littlewood maximal operator is bounded on $L^p_w(\mathbb{R}^n)$, see [20].

Hastö and Diening defined the class $A_{q(.)}$ to consist of those weights w for which

$$\|w\|_{A_{q(.)}} := \sup_{B \in \varkappa} |B|^{-q_B} \|w\|_{L^1(B)} \left\| \frac{1}{w} \right\|_{L^{\frac{p(.)}{q(.)}}(B)} < \infty,$$

where $\varkappa$ denotes the set of all balls in $\mathbb{R}^n$, $q_B = \left(\frac{1}{|B|} \int\limits_B \frac{1}{q(x)} dx \right)^{-1}$ and $\frac{1}{q(.)} + \frac{1}{p(.)} = 1$. It is clear that the Hardy-Littlewood maximal operator is bounded on $L^{q(.)}_\vartheta(\mathbb{R}^n)$ if and only if $q \in P^{\log}(\mathbb{R}^n)$, $1 < q^- \leq q^+ < \infty$, $\vartheta \in A_{q(.)}$, see [3].

Theorem 10. Let $1 < p < \infty$, $q \in P^{\log}(\mathbb{R}^n)$ and $1 < q^- \leq q^+ < \infty$. Then, the maximal operator $M : A^{p,q(.)}_{w,\vartheta}(\mathbb{R}^n) \longrightarrow A^{p,q(.)}_{w,\vartheta}(\mathbb{R}^n)$ is bounded if and only if $w \in A_p$ and $\vartheta \in A_{q(.)}$.

Proof. It is known that the maximal operators $M : L^p_w(\mathbb{R}^n) \longrightarrow L^p_w(\mathbb{R}^n)$ and $M : L^{q(.)}_\vartheta(\mathbb{R}^n) \longrightarrow L^{q(.)}_\vartheta(\mathbb{R}^n)$ are bounded if and only if $w \in A_p$ and $\vartheta \in A_{q(.)}$, respectively. Using the definition of the intersection space, the proof is completed. ■

Definition 1. Let $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a nonnegative, radial, decreasing function belonging to $C_0^\infty(\mathbb{R}^n)$ and having the properties:

(i) $\varphi(x) = 0$ if $|x| \geq 1$,

(ii) $\int\limits_{\mathbb{R}^n} \varphi(x)\, dx = 1$.

Let $\varepsilon > 0$. If the function $\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi\left(\frac{x}{\varepsilon}\right)$ is nonnegative, belongs to $C_0^\infty(\mathbb{R}^n)$ and satisfies

(i) $\varphi_\varepsilon(x) = 0$ if $|x| \geq \varepsilon$,

(ii) $\int\limits_{\mathbb{R}^n}\varphi_\varepsilon(x)\,dx=1,$

then φ_ε is called a mollifier and we define the convolution by $(\varphi_\varepsilon * f)(x)=\int\limits_{\mathbb{R}^n}\varphi_\varepsilon(x-y)\,f(y)\,dy.$ The following proposition was proved in [9].

Proposition 2. Let φ_ε be a mollifier and $f\in L^1_{loc}(\mathbb{R}^n)$. Then $\sup\limits_{\varepsilon>0}|(\varphi_\varepsilon * f)(x)|\leq Mf(x)$.

Proposition 3. If $\vartheta\in A_{q(.)}$ and $f\in L^{q(.)}_\vartheta(\mathbb{R}^n)$, then $\varphi_\varepsilon * f\rightharpoonup f$ in $L^{q(.)}_\vartheta(\mathbb{R}^n)$ as $\varepsilon\longrightarrow 0^+$, see [3].

Theorem 11. If $w\in A_p$ and $\vartheta\in A_{q(.)}$ and $f\in A^{p,q(.)}_{w,\vartheta}(\mathbb{R}^n)$, then $\varphi_\varepsilon * f\rightharpoonup f$ in $A^{p,q(.)}_{w,\vartheta}(\mathbb{R}^n)$ as $\varepsilon\longrightarrow 0^+$.

Proof. Let $f\in A^{p,q(.)}_{w,\vartheta}(\mathbb{R}^n)$ and $\varepsilon>0$ be given. If $w\in A_p$ and $\vartheta\in A_{q(.)}$, then the inequality

$$\begin{aligned}\|f-\varphi_\varepsilon * f\|^{p,q(.)}_{w,\vartheta} &= \|f-\varphi_\varepsilon * f\|_{p,w}+\|f-\varphi_\varepsilon * f\|_{q(.),\vartheta}\\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\end{aligned}$$

is satisfied by [1], [3]. That is the desired. ∎

Corollary 2. Let $w\in A_p$ and $\vartheta\in A_{q(.)}$. The class $C_0^\infty(\mathbb{R}^n)$ is dense in $A^{p,q(.)}_{w,\vartheta}(\mathbb{R}^n)$.

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