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Exponentiated UEHL Distribution: Properties and Applications

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ABSTRACT

In this paper, we propose a distribution for modeling data defined on a unit interval using an exponentiated transformation. The new distribution is based on the unit exponential half-logistic distribution, a member of proportional hazard models. Several measures of the statistical characterization of the distribution are discussed. The statistical inference of the parameters of the proposed distribution is studied by the maximum likelihood method. To explore the properties of the maximum likelihood estimates of the parameters, simulation studies are carried out under various scenarios. Furthermore, a real dataset is analyzed to demonstrate the performance of the distribution.

Keywords: Exponentiated distribution, UEHL distribution, maximum likelihood estimator, goodness of fit

Üstellenmiş UEHL Dağılımı: Özellikler ve Uygulamalar

ÖZ

Bu makalede, birim aralıkta tanımlanan verilerin üstel bir dönüşüm kullanılarak modellenmesi için bir dağılım önerilmiştir. Yeni dağılım, orantılı tehlike modellerinin bir üyesi olan birim üstel yarı lojistik dağılıma dayanmaktadır. Dağılımın istatistiksel karakterizasyonuna ilişkin çeşitli ölçütler tartışılmıştır. Önerilen dağılımın parametrelerinin istatistiksel çıkarımı en çok olabilirlik yöntemi ile incelenmiştir. Parametrelerin en çok olabilirlik tahminlerinin özelliklerini araştırmak için çeşitli senaryolar altında simülasyon çalışmaları gerçekleştirilmiştir. Ayrıca, dağılımın performansını göstermek için gerçek bir veri kümesi analiz edilmiştir

Anahtar Kelimeler: Üstellenmiş dağılım, UEHL dağılımı, en çok olabilirlik tahmin edicisi, uyum iyiliği

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1. Introduction

The well-known Weibull distribution was proposed as an enhanced extension of the exponential distribution. The Weibull distribution is more flexible than the exponential distribution in shape. Due to this characteristic, the Weibull distribution has applications in many areas such as business, economics and survival analysis (Murthy et al., 2004; McCool, 2012; Aslam et al., 2015; Dokur and Kurban, 2015), manufacturing industry (Sürücü et al., 2009; Arenas et al., 2010; Barman et al., 2023; Periyasamypandian and Balamurali, 2023) and medicine (Feronze et al., 2022; Ghazal and Radwan, 2022).

In reliability theory, the properties of the hazard function of the underlying distribution are crucial for the success of statistical modeling. The Weibull distribution is commonly used in reliability theory to model the time to failure. The distribution generalizes the exponential model to cover nonconstant hazard rate functions and is appropriate for modeling monotone hazard rates. In this respect, the Weibull distribution and its extensions are useful for modeling a wide range of real-life data (Ijaz et al., 2020; Khalil et al., 2021; Alotaibi, 2023). However, the distribution is not convenient for modeling non-monotonic hazard rates such as bathtub-shaped failure rates (Almalki and Nadarajah, 2014; Carrasco et al., 2008; Lai, 2014). To improve the capability of the Weibull distribution in modeling bathtube-shaped failure rates various generalizations and modifications of the Weibull distribution were proposed in the literature (Lai, 2014).

Dombi et al. (2019) proposed the omega distribution as an alternative to the Weibull distribution and explored its applications in reliability theory. Dombi et al. (2019) proved that the asymptotic hazard rate function of the omega distribution is the Weibull hazard rate function. Hence, the asymptotic omega distribution is the Weibull distribution. Also, the omega distribution belongs to the class of proportional hazard rate models. Based on the omega distribution, Özbilen and Genç (2022) proposed the unit exponentiated half-logistic (UEHL) distribution with a special case of the parameter of the omega distribution. The UEHL distribution reduces to an exponentiated half-logistic distribution with a simple transformation, which has many uses in reliability theory (Seo and Kang, 2015; Gui, 2017).

In distribution theory, generating new distributions using some baseline distributions is a popular practice (Gupta et al., 1998; Cordeiro and de Castro, 2011). In this context, Lehmann (1953) introduced exponentiated G-family of distributions (EG). If $G(x)$ is the cumulative distribution function (cdf) of the baseline distribution, then an exponentiated G-family of distribution is defined by taking the α-th power of $G(x)$ as

$$F(x) = G(x)^{\alpha}, x \in D \tag{1}$$

where $\alpha > 0$ is a shape parameter and D is the domain of the baseline distribution. The corresponding probability density function (pdf) is given by

$$f(x) = \alpha g(x) G(x)^{\alpha-1} \tag{2}$$

where $g(x)$ is the pdf corresponding to $G(x)$. There are various studies in the literature in the context of exponentiated transformation. Mudholkar and Srivastava (1993) proposed the exponentiated Weibull distribution for modeling lifetime data with bathtub failure rate and for testing the goodness-of-fit of the Weibull distribution. Also, Mudholkar et al. (1995) used the exponentiated Weibull distribution in reanalyzing classical data sets on bus-motor failure to illustrate the flexibility of the family. Surles and Padgett (2001) proposed a scaled BurrX distribution and studied the inference of the distribution. Also, Kundu and Raqab (2005) examined the distribution in the scope of the lifetime data analysis by describing it as the generalized Rayleigh distribution. Cordeiro et al. (2014) proposed the exponentiated half-logistic family of distributions based on the half-logistic distribution and investigated various characteristics. Ashour and Eltehiwy (2015) proposed the exponentiated power Lindley distribution as a new generalization of the Lindley distribution to obtain a more flexible model compared to the power Lindley distribution. Arshad et al. (2020) introduced exponentiated power function distribution and applied it to lifetime datasets from engineering. El-Monsef et al. (2021) proposed the exponentiated power Lomax distribution and used it to model various real-life datasets. Rather et al. (2022) introduced the Exponentiated Ailamujia distribution and utilized the distribution for a medical data analysis. Sharma et al. (2022) proposed two-parameter exponentiated Teissier distribution, whose hazard rate function has increasing, decreasing and bathtub shapes, and used several techniques for parameter estimation of the distribution. Alotaibi et al. (2023) proposed exponentiated-Chen distribution and used the distribution to model the number of vehicle fatalities.

In this paper, we use the concept of exponentiated distributions to introduce a new distribution called the exponentiated UEHL (EUEHL) distribution by considering the UEHL distribution as the baseline distribution in the G-family defined in Equation (1).

The contents of this paper are organized as follows. Section 2 introduces the EUEHL distribution based on the exponentiated transformation and the survival and hazard rate functions are provided. Section 3 handles some analytical characteristics of the proposed distribution including the quantile function, moments, moment generating function, order statistics, stress-strength reliability, and the maximum likelihood estimation of the method. Section 4 presents a simulation study to evaluate the performance of the maximum likelihood estimates. Section 5 provides a numerical example to illustrate the performance of the proposed distribution in modeling a real-life data set. Finally, Section 6 concludes the paper.

2. Exponentiated UEHL Distribution

Recently, Özbilen and Genç (2022) introduced the UEHL distribution based on the omega distribution, which is a member of the class of proportional hazard rate models. The UEHL distribution has various applications in reliability theory via exponentiated half-logistic distribution (Kang, 2011; Rastogi, 2014).

In this section, the three-parameter EUEHL distribution will be defined by applying the G-family transformation given by Equation (1) to the UEHL distribution. By introducing the shape parameter α in the EUEHL distribution, we aim to achieve a more useful model than the UEHL model in terms of flexibility and performance of fitting data. The cdf and pdf of the UEHL distribution are given, respectively, by

$$F_{UEHL}(x,\theta,\lambda) = 1 - \left(\frac{1-x^{\theta}}{1+x^{\theta}}\right)^{\lambda}, 0 < x < 1 \tag{3}$$

and

$$f_{UEHL}(x,\theta,\lambda) = 2\theta\lambda x^{\theta-1}\frac{\left(1-x^{\theta}\right)^{\lambda-1}}{\left(1+x^{\theta}\right)^{\lambda+1}}, 0 < x < 1 \tag{4}$$

where $\theta > 0$ and $\lambda > 0$ are the scale and shape parameters of the distribution, respectively. By applying the transformations in Equations (1) and (2) to Equations (3) and (4) the cdf and pdf of the EUEHL distribution are obtained, respectively, as

$$F(x;\theta,\lambda,\alpha) = \left[1 - \left(\frac{1-x^{\theta}}{1+x^{\theta}}\right)^{\lambda}\right]^{\alpha} \tag{5}$$

and

$$f(x;\theta,\lambda,\alpha) = 2\theta\lambda\alpha\frac{x^{\theta-1}}{\left(1+x^{\theta}\right)^{2}}\left(\frac{1-x^{\theta}}{1+x^{\theta}}\right)^{\lambda-1}\left[1 - \left(\frac{1-x^{\theta}}{1+x^{\theta}}\right)^{\lambda}\right]^{\alpha-1} \tag{6}$$

with parameters θ, λ and α. Henceforth, we denote the random variable X having the pdf in Equation (6) by EUEHL(θ, λ, α).

Figure 1 illustrates the pdf of the EUEHL(θ, λ, α) distribution for selected values of the θ, λ and α parameters. Hence, the EUEHL(θ, λ, α) distribution shows unimodal, increasing, decreasing and U-shaped functions depending on the parameter values.

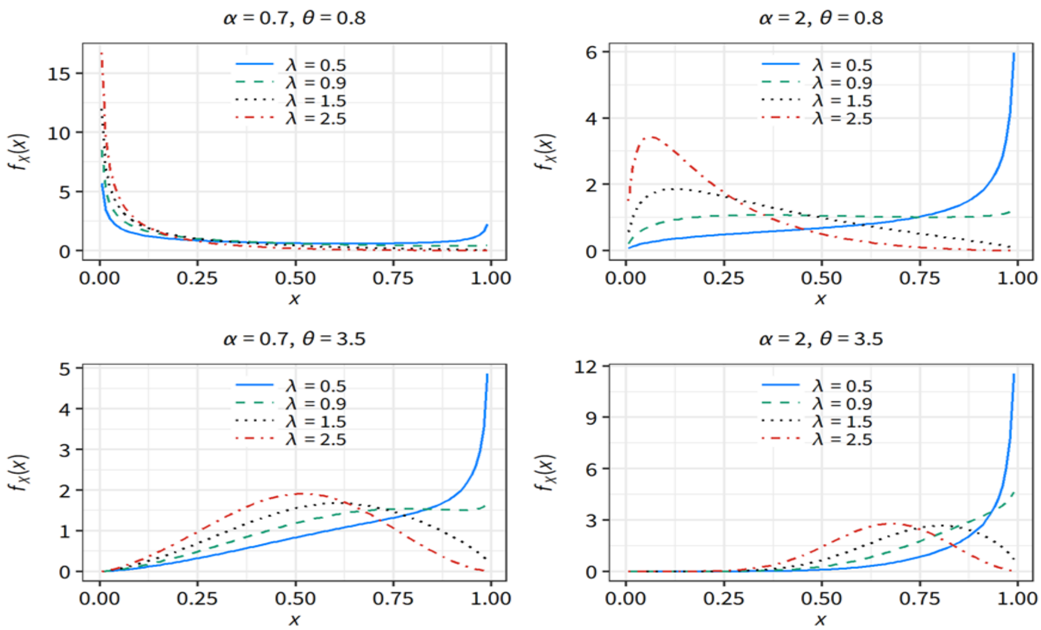

Figure 1. Pdf plots of the EUEHL$(\theta, \lambda, \alpha)$ distribution for selected values of the parameters.

Also, the survival and the hazard rate functions of the EUEHL distribution can be obtained by

$$S(x;\theta,\lambda,\alpha) = 1 - \left[1 - \left(\frac{1-x^\theta}{1+x^\theta}\right)^\lambda\right]^\alpha \tag{7}$$

and

$$h(x;\theta,\lambda,\alpha) = \frac{2\theta\lambda\alpha\frac{x^{\theta-1}}{(1+x^\theta)^2}\left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda-1}\left[1-\left(\frac{1-x^\theta}{1+x^\theta}\right)^\lambda\right]^{\alpha-1}}{1-\left[1-\left(\frac{1-x^\theta}{1+x^\theta}\right)^\lambda\right]^\alpha} \tag{8}$$

3. Statistical Characteristics of the EUEHL Distribution

3.1. Quantile Function

The quantile function of the EUEHL$(\theta, \lambda, \alpha)$ distribution is given by

$$(u;\theta,\lambda,\alpha) = \left[\frac{1-\left(1-u^{1/\alpha}\right)^{1/\lambda}}{1+\left(1-u^{1/\alpha}\right)^{1/\lambda}}\right]^{1/\theta} \tag{9}$$

Hence, the median of the EUEHL$(\theta, \lambda, \alpha)$ distribution is obtained as a function of the θ, λ, α parameters by

$$Q(0.5;\theta,\lambda,\alpha) = \left[\frac{1-\left(1-0.5^{1/\alpha}\right)^{1/\lambda}}{1+\left(1-0.5^{1/\alpha}\right)^{1/\lambda}}\right]^{1/\theta}$$

Based on the quantile function given by Equation (9) the random number generation process for EUEHL distribution is given as follows:

Step 1. Generate a uniform random number from the interval $[0, 1]$.

Step 2. Run the quantile function in Equation (9) on the uniform random number in Step 1.

3.2. Moments

The moments are useful tools for comprehending the various features of a statistical distribution. In this context, we consider the moments of an EUEHL$(\theta, \lambda, \alpha)$ random variable. Let X be an EUEHL$(\theta, \lambda, \alpha)$ random variable with pdf given by Equation (6). Then, the r-th raw moment of X is

$$E(X^r) = \int_0^1 2\theta\lambda\alpha \frac{x^{r+\theta-1}}{(1+x^\theta)^2}\left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda-1}\left[1 - \left(\frac{1-x^\theta}{1+x^\theta}\right)^\lambda\right]^{\alpha-1} dx \tag{10}$$

for $r \in \{1,2,3,\dots\}$. By using the binomial expansion of the last factor in the integrand in Equation (10),

$$\left[1 - \left(\frac{1-x^\theta}{1+x^\theta}\right)^\lambda\right]^{\alpha-1} = \sum_{j=0}^{\infty}\binom{\alpha-1}{j}(-1)^j\left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda j} \tag{11}$$

we get

$$E(X^r) = \int_0^1 2\theta\lambda\alpha \frac{x^{r+\theta-1}}{(1+x^\theta)^2} \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j \left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda(1+j)-1} dx$$

$$= \sum_{j=0}^{\infty} 2\lambda\alpha \binom{\alpha-1}{j} (-1)^j \int_0^1 \theta \frac{x^{r+\theta-1}}{(1+x^\theta)^2} \left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda(1+j)-1} dx \tag{12}$$

Substituting $u = x^\theta$ in Equation (12), the expectation becomes

$$E(X^r) = \sum_{j=0}^{\infty} 2\lambda\alpha \binom{\alpha-1}{j} (-1)^j \int_0^1 u^{r/\theta} (1-u)^{\lambda(1+j)-1} (1+u)^{-\lambda(1+j)-1} du \tag{13}$$

By applying the equation

$$\int_0^1 t^{\lambda-1}(1-t)^{\mu-1}(1-\beta t)^{-\nu} dt = B(\lambda,\mu)\,{}_2F_1(\nu,\lambda;\lambda+\mu;\beta) \tag{14}$$

provided in Gradshteyn and Ryzhik (2007), we obtain the r-th raw moment as

$$E(X^r) = 2\lambda\alpha \sum_{j=0}^{\infty} (-1)^j \binom{\alpha-1}{j} B\left(1+\frac{r}{\theta}, \lambda(1+j)\right)$$

$$\times\ {}_2F_1\left(\lambda(1+j)+1, 1+\frac{r}{\theta}; 1+\frac{r}{\theta}+\lambda(1+j); -1\right)$$

where

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt$$

and

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!}$$

are the beta and Gauss hypergeometric functions, respectively, and $(a)_n = a(a+1)\cdots(a+n+1)$.

3.3. Moment Generating Function

The moment generating function (mgf) provides an alternative route to analytical results rather than working directly with pdfs. Let X be an EUEHL(θ,λ,α) random variable with pdf given by Equation (6). Then, the mgf of X is

$$M_X(t) = \int_0^1 2\theta\lambda\alpha \frac{x^{\theta-1}}{(1+x^\theta)^2} \left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda-1} \left[1 - \left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda}\right]^{\alpha-1} e^{tx}\, dx.$$

Using the binomial expansion in Equation (11), we write

$$M_X(t) = 2\lambda\alpha \sum_{j=0}^{\infty} \left[\binom{\alpha-1}{j} (-1)^j \int_0^1 \frac{\theta x^{\theta-1}}{(1+x^\theta)^2} \left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda(1+j)-1} e^{tx}\, dx\right] \tag{15}$$

By the binomial expansion of e^{tx} in Equation(15), we get

$$Mx(t) = 2\lambda\alpha \sum_{j=0}^{\infty} \left[\binom{\alpha-1}{j} (-1)^j \int_0^1 \frac{\theta x^{\theta-1}}{(1+x^\theta)^2} \left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda(1+j)-1} \sum_{i=0}^{\infty} \frac{t^i x^i}{i!}\, dx\right]$$

$$= 2\lambda\alpha \sum_{j=0}^{\infty} \left[\binom{\alpha-1}{j} (-1)^j \sum_{i=0}^{\infty} \left(\frac{t^i}{i!} \int_0^1 \frac{\theta x^{\theta+i-1}}{(1+x^\theta)^2} \left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda(1+j)-1} dx\right)\right] \tag{16}$$

Applying Equation (14) after the transformation $u = x^\theta$ in Equation (16), we obtain the mgf of the random variable having the distribution EUEHL(θ,λ,α) as

$$M_X(t) = 2\lambda\alpha \sum_{j=0}^{\infty} \left[\binom{\alpha-1}{j} (-1)^j \sum_{i=0}^{\infty} \left(\frac{t^i}{i!} B\left(1+\frac{i}{\theta}, \lambda(1+j)\right) {}_2F_1\left(\lambda(1+j)+1, 1+\frac{i}{\theta}; 1+\frac{i}{\theta}+\lambda(1+j); -1\right)\right)\right]$$

$$= 2\lambda\alpha \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \left[\binom{\alpha-1}{j} \frac{(-1)^j t^i}{i!} B\left(1+\frac{i}{\theta}, \lambda(1+j)\right) {}_2F_1\left(\lambda(1+j)+1, 1+\frac{i}{\theta}; 1+\frac{i}{\theta}+\lambda(1+j); -1\right)\right]$$

3.4. Order Statistics

In statistical modeling, information obtained from the ordered values of the sample can be useful. Thus, it is important to obtain the order statistics of the distributions. In this section, the order statistics of the EUEHL distribution will be provided. Consider $X_1, X_2, \ldots, X_n$ be a random sample from EUEHL(θ,λ,α) distribution and $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ represents the associated order statistics. The pdf of the rth order statistics $Y = X_{(r)}$ are obtained by

$$f_Y(x) = 2\theta\lambda\alpha \frac{n!}{(r-1)!(n-r)!} \cdot \frac{x^{\theta-1}}{(1+x^\theta)^2} \left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda-1} \left[1 - \left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda}\right]^{\alpha r-1} \left[1 - \left(1 - \left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda}\right)^{\alpha}\right]^{n-r}$$

In addition, the pdf of the smallest and largest order statistics are given, respectively, as the following:

$$f_{X_{(1)}}(x) = 2n\theta\lambda\alpha \frac{x^{\theta-1}}{(1+x^\theta)^2} \left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda-1} \left[1 - \left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda}\right]^{\alpha-1} \times \left[1 - \left(1 - \left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda}\right)^{\alpha}\right]$$

and

$$f_{X_{(n)}}(x) = 2n\theta\lambda\alpha \frac{x^{\theta-1}}{(1+x^\theta)^2} \left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda-1} \left[1 - \left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda}\right]^{\alpha n-1}$$

3.5. Stress-Strength Reliability

Given the stress and strength random variables, Y and X, the stress-strength reliability is defined as $R = P(Y < X)$. In this section, we obtain the stress-strength reliability for the EUEHL$(\theta, \lambda, \alpha)$ model.

Proposition 1. Given Y and X independent stress and strength random variables following EUEHL distribution with parameters $(\theta, \lambda_1, \alpha_1)$ and $(\theta, \lambda_2, \alpha_2)$, respectively, the stress-strength reliability is

$$R = \alpha_2 \Gamma(\alpha_2) \sum_{j=0}^{\infty} (-1)^j \binom{\alpha_1}{j} \frac{\Gamma(1 + j\lambda_1/\lambda_2)}{\Gamma(1 + \alpha_2 + j\lambda_1/\lambda_2)}$$

where $\lambda_1 > 0$ and $\lambda_2 > 0$, Γ is the gamma function and $\gamma(a, x) = \int_a^x t^{a-1} e^{-t}\, dt$ is the incomplete gamma function.

Proof: By definition, stress-strength reliability can be written as

$$R = \int_0^1 P(Y < X \mid X = x) f_X(x)\, dx$$

$$= \int_0^1 2\theta\lambda_2\alpha_2 \frac{x^{\theta-1}}{(1+x^\theta)^2} \left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda_2-1} \left(1 - \left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda_1}\right)^{\alpha_1} \left(1 - \left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda_2}\right)^{\alpha_2-1} dx$$

By substituting $t = \left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda_2}$ and applying the binomial expansion $\left(1 - t^{\lambda_1/\lambda_2}\right)^{\alpha_1} = \sum_{j=0}^{\infty} \binom{\alpha_1}{j} (-1)^j t^{j\lambda_1/\lambda_2}$, we obtain the stress-strength reliability as

$$R = \alpha_2 \int_0^1 (1-t)^{\alpha_2-1} \left(1 - t^{\lambda_1/\lambda_2}\right)^{\alpha_1} dt$$

$$= \alpha_2 \sum_{j=0}^{\infty} \left\{ \binom{\alpha_1}{j} (-1)^j t^{j\lambda_1/\lambda_2} \int_0^1 t^{j\lambda_1/\lambda_2} (1-t)^{\alpha_2-1}\, dt \right\}$$

$$= \alpha_2 \Gamma(\alpha_2) \sum_{j=0}^{\infty} \binom{\alpha_1}{j} (-1)^j \frac{\Gamma(1 + j\lambda_1/\lambda_2)}{\Gamma(1 + \alpha_2 + j\lambda_1/\lambda_2)}$$

3.6. Maximum Likelihood Estimation

Let $X_1, X_2, \ldots, X_n$ be an identically independent sample from EUEHL$(\theta, \lambda, \alpha)$ distribution, then, the log-likelihood function is written as

$$\ell(\theta, \lambda, \alpha) = n \log 2 + n \log \theta + n \log \lambda + n \log \alpha + (\theta - 1) \sum_{i=1}^{n} \log x_i - 2 \sum_{i=1}^{n} \log\left(1 + x_i^\theta\right) + (\lambda - 1) \sum_{i=1}^{n} \log\left(\frac{1-x_i^\theta}{1+x_i^\theta}\right) + (\alpha - 1) \sum_{i=1}^{n} \log\left(1 - \left(\frac{1-x_i^\theta}{1+x_i^\theta}\right)^{\lambda}\right)$$

Differentiating the log-likelihood function with respect to the parameters θ, λ and α, we obtain the log-likelihood equations, respectively, as

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \log x_i - 2 \sum_{i=1}^{n} \frac{x_i^\theta \log x_i}{1 + x_i^\theta} + (\lambda - 1) \sum_{i=1}^{n} \frac{-2x_i^\theta \log x_i}{1 - x_i^{2\theta}}$$

$$+ (\alpha - 1) \sum_{i=1}^{n} \frac{-2\lambda x_i^\theta \log x_i (1-x^\theta)^\lambda}{\left(1 - x_i^{2\theta}\right)\left((1-x^\theta)^\lambda - (1+x^\theta)^\lambda\right)}, \tag{17}$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} \log\left(\frac{1-x_i^\theta}{1+x_i^\theta}\right) + (\alpha - 1) \sum_{i=1}^{n} \frac{(1-x^\theta)^\lambda \left(\log(1-x^\theta) - \log(1+x^\theta)\right)}{\left((1-x^\theta)^\lambda - (1+x^\theta)^\lambda\right)} \tag{18}$$

and

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log\left(1 - \left(\frac{1-x_i^\theta}{1+x_i^\theta}\right)^{\lambda}\right) \tag{19}$$

Since Equations (17), (18) and (19) do not have a closed-form solution, some iterative methods are required to get the MLE estimates of the parameters of the EUEHL$(\theta, \lambda, \alpha)$ model. Here, we use the optim procedure in R to obtain the solution of the underlying equation system.

4. Simulation Study

In this section, we carry out a simulation study to investigate the properties of the MLE, which we handle in detail in Section 3. Table 1 shows the biases and mean squared errors (MSEs) of the parameter estimates based on 5000 replications of the experiments for several values of the distribution parameters θ, λ and α and sample size, n. According to the outputs in Table 1, the MLEs have positive or negative bias depending on the distribution parameters and sample sizes. Also, the parameter estimates are asymptotically unbiased. Furthermore, the MSEs of the MLEs decrease to zero as the sample size increases as expected

5. Real Data Application

In this section, we consider the flexibility performance of the EUEHL$(\theta, \lambda, \alpha)$ distribution based on a real data application. The data is called the reservoir data that is obtained from the monthly water capacity of the Shasta Reservoir in California (Nadar et al., 2013). We compare the performance of the EUEHL$(\theta, \lambda, \alpha)$ distribution with the Weibull, Beta, Kumaraswamy (Kumaraswamy, 1980), UEHL, and DUS-UEHL (Genç and Özbilen, 2023) distributions. The pdfs of the distributions that are used for comparison are given as follows:

Weibull distribution

$$f_{Weibull}(x;\theta,\lambda) = \frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1} e^{-(x/\lambda)^{\theta}}, \ \ \theta,\lambda > 0$$

Beta distribution

$$f_{Beta}(x;\theta,\lambda) = \frac{1}{B(\theta,\lambda)} x^{\theta-1}(1-x)^{\lambda-1}, \ \ \theta,\lambda > 0$$

Kumaraswamy distribution

$$f_{Kw}(x;\theta,\lambda) = \theta\lambda x^{\theta-1}\left(1-x^{\theta}\right)^{\lambda-1}, \ \ \theta,\lambda > 0$$

DUS-UEHL distribution

$$f_{DUS-UEHL}(x) =$$

$$\frac{1}{e-1} 2\lambda\theta x^{\theta-1} \frac{\left(1-x^{\theta}\right)^{\lambda-1}}{\left(1+x^{\theta}\right)^{\lambda+1}} e^{1-\left(\frac{1-x^{\theta}}{1+x^{\theta}}\right)^{\lambda}}, \ \ \theta,\lambda > 0$$

We employ the maximum likelihood estimation method to obtain the parameter estimates of the underlying distributions. To assess the goodness of fits of the models, we used the Kolmogorov-Smirnov test statistic (K-S (stat)) and associated p-value (K-S (p-value)). We report the Akaike information criterion (AIC) and Bayesian information criterion (BIC) for comparison of the models.

Table 1. Bias and MSEs of MLEs for selected parameter values.

				Bias			**MSE**		
$\boldsymbol{\theta}$	$\boldsymbol{\lambda}$	$\boldsymbol{\alpha}$	$\boldsymbol{n}$	$\hat{\boldsymbol{\theta}}$	$\hat{\boldsymbol{\lambda}}$	$\hat{\boldsymbol{\alpha}}$	$\hat{\boldsymbol{\theta}}$	$\hat{\boldsymbol{\lambda}}$	$\hat{\boldsymbol{\alpha}}$
2	2	0.7	50	-0.14998	0.13959	0.49978	0.77478	0.40058	0.85349
			100	-0.15627	0.03831	0.32564	0.44643	0.15492	0.43831
			200	-0.30196	-0.05485	0.27518	0.28597	0.07120	0.17299
			300	-0.05199	0.01988	0.14195	0.28219	0.06269	0.15892
0.8	1.5	2.5	50	0.50013	0.14801	-0.81067	0.66485	0.16213	1.21904
			100	0.43258	0.07978	-0.88063	0.42064	0.06062	1.10793
			200	0.41690	0.04524	-1.08156	0.21043	0.02225	1.25049
			300	0.34341	0.03793	-0.85480	0.22409	0.01507	0.94521
1.5	0.8	2	50	0.37225	0.00734	-0.16906	0.61365	0.01855	0.41367
			100	0.30107	-0.00611	-0.24477	0.30830	0.00855	0.25049
			200	0.20705	-0.00577	-0.17520	0.17167	0.00407	0.21506
			300	0.29907	-0.01384	-0.32148	0.20349	0.00254	0.17557

AIC and BIC are computed as

$$\mathrm{AIC} = 2k - 2\ell(\theta,\lambda) \text{ and } \mathrm{BIC} = k\log n - 2\ell(\theta,\lambda)$$

where k is the number of parameters, n is the sample size, and ℓ is the maximum value of the likelihood function for the underlying distribution.

The parameter estimates, comparison criteria and goodness-of-fit test results of all the models for reservoir data are reported in Table 2. Based on the K-S (stat) and K-S (p-value) given in Table 2, the compared distributions are appropriate for modeling the dataset. Table 2 shows that the EUEHL(θ,λ,α) model yields the smallest -2ℓ, AIC and BIC followed by the DUS-UEHL(θ,λ) model for the data set. Therefore, the EUEHL (θ,λ,α) distribution outperforms the compared distributions in modeling the reservoir data. We give the histogram of the reservoir data set and plots of the fitted UEHL, DUS-UEHL and EUEHL models in Figure 2 for illustrative purposes.

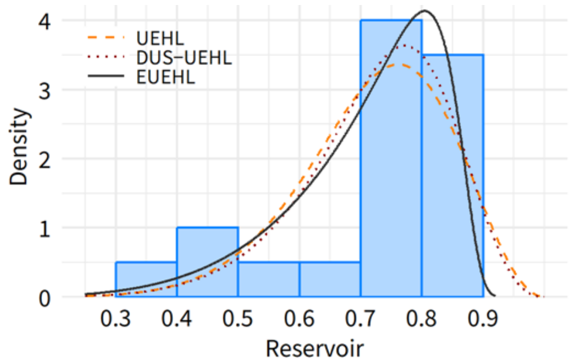

Figure 2. The histogram of the reservoir data set and the fitted models.

Table 2. Parameter estimates, comparison criteria and goodness-of-fit test results for reservoir data.

Distribution	θ	λ	α	AIC	BIC	-2ℓ	K-S (stat)	K-S (p-val)
Weibull	7.299	0.775	-	-20.5347	-18.5432	-24.5347	0.2220	0.2396
Beta	7.316	2.910	-	-21.1238	-19.1324	-25.1238	0.2359	0.1834
Kw	6.348	4.489	-	-22.9494	-20.9580	-26.9494	0.2209	0.2447
UEHL	6.901	2.888	-	-21.2699	-19.2784	-25.2699	0.2254	0.2248
DUS-UEHL	6.432	3.268	-	-23.0543	-21.0629	-27.0543	0.2046	0.3267
EUEHL	25.390	32.859	0.202	-27.4472	-24.4600	-33.4472	0.2087	0.3044

6. Conclusions

In this study, we propose the EUEHL distribution based on the exponentiated transformation of the UEHL distribution, which is convenient for modeling data with values in unit intervals. Some analytical properties including structural and reliability measures of the distribution along with estimation issues are discussed. Simulation studies are conducted to evaluate the performance of the maximum likelihood method used in parameter estimation. The real data analysis on a record data with values in unit interval reveals that the EUEHL model performs better on the data than the other well-known models in terms of comparison criteria.

The idea of obtaining the EUEHL distribution can be applied to various G-Family distributions such as the Marshall-Olkin and McDonald distributions by considering the UEHL distribution as the baseline distribution and, hence, can be used to form new statistical distributions appropriate for various real-life data sets.

Author Contributions

All the authors equally contributed to this work. The authors read and approved the final version of the paper.

Conflict of Interest

All the authors declare no conflict of interest.

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