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Iterative Methods For Solving Nonlinear Lane-Emden Equations

Lineer Olmayan Lane-Emden Denklemlerinin Çözümünde İteratif Yöntemler

Faruk UÇAR [1], **Volkan YAMAN** , **Bülent YILMAZ** [2]

[1] *Marmara University, Department of Mathematics, Kadikoy, 34722, Istanbul, Turkey*

[2] *Marmara University, Department of Mathematics, Kadikoy, 34722, Istanbul, Turkey*

Abstract

In this paper we derive solutions for the nonlinear Lane-Emden type of equations with iterative methods, Daftardar-Jafari Method (DJM), Adomian Decomposition Method (ADM) and Differential Transformation Method (DTM). The difficulty of Lane-Emden type of equations for implementing these iterative methods is due to the singularity at $x = 0$. We compare the efficiency of DJM – which is the authentic part of the paper – with ADM and DTM in solving the nonlinear, singular, initial value Lane-Emden type of equations.

Keywords: Daftardar-Jafari Method (DJM), Adomian Decomposition Method (ADM), Differential Transformation Method (DTM), Lane-Emden type of equations, e nonlinear, singular, initial value Lane-Emden type of equations

Öz

Bu çalışmada lineer olmayan Lane-Emden tip denklemlerin Daftardar-Jafari Method (DJM), Adomian Decomposition Method (ADM) ve Differential Transformation Method (DTM) isimli yinelemeli yöntemlerle çözümlerini elde ettik. Bu yinelemeli yöntemlerin Lane-Emden tip denklemlere uygulanmasındaki zorluk, denklemlerin da tekil noktalarının olmasındandır. Bu calışmanın özgün tarafı olarak, tekil, lineer olmayan, başlangıç değerli Lane-Emden tip denklemlerin çözümünde DJM yönteminin ADM ve DTM yöntemleriyle karşılaştırması yapılmıştır.

Anahtar Kelimeler: Daftardar-Jafari Metodu (DJM), Adomian Ayrıştırma Metodu (ADM), Diferansiyel Dönüşüm Yöntemi (DTM), Lane-Emden tip denklemler.

I. INTRODUCTION

Lane-Emden equations are singular initial value problems that arise in the study of stellar structures. In this paper we consider Lane-Emden equations of the first kind;

(1)

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y^n = 0, \qquad y(0) = 1, \frac{dy}{dx}|_{x=0} = 0$$

and Lane-Emden equations of the second kind;

(2)

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + e^y = 0, \qquad y(0) = 1, \frac{dy}{dx}|_{x=0} = 0$$

Sorumlu Yazar/Corresponding Author: Faruk UÇAR, **Tel:** +902163482195, **e-posta:** fucar@marmara.edu.tr
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Lane - Emden equation of the first kind can be solved analytically for $n = 0, 1$ and 5, for the remaining values of n numerical methods need to be applied [13]. There are some decomposition methods in literature that have been proposed to solve nonlinear problems without simplifying the original problem.

ADM, DTM and DJM have been shown to solve effectively and accurately large class of linear and nonlinear equations [1, 2, 3]. In literature first and second kind of Lane-Emden equations have been studied with ADM and DTM [5, 6, 7, 8, 9], but not with DJM so far.

In this paper we investigate the efficiency of DJM compared with ADM and DTM applied both to first and second kind of Lane-Emden equations through two numerical examples.

The paper is organized as Section 2 to illustrate briefly the theory of DJM, ADM and DTM, Section 3 to show the application of these 3 methods to two numerical problems and analysis of the results, and Section 4 to give a discussion and conclusion.

II.THEORY OF METHODS

1.1. Daftardar-Jafari Method (DJM). Let

(3)

$$y = N(y) + g$$

where N might be a nonlinear operator and g is a known function. DJM decomposes the solution y into series as

(4)

$$y = \sum_{n=0}^{\infty} y_n$$

With the series expansion (4), nonlinear operator $N(y)$ in (3) can be decomposed as

(5)

$$N\left(\sum_{n=0}^{\infty} y_n\right) = N(y_0) + \sum_{n=1}^{\infty}\left(N(\sum_{m=0}^{n} y_m) - N(\sum_{m=0}^{n-1} y_m)\right)$$

From (4) and (5), (3) can be rewritten as

(6)

$$y_0 + y_1 + \sum_{n=2}^{\infty} y_n = g + N(y_0) + \sum_{n=1}^{\infty}\left(N(\sum_{m=0}^{n} y_m) - N(\sum_{m=0}^{n-1} y_m)\right)$$

DJM considers $y_0 = g$ and $y_1 = N(y_0)$, and obtain the remaining y_n terms by the following recurrence relation;

(7)

$$y_0 = g$$

$$y_1 = N(y_0)$$

$$\vdots$$

$$y_{m+1} = N(y_0 + y_1 + \cdots + y_m) - N(y_0 + y_1 + \cdots + y_{m-1}), \qquad m =$$

This yields

(8)

$$y_1 + y_2 + \cdots + y_{m+1} = N(y_0 + y_1 + \cdots + y_m), \qquad m = 1, 2, \ldots$$

From this y can be written as follows

(9)

$$y = g + \sum_{n=0}^{\infty} y_n$$

It is shown that if N is a contraction operator then $y = \sum_{n=0}^{\infty} y_n$ converges absolutely and uniformly to the unique y in view of Banach fixed point theorem [4]. The practical solution will be the *k-term* approximation to y .

(10)

$$y \approx \sum_{n=0}^{k} y_n$$

2.2. Adomian Decomposition Method (ADM). Let

(11)

$$Ly + Ry + Ny = g$$

where L is the highest order linear differential operator ($L = \frac{d^n}{dt^n}(.)$; with invertibility assumption), R is the remainder linear operator and N is the nonlinear operator and g is any function. By applying the inverse operator L^{-1} we get the following

(12)

$$L^{-1}Ly = L^{-1}g - L^{-1}Ry - L^{-1}Ny$$

where for initial value problems the inverse operator L^{-1} is conveniently defined as the *n-fold* definite integration operator from 0 to t. ADM decomposes the solution into a series

(13)

$$y = \sum_{n=0}^{\infty} y_n$$

Then

(14)

$$L^{-1}Ly = L^{-1}L\sum_{n=0}^{\infty} y_n = L^{-1}g - L^{-1}R\sum_{n=0}^{\infty} y_n - L^{-1}N$$

As N is a nonlinear operator, Ny can not be evaluated as $Ny0 + Ny1 + \ldots$. ADM replaces Ny with a series of "Adomian" polynomials ($An, n = 0, 1, 2 \ldots$) which are generated for the particular nonlinearity of the operator N. Thus we have

(15)

$$Ny = \sum_{n=0}^{\infty} A_n$$

With the definition of L^{-1} and by taking y_0 (the first term of the series $< y_n >$) as the sum of $L^{-1}g$ and with the terms resulting from the initial conditions, (14) can be written as

(16)

$$y = \sum_{n=0}^{\infty} y_n = y_0 - L^{-1}R\sum_{n=0}^{\infty} y_n - L^{-1}\sum_{n=0}^{\infty} A_n$$

Where A_n can be formulated as follows

(17)

$$A_0 = f(y_0)$$

(18)

$$A_n = \sum_{v=1}^{n} c\,(v,n) f^{(v)}\,(y_0), \quad n = 1, 2, \ldots$$

Where $f^{(v)}(y_0)$ is the v$^{\text{th}}$ derivative of the nonlinear term evaluated at y_0 and $c(v,n)$ is the function defined in [1]. A_1, A_2 and A_3 are calculated as given below

(19)

$$A_1 = c(1,1)\, f'(y_0) = y_1\, f'(y_0)$$

$$A_2 = c(1,2)\, f'(y_0) + c(2,2)\, f''(y_0) = y_2\, f'(y_0) + \frac{y_1^2}{2}\, f''(y_0)$$

$$A_3 = c(1,3)\, f'(y_0) + c(2,3)\, f''(y_0) + c(3,3)\, f^{(3)}(y_0) = y_3\, f'(y_0) + y_1 y_2 f''(y_0) \frac{y_1^3}{3!}\, f^{(3)}(y_0)$$

Convergence of $\sum_{n=0}^{\infty} A_n$ has been shown based on the assumption that nonlinear operator N is a contraction in Banach space [10].

Using (18) following recursive relation is obtained

(20)

$$y_1 = -L^{-1}Ry_0 - L^{-1}A_0$$

$$\vdots$$

$$y_n = -L^{-1}Ry_{n-1} - L^{-1}A_{n-1}$$

As y_0 is calculated from the initial conditions and A_n depends only on $y_0, y_1, \ldots, y_n$ we can find all y_n and A_n respectively.

The practical solution will be the *k-term* approximation to y.

(21)

$$y \approx \sum_{n=0}^{k} y_n$$

Convergence of the series $y = \sum_{n=0}^{\infty} y_n$ can be evaluated with the ratio test.

For $n = 0, 1, 2, \ldots$ we define

(22)

$$\alpha_n = \frac{\|y_{n+1}\|}{\|y_n\|}$$

then the series $y = \sum_{n=0}^{\infty} y_n$ converges when $\alpha_n < 1$ for all $n = 0, 1, 2, \ldots$ [11]

By the same ratio test, domain of convergence can also be determined as the interval where each α_n is less than 1 [12].

2.3 Differential Transformation Method (DTM). An arbitrary analytic function $y(x)$ can be expanded by a Taylor series about any point x_0 as

(23)

$$y(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k y(x)}{dx^k}|_{x=x_0} \ (x - x_0)^k$$

Differential Transformation of $y(x)$ is defined as

(24)

$$Y(k) = \frac{1}{k!} \frac{d^k y(x)}{dx^k}|_{x=x_0} \qquad , k = 0, 1, 2, \ldots$$

and Differential inverse transform of $Y(k)$ is defined as

(25)

$$y(x) = \sum_{k=0}^{\infty} Y(k) \ (x - x_0)^k$$

Thus, with *n-terms* approximation we obtain

(26)

$$y(x) \approx \sum_{k=0}^{n} Y(k) \ (x - x_0)^k$$

The fundamental operations of DTM performed at $x = 0$ is shown in Table 1.

Table 1 - *Most used Diff Transform Operators*

Original Function	Transformed Function
$y(x) = u(x) \pm v(x)$	$Y(k) = U(k) \pm V(k)$
$y(x) = c\, u(x)$	$Y(k) = c\, U(k)$
$y(x) = u(x)v(x)$	$Y(k) = \sum_{l=0}^{k} U(l)V(k-l)$
$y(x) = \frac{d^n u(x)}{dx^n}$	$Y(k) = \frac{(k+n)!}{k!} U(k+n)$
$y(x) = x^n$	$Y(k) = \delta(k-n) = \begin{cases} 1, & if \ k = n \\ 0, & if \ k \neq n \end{cases}$
$y(x) = e^{\lambda x}$	$Y(k) = \frac{\lambda^k}{k!}$

III.NUMERICAL APPLICATIONS

In this section we consider numerical examples to compare the efficiency of DJM, ADM and DTM in solving two Lane-Emden type of equations, one of first kind and other is of second kind.

Example 1. Lane-Emden equation of first kind;

(27)

$$\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}+y^5=0$$

$$y(0)=1,\ \ y'(0)=0$$

Exact solution is $\left(1+\frac{x^2}{3}\right)^{-\frac{1}{2}}$ [13].

Solving with ADM:

To interpret the equation in operator form in a way to rescue from the singularity at $x=0$ we define the operator L as follows as shown in [5];

$$L\,(.)=x^{-2}\frac{d}{dx}\left(x^2\frac{d}{dx}\right)(.)$$

$$Ly=x^{-2}\frac{d}{dx}\left(x^2\frac{d}{dx}\right)y=\frac{2}{x}\frac{dy}{dx}+\frac{d^2y}{dx^2}$$

So (27) can be interpreted as

(28)

$$Ly=\ -y^5$$

Conveniently we define the inverse operator

$$L^{-1}(.)=\int_0^x x^{-2}\int_0^x x^2(.)\,dx\,dx$$

Applying L^{-1} to (28) we obtain;

(29)

$$y=y(0)-L^{-1}y^5$$

Taking into consideration the initial values and series expansion of y we obtain the following recursive relation with $y_0=1$.

(30)

$$y_i=-\int_0^x x^{-2}\ \left(\int_0^x x^2A_{i-1}\,dx\right)dx$$

for $i=1,2,\dots$ where the A_i are the Adomian polynomials for the nonlinear term y^5.

The ADM solution for the first 13 terms of y_i is as follows;

$$y=\frac{676039\,x^{24}}{2229025112064}-\frac{29393\,x^{22}}{30958682112}+\frac{46189x^{20}}{15479341056}-\frac{12155\,x^{18}}{1289945088}+\frac{715\,x^{16}}{23887872}-\frac{143\,x^{14}}{1492992}+\frac{77\,x^{12}}{248832}-\frac{7\,x^{10}}{6912}+\frac{35\,x^8}{10368}-\frac{5\,x^6}{432}+\frac{x^4}{24}-\frac{x^2}{6}+1$$

We now check the region of convergence of this solution y by the ratio test defined in (2.20) through the first 12 terms of the series. We are looking for the upper limit of x that make all $\alpha_i < 1$ $(i = 0 \; to \; 12)$.

As shown on the graph upper limit of x gets almost steady starting from 8^{th} term which means the upper limit of the region of convergence will not be less than 1.5.

Graph of the exact solution and Adomian solution $y(x)$ is shown below in the region of convergence $x \in (0, 1.5)$ $x \in (0, 1.5)$;

Solving with DTM:

To remove singularity at $x = 0$ both sides of (27) is multiplied by x. So we have

(31)

$$x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + x\, y^5 = 0$$

Applying Differential Transform to (31) we obtain

(32)

$$\sum_{l=0}^{k} \delta(l-1)\,(k-l+1)\,(k-l+2)\,Y(k-l+2) + 2\,(k+1)\,Y(k+1)$$

$$+ \sum_{l=0}^{k} \delta(l-1) \left(\sum_{m=0}^{k-l} Y(m) \left(\sum_{t=0}^{k-l-m} Y(t) \left(\sum_{p=0}^{k-l-m-t} Y(p) \left(\sum_{r=0}^{k-l-m-t-p} Y(r)\, Y(k-l-m-t-p - r) \right)\right)\right)\right)$$

From initial conditions we get $Y(0) = 1$ and $Y(1) = 0$. The remaining $Y(k)$ are recursively obtained from (3.6). The DTM solution for the 23 terms of y_i $(i = 0 \; to \; 22)$ is as follows;

$$y = \frac{46189\,x^{20}}{15479341056} - \frac{12155\,x^{18}}{1289945088} + \frac{715\,x^{16}}{23887872} - \frac{143\,x^{14}}{1492992} + \frac{77\,x^{12}}{248832} - \frac{7\,x^{10}}{6912} + \frac{35\,x^{8}}{10368} - \frac{5\,x^{6}}{432} + \frac{x^4}{24} - \frac{x^2}{6} + 1$$

Solving with DJM:

We use the same inverse operator L^{-1} as used in ADM solution.

(33)

$$Ly = -y^5$$

(34)

$$y = y(0) - L^{-1} y^5$$

By applying the DJM on (34) we get the following

(35)

$$y_{i+1} = -\int_0^x x^{-2} \left(\int_0^x x^2 \left(\left(\sum_{k=0}^{i} y_k \right)^5 - \left(\sum_{k=0}^{i-1} y_k \right)^5 \right) dx \right) dx, \quad i = 0,1, \ldots$$

where from the initial conditions we have $y_0 = y(0) = 1$. y_1, y_2 and y_3 are shown below, and remaining y_i are found recursively.

$$y_1 = -\frac{x^2}{6}$$

$$y_2 = \frac{x^{12}}{1213056} - \frac{x^{10}}{28512} + \frac{5x^8}{7776} - \frac{5x^6}{756} + \frac{x^4}{24}$$

$$\begin{aligned} y_3 = &-\frac{x^{62}}{102597431937677823698299538437570 56} \\ &+ \frac{x^{60}}{451920627241336928732646620528 64} \\ &- \frac{170655x^{58}}{697180174205672322596487052866 3552} \\ &+ \frac{1222645x^{56}}{695482384963986972112860306682 6752} \\ &- \frac{787296635x^{54}}{854188327698250164621321286440 71424} \\ &+ \frac{45537508673x^{52}}{121097892548167564490553573324 619776} \\ &- \frac{132931480411x^{50}}{106953294398244470706011914255 728640} \\ &+ \frac{1803028635x^{48}}{526969474662040286168510606868 48} \\ &- \frac{1329643780895x^{46}}{166079883996987040626285704380 416} \\ &+ \frac{6697763153581x^{44}}{414047444377317367703829762539 52} \\ &- \frac{37659458624195x^{42}}{} \end{aligned}$$

$$\begin{aligned} &+ \frac{5801288321401x^{40}}{1299046925273187333628483338 24} - \frac{189472789416445x^{38}}{3043431075606152980817950801 92} \\ &+ \frac{123223324205395x^{36}}{1578110476762455531940793548 8} \\ &- \frac{10426114956665x^{34}}{1174894561653730031912878 08} + \frac{46601856508615x^{32}}{5068172618898443274918297 6} \\ &- \frac{5398476088321x^{30}}{618313070389026924527616} + \frac{17160622399925x^{28}}{224941851772352089030656} \\ &- \frac{39894960567365x^{26}}{64823144762689517649 92} + \frac{119572763x^{24}}{26018783219810304} \end{aligned}$$

$$\begin{aligned} &- \frac{50429974571x^{22}}{1584736629814001664} + \frac{407717491x^{20}}{1993021273571328} - \frac{708048755x^{18}}{579603125477376} + \frac{159505x^{16}}{23538138624} \\ &- \frac{638941x^{14}}{18307441152} + \frac{92975x^{12}}{560431872} - \frac{421x^{10}}{598752} + \frac{65x^8}{27216} - \frac{5x^6}{1008} \end{aligned}$$

$\vdots$

The graph of DJM solution for the 6 terms of y_i ($i = 0\ to\ 5$) is as follows;

Example 2. Lane-Emden equation of second kind;

(36)

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + 4\left(2e^y + e^{\frac{y}{2}}\right) = 0$$

With initial conditions $y(0) = 0$ and $y'(0) = 0$.

Exact solution is $-2\ \ln(1 + x^2)$

Solving with ADM:

We use the same operator L and the inverse operator L^{-1} used in the first example

(37)

$$Ly = -4\left(2e^y + e^{\frac{y}{2}}\right)$$

(38)

$$y = y(0) - L^{-1}4\left(2e^y + e^{\frac{y}{2}}\right)$$

(39)

$$y_i = -\int_0^x x^{-2}\left(\int_0^x x^2 A_{i-1}\,dx\right)dx$$

for $i = 1, 2, \ldots$ where the A_i are the Adomian polynomials for the nonlinear term $4\left(2e^y + e^{\frac{y}{2}}\right)$.

The ADM solution for the first 13 terms of $y_i(i = 0\ to\ 12)$ is as follows

$$y = \frac{x^{24}}{6} - \frac{2x^{22}}{11} + \frac{x^{20}}{5} - \frac{2x^{18}}{9} + \frac{x^{16}}{4} - \frac{2x^{14}}{7} + \frac{x^{12}}{3} - \frac{2x^{10}}{5} + \frac{x^8}{2} - \frac{2x^6}{3} + x^4 - 2x^2$$

For the convergence region of the series y we check the upper limit of the domain x that ensure $\alpha_i < 1$ for each $i = 0\ to\ 11$ by the ratio test defined at (2.20).

As shown on the graph upper limit of x gets steady at the value 1 starting from the 8^{th} term which means the upper limit of the region of convergence will be around 1.0.

Graph of exact solution and Adomian solution $y(x)$ is shown below for $x \in (0, 1.5)$;

We see on this graph that the series solution y is not a good approximation to the exact solution when x is out of region of convergence as stated above ($x > 1$)

Solving with DTM:

To overcome the singularity at $x = 0$ both sides of (36) is multiplied by x, obtaining

(40)

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + x4\left(2e^y + e^{\frac{y}{2}}\right) = 0$$

Table 1 does not include the differential transformation of functions in the form of e^y. This transformation can be handled by using Maclaurin series expansion of $e^{\alpha y(x)}$ where α is a scalar.

(41)

$$e^{\alpha y(x)} = e^{\alpha y(0)} + \alpha y'(0)e^{\alpha y(0)}x + \frac{\left(\alpha y''(0)e^{\alpha y(0)} + (\alpha y'(0))^2 e^{\alpha y(0)}\right)}{2}x^2 +$$

$$\frac{\left(\alpha y^{(3)}(0)e^{\alpha y(0)} + e^{\alpha y(0)}\alpha y''(0)y'(0) + 2\alpha y'(0)e^{\alpha y(0)}y''(0) + e^{\alpha y(0)}\left(\alpha y'(0)\right)^3\right)}{3!}x^3 + \cdots$$

We define $g(x) = e^{y(x)}$ whose differential transform is $G(k)$ and $q(x) = e^{\frac{y(x)}{2}}$ whose differential transform is $Q(k)$.

We then apply differential transformation to (41)

(42)

$$\sum_{l=0}^{k} \delta(l-1)\,(k-l+1)\,(k-l+2)\,Y(k-l+2) + 2\,(k+1)\,Y(k+1)$$

$$+8\sum_{l=0}^{k} \delta(l-1)\,G(k-l) + 4\sum_{l=0}^{k} \delta(l-1)\,Q(k-l) = 0$$

where from the initial conditions we have $Y(0) = 0$, and $Y(1) = 0$.

$G(k)(k = 0,1,\ldots)$ is derived from the following illustration of $g(x)$ again with the initial conditions of $Y(0) = 0$ and $Y(1) = 0$ and $y^{(k)}(0) = k!\,Y(k)$. As seen $G(k)$ s depend only on $Y(j)$ s with $j \leq k$.

$$g(x) = 1 + Y(2)\,x^2 + Y(3)\,x^3 + \frac{1}{4}[2\,Y(2)^2 + 4\,Y(4)]\,x^4 + \frac{1}{5}[5\,Y(2)\,Y(3) + 5\,Y(5)]\,x^5$$

$$+\frac{1}{6}\left[3\,Y(3)^2 + 4\,Y(2)\,Y(4) + \frac{1}{2}\,Y(2)\left(2\,Y(2)^2 + 4\,Y(4)\right) + c6\,Y(6)\right]\,x^6$$

$$+\frac{1}{7}\Big[4\,Y(3)\,Y(4) + \frac{3}{4}\,Y(3)\left(2\,Y(2)^2 + 4\,Y(4)\right) + 5\,Y(2)\,Y(5) + \frac{2}{5}Y(2)\left(5\,Y(2)\,Y(3) + 5\,Y(5)\right)$$
$$+ 7\,Y(7)\Big]\,x^7$$

$$+\frac{1}{8}\Big[Y(4)\left(2\,Y(2)^2 + 4Y(4)\right) + 5\,Y(3)\,Y(5) + \frac{3}{5}\,Y(3)\left(5\,Y(2)\,Y(3) + 5\,Y(5)\right) + 6\,Y(2)\,Y(6)$$
$$+\frac{1}{3}Y(2)\left(3\,Y(3)^2 + 4\,Y(2)Y(4) + \frac{1}{2}\,Y(2)\left(2\,Y(2)^2 + 4\,Y(4)\right) + 6\,Y(6)\right)$$
$$+ 8\,Y(8)\Big]\,x^8$$

$$+\frac{1}{9}\Big[\frac{5}{4}\,(2\,Y(2)^2+4\,Y(4)\,)\,Y(5)+\frac{4}{5}\,Y(4)\big(5\,Y(2)\,Y(3)+5\,Y(5)\big)+6\,Y(3)\,Y(6)$$
$$+\frac{1}{2}\,Y(3)\left(3\,Y(3)^2\,4\,Y(2)\,Y(4)+\frac{1}{2}\,Y(2)\big(2\,Y(2)^2+4\,Y(4)\big)+6\,Y(6)\right)$$
$$+7\,Y(2)Y(7)$$
$$+\frac{2}{7}\,Y(2)\Big(4\,Y(3)\,Y(4)+\frac{3}{4}\,Y(3)\big(2\,Y(2)^2+4\,Y(4)\big)+5\,Y(2)\,Y(5)$$
$$+\frac{2}{5}\,Y(2)\big(5\,Y(2)\,Y(3)+5\,Y(5)\big)+7\,Y(7)\Big)+9\,Y(9)\Big]\,x^9$$

$$+\frac{1}{10}\Big[Y(5)\,\big(5\,Y(2)\,Y(3)+5\,Y(5)\big)+\frac{3}{2}\,\big(2\,Y(2)^2+4\,Y(4)\big)\,Y(6)$$
$$+\frac{2}{3}\,Y(4)\left(3\,Y(3)^2+4\,Y(2)\,Y(4)+\frac{1}{2}\,Y(2)\big(2\,Y(2)^2+4\,Y(4)\big)+6\,Y(6)\right)$$
$$+7\,Y(3)\,Y(7)$$
$$+\frac{3}{7}\,Y(3)\,\Big(4\,Y(3)\,Y(4)+\frac{3}{4}\,Y(3)\,\big(2\,Y(2)^2+4\,Y(4)\big)+5\,Y(2)\,Y(5)$$
$$+\frac{2}{5}\,Y(2)\big(5\,Y(2)\,Y(3)+5\,Y(5)\big)+7\,Y(7)\Big)+8\,Y(2)\,Y(8)$$
$$+\frac{1}{4}\,Y(2)\Big(Y(4)\,\big(2\,Y(2)^2+4\,Y(4)\big)+5\,Y(3)\,Y(5)$$
$$+\frac{3}{5}\,Y(3)\big(5\,Y(2)\,Y(3)+5\,Y(5)\big)+6\,Y(2)\,Y(6)$$
$$+\frac{1}{3}\,Y(2)\left(3\,Y(3)^2+4\,Y(2)\,Y(4)+\frac{1}{2}\,Y(2)\,\big(2\,Y(2)^2+4\,Y(4)\big)+6\,Y(6)\right)$$
$$+8\,Y(8)\Big)+10\,Y(10\Big]\,x^{10}+\cdots$$

$Q(k)\,(k=0,1,\ldots)$ is derived from the similar illustration of $q(x)$.

With these $G(k)$ and $Q(k)$ and initial conditions, (42) can be solved for each *k* up to the desired approximation term.

The DTM solution for the 19 terms of y_i $(i = 0\ to\ 18)$ is found as

$$y = 0.25\,x^{16} - 0.285714\,x^{14} + 0.333333\,x^{12} - 0.4\,x^{10} + 0.5\,x^8 - 0.666667\,x^6 + x^4 - 2\,x^2$$

Solving with DJM:

We use the same operator L and L^{-1} used in the first example.

(43)

$$Ly = -N(y)$$

(44)

$$y = y_0 - \int_0^x x^{-2}\,\left(\int_0^x x^2\,4\left(2e^y + e^{\frac{y}{2}}\right)dx\right)dx$$

By applying DJM we obtain the following recurrent relation

(45)

$$y_{i+1} = -\int_0^x x^{-2} \int_0^x x^2 \left[8\, e^{\sum_{k=0}^{i} y_k} + 4\, e^{\sum_{k=0}^{i} \frac{y_k}{2}} - \left(8\, e^{\sum_{k=0}^{i-1} y_k} + 4\, e^{\sum_{k=0}^{i-1} \frac{y_k}{2}}\right)\right] dx\, dx$$

With $y_0 = 1$ from the initial conditions.

Integral in (45) can`t be solved analytically for $i > 0$.

However we can use Maclaurin series expansion of e^y by agreeing with the truncation error which would definitely converge to zero as the number of the series terms (n) increase. So (45) can be rewritten as follows

(46)

$$y_{i+1} = -\int_0^x x^{-2} \int_0^x x^2 \left[8 \sum_{r=0}^{\infty} \frac{\left(\sum_{k=0}^{i} y_k\right)^r}{r!} + 4 \sum_{r=0}^{\infty} \frac{\left(\sum_{k=0}^{i} \frac{y_k}{2}\right)^r}{r!} - \left(8 \sum_{r=0}^{\infty} \frac{\left(\sum_{k=0}^{i-1} y_k\right)^r}{r!} + 4 \sum_{r=0}^{\infty} \frac{\left(\sum_{k=0}^{i-1} \frac{y_k}{2}\right)^r}{r!}\right)\right] dx\, dx$$

For the practical solution we take *n* as a finite number. Using (46) we get

$$y_1 = -2\, x^2$$

$$y_2 = -4.463632181023485\, x^{22} + 0.0000269012\, x^{20} - 0.00014881\, x^{18} + 0.000749883\, x^{16} - 0.0034127\, x^{14} + 0.0138889\, x^{12} - 0.05\, x^{10} + 0.157407\, x^8 - 0.428571\, x^6 + x^4$$

$\vdots$

The DJM solution for 21 terms of y_i ($i = 0\ to\ 20$) is as follows

$$y = -0.181818\, x^{22} + 0.2\, x^{20} - 0.222222\, x^{18} + 0.25\, x^{16} - 0.285714\, x^{14} + 0.333333\, x^{12} - 0.4\, x^{10} + 0.5\, x^8 - 0.666667\, x^6 + x^4 - 2\, x^2$$

IV.CONCLUSION

We analyzed 3 iterative methods for solving singular nonlinear Lane-Emden equations. As can easily be seen from solution plots Lane-Emden equation of first kind is solved very effectively with all ADM, DTM and DJM. Second kind Lane-Emden equation can be solved by ADM and DTM directly whereas DJM requires to use the series expansion of the exponential nonlinear term. We also find out second kind Lane-Emden equation with the exponential nonlinear term can be approximated within a narrower convergence region compared with the first kind equation.

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