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On Two Dimensional Dynamical Analysis of a Pre-Stressed Anisotropict Plate-Strip with Finite Length

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Abstract: *In this study, the influence of initial stress on a pre-stressed orthotropic plate-strip with finite length resting on a rigid half plane is investigated by utilizing Three-Dimensional Linearized Theory of Elastic Waves in Initially Stressed Bodies. The material of the plate-strip is assumed to be anisotropic. The total energy functional of the considered problem is developed. Also, finite element modeling is developed for the considered boundary-value problem.*

Keywords: *Anisotropic material; Initial stress; Finite element method; Forced vibration; Time-harmonic load*

Sonlu Uzunluğa Sahip Öngerilmeli Anizotrop Şerit-Plağın İki Boyutlu Dinamik Analizi Üzerine

Özet: *Bu çalışmada, rijit zemin üzerinde oturan sonlu uzunluğa sahip öngerilmeli anizotrop bir şerit-plakta öngerilmenin etkisi öngerilmeli ortamlardaki elastik dalgaların doğrusallaştırılmış üç boyutlu teorisi kullanılarak incelenmiştir. Şerit-plağın anizotrop malzemeden yapıldığı kabul edilmiştir. Ele alınan problemin toplam potansiyel enerji fonksiyoneli oluşturulmuştur. Ayrıca, ilgili sınır değer probleminin sonlu eleman modellemesi yapılmıştır.*

Anahtar Kelimeler: *Anizotrop madde; Öngerilme; Sonlu elemanlar metodu; Zorlanmış titreşim; Harmonik yük*

1. INTRODUCTION

Problems regarding non-linear effects in the dynamics of the elastic medium can be divided into two groups: the problems related to the influence of initial stresses on the speed of waves in elastic mediums and the ones related to wave propagation. Some examples of studies on the influence of initial stresses on the speed of waves are given in [1-4].

Studies related to wave propagation are reviewed by Guz [5] and Babich et al. [6]. Some of the studies related to wave propagation problems before 2000 are [7-10].

Initial stresses are present in composites, in rocks, in the Earth's crust, and so on. After manufacturing and assembly process of structural elements, initial stresses also occur. From this point of view, problems involving initially stressed bodies have a wide range of applications. That is why they must be taken into account during the studies in elastodynamics.

The above-stated problems involving initially stressed bodies can be solved using the Three Dimensional Linearized Theory of Elastic Waves in Initially Stressed Bodies (TLTEWISB). The fundamentals of TLTEWISB are given in [11-16]. A systematic analysis of the results obtained before 2002 was made in [17]. Some applications of wave propagation problems for pre-stressed bodies were investigated in [18-22]. The mentioned studies were made with the use of TLTEWISB, and the effect of initial stresses on the time-harmonic dynamical stress state in the layered medium was investigated therein. The problems considered in the studies [18-22] all have layers with infinite lengths along one or more directions. It is well known that problems involving layers with finite length and width are encountered frequently. But, the integral transformation method is not applicable in case where the length of the layers is finite. Therefore, numerical methods are to be utilized to the problems having layers with finite length and width.

Recently, Akbarov et al. developed the finite element modeling of the stress field problem for layered medium of finite length and width in [23,24]. In the papers [23,24], the time-harmonic external force is perpendicular to the face plane of the layers. Eröz [25] developed the investigations in [26,27] for arbitrary inclined linearly located time-harmonic force.

In order to investigate the influence of initial stress on a pre-stressed anisotropic plate-strip with finite length resting on a rigid foundation, the total energy functional of the considered problem is developed in this study. Also, finite element modeling of the considered boundary-value problem is developed.

Boundary value problem of the considered case

Considering the plate-strip with geometry given in Fig. 1, we determine the positions of points of the plate-strip by the Lagrangian coordinates in the Cartesian system of coordinates $Ox_1x_2x_3$. The Ox_3-axis is directed normal to the plane of Fig. 1. It is assumed that the length of the plate in the direction of the Ox_3-axis is infinite. All investigations in this study will be made for the plane-strain state in the Ox_1x_2-plane for the region

$$D=\{(x_1,x_2):-\ell\le x_1\le \ell, 0\le x_2\le h\} \quad (1)$$

The linear elastic material of the plate-strip is supposed to be homogeneous and anisotropic, especially orthotropic with the principal axes Ox_1, Ox_2, and Ox_3. We also assume that the material of the plate-strip is relatively rigid and determine the initial stresses in the plate-strip within the scope of the linear theory of elasticity as follows:

$$\sigma_{11}^0=q, \sigma_{ij}^0=0 \text{ for all } ij\ne 11 \quad (2)$$

In Eq. (2), the upper index "0" denotes the values regarding the initial stress state. After contacting the initially stressed plate-strip with rigid foundation, a time-harmonic linear located normal load, $p\delta(x_1)\mathbf{e}^{i\omega t}$, acts perpendicularly on the upper free face of the plate. Here, $\delta(\cdot)$ stands for the Dirac delta function.

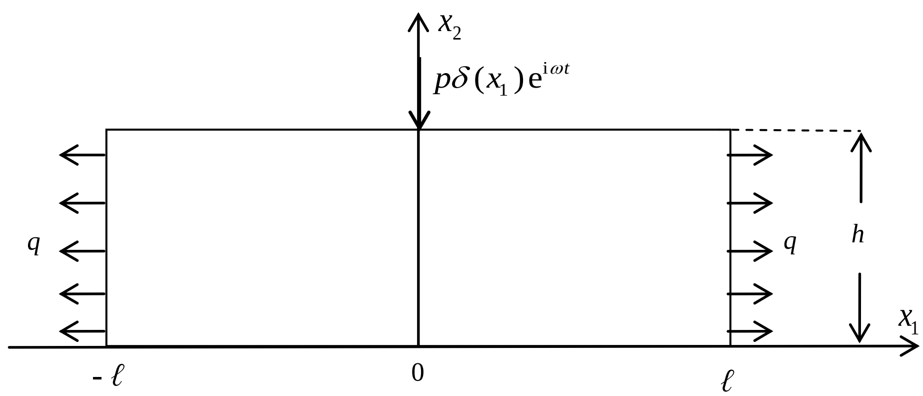

Figure 1. The geometry of the pre-stressed plate-strip.

The equations of motion of TLTEWISB for small initial deformation are

$$\frac{\partial\sigma_{ij}}{\partial x_j}+\sigma_{11}^0\frac{\partial^2 u_i}{\partial x_1{}^2}=\rho\frac{\partial^2 u_i}{\partial t^2}, i=1,2, j=1,2 \quad (3)$$

as given by Guz in [16]. In Eq. (3), ρ stands for the density of the material in the natural state, $u_1=u_1(x_1,x_2,t)$ and $u_2=u_2(x_1,x_2,t)$ stand for the perturbation of the components of the displacement vector in the directions of the Ox_1 and Ox_2 axes, respectively. Also, in Eq. (3), σ_{ij} are the perturbations of the components of the stress tensor.

According to Lekhnitskii [26], the relations between the strains and the stresses for orthotropic body under plane-strain state are

$$\sigma_{11} = A_{11}\varepsilon_{11} + A_{12}\varepsilon_{22}, \quad \sigma_{22} = A_{12}\varepsilon_{11} + A_{22}\varepsilon_{22}, \quad \sigma_{12} = G_{12}\varepsilon_{12} \tag{4}$$

Where

$$A_{11} = \frac{a_{22}}{a_{11}a_{22} - a_{12}^2}, \quad A_{12} = \frac{-a_{12}}{a_{11}a_{22} - a_{12}^2}, \quad A_{22} = \frac{a_{11}}{a_{11}a_{22} - a_{12}^2} \tag{5}$$

and

$$a_{11} = \frac{1}{E_1}\left(1 - \nu_{13}^2 \frac{E_1}{E_3}\right), \quad a_{12} = \frac{1}{E_2}\left(-\nu_{12} - \nu_{13}\nu_{23}\frac{E_2}{E_3}\right), \quad a_{22} = \frac{1}{E_2}\left(1 - \nu_{23}^2 \frac{E_2}{E_3}\right) \tag{6}$$

The components ε_{11}, ε_{12}, and ε_{22} of the strain tensor given in Eq. (4) are related to the components u_1 and u_2 of the displacement vector by the equations

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \tag{7}$$

In Eqs. (4)-(6), $G_{12} = 2\mu_{12}$ is the shear modulus; E_1, E_2, and E_3 are the modulus of elasticity of the material in the direction of the principal axes Ox_1, Ox_2, and Ox_3, respectively. In Eq.(6), ν_{12}, ν_{13}, ν_{23}, ν_{21}, ν_{31}, and ν_{32} are the Poisson's ratios for the material of the plate-strip.

For the perturbed state, let the following boundary conditions hold:

$$u_1\big|_{x_2=0} = 0, \quad u_2\big|_{x_2=0} = 0, \quad \left.\left(\sigma_{11} + q\frac{\partial u_1}{\partial x_1}\right)\right|_{x_1=\pm\ell} = 0, \quad \left.\left(\sigma_{12} + q\frac{\partial u_2}{\partial x_1}\right)\right|_{x_1=\pm\ell} = 0,$$
$$\sigma_{21}\big|_{x_2=h} = 0, \quad \sigma_{22}\big|_{x_2=h} = -p\delta(x_1)\mathbf{e}^{\mathbf{i}\omega t}. \tag{8}$$

The applied lineal located load is time-harmonic, so the dependent variables are time-harmonic. Hence, the dependent variables can be represented as

$$\left\{u_i, \sigma_{ij}, \varepsilon_{ij}\right\} = \left\{\bar{u}_i, \bar{\sigma}_{ij}, \bar{\varepsilon}_{ij}\right\}\mathbf{e}^{\mathbf{i}\omega t} \tag{9}$$

In Eq. (9), the superposed bar denotes the amplitude of the corresponding quantity. Hereafter, throughout the paper the superposed bars will be omitted. Thus, the mathematical formulation of the considered problem is given.

Derivation of the total energy functional

The problem given by (2)-(8) will be investigated by employing Finite Element Method (FEM) since analytical solution to the problem cannot be obtained. We introduce the dimensionless coordinate system by the transformations

$$\hat{x}_1 = x_1 / h \ , \ \hat{x}_2 = x_2 / h \,. \tag{10}$$

With the relation (9) and the transformations (10), the equations of motion given in Eq. (3) and the boundary conditions (8) will be as follows:

$$\frac{1}{h}\frac{\partial \sigma_{11}}{\partial \hat{x}_1} + \frac{1}{h}\frac{\partial \sigma_{12}}{\partial \hat{x}_2} + \frac{\sigma_{11}^0}{h^2}\frac{\partial^2 u_1}{\partial \hat{x}_1^2} = -\rho\omega^2 u_1 \,, \tag{11}$$

$$\frac{1}{h}\frac{\partial \sigma_{21}}{\partial \hat{x}_1} + \frac{1}{h}\frac{\partial \sigma_{22}}{\partial \hat{x}_2} + \frac{\sigma_{11}^0}{h^2}\frac{\partial^2 u_2}{\partial \hat{x}_1^2} = -\rho\omega^2 u_2 \,, \tag{12}$$

$$u_1\Big|_{\hat{x}_2=0} = 0, \quad u_2\Big|_{\hat{x}_2=0} = 0, \quad \left(\sigma_{11} + \sigma_{11}^0 \frac{\partial u_1}{\partial \hat{x}_1}\right)\Bigg|_{\hat{x}_1=\pm\ell/h} = 0, \quad \left(\sigma_{12} + \sigma_{11}^0 \frac{\partial u_2}{\partial \hat{x}_1}\right)\Bigg|_{\hat{x}_1=\pm\ell/h} = 0, \tag{13}$$

$$\sigma_{21}\Big|_{\hat{x}_2=1} = 0, \quad \sigma_{22}\Big|_{\hat{x}_2=1} = -p\delta\left(h\hat{x}_1\right).$$

To have the variational formulation of the current boundary-value problem, we multiply Eqs. (11) and (12) by the test functions $v_1 = v_1(\hat{x}_1 \,,\, \hat{x}_2)$ and $v_2 = v_2(\hat{x}_1 \,,\, \hat{x}_2)$, and add the resultant equations. Then, this equation is integrated over the domain

$$\hat{D} = \left\{\left(\hat{x}_1, \hat{x}_2\right) : -\ell / h \le \hat{x}_1 \le \ell / h, 0 \le \hat{x}_2 \le 1\right\} \tag{14}$$

We get the equation

$$\begin{aligned}&\int_0^1\int_{-\ell/h}^{\ell/h}\left[\frac{1}{h}\frac{\partial \sigma_{11}}{\partial \hat{x}_1}v_1 + \frac{1}{h}\frac{\partial \sigma_{21}}{\partial \hat{x}_1}v_2 + \frac{1}{h}\frac{\partial \sigma_{12}}{\partial \hat{x}_2}v_1 + \frac{1}{h}\frac{\partial \sigma_{22}}{\partial \hat{x}_2}v_2 + \frac{\sigma_{11}^0}{h^2}\frac{\partial^2 u_1}{\partial \hat{x}_1{}^2}v_1 + \frac{\sigma_{11}^0}{h^2}\frac{\partial^2 u_2}{\partial \hat{x}_1{}^2}v_2\right]\mathrm{d}\hat{x}_1\mathrm{d}\hat{x}_2 \\ &= -\int_0^1\int_{-\ell/h}^{\ell/h}\rho\omega^2(u_1v_1 + u_2v_2)\mathrm{d}\hat{x}_1\mathrm{d}\hat{x}_2.\end{aligned} \tag{15}$$

Applying integration by parts to Eq. (15) and using the boundary conditions (13), we get

$$
\begin{aligned}
&\int_0^1 \int_{-\ell/h}^{\ell/h} \left[\begin{aligned} &\frac{1}{h}\sigma_{11}\frac{\partial v_1}{\partial \hat{x}_1} + \frac{1}{h}\sigma_{21}\frac{\partial v_2}{\partial \hat{x}_1} + \frac{1}{h}\sigma_{12}\frac{\partial v_1}{\partial \hat{x}_2} + \frac{1}{h}\sigma_{22}\frac{\partial v_2}{\partial \hat{x}_2} \\ &\qquad + \frac{\sigma_{11}^0}{h^2}\frac{\partial u_1}{\partial \hat{x}_1}\frac{\partial v_1}{\partial \hat{x}_1} + \frac{\sigma_{11}^0}{h^2}\frac{\partial u_2}{\partial \hat{x}_1}\frac{\partial v_2}{\partial \hat{x}_1} - \rho\omega^2(u_1 v_1 + u_2 v_2) \end{aligned} \right] \mathrm{d}\hat{x}_1 \mathrm{d}\hat{x}_2 \\
&= -\int_{-\ell/h}^{\ell/h} \frac{1}{h}\sigma_{22}\Big|_{\hat{x}_2=1} v_2\Big|_{\hat{x}_2=1} \mathrm{d}\hat{x}_1.
\end{aligned}
\tag{16}
$$

The relations (4) under new coordinate system are as follows:

$$
\sigma_{11} = \frac{A_{11}}{h}\frac{\partial u_1}{\partial \hat{x}_1} + \frac{A_{12}}{h}\frac{\partial u_2}{\partial \hat{x}_2},\ \sigma_{22} = \frac{A_{12}}{h}\frac{\partial u_1}{\partial \hat{x}_1} + \frac{A_{22}}{h}\frac{\partial u_2}{\partial \hat{x}_2}, \sigma_{12} = \frac{\mu_{12}}{h}\left(\frac{\partial u_1}{\partial \hat{x}_2} + \frac{\partial u_2}{\partial \hat{x}_1}\right).
\tag{17}
$$

Using certain properties of the variational operator $\delta(\cdot)$ and relations (17), we can rewrite Eq. (16) as

$$
\begin{aligned}
&\int_0^1 \int_{-\ell/h}^{\ell/h} \left\{ \left[\left(\frac{\sigma_{11}^0}{\mu_{12}} + \frac{A_{11}}{\mu_{12}} \right) \frac{\partial u_1}{\partial \hat{x}_1} + \frac{A_{12}}{\mu_{12}}\frac{\partial u_2}{\partial \hat{x}_2} \right] \frac{\partial v_1}{\partial \hat{x}_1} + \left[\frac{\partial u_1}{\partial \hat{x}_2} + \left(\frac{\sigma_{11}^0}{\mu_{12}} + 1 \right) \frac{\partial u_2}{\partial \hat{x}_1} \right] \frac{\partial v_2}{\partial \hat{x}_1} \right. \\
&\left. + \left(\frac{\partial u_1}{\partial \hat{x}_2} + \frac{\partial u_2}{\partial \hat{x}_1} \right) \frac{\partial v_1}{\partial \hat{x}_2} + \left(\frac{A_{12}}{\mu_{12}}\frac{\partial u_1}{\partial \hat{x}_1} + \frac{A_{22}}{\mu_{12}}\frac{\partial u_2}{\partial \hat{x}_2} \right) \frac{\partial v_2}{\partial \hat{x}_2} - \Omega^2 (u_1 v_1 + u_2 v_2) \right\} \mathrm{d}\hat{x}_1 \mathrm{d}\hat{x}_2 \\
&= -\int_{-\ell/h}^{\ell/h} \frac{p}{\mu_{12}} \delta(\hat{x}_1) v_2\Big|_{\hat{x}_2=1} \mathrm{d}\hat{x}_1.
\end{aligned}
\tag{18}
$$

In Eq. (18), we introduce the dimensionless frequency Ω as $\Omega^2 = \rho\omega^2 h^2 / \mu_{12}$. Note that, both sides of Eq. (18) is multiplied by the quantity h^2 / μ_{12}, where μ_{12} is Lamé constant. Thus, according to the variational principle of TLTEWISB given in [16], using Eq. (18), the total energy functional $\mathbf{J}(\mathbf{u})=\mathbf{A}(\mathbf{u},\mathbf{u})/2-\mathbf{l}(\mathbf{u})$, is introduced as

$$
\begin{aligned}
\mathbf{J}(u_1, u_2) = &\frac{1}{2}\int_0^1 \int_{-\ell/h}^{\ell/h} \left[\left(\frac{\sigma_{11}^0}{\mu_{12}} + \frac{A_{11}}{\mu_{12}} \right) \left(\frac{\partial u_1}{\partial \hat{x}_1} \right)^2 + \frac{A_{22}}{\mu_{12}} \left(\frac{\partial u_2}{\partial \hat{x}_2} \right)^2 + 2\frac{A_{12}}{\mu_{12}}\frac{\partial u_1}{\partial \hat{x}_1}\frac{\partial u_2}{\partial \hat{x}_2} \right. \\
&\left. + \left(\frac{\partial u_1}{\partial \hat{x}_2} + \frac{\partial u_2}{\partial \hat{x}_1} \right)^2 + \frac{\sigma_{11}^0}{\mu_{12}} \left(\frac{\partial u_2}{\partial \hat{x}_1} \right)^2 - \Omega^2 \left(u_1^2 + u_2^2 \right) \right] \mathrm{d}\hat{x}_1 \mathrm{d}\hat{x}_2 + \int_{-\ell/h}^{\ell/h} \frac{p}{\mu_{12}} \delta(\hat{x}_1) u_2\Big|_{\hat{x}_2=1} \mathrm{d}\hat{x}_1,
\end{aligned}
\tag{19}
$$

where $\mathbf{u}=\mathbf{u}(u_1, u_2)$. In the functional $\mathbf{J}(\mathbf{u})$, $\mathbf{A}(\mathbf{u},\mathbf{v})$ is a bilinear form and is defined as the left hand side of Eq. (18), and $\mathbf{l}(\mathbf{u})$ is a linear form and is defined as the right hand side of Eq. (18). Here, $\mathbf{v}=\mathbf{v}(v_1, v_2)$.

Equating the first variational of the total energy functional $\mathbf{J}(\mathbf{u})$ to zero, namely setting the equation $\delta\mathbf{J}(\mathbf{u})=0$, Eqs. (11) and (12), and boundary conditions (13) are obtained with the help of the principles from calculus of variation. By the way the validity of the total energy functional given in Eq. (19) is proven.

Finite element modeling

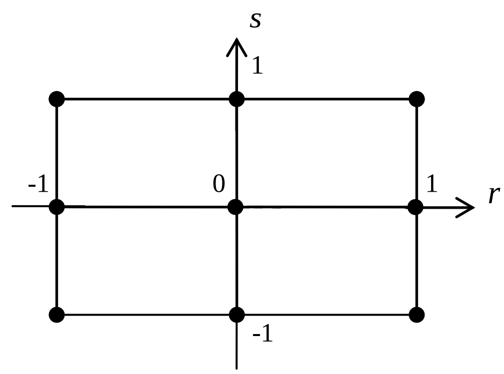

Figure 2. The order of the nodes of a finite element define over a pilot element.

Figure 2. The order of the nodes of a finite element define over a pilot element.By the use of the Rayleigh-Ritz technique given in [27], the FEM modeling of the equation $\delta\mathbf{J}(\mathbf{u})=0$ is constructed as follows: First, the domain $\hat{D}$ is divided into a finite number of subdomains $\hat{D}_j$. Displacement based FEM given in [27] is utilized. So, let the displacement over the k th finite element be

$$u_1^k=\sum_{j=1}^{M}a_{1j}^k N_j^k\left(r,s\right),\quad u_2^k=\sum_{j=1}^{M}a_{2j}^k N_j^k\left(r,s\right), \tag{20}$$

where a_{1j}^k and a_{2j}^k are unknown coefficients and are to be determined. In Eq. (20), M is the number of nodes over a finite element, $N_j^k\left(r,s\right)$ are the shape functions of the k th finite element, and r and s are the local normalized coordinates of the points of the k th finite element given in Fig. 2.

As an example, rectangular Lagrange family of quadratic finite elements [36] can be used for which the shape functions $N_j^k\left(r,s\right)$ of the k th finite element are defined on the unit square $\left[-1,1\right]\times\left[-1,1\right]$. According to Rayleigh-Ritz method, substituting approximate solutions (20) into the equation $\delta\mathbf{J}(\mathbf{u})=0$, the equation

$$\mathbf{K}\mathbf{w}=\mathbf{f}, \tag{21}$$

is obtained with respect to nodal displacements. In Eq. (21), $\mathbf{K}$ is a stiffness matrix, $\mathbf{w}$ is a vector whose components are unknown coefficients a_{1j}^k and a_{2j}^k for the kth finite element, and $\mathbf{f}$ is a vector the components of which are nodal forces. The force vector $\mathbf{f}$ has only one nonzero component, $\frac{\rho}{\mu_{12}} N_{ij}\big|_{\hat{x}_1=0,\hat{x}_2=0}$, in the component where the external force acts. The stiffness matrix $\mathbf{K}$ is set up as follows:

$$\mathbf{K} = \sum_{k=1}^{R} \mathbf{K}^k , \tag{22}$$

where $\mathbf{K}^k$ is local stiffness matrix for the kth finite element and is given by

$$\mathbf{K}^k = \begin{bmatrix} \left[\mathbf{K}_{11}^k\right] & \left[\mathbf{K}_{12}^k\right] \\ \left[\mathbf{K}_{21}^k\right] & \left[\mathbf{K}_{22}^k\right] \end{bmatrix} \tag{23}$$

The block matrices given in Eq. (23) are evaluated by the equations

$$\left[\mathbf{K}_{11}^k\right] = \left[\int_0^1 \int_{-\ell/h}^{\ell/h} \left\{ \left(\frac{\sigma_{11}^0}{\mu_{12}} + \frac{A_{11}}{\mu_{12}} \right) \frac{\partial N_{mn}}{\partial \hat{x}_1} \frac{\partial N_{ij}}{\partial \hat{x}_1} - \frac{\partial N_{mn}}{\partial \hat{x}_2} \frac{\partial N_{ij}}{\partial \hat{x}_2} - \Omega^2 N_{mn} N_{ij} \right\} \mathrm{d}\hat{x}_1 \mathrm{d}\hat{x}_2 \right], \tag{24}$$

$$\left[\mathbf{K}_{12}^k\right] = \left[\int_0^1 \int_{-\ell/h}^{\ell/h} \left\{ \frac{A_{12}}{\mu_{12}} \frac{\partial N_{mn}}{\partial \hat{x}_1} \frac{\partial N_{ij}}{\partial \hat{x}_2} + \frac{\partial N_{mn}}{\partial \hat{x}_2} \frac{\partial N_{ij}}{\partial \hat{x}_1} \right\} \mathrm{d}\hat{x}_1 \mathrm{d}\hat{x}_2 \right], \tag{25}$$

$$\left[\mathbf{K}_{21}^k\right] = \left[\int_0^1 \int_{-\ell/h}^{\ell/h} \left\{ \frac{A_{12}}{\mu_{12}} \frac{\partial N_{mn}}{\partial \hat{x}_2} \frac{\partial N_{ij}}{\partial \hat{x}_1} + \frac{\partial N_{mn}}{\partial \hat{x}_1} \frac{\partial N_{ij}}{\partial \hat{x}_2} \right\} \mathrm{d}\hat{x}_1 \mathrm{d}\hat{x}_2 \right], \tag{26}$$

$$\left[\mathbf{K}_{22}^k\right] = \left[\int_0^1 \int_{-\ell/h}^{\ell/h} \left\{ \frac{A_{22}}{\mu_{12}} \frac{\partial N_{mn}}{\partial \hat{x}_2} \frac{\partial N_{ij}}{\partial \hat{x}_2} + \left(1 + \frac{\sigma_{11}^0}{\mu_{12}} \right) \frac{\partial N_{mn}}{\partial \hat{x}_1} \frac{\partial N_{ij}}{\partial \hat{x}_1} - \Omega^2 N_{mn} N_{ij} \right\} \mathrm{d}\hat{x}_1 \mathrm{d}\hat{x}_2 \right], \tag{27}$$

where $i, j, m, n = 1, ..., M$. The meaning of the indices used in Eqs. (24)-(27) is obvious. In Eq. (22), R is the number of finite elements used during this study and is determined through the convergence requirements. Solution of Eq. (21) gives the unknown displacements at the corresponding nodes. The components of the stress tensor $\boldsymbol{\sigma}^k = \{\sigma_{11}, \sigma_{22}, \sigma_{12}\}^T$ are calculated by

$$\boldsymbol{\sigma}^k = \mathbf{D}^k \mathbf{B}^k \mathbf{w}^k , \tag{28}$$

where $\mathbf{w}^k = \left\{a_{11}^k, a_{12}^k, ..., a_{19}^k, a_{21}^k, a_{22}^k, ..., a_{29}^k\right\}^T$,

$$\mathbf{D}^k = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & 2\mu_{12} \end{bmatrix}, \tag{29}$$

and

$$\mathbf{B}^k = \begin{bmatrix} \frac{\partial N_1}{\partial \hat{x}_1} & \dots & \frac{\partial N_9}{\partial \hat{x}_1} & 0 & \dots & 0 \\ 0 & \dots & 0 & \frac{\partial N_1}{\partial \hat{x}_2} & \dots & \frac{\partial N_9}{\partial \hat{x}_2} \\ \frac{\partial N_1}{\partial \hat{x}_1} & \dots & \frac{\partial N_9}{\partial \hat{x}_1} & \frac{\partial N_1}{\partial \hat{x}_2} & \dots & \frac{\partial N_9}{\partial \hat{x}_2} \end{bmatrix} \tag{30}$$

Thus, the FEM modeling for the considered problem is developed.

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