STABILIZED FEM SOLUTIONS OF MHD EQUATIONS AROUND A SOLID AND INSIDE A CONDUCTING MEDIUM

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ABSTRACT. In this study, the numerical solution of the magnetohydrodynamic (MHD) flow is considered in a circular pipe around a conducting solid and in an insulating or conducting medium. An external magnetic field is applied through axis of the pipe with an angle α with through the x-axis. The mathematical model of the considered physical problem can be defined in terms of coupled MHD equations in the pipe domain and the Laplace equations on the solid and external mediums. The coupled equations are transformed into decoupled inhomogeneous convection-diffusion type equations in order to apply stabilization in the finite element method solution procedure. Obtained stabled solutions for the high values of the problem parameters display the well-known characteristics of the MHD pipe flow.

INTRODUCTION

The electrically conducting, viscous and incompressible fluid is driven down a straight pipe of sufficient length and of circular cross-section by a constant pressure gradient under an external magnetic field applied perpendicular to the axis of the pipe [1]. MHD pipe flow finds some engineering and biomedical applications as MHD generators, pumps and instruments for measuring blood pressure. In this study, a sufficiently long annular pipe around a cylindrical conducting solid and inside a medium is considered which may be electrically insulated or conducting. There is an electrically conducting, viscous and incompressible fluid in the annular region. The fluid moves by a constant pressure gradient under an external magnetic field applied perpendicular to the axis of the pipe. The considered problem models the MHD turbines and nuclear fusion apparatus. Therefore there are many studies, most of them are numerical in the literature. Hartmann firstly investigated the MHD flow of viscous, incompressible flow between two plates [2]. Then, many

number of authors have interested in the solutions of the MHD pipe flow equations both analytically and numerically.

The MHD pipe flow problem without a solid in the pipe but in a conducting medium has been solved by using BEM [3] for square and circular pipes, by using DRBEM [4] and FEM [5] for a circular pipe with the assumption that the induced magnetic field of outside conducting medium tends to zero at infinity.

The standard Galerkin FEM is one of the most usable numerical solution technique for the linear/nonlinear or coupled boundary value problems. However, it is well known that numerical solution of the MHD problems for the high values of the problem parameters (Reynolds number, Hartmann number, etc.) bring some well-known numerical instabilities. In order to eliminate these numerical difficulties, either a finer mesh should be used or a stabilized finite element methods should be considered in the numerical solution procedure. Using the finer mesh increases the size of the resulting system of equations so the computational time and computational cost which is not preferred. Therefore, as an alternative, some stabilization techniques should be used in the solution procedure. Streamline Upwind Petrov-Galerkin (SUPG) method [6] is the most popular and widely used. In 1979, Hughes and Brooks [6] proposed the SUPG method in order to solve the incompressible Navier-Stokes equations as a first application. The proposed method has better stability and accuracy futures compared to previously used standard method. The method has been already applied for many different problems such as convection-diffusion typed equations, Navier-Stokes equations, MHD equations, natural convection equations. Also some variants of the method are developed such as GLS (Galerkin Least Square), DWG(Douglas-Wang), RFB(Residual Free Bubble), TLFEM(Two Level Finite Element Method), SSM(Stabilizing Subgrid Method). The SUPG method reduces the oscillations in the standard Galerkin FEM solutions by adding the mesh-dependent perturbation terms to the formulation [8]. In this study we have used SUPG typed stabilization for the considered problem.

1. Mathematical Modeling

Mathematical problem is derived from Navier-Stokes equations of continuum mechanics and Maxwell's equations of electromagnetic field through Ohm's law. Assume that the cross-section of the pipe contains solid Ω_s, annular fluid domain Ω_f and conducting outside region is Ω_{ex} (see Fig. 1). Also, we assumed that magnetic field is applied through with an angle α with x-direction [1].

Therefore, governing coupled differential equations in non-dimensional form are written as [3]

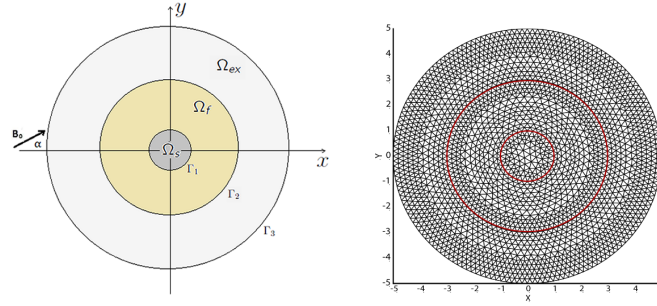$$\nabla^2 B^s = 0 \qquad \text{in } \Omega_s \qquad (1.1)$$

FIGURE 1. Domain of the problem and sample mesh used in the numerical scheme

$$\left.\begin{array}{rcl} \nabla^2 V^f + Re \cdot Rh \cdot \left[\cos\alpha \, \dfrac{\partial B^f}{\partial x} + \sin\alpha \, \dfrac{\partial B^f}{\partial y} \right] &=& -1 \\[2em] \nabla^2 B^f + Rm_f \cdot \left[\cos\alpha \, \dfrac{\partial V^f}{\partial x} + \sin\alpha \, \dfrac{\partial V^f}{\partial y} \right] &=& 0 \end{array}\right\} \quad \text{in } \Omega_f \quad (1.2)$$

$$\nabla^2 B^{ex} = 0 \qquad \text{in } \Omega_{ex} \qquad (1.3)$$

with boundary conditions

$$V^f(x,y) = 0$$

$$B^f(x,y) = B^s(x,y) \qquad (x,y) \in \Gamma_1 \qquad (1.4)$$

$$\frac{1}{Rm_f} \frac{\partial B^f}{\partial n'} = \frac{1}{Rm_s} \frac{\partial B^s}{\partial n''}$$

and

$$V^f(x,y) = 0$$

$$B^f(x,y) = B^{ex}(x,y) \qquad (x,y) \in \Gamma_2 \qquad (1.5)$$

$$\frac{1}{Rm_f} \frac{\partial B^f}{\partial n'''} = \frac{1}{Rm_{ex}} \frac{\partial B^{ex}}{\partial n''''}$$

where $V^f(x,y), B^s(x,y), B^f(x,y)$ and $B^{ex}(x,y)$ are the velocity of the fluid, induced magnetic field of the solid, induced magnetic field of the fluid and induced magnetic field of the external medium, respectively. The superscripts s, f and ex correspond to solid, fluid and external medium, respectively. The parameters Re, Rh, Rm_s, Rm_f and Rm_{ex} are the Reynolds number, magnetic pressure, magnetic Reynolds number of the solid, magnetic Reynolds number of the fluid and magnetic Reynolds number of the external medium, respectively. The unit normal

vectors n' and n''' are the inward normal vectors, and n'' and n'''' are the outward normal vectors.

In order to apply FEM, an artificial boundary (Γ_3) should be defined for the unbounded external region. Using the behavior of the real potential solution of B^{ex}, we can assume that $B^{ex} \to 0$ as $x^2 + y^2 \to \infty$. Therefore one can define the following boundary condition for the external region.

$$B^{ex}(x, y) = 0 \qquad (x, y) \in \Gamma_3. \tag{1.6}$$

Alternatively, one can also use the free exit condition for B^{ex} at infinity. Therefore, it is also possible to use

$$\frac{\partial B^{ex}(x, y)}{\partial n''''} = 0 \qquad (x, y) \in \Gamma_3 \tag{1.7}$$

as a boundary condition. However, if this type of boundary condition is used, induced magnetic field solutions will be obtained up to a constant. Therefore, these values should be normalized by using the identity.

$$\int_\Omega \int B d\Omega = 0. \tag{1.8}$$

As a more simple case, if there is no solid inside the fluid, then equation (1.3) and the boundary conditions (1.4) are not considered.

2. FEM Formulation

It is seen that both equations and also boundary conditions are in a coupled form. Therefore all of the equations should be solved simultaneously. However, as indicated in the introduction using the standard Galerkin finite element method for these coupled equations, brings some numerical instabilities. Therefore we should consider the SUPG typed stabilization technique.

In order to apply SUPG stabilization to the coupled MHD equations (1.2), they should be in decoupled convection-diffusion typed equations. Therefore, they are decoupled first denoting $V = V^f$ and $B_1 = \frac{ReRhB^f}{M}$, where $M = \sqrt{ReRhRm_f}$ is the Hartmann number of the fluid. Then the coupled equations are rewritten as

$$\nabla^2 V + M_x \frac{\partial B_1}{\partial x} + M_y \frac{\partial B_1}{\partial y} = -1$$

$$\nabla^2 B_1 + M_x \frac{\partial V}{\partial x} + M_y \frac{\partial V}{\partial y} = 0 \tag{2.1}$$

where $M_x = M \cos\alpha$ and $M_y = M \sin\alpha$.

In order to decouple the equations, the new variables $U_1(x, y)$ and $U_2(x, y)$ are defined as

$$U_1 = V + B_1$$
$$U_2 = V - B_1 \tag{2.2}$$

then equations become

$$\nabla^2 U_1 + M_x \frac{\partial U_1}{\partial x} + M_y \frac{\partial U_1}{\partial y} = -1$$

$$\nabla^2 U_2 - M_x \frac{\partial U_2}{\partial x} - M_y \frac{\partial U_2}{\partial y} = -1.$$

(2.3)

Now, it is possible to use the SUPG type stabilized FEM technique to these decoupled equations. Before obtaining the SUPG typed formulation, let's write standard Galerkin FEM type weak formulation of the equations (1.1), (2.3) and (1.3) by employing the linear function space $L = (H_0^1(\Omega))^2$ as: Find $\{B^s, U_1, U_2, B^{ex}\} \in \{L \times L \times L \times L\}$ such that

$$B(B^s; U_1; U_2; B^{ex}, s; v_1; v_2; e) - \ell(U_1; U_2, v_1; v_2) = (1, v_1) + (1, v_2) \qquad (2.4)$$

$\forall \{s, v_1, v_2, e\} \in \{L \times L \times L \times L\}$ where

$$\begin{aligned}
B(B^s; U_1; U_2; B^{ex}, s; v_1; v_2; e) \;=\;\; & (\nabla B^s, \nabla s) \\
+ \;\; & (\nabla U_1, \nabla v_1) - (M_x \frac{\partial U_1}{\partial x} + M_y \frac{\partial U_1}{\partial y}, v_1) \\
+ \;\; & (\nabla U_2, \nabla v_2) + (M_x \frac{\partial U_2}{\partial x} + M_y \frac{\partial U_2}{\partial y}, v_2) \\
+ \;\; & (\nabla B^{ex}, \nabla e)
\end{aligned}$$

and

$$\ell(U_1; U_2, v_1; v_2) = (\frac{\partial U_1}{\partial n}, v_1) + (\frac{\partial U_2}{\partial n}, v_2)$$

where $B(u, v)$ and $\ell(u, v)$ are the usual bi-linear and linear forms for the domain and boundary integrals as follows;

$$B(u, v) = \int_\Omega uv d\Omega \qquad \text{and} \qquad \ell(u, v) = \int_{\partial\Omega} uv ds.$$

Then, the variational formulation is written by the choice of finite dimensional subspaces $L_h \subset L$, defined by triangulation of the domain. Therefore, specifying a finite element method [7]: Find $\{B_h^s, U_{1_h}, U_{2_h}, B_h^{ex}\} \in \{L_h \times L_h \times L_h \times L_h\}$ such that

$$B(B_h^s; U_{1_h}; U_{2_h}; B_h^{ex}, s_h; v_{1_h}; v_{2_h}; e_h) - \ell(U_{1_h}; U_{2_h}, v_{1_h}; v_{2_h}) = (1, v_{1_h}) + (1, v_{2_h})$$

(2.5)

$\forall \{s_h, v_{1_h}, v_{2_h}, e_h\} \in \{L \times L \times L \times L\}.$

Now, it is ready to write the SUPG typed variational formulation of these equations using linear elements as [6];

$$
\begin{aligned}
B(B_h^s; U_{1_h}; U_{2_h}; B_h^{ex}, s_h; v_{1_h}; v_{2_h}; e_h) + \ell(U_{1_h}; U_{2_h}, v_{1_h}; v_{2_h}) \\
+\tau_K \left\{ \left(M_x \frac{\partial U_{1_h}}{\partial x} + M_y \frac{\partial U_{1_h}}{\partial y} + 1, M_x \frac{\partial v_{1_h}}{\partial x} + M_y \frac{\partial v_{1_h}}{\partial y} \right) \right. \\
\left. + \left(M_x \frac{\partial U_{2_h}}{\partial x} + M_y \frac{\partial U_{2_h}}{\partial y} - 1, M_x \frac{\partial v_{2_h}}{\partial x} + M_y \frac{\partial v_{2_h}}{\partial y} \right) \right\} \\
= (1, v_{1_h}) + (1, v_{2_h})
\end{aligned}
\tag{2.6}
$$

with the stabilization parameter

$$
\tau_K = \begin{cases}
\dfrac{h_K}{2M} & \text{if } Pe_k \geq 1 \\[4mm]
\dfrac{h_K^2}{12} & \text{if } Pe_k < 1
\end{cases}
\tag{2.7}
$$

where h_K is the diameter of the element K, $Pe_K = h_K \dfrac{M}{6}$ is the Peclet number. For the value of h_K there are two different approaches in the literature which are the area of the element or the longest edge. In this study the value is taken as the length of the longest edge. The Peclet number is mesh element size dependent positive number and controls the convective and diffusive effects as a ratio. If the flow is convective dominated or small diffusive the value of the Peclet number is larger than 1.

Let's turn back to original unknowns with inverse transformations

$$
\begin{aligned}
V &= \frac{U_1 + U_2}{2} \\[3mm]
B_1 &= \frac{U_1 - U_2}{2} \quad \rightarrow \quad B_f = \frac{M}{ReRh} B_1
\end{aligned}
\tag{2.8}
$$

and use the relations in the coupled boundary conditions (1.4) and (1.5) as

$$
\frac{\partial B^f}{\partial n} = \frac{Rm_f}{Rm_s} \frac{\partial B^s}{\partial n} \quad \text{on } \Gamma_1
$$

and

$$
\frac{\partial B^f}{\partial n} = \frac{Rm_f}{Rm_{ex}} \frac{\partial B^{ex}}{\partial n} \quad \text{on } \Gamma_2,
$$

then one gets the final variational form of the equations as

$$(\nabla B_h^s, \nabla s_h) + (\nabla V_h, \nabla v_{1_h}) - ReRh(\cos\alpha\frac{\partial B_h^f}{\partial x} + \sin\alpha\frac{\partial B_h^f}{\partial y}, v_{1_h})$$

$$+\tau_K(M_x\frac{\partial V_h}{\partial x} + M_y\frac{\partial V_h}{\partial y}, M_x\frac{\partial v_{1_h}}{\partial x} + M_y\frac{\partial v_{1_h}}{\partial y}) + (\nabla B_h^f, \nabla v_{2_h})$$

$$-Rm_f(\cos\alpha\frac{\partial V_h}{\partial x} + \sin\alpha\frac{\partial V_h}{\partial y}, v_{2_h}) + \tau_K(M_x\frac{\partial B_h^f}{\partial x} + M_y\frac{\partial B_h^f}{\partial y}, M_x\frac{\partial v_{2_h}}{\partial x} + M_y\frac{\partial v_{2_h}}{\partial y})$$

$$-\frac{Rm_f}{Rm_s}(\frac{\partial B_h^s}{\partial n}, v_{2_h}) - \frac{Rm_f}{Rm_{ex}}(\frac{\partial B_h^{ex}}{\partial n}, v_{2_h}) + (\nabla B_h^{ex}, \nabla e_h)$$

$$= (1, v_{1_h}) - Rm_f\tau_K(1, \cos\alpha\frac{\partial v_{2_h}}{\partial x} + \sin\alpha\frac{\partial v_{2_h}}{\partial y}).$$

$$(2.9)$$

We will obtain the system of linear equations in matrix-vector form as

$$\begin{bmatrix} K & 0 & 0 & 0 \\ 0 & K+S & -ReRhC & 0 \\ -\frac{Rm_f}{Rm_s}Q & Rm_fC & K+S & -\frac{Rm_f}{Rm_{ex}}Q \\ 0 & 0 & 0 & K \end{bmatrix} \left\{ \begin{array}{c} B^s \\ V^f \\ B^f \\ B^{ex} \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ R_1 \\ R_2 \\ 0 \end{array} \right\}$$

$$(2.10)$$

where K, C, S and Q are the matrices and R_1 and R_2 are the vectors with the entries

$$K_{ij} = \int_{\Omega_f} \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} \right) d\Omega,$$

$$C_{ij} = \int_{\Omega_f} \left(\cos\alpha\frac{\partial N_i}{\partial x} + \sin\alpha\frac{\partial N_i}{\partial y} \right) N_j d\Omega,$$

$$S_{ij} = \int_{\Omega_f} \tau_K \left(M_x\frac{\partial N_i}{\partial x} + M_y\frac{\partial N_i}{\partial y} \right) \left(M_x\frac{\partial N_j}{\partial x} + M_y\frac{\partial N_j}{\partial y} \right) d\Omega,$$

$$Q_{ij} = \int_{\Gamma_f} \frac{\partial N_i}{\partial n} N_j d\Gamma, \quad R_{1_i} = \int_{\Omega_f} N_i d\Omega.$$

$$R_{2_i} = \int_{\Omega_f} \tau_K \left(M_x\frac{\partial N_i}{\partial x} + M_y\frac{\partial N_i}{\partial y} \right) d\Omega.$$

From the solution of the system 2.10, the induced magnetic field of the solid, the velocity of the fluid, the induced magnetic field of the fluid and the induced magnetic field of the external medium are obtained at the discretization points.

3. Numerical Results and Discussion

In this section, some test results of the considered stabilized FEM formulation are given in terms of contour plots of the velocity and induced magnetic field. For the small value of the Hartmann number ($M = 100$) 17652 linear triangular elements obtained from 8985 discretization points are used. However, for the large value of the Hartmann number ($M = 1000$) the number of the linear triangular elements is 31530 (from 15976 discretization points). The obtained system of linear equations are solved by using the linear system solver from Netlib library called dgsev.

3.1. **Case 1: No solid inside the pipe.** As the first test problem, it is assumed that there is no solid inside the pipe. The radius of the circular pipe is taken as 1 and the radius of the artificial boundary is 3. Both of the pipe domain and the external region is discretized using linear triangular elements.
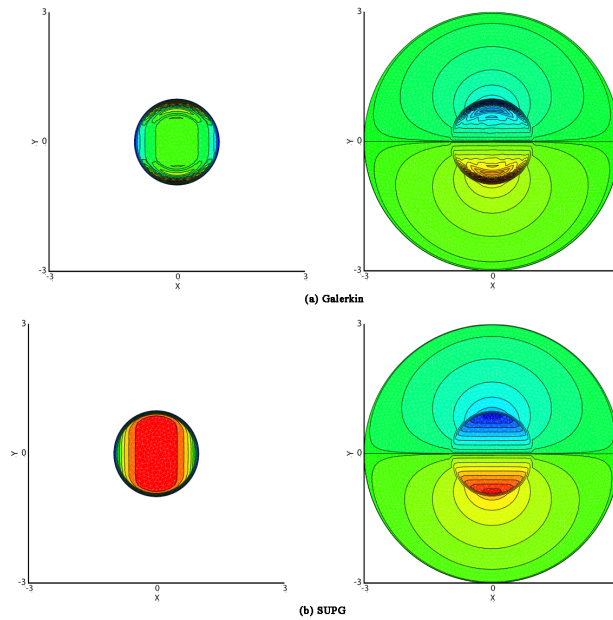

Figure 2. The effect of the stabilization velocity (left) and induced magnetic fields (right) for $Rm_f = 100, Rm_{ex} = 1, Re = 10, Rh = 10$

In Figure 2, the effect of the stabilization in both velocity and induced magnetic fields for the values $Rm_f = 100, Rm_{ex} = 1, Re = 10$ and $Rh = 10$, so Hartmann number becomes $M = 100$. In Figure 2(a), standard Galerkin FEM solutions,

in Figure 2(b), SUPG typed stabilized solutions are displayed. The stabilization effect is clearly seen in fluid velocity and induced current values inside the pipe domain. Since the Laplace equation is solved in the external region, both Galerkin and SUPG solutions are regular in the external region.

The moderate Hartmann value ($M = 1000$) results are displayed in Figure 3. The accuracy of the proposed numerical scheme is seen clearly even for the very large values of the problem parameters. Also, the well known characteristic of MHD flow which is boundary layer formation is seen both in the fluid velocity and in the induced magnetic field as Rm_f getting larger. The velocity values are getting smaller inside the pipe center for the large values of M.
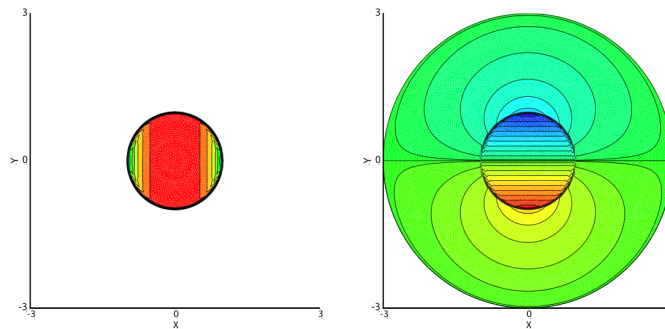

FIGURE 3. Velocity of the fluid and induced magnetic fields for $Rm_f = 1000, Rm_{ex} = 1, Re = 100, Rh = 10$

In Figure 4, the problem is solved with the same parameters as in Figure 3 but with the angle $\alpha = \pi/4$. It is seen that, the same behaviors are obtained with rotating angle α.

The normal derived boundary condition for B^{ex} on Γ_3 is also tested and displayed in Figure 5. As expected, induced magnetic field contours are perpendicular to the artificial external boundary, and they have tendency to become zero at infinity.

3.2. **Case 2: A conducting solid inside the pipe.** In the second case, let's assume that there is a conducting solid inside the pipe with unit circular cross section and outer radius of the circular pipe is 3. Therefore the fluid moves inside the annular pipe domain. Similar to Case 1, we have tested the proposed numerical scheme for the values $Rm_f = 1000, Rm_s = 10, Rm_{ex} = 1, Re = 10, Rh = 10$ so $M = 100$ using boundary condition 1.6 in Figure 6 and boundary condition 1.7 in Figure 7 with $\alpha = \pi/2$. The continuation of the induced magnetic contours and the effect of ratios $\frac{Rm_f}{Rm_s}$ and $\frac{Rm_f}{Rm_{ex}}$ is clearly seen from the figure. Existence of different circulations in the velocity is also observed.
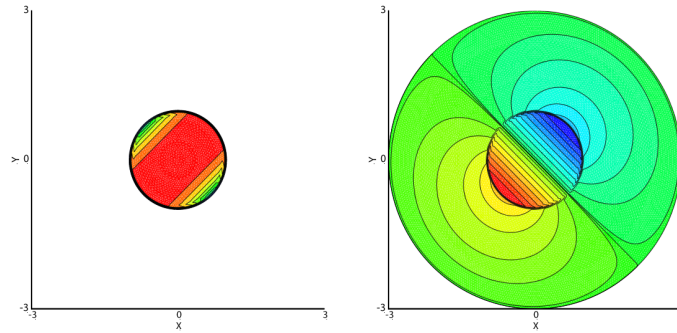
FIGURE 4. Velocity of the fluid and induced magnetic fields for $Rm_f = 1000, Rm_{ex} = 1, Re = 100, Rh = 10, \alpha = \pi/4$
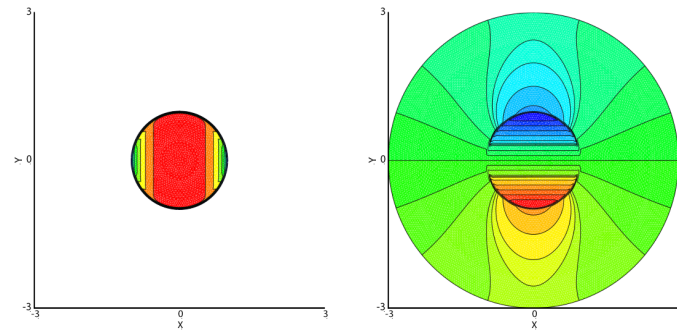

FIGURE 5. Velocity of the fluid and induced magnetic fields for the normal derivative boundary condition and for $Rm_f = 500, Rm_{ex} = 1, Re = 50, Rh = 10, M = 500$

4. CONCLUSION

The magnetohydrodynamic flow inside a circular cross-section duct around a conducting solid in an insulating or conducting medium is solved using stabilized finite element method. Even the considered problem has been already solved with some other numerical solution procedures, the proposed formulation enables to obtain the stable solution for the high values of the problem parameters. Obtained solutions are in good agreement with the previously obtained results and display the well-known characteristics of the MHD flow [3, 4].
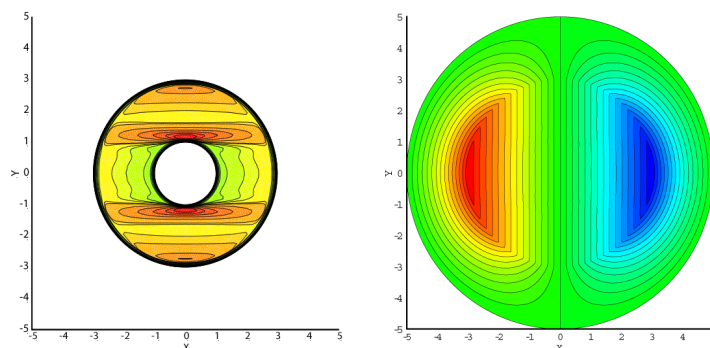
FIGURE 6. Velocity of the fluid and induced magnetic fields for $Rm_f = 1000, Rm_s = 10, Rm_{ex} = 1, Re = 10, Rh = 10, \alpha = \pi/2$
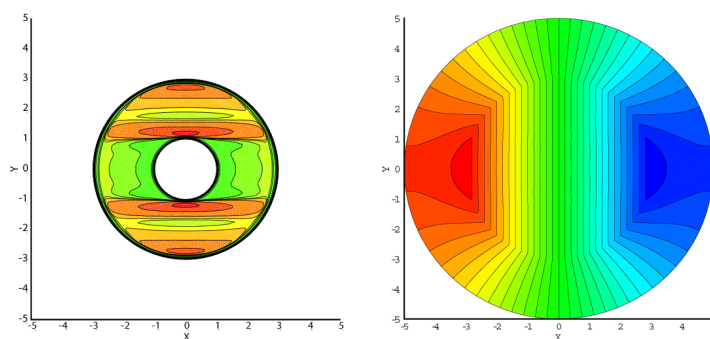

FIGURE 7. Velocity of the fluid and induced magnetic fields for the normal derivative boundary condition and for $Rm_f = 500, Rm_{ex} = 1, Re = 50, Rh = 10, \alpha = \pi/2$

ACKNOWLEDGEMENTS

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