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New results on IBVP for a class of nonlinear parabolic equations

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Abstract

In this article we propose a new approach for investigation the local existence of classical solutions of IBVP for a class of nonlinear parabolic equations.

Keywords: Nonlinear parabolic equation, Local existence, Classical solutions, IBVP.
2010 MSC: 35K20, 35K58, 35K61.

1. Introduction

In this article we investigate the IBVP

$$\begin{aligned} &u_t - \triangle u = f(t, x, u, Du), \quad t \geq 0, x_1 \geq 0, (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}, \\ &u(0, x) = u_0(x), \quad x_1 \geq 0, (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}, \\ &u(t, 0, x_2, \ldots, x_n) = v(t, x_2, \ldots, x_n), \quad t \geq 0, (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}, \end{aligned} \tag{1.1}$$

where $n \geq 2$, $\triangle u = \sum_{i=1}^{n} u_{x_i x_i}$, $Du = (u_{x_1}, u_{x_2}, \ldots, u_{x_n})$, $u : \mathbb{R}^{n+1} \longmapsto \mathbb{R}$ is unknown function, $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \longmapsto \mathbb{R}$, $u_0 : \mathbb{R}^n \longmapsto \mathbb{R}$ and $v : \mathbb{R}^n \longmapsto \mathbb{R}$ are given functions which satisfy the following conditions

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(H1)

$$f \in \mathcal{C}(\mathbb{R}, \mathbb{R}^n, \mathbb{R}, \mathbb{R}^n),$$

$$|f(t,x,u,Du)| \le \sum_{i=1}^{m_2} \Big(a_i^0(t,x)|u(t,x)|^{p_i^0} + a_i^1(t,x)|u_{x_1}(t,x)|^{p_i^1}$$

$$+ \cdots + a_i^n(t,x)|u_{x_n}(t,x)|^{p_i^n} \Big),$$

where $a_i^j(t,x) : \mathbb{R}^{n+1} \longmapsto \mathbb{R}$, $i = 1, \ldots, m_2, j = 0, 1, \ldots, n$, are positive continuous functions on $\mathbb{R}^{n+1}$ for which $\sup_{\mathbb{R}\times\mathbb{R}^n} a_i^j(t,x) < \infty, p_i^j \in \mathbf{N} \cup \{0\}, m_2 \in \mathbb{N}$.

(H2)

$$u_0 \in \mathcal{C}^2(\mathbb{R}^n),$$

$$||u_0||_{\mathcal{C}^2(\mathbb{R}^n)} \le P \quad \text{for} \quad \forall\, x \in \mathbb{R}^n,$$

$$v \in \mathcal{C}^1(\mathbb{R}^1, \mathcal{C}^2(\mathbb{R}^{n-1})),$$

$$||v||_{C^1(\mathbb{R}^1,\mathcal{C}^2(\mathbb{R}^{n-1}))} \le P \quad \text{for} \quad \forall\,(t, x_2, \ldots, x_n) \in \mathbb{R}^n,$$

$$v(0, x_2, \ldots, x_n) = u_0(0, x_2, \ldots, x_n) \quad \text{for} \quad \forall\,(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}.$$

Here P is a fixed positive constant.

For $O_1 \subset [0,\infty)$ and $O_2 \subset \mathbb{R}^n$ with $\mathcal{C}^1(O_1, \mathcal{C}^2(O_2))$ we denote the space of all continuous functions $u : O_1 \times O_2 \longmapsto \mathbb{R}$ such that u_t, u_{x_i}, $u_{x_i x_i}$, $i \in \{1, \ldots, n\}$, exist and are continuous on $O_1 \times O_2$.
As an example for the function f that satisfies (H1) we consider the function $f = \lambda|u|^{p-1}u, p \in \mathbb{N}$, $\lambda \in \mathbb{R}$.
With $\mathbb{R}^n_{1+}$ we will denote the space $\{(x_1, x_2, \ldots, x_n) : x_1 \ge 0, x_i \in \mathbb{R}, i = 2, \ldots, n\}$.
The main question which we consider here is local existence of classical solutions to the problem (1.1).

A lot of articles have been devoted to the investigation of initial boundary value problems for parabolic equations and systems (see, for example, [1]-[12] and the references therein). We note that in the references the IBVP (1.1) is connected with the dimension n, Fujita exponent, Sobolev critical exponents, bounded and unbounded domain. In this article we propose new idea which tell us that the local existence of classical solutions of the IBVP is connected with the integral representation of the solutions, it is not connected with the dimension n and if the domain is bounded or not.
At this moment the problem for existence of classical solutions for the problem (1.1) for arbitrary dimension $n \ge 2$ was opened. Here we propose its proof.
Our main results for local existence are as follows.

Theorem 1.1. *Let $n \ge 2$ be fixed, f satisfy* (H1)*, u_0 and v satisfy* (H2)*. Then there exist positive constants m, A_1, ..., A_n so that there exists a solution $u \in \mathcal{C}^1([0,m], \mathcal{C}^2([0,A_1] \times \cdots \times [0,A_n]))$ to the problem* (1.1).

Theorem 1.2. *Let $n \ge 2$ be fixed, f satisfy* (H1)*, u_0 and v satisfy* (H2)*. Then there exist positive constant m such that there exists a solution $u \in \mathcal{C}^1([0,m], \mathcal{C}^2(\mathbb{R}^n_{1+}))$ to the problem* (1.1).

Example 1.3. Consider the IBVP problem

$$u_t - \triangle u = -|u|^2 u, \quad t \ge 0, \quad x_1 \ge 0, \quad (x_2, \ldots, x_n) \in \mathbb{R}^{n-1},$$
$$u(0,x) = \frac{1}{\sqrt{2}}, \qquad x_1 \ge 0, \quad (x_2, \ldots, x_n) \in \mathbb{R}^{n-1},$$
$$u(t, 0, x_2, \ldots, x_n) = \frac{1}{\sqrt{2(t+1)}}, \quad t \ge 0, \quad (x_2, \ldots, x_n) \in \mathbb{R}^{n-1},$$

Then

$$u(t,x)=\frac{1}{\sqrt{2}}\frac{1}{\sqrt{t+1}}$$

is a $\mathcal{C}^1\left([0,1],\mathcal{C}^2(\mathbb{R}^n_{1+})\right)$-solution to the considered problem. Really, for $u(t,x)=\frac{1}{\sqrt{2}}\frac{1}{\sqrt{t+1}}$, we have

$$\begin{aligned}
&u_t=-\frac{1}{2\sqrt{2}(t+1)^{\frac{3}{2}}},\\
&u_t-\triangle u=-|u|^2u \iff\\
&-\frac{1}{2\sqrt{2}(t+1)^{\frac{3}{2}}}=-\frac{1}{2(t+1)}\frac{1}{\sqrt{2}\sqrt{t+1}}.
\end{aligned}$$

The paper is organized as follows. In Section 2 we prove Theorem 1.1. In Section 3 we prove Theorem 1.2.

2. Proof of Theorem 1.1

Let $\epsilon\in(0,1)$ be fixed and $G=\max_{i\in\{1,\dots,m_2\},j\in\{0,1,\dots,n\}}\sup_{\mathbb{R}^{n+1}}a_i^j(t,x)$. We choose the positive constants m, A_i, $i=1,2,\dots,n$, in the following way

$$\begin{aligned}
&\epsilon P+2\left(A_1A_2\dots A_n\right)^2P+2P\left(A_2\dots A_n\right)^2\left(1+A_1+\frac{A_1^2}{2}\right)m\\
&+2\textstyle\sum_{i=2}^n\left(A_1\dots A_{i-1}A_{i+1}\dots A_n\right)^2\left(1+A_i+\frac{A_i^2}{2}\right)Pm\\
&+\left(A_1\dots A_n\right)^2Gm\textstyle\sum_{i=1}^{m_2}\left(P^{p_i^0}+P^{p_i^1}+\dots+P^{p_i^n}\right)\le P,
\end{aligned}\tag{2.1}$$

$$\begin{aligned}
&\epsilon P+\left(A_1A_2\dots A_n\right)^2P+2P(A_2\dots A_n)^2\left(1+A_1+\frac{A_1^2}{2}\right)\\
&+2\textstyle\sum_{i=2}^n\left(A_1\dots A_{i-1}A_{i+1}\dots A_n\right)^2\left(1+A_i+\frac{A_i^2}{2}\right)P\\
&+\left(A_1\dots A_n\right)^2G\textstyle\sum_{i=1}^{m_2}\left(P^{p_i^0}+P^{p_i^1}+\dots+P^{p_i^n}\right)\le P,
\end{aligned}\tag{2.2}$$

$$\begin{aligned}
&\epsilon P+2A_1\left(A_2\dots A_n\right)^2P+2\left(A_2\dots A_n\right)^2\left(1+A_1+\frac{A_1^2}{2}\right)mP\\
&+2\textstyle\sum_{i=2}^n A_1\left(A_2\dots A_{i-1}A_{i+1}\dots A_n\right)^2\left(1+A_i+\frac{A_i^2}{2}\right)mP\\
&+A_1\left(A_2\dots A_n\right)^2Gm\textstyle\sum_{i=1}^{m_2}\left(P^{p_i^0}+P^{p_i^1}+\dots+P^{p_i^n}\right)\le P,
\end{aligned}\tag{2.3}$$

$$
\begin{aligned}
&\epsilon P + 2A_i \left(A_1 \dots A_{i-1}A_{i+1} \dots A_n\right)^2 P \\
&\quad +2\left(A_2 \dots A_{i-1}A_{i+1} \dots A_n\right)^2 \left(1 + A_1 + \tfrac{A_1^2}{2}\right) mPA_i \\
&\quad +2\textstyle\sum_{j=2, j\neq i}^{n} A_i \left(A_1 \dots A_{i-1}A_{i+1} \dots A_{j-1}A_{j+1} \dots A_n\right)^2 \left(1 + A_j + \tfrac{A_j^2}{2}\right) mP \\
&\quad +2(A_1 \dots A_{i-1}A_{i+1} \dots A_n)^2 \left(1 + A_i + \tfrac{A_i^2}{2}\right) mP \\
&\quad +A_i \left(A_1 \dots A_{i-1}A_{i+1} \dots A_n\right) mG \textstyle\sum_{i=1}^{m_2} \left(P^{p_i^0} + P^{p_i^1} + \dots + P^{p_i^n}\right) \leq P, \\
&\qquad i = 2, \dots, n,
\end{aligned} \tag{2.4}
$$

$$
\begin{aligned}
&\epsilon P + 2(A_2 \dots A_n)^2 P + (A_2 \dots A_n)^2 \left(1 + A_1 + \frac{A_1^2}{2}\right) mP \\
&\quad +2m \textstyle\sum_{i=2}^{n} (A_2 \dots A_{i-1}A_{i+1} \dots A_n)^2 \left(1 + A_i + \tfrac{A_i^2}{2}\right) P \\
&\quad +m(A_2 \dots A_n)^2 G \textstyle\sum_{i=1}^{m_2} \left(P^{p_i^0} + \dots + P^{p_i^n}\right) \leq P,
\end{aligned} \tag{2.5}
$$

$$
\begin{aligned}
&\epsilon P + 2(A_1 \dots A_{j-1}A_{j+1} \dots A_n)^2 P \\
&\quad +2(A_2 \dots A_{j-1}A_{j+1} \dots A_n)^2 \left(1 + A_1 + \tfrac{A_1^2}{2}\right) mP \\
&\quad +2\textstyle\sum_{i=2, i\neq j}^{n} (A_1 \dots A_{i-1}A_{i+1} \dots A_{j-1}A_{j+1} \dots A_n)^2 \left(1 + A_i + \tfrac{A_i^2}{2}\right) mP \\
&\quad +2(A_1 \dots A_{j-1}A_{j+1} \dots A_n)^2 \left(1 + A_j + \tfrac{A_j^2}{2}\right) mP \\
&\quad +(A_1 \dots A_{j-1}A_{j+1} \dots A_n)^2 mG \textstyle\sum_{i=1}^{m_2} \left(P^{p_i^0} + \dots + P^{p_i^n}\right) \leq P.
\end{aligned} \tag{2.6}
$$

Example 2.1. The constants $m = \sqrt[n]{1-\epsilon}$,

$$
A_i = \frac{\sqrt[2n]{1-\epsilon}}{n(1+P)(1+G)\left(1 + \sum_{i=1}^{m_2}\left(P^{p_i^0} + \dots + P^{p_i^n}\right)\right)\left(1 + \sum_{i=1}^{n} A_i + \frac{1}{2}\sum_{i=1}^{n} A_i^2\right)},
$$

$i \in \{1, \dots, n\}$, satisfy the conditions (2.1)-(2.6).

For fixed positive constants m, $A_1, A_2, \dots, A_n$ that satisfy (2.1)-(2.6) we denote the set

$$
B_1 = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : 0 \leq x_i \leq A_i, \quad i = 1, \dots, n\}.
$$

In this section we will prove that the IBVP

$$
\begin{aligned}
& u_t - \triangle u = f(t,x,u,Du), \quad t \in [0,m], \quad x \in B_1, \\
& u(0,x) = u_0(x), \quad x \in B_1, \\
& u(t,0,x_2,\dots,x_n) = v(t,x_2,\dots,x_n), \quad t \in [0,m], \quad 0 \le x_i \le A_i, \\
& i = 2,\dots,n,
\end{aligned}
\tag{2.7}
$$

has a solution $u \in \mathcal{C}^1([0,m],\mathcal{C}^2(B_1))$.
The main key of our proof is the following lemma.

Lemma 2.2. *Let $u \in \mathcal{C}^1([0,m],\mathcal{C}^2(B_1))$ satisfies the integral equation*

$$
\begin{aligned}
0 &= \textstyle\int_0^x \int_0^s u(t,\sigma)d\sigma ds - \int_0^x \int_0^s u_0(\sigma)d\sigma ds \\
&\quad - \textstyle\int_0^t \int_0^{\overline{x}_1} \int_0^{\overline{s}_1} \left(u(\tau,\tilde{\sigma}_1) - v(\tau,\overline{\sigma}_1)\right) d\overline{\sigma}_1 d\overline{s}_1 d\tau \\
&\quad - \textstyle\sum_{i=2}^n \int_0^t \int_0^{\overline{x}_i} \int_0^{\overline{s}_i} u(\tau,\tilde{\sigma}_i)d\overline{\sigma}_i d\overline{s}_i d\tau \\
&\quad - \textstyle\int_0^t \int_0^x \int_0^s f(\tau,\sigma,u,Du)d\sigma ds d\tau.
\end{aligned}
\tag{2.8}
$$

Then u satisfies the IBVP (2.7).

Here

$$
\begin{aligned}
\int_c^x &= \int_{c_1}^{x_1} \cdots \int_{c_n}^{x_n}, \\
\int_c^{\overline{x}_i} &= \int_{c_1}^{x_1} \cdots \int_{c_{i-1}}^{x_{i-1}} \int_{c_{i+1}}^{x_{i+1}} \cdots \int_{c_n}^{x_n}, \\
c &= (c_1,\dots,c_n), \\
\overline{s}_i &= (s_1,\dots,s_{i-1},s_{i+1},\dots,s_n), \\
\tilde{\sigma}_i &= (\sigma_1,\dots,\sigma_{i-1},x_i,\sigma_{i+1},\dots,\sigma_n), \\
d\sigma &= d\sigma_n d\sigma_{n-1}\dots d\sigma_1, \\
d\overline{\sigma}_i &= d\sigma_n \dots d\sigma_{i+1} d\sigma_{i-1} \dots d\sigma_1, \\
i &= 1,\dots,n.
\end{aligned}
$$

Proof. For $t \in [0,m]$, $x \in B_1$, after we differentiate once in t, then twice in x_1, twice in x_2 and etc., twice in x_n, the equation (2.8) and we obtain

$$u_t - \triangle u = f(t,x,u,Du).$$

Now we put $t = 0$ in the equation (2.8) and we obtain

$$\int_0^x \int_0^s (u(0,\sigma) - u_0(\sigma))\, d\sigma ds = 0,$$

which we differentiate twice in x_1, twice in x_2 and etc., twice in x_n, and we get

$$u(0,x) = u_0(x).$$

We put $x_1 = 0$ in the equation (2.8) and we obtain

$$\int_0^t \int_0^{x_2} \cdots \int_0^{x_n} (u(\tau, 0, s_2, \ldots, s_n) - v(\tau, s_2, \ldots, s_n))\, ds_n \ldots ds_2 d\tau = 0.$$

The last equality we differentiate once in t, twice in x_2, and etc., twice in x_n, and we find

$$u(t, 0, x_2, \ldots, x_n) = v(t, x_2, \ldots, x_n).$$

□

The above lemma motivate us to define the operator

$$\begin{aligned}
L_{11}(u)(t,x) &= u(t,x) + \int_0^x \int_0^s u(t,\sigma) d\sigma ds - \int_0^x \int_0^s u_0(\sigma) d\sigma ds \\
&- \int_0^t \int_0^{\overline{x}_1} \int_0^{\overline{s}_1} (u(\tau, \tilde{\sigma}_1) - v(\tau, \overline{\sigma}_1))\, d\overline{\sigma}_1 d\overline{s}_1 d\tau \\
&- \sum_{i=2}^n \int_0^t \int_0^{\overline{x}_i} \int_0^{\overline{s}_i} u(\tau, \tilde{\sigma}_i) d\overline{\sigma}_i d\overline{s}_i d\tau \\
&- \int_0^t \int_0^x \int_0^s f(\tau, \sigma, u, Du) d\sigma ds d\tau, \\
&(t,x) \in [0,m] \times B_1, \quad u \in \mathcal{C}^1([0,m], \mathcal{C}^2(B_1)),
\end{aligned}$$

Note that every fixed point of the operator L_{11} satisfies the equation (2.8) and from here it follows that every fixed point of the operator L_{11} is a solution to the IBVP (2.7).

To prove that the operator L_{11} has a fixed point we will use the following fixed point theorem.

Theorem 2.3 ([13], Corollary 2.4, pp. 3231)**.** *Let X be a nonempty closed convex subset of a Banach space Y. Suppose that T and S map X into Y such that*

1. *S is continuous, $S(X)$ resides in a compact subset of Y.*
2. *$T : X \longmapsto Y$ is expansive and onto.*

Then there exists a point $x^* \in X$ with $Sx^* + Tx^* = x^*$.

Here we will use the following definition for expansive operator.

Definition 2.4 ([13], pp. 3230)**.** Let (X, d) be a metric space and M be a subset of X. The mapping $T : M \longmapsto X$ is said to be expansive, if there exists a constant $h > 1$ such that

$$d(Tx, Ty) \geq h d(x, y) \quad \text{for} \quad \forall\, x, y \in M.$$

We represent the operator L_{11} in the following way

$$L_{11}(u)(t,x) = M_{11}(u)(t,x) + N_{11}(u)(t,x),$$

where

$$\begin{aligned}
M_{11}(u)(t,x) &= (1+\epsilon)u(t,x),\\
N_{11}(u)(t,x) &= -\epsilon u(t,x) + \int_0^x \int_0^s u(t,\sigma)d\sigma ds - \int_0^x \int_0^s u_0(\sigma)d\sigma ds\\
&\quad - \int_0^t \int_0^{\overline{x}_1} \int_0^{\overline{s}_1} \left(u(\tau,\tilde{\sigma}_1) - v(\tau,\overline{\sigma}_1)\right) d\overline{\sigma}_1 d\overline{s}_1 d\tau\\
&\quad - \sum_{i=2}^n \int_0^t \int_0^{\overline{x}_i} \int_0^{\overline{s}_i} u(\tau,\tilde{\sigma}_i) d\overline{\sigma}_i d\overline{s}_i d\tau\\
&\quad - \int_0^t \int_0^x \int_0^s f(\tau,\sigma,u,Du)d\sigma ds d\tau,\\
&\quad (t,x) \in [0,m]\times B_1, \quad u \in \mathcal{C}^1([0,m],\mathcal{C}^2(B_1)).
\end{aligned}$$

We define the sets

$$\begin{aligned}
K_{11} &= \Big\{u \in \mathcal{C}^1([0,m],\mathcal{C}^2(B_1)) : \max_{t\in[0,m],x\in B_1} |u(t,x)| \leq P,\\
&\quad \max_{t\in[0,m],x\in B_1} |u_{x_i}(t,x)| \leq P, \quad i = 0,1,\ldots,n,\\
&\quad \max_{t\in[0,m],x\in B_1} |u_{x_i x_i}(t,x)| \leq P, \quad i = 1,\ldots,n\Big\},\\
\tilde{K}_{11} &= \Big\{u \in \mathcal{C}^1([0,m],\mathcal{C}^2(B_1)) : \max_{t\in[0,m],x\in B_1} |u(t,x)| \leq (1+\epsilon)P,\\
&\quad \max_{t\in[0,m],x\in B_1} |u_{x_i}(t,x)| \leq (1+\epsilon)P, \quad i = 0,1,\ldots,n,\\
&\quad \max_{t\in[0,m],x\in B_1} |u_{x_i x_i}(t,x)| \leq (1+\epsilon)P, \quad i = 1,\ldots,n\Big\},
\end{aligned}$$

where $u_{x_0} = u_t$.

In these sets we define a norm as follows

$$\begin{aligned}
\|u\| &= \sup\Big\{\max_{t\in[0,m],x\in B_1} |u(t,x)|, \quad \max_{t\in[0,m],x\in B_1} |u_t(t,x)|,\\
&\quad \max_{t\in[0,m],x\in B_1} |u_{x_i}(t,x)|, \quad \max_{t\in[0,m],x\in B_1} |u_{x_i x_i}(t,x)|,\\
&\quad i \in \{1,\ldots,n\}\Big\}.
\end{aligned}$$

Here $D^\alpha u = \left(\partial_t^{\alpha_0}u, \partial_{x_1}^{\alpha_1}u, \ldots, \partial_{x_n}^{\alpha_n}u\right)$. The sets K_{11} and $\tilde{K}_{11}$ are completely normed spaces with respect to this norm. Let K_{11}^1 and $\tilde{K}_{11}^1$ denote the sets of all equi-continuous families in K_{11} and $\tilde{K}_{11}$, respectively. Let also, $D_{11} = \overline{K_{11}^1}$ and $\tilde{D}_{11} = \overline{\tilde{K}_{11}^1}$, where $\overline{K_{11}^1}$ and $\overline{\tilde{K}_{11}^1}$are the closures of K_{11}^1 and $\tilde{K}_{11}^1$, respectively. Note that that D_{11} and $\tilde{D}_{11}$ are compact sets in $\mathcal{C}^1([0,m],\mathcal{C}^2(B_1))$.

Lemma 2.5. *The operator $M_{11} : D_{11} \longmapsto \tilde{D}_{11}$ is an expansive operator and onto.*

Proof. Let $u \in D_{11}$. Then $u \in \mathcal{C}^1([0,m],\mathcal{C}^2(B_1))$. From here, $(1+\epsilon)u \in \mathcal{C}^1([0,m],\mathcal{C}^2(B_1))$ and $M_{11}(u) \in \mathcal{C}^1([0,m],\mathcal{C}^2(B_1))$. Also,

$$\max_{t\in[0,m],x\in B_1} |u(t,x)| \le P,$$

$$\max_{t\in[0,m],x\in B_1} |u_{x_i}(t,x)| \le P, \quad i = 0,1,\dots,n,$$

$$\max_{t\in[0,m],x\in B_1} |u_{x_i x_i}(t,x)| \le P, \quad i = 1,\dots,n.$$

Therefore

$$\max_{t\in[0,m],x\in B_1} |M_{11}(u)(t,x)| \le (1+\epsilon)P,$$

$$\max_{t\in[0,m],x\in B_1} |M_{11}(u)_{x_i}(t,x)| \le (1+\epsilon)P, \quad i = 0,1,\dots,n,$$

$$\max_{t\in[0,m],x\in B_1} |M_{11}(u)_{x_i x_i}(t,x)| \le (1+\epsilon)P, \quad i = 1,\dots,n.$$

Consequently

$$M_{11} : D_{11} \longmapsto \tilde{D}_{11}.$$

For $u, v \in D_{11}$ we have that

$$\|M_{11}(u) - M_{11}(v)\| = (1+\epsilon)\|u - v\|,$$

from where it follows that the operator M_{11} is an expansive operator with constant $1+\epsilon$. Also, for $v \in \tilde{D}_{11}$, we take $u = \frac{v}{1+\epsilon}$. Then $|u(t,x)| \le P$,$|u_{x_i}(t,x)| \le P$, $|u_{x_j x_j}(t,x)| \le P$, $i = 0,1,\dots,n$, $j = 1,\dots,n$, for every $t \in [0,m]$ and $x \in B_1$, i.e., $u \in D_{11}$ and $M_{11}(u) = v$. Consequently $M_{11} : D_{11} \longmapsto \tilde{D}_{11}$ is onto. □

Lemma 2.6. *The operator*

$$N_{11} : D_{11} \longmapsto D_{11}$$

is continuous.

Proof. First, we will prove that

$$N_{11} : D_{11} \longmapsto D_{11}$$

Let $u \in D_{11}$.

1.

$$\begin{aligned}
|N_{11}(u)(t,x)| \le\; & \epsilon|u(t,x)| + \textstyle\int_0^x \int_0^s |u(t,\sigma)|d\sigma ds + \int_0^x \int_0^s |u_0(\sigma)|d\sigma ds \\
& + \textstyle\int_0^t \int_0^{\overline{x}_1} \int_0^{\overline{s}_1} \left(|u(\tau,\tilde{\sigma}_1)| + |v(\tau,\overline{\sigma}_1)|\right) d\overline{\sigma}_1 d\overline{s}_1 d\tau \\
& + \textstyle\sum_{i=2}^n \int_0^t \int_0^{\overline{x}_i} \int_0^{\overline{s}_i} |u(\tau,\tilde{\sigma}_i)| d\overline{\sigma}_i d\overline{s}_i d\tau \\
& + \textstyle\int_0^t \int_0^x \int_0^s \sum_{i=1}^{m_2} \Big(a_i^0(\tau,\sigma)|u(\tau,\sigma)|^{p_i^0} + a_i^1(\tau,\sigma)|u_{x_1}(\tau,\sigma)|^{p_i^1} \\
& + \dots + a_i^n(\tau,\sigma)|u_{x_n}(\tau,\sigma)|^{p_i^n} \Big) d\sigma ds d\tau \\
\le\; & \epsilon P + 2(A_1 \dots A_n)^2 P + 2P(A_2 \dots A_n)^2 m \\
& + \textstyle\sum_{i=2}^n (A_1 \dots A_{i-1} A_{i+1} \dots A_n)^2 P m \\
& + (A_1 \dots A_n)^2 G m \textstyle\sum_{i=1}^{m_2} \left(P^{p_i^0} + P^{p_i^1} + \dots + P^{p_i^n} \right) \\
\le\; & P, \quad (t,x) \in [0,m] \times B_1.
\end{aligned}$$

In the last inequality we have used the conditions (H1) for the function f and the inequality (2.1).

2.

$$\begin{aligned} |(N_{11}(u))_t(t,x)| &\leq \epsilon|u_t(t,x)| + \int_0^x\int_0^s |u_t(t,\sigma)|d\sigma ds \\ &+ \int_0^{\overline{x}_1}\int_0^{\overline{s}_1}\left(|u(t,\tilde{\sigma}_1)| + |v(t,\overline{\sigma}_1)|\right)d\overline{\sigma}_1 d\overline{s}_1 \\ &+ \sum_{i=2}^n \int_0^{\overline{x}_i}\int_0^{\overline{s}_i}|u(t,\tilde{\sigma}_i)|d\overline{\sigma}_i d\overline{s}_i \\ &+ \int_0^x\int_0^s \sum_{i=1}^{m_2}\Big(a_i^0(t,\sigma)|u(t,\sigma|^{p_i^0} + a_i^1(t,\sigma)|u_{x_1}(t,\sigma)|^{p_i^1} \\ &+ \cdots + a_i^n(t,\sigma)|u_{x_n}(t,\sigma)|^{p_i^n}\Big)d\sigma ds \\ &\leq \epsilon P + (A_1\dots A_n)^2 P + 2(A_2\dots A_n)^2 P \\ &+ \sum_{i=2}^n (A_1\dots A_{i-1}A_{i+1}\dots A_n)^2 P \\ &+ (A_1\dots A_n)^2 G \sum_{i=1}^{m_2}\left(P^{p_i^0} + P^{p_i^1} + \cdots + P^{p_i^n}\right) \\ &\leq P, \quad (t,x)\in[0,m]\times B_1. \end{aligned}$$

In the last inequality we have used the inequality (2.2) and the conditions (H1) for the function f.

3.

$$\begin{aligned} |N_{11}(u)_{x_1}(t,x)| &\leq \epsilon|u_{x_1}(t,x)| \\ &+ \int_0^{x_2}\cdots\int_0^{x_n}\int_0^{x_1}\int_0^{s_2}\cdots\int_0^{s_n}|u(t,\sigma)|d\sigma d\overline{s}_1 \\ &+ \int_0^{x_2}\cdots\int_0^{x_n}\int_0^{x_1}\int_0^{s_2}\cdots\int_0^{s_n}|u_0(\sigma)|d\sigma d\overline{s}_1 \\ &+ \int_0^t\int_0^{\overline{x}_1}\int_0^{\overline{s}_1}|u_{x_1}(\tau,\tilde{\sigma}_1)|d\overline{\sigma}_1 d\overline{s}_1 d\tau \\ &+ \sum_{i=2}^n \int_0^t\int_0^{x_2}\cdots\int_0^{x_{i-1}}\int_0^{x_{i+1}}\cdots\int_0^{x_n}\int_0^{x_1}\int_0^{s_2} \\ &\cdots\int_0^{s_{i-1}}\int_0^{s_{i+1}}\cdots\int_0^{s_n}|u(\tau,\tilde{\sigma}_i)|d\overline{\sigma}_i ds_n\cdots ds_{i+1}ds_{i-1}\cdots ds_2 d\tau \\ &+ \int_0^t\int_0^{x_2}\cdots\int_0^{x_n}\int_0^{x_1}\int_0^{s_2}\cdots\int_0^{s_n}\sum_{i=1}^{m_2}\Big(a_i^0(\tau,\sigma)|u(\tau,\sigma)|^{p_i^0} \\ &+ a_i^1(\tau,\sigma)|u_{x_1}(\tau,\sigma)|^{p_i^1} + \cdots + a_i^n(\tau,\sigma)|u_{x_n}(\tau,\sigma)|^{p_i^n}\Big)d\sigma d\overline{s}_1 d\tau \\ &\leq \epsilon P + 2A_1(A_2\dots A_n)^2 P + (A_2\dots A_n)^2 Pm \\ &+ \sum_{i=2}^n A_1(A_2\dots A_{i-1}A_{i+1}\dots A_n)^2 Pm \\ &+ A_1(A_2\dots A_n)^2 Gm\sum_{i=1}^{m_2}\left(P^{p_i^0} + P^{p_i^1} + \cdots + P^{p_i^n}\right) \\ &\leq P, \quad (t,x)\in[0,m]\times B_1. \end{aligned}$$

In the last inequality we have used the conditions (H1) for the function f and the inequality (2.3).

4. For $j = 2, \ldots, n$, we have

$$|N_{11}(u)_{x_j}| \quad \leq \epsilon |u_{x_j}(t,x)| + \int_0^{\overline{x}_j} \int_0^{s_1} \cdots \int_0^{s_{j-1}} \int_0^{x_j} \int_0^{s_{j+1}} \cdots \int_0^{s_n} |u(t,\sigma)| d\sigma d\overline{s_j}$$

$$+ \int_0^{\overline{x}_j} \int_0^{s_1} \cdots \int_0^{s_{j-1}} \int_0^{x_j} \int_0^{s_{j+1}} \cdots \int_0^{s_n} |u_0(\sigma)| d\sigma d\overline{s_j}$$

$$+ \int_0^t \int_0^{x_2} \cdots \int_0^{x_{j-1}} \int_0^{x_{j+1}} \cdots \int_0^{x_n} \int_0^{s_2} \cdots \int_0^{s_{j-1}} \int_0^{x_j} \int_0^{s_{j+1}} \cdots \int_0^{s_n}$$

$$\Big(|u(\tau, x_1, \sigma_2, \ldots, \sigma_{j-1}, \sigma_j, \sigma_{j+1}, \ldots, \sigma_n)| + |v(\tau, \sigma_2, \ldots, \sigma_{j-1}, \sigma_j, \sigma_{j+1}, \ldots, \sigma_n)| \Big)$$

$$d\overline{\sigma}_1 ds_n \ldots ds_{j+1} ds_{j-1} \ldots ds_2 d\tau$$

$$+ \sum_{i=2, i\neq j}^{n} \int_0^t \int_0^{x_1} \cdots \int_0^{x_{i-1}} \int_0^{x_{i+1}} \cdots \int_0^{x_{j-1}} \int_0^{x_{j+1}} \cdots \int_0^{x_n}$$

$$\int_0^{s_1} \cdots \int_0^{s_{i-1}} \int_0^{s_{i+1}} \cdots \int_0^{s_{j-1}} \int_0^{x_j} \int_0^{s_{j+1}} \cdots \int_0^{s_n}$$

$$|u(\tau, \tilde{\sigma}_i)| d\overline{\sigma}_i ds_n \ldots ds_{j+1} ds_{j-1} \ldots ds_{i+1} ds_{i-1} \ldots ds_1 d\tau + \int_0^t \int_0^{\overline{x}_j} \int_0^{\overline{s}_j} |u_{x_j}(\tau, \tilde{\sigma}_j)| d\overline{\sigma}_j d\overline{s}_j d\tau$$

$$+ \int_0^t \int_0^{\overline{x}_j} \int_0^{s_1} \cdots \int_0^{s_{j-1}} \int_0^{x_j} \int_0^{s_{j+1}} \cdots \int_0^{s_n} |f(\tau, \sigma, u, Du)| d\sigma d\overline{s}_j d\tau$$

$$\leq \epsilon P + 2A_j (A_1 \ldots A_{j-1} A_{j+1} \ldots A_n)^2 P + 2(A_2 \ldots A_{j-1} A_{j+1} \ldots A_n)^2 mP$$

$$+ \sum_{i=2, i\neq j}^{n} (A_1 \ldots A_{i-1} A_{i+1} \ldots A_{j-1} A_{j+1} \ldots A_n)^2 A_j mP + (A_1 \ldots A_{j-1} A_{j+1} \ldots A_n)^2 mP$$

$$+ (A_1 \ldots A_{j-1} A_{j+1} \ldots A_n)^2 A_j mG \sum_{i=1}^{m_2} \Big(P^{p_i^0} + \cdots + P^{p_i^n} \Big) \leq P, \quad (t,x) \in [0,m] \times B_1.$$

In the last inequality we have used the inequality (2.4) and the conditions (H1) for the function f.

5.

$$|N_{11}(u)_{x_1 x_1}(t,x)| \quad \leq \epsilon |u_{x_1 x_1}(t,x)| + \int_0^{\overline{x}_1} \int_0^{\overline{s}_1} |u(t, \tilde{\sigma}_1)| d\overline{\sigma}_1 d\overline{s}_1 + \int_0^{\overline{x}_1} \int_0^{\overline{s}_1} |u_0(\tilde{\sigma}_1)| d\overline{\sigma}_1 d\overline{s}_1$$

$$+ \int_0^t \int_0^{\overline{x}_1} \int_0^{\overline{s}_1} |u_{x_1 x_1}(\tau, \tilde{\sigma}_1) d\overline{\sigma}_1 d\overline{s}_1 d\tau + \sum_{i=2}^{n} \int_0^t \int_0^{x_2} \cdots \int_0^{x_{i-1}} \int_0^{x_{i+1}} \cdots \int_0^{x_n} \int_0^{s_2} \cdots \int_0^{s_{i-1}} \int_0^{s_{i+1}} \cdots \int_0^{s_n}$$

$$|u(\tau, x_1, \sigma_2, \ldots, \sigma_{i-1}, x_i, \sigma_{i+1}, \ldots, \sigma_n)| d\sigma_n \ldots d\sigma_{i+1} d\sigma_{i-1} \ldots d\sigma_2 ds_n \ldots ds_{i+1} ds_{i-1} \ldots ds_2 d\tau$$

$$+ \int_0^t \int_0^{\overline{x}_1} \int_0^{\overline{s}_1} |f(\tau, \tilde{\sigma}_1, u, Du)| d\overline{\sigma}_1 d\overline{s}_1 d\tau$$

$$\leq \epsilon P + 2(A_2 \ldots A_n)^2 P + (A_2 \ldots A_n)^2 mP + m \sum_{i=2}^{n} (A_2 \ldots A_{i-1} A_{i+1} \ldots A_n)^2 P$$

$$+m(A_2 \ldots A_n)^2 G \sum_{i=1}^{m_2} \left(P^{p_i^0} + \cdots + P^{p_i^n} \right) \leq P, \quad (t,x) \in [0,m] \times B_1.$$

In the last inequality we have used the inequality (2.5) and the conditions (H1) for the function f.

6. For $j = 2, \ldots, n$, we have

$$\begin{aligned}
|N_{11}(u)_{x_j x_j}| &\leq \epsilon |u_{x_j x_j}(t,x)| + \int_0^{\overline{x}_j} \int_0^{\overline{s}_j} |u(t, \tilde{\sigma}_j)| d\overline{\sigma}_j d\overline{s_j} \\
&+ \textstyle\int_0^{\overline{x}_j} \int_0^{\overline{s}_j} |u_0(\tilde{\sigma}_j)| d\overline{\sigma}_j d\overline{s}_j \\
&+ \textstyle\int_0^t \int_0^{x_2} \ldots \int_0^{x_{j-1}} \int_0^{x_{j+1}} \ldots \int_0^{x_n} \\
&\textstyle\int_0^{s_2} \ldots \int_0^{s_{j-1}} \int_0^{s_{j+1}} \ldots \int_0^{s_n} \\
&\Big(|u(\tau, x_1, \sigma_2, \ldots, \sigma_{j_1}, x_j, \sigma_{j+1}, \ldots, \sigma_n)| \\
&+ |v(\tau, \sigma_2, \ldots, \sigma_{j-1}, x_j, \sigma_{j+1}, \ldots, \sigma_n)| \Big) \\
&d\sigma_n \ldots d\sigma_{j+1} d\sigma_{j-1} \ldots d\sigma_2 ds_n \ldots ds_{j+1} ds_{j-1} \ldots, ds_2 d\tau \\
&+ \textstyle\sum_{i=2, i \neq j}^{n} \int_0^t \int_0^{x_1} \ldots \int_0^{x_{i-1}} \int_0^{x_{i+1}} \ldots \int_0^{x_{j-1}} \int_0^{x_{j+1}} \ldots \int_0^{x_n} \\
&\textstyle\int_0^{s_1} \ldots \int_0^{s_{i-1}} \int_0^{s_{i+1}} \ldots \int_0^{s_{j-1}} \int_0^{s_{j+1}} \ldots \int_0^{s_n} \\
&|u(\tau, \sigma_1, \ldots, \sigma_{i-1}, x_i, \sigma_{i+1}, \ldots, \sigma_{j-1}, x_j, \sigma_{j+1}, \ldots, \sigma_n)| \\
&d\sigma_n \ldots d\sigma_{j+1} d\sigma_{j-1} \ldots d\sigma_{i+1} d\sigma_{i-1} \ldots d\sigma_1 \\
&ds_n \ldots ds_{j+1} ds_{j-1} \ldots ds_{i+1} ds_{i-1} \ldots ds_1 d\tau \\
&+ \textstyle\int_0^t \int_0^{\overline{x}_j} \int_0^{\overline{s}_j} |u_{x_j x_j}(\tau, \tilde{\sigma}_j) d\overline{\sigma}_j d\overline{s}_j d\tau \\
&+ \textstyle\int_0^t \int_0^{\overline{x}_j} \int_0^{\overline{s}_j} |f(\tau, \tilde{\sigma}_j, u, Du)| d\overline{\sigma}_j d\overline{s}_j d\tau \\
\leq \quad & \epsilon P + 2(A_1 \ldots A_{j-1} A_{j+1} \ldots A_n)^2 P \\
&+2(A_2 \ldots A_{j-1} A_{j+1} \ldots A_n)^2 mP \\
&+ \textstyle\sum_{i=2, i \neq j}^{n} (A_1 \ldots A_{i-1} A_{i+1} \ldots A_{j-1} A_{j+1} \ldots A_n)^2 mP \\
&+(A_1 \ldots A_{j-1} A_{j+1} \ldots A_n)^2 mP \\
&+(A_1 \ldots A_{j-1} A_{j+1} \ldots A_n)^2 mG \textstyle\sum_{i=1}^{m_2} \left(P^{p_i^0} + \cdots + P^{p_i^n} \right) \\
\leq \quad & P, \quad (t,x) \in [0,m] \times B_1.
\end{aligned}$$

In the last inequality we have used the inequality (2.6) and the conditions (H1) for the function f.

Consequently

$$N_{11} : D_{11} \longmapsto D_{11}.$$

From the above estimates it follows that if $u_n \to u$, $u_n \in D_{11}$, $u \in D_{11}$, as $n \to \infty$, in the sense of the topology in D_{11}, since f is a continuous function of its arguments, we have $N_{11}(u_n) \to N_{11}(u)$, as $n \to \infty$, in the sense of the topology of the space D_{11}. Therefore $N_{11} : D_{11} \to D_{11}$ is a continuous operator. □

From Lemma 2.2, Theorem 2.3, Lemma 2.5 and Lemma 2.6 it follows that the IBVP (2.7) has a solution $u \in \mathcal{C}^1([0, m], \mathcal{C}^2(B_1))$.

3. Proof of Theorem 1.2

Let u^{11} is the solution which is obtained in the previous section.

Now we define the set

$$B_2 = \{x \in \mathbb{R}^n : A_1 \le x_1 \le 2A_1, \quad 0 \le x_i \le A_i, \quad i = 2, \ldots, n\},$$

the operators

$$\begin{aligned}
L_{12}(u)(t,x) &= u(t,x) + \textstyle\int_A^x \int_A^s u(t,\sigma) d\sigma ds - \int_A^x \int_A^s u_0(\sigma) d\sigma ds \\
&\quad - \textstyle\int_0^t \int_A^{\overline{x}_1} \int_A^{\overline{s}_1} \left(u(\tau,\tilde{\sigma}_1) - \tilde{u}^{11}(\tau,\tilde{\sigma}_1)\right) d\overline{\sigma}_1 d\overline{s}_1 d\tau \\
&\quad - \textstyle\sum_{i=2}^n \int_0^t \int_A^{\overline{x}_i} \int_A^{\overline{s}_i} u(\tau,\tilde{\sigma}_i) d\overline{\sigma}_i d\overline{s}_i d\tau - \int_0^t \int_A^x \int_A^s f(\tau,\sigma,u,Du) d\sigma ds d\tau, \\
M_{12}(u)(t,x) &= (1+\epsilon) u(t,x), \\
N_{12}(u)(t,x) &= -\epsilon u(t,x) + \textstyle\int_A^x \int_A^s u(t,\sigma) d\sigma ds - \int_A^x \int_A^s u_0(\sigma) d\sigma ds \\
&\quad - \textstyle\int_0^t \int_A^{\overline{x}_1} \int_A^{\overline{s}_1} \left(u(\tau,\tilde{\sigma}_1) - \tilde{u}^{11}(\tau,\tilde{\sigma}_1)\right) d\overline{\sigma}_1 d\overline{s}_1 d\tau \\
&\quad - \textstyle\sum_{i=2}^n \int_0^t \int_A^{\overline{x}_i} \int_A^{\overline{s}_i} u(\tau,\tilde{\sigma}_i) d\overline{\sigma}_i d\overline{s}_i d\tau - \int_0^t \int_A^x \int_A^s f(\tau,\sigma,u,Du) d\sigma ds d\tau, \\
L_{12}(u)(t,x) &= M_{12}(u)(t,x) + N_{12}(u)(t,x), (t,x) \in [0,m] \times B_2, \quad u \in \mathcal{C}^1([0,m],\mathcal{C}^2(B_2)), \\
A &= (A_1, 0, \ldots, 0), \\
\tilde{u}^{11}(t,x) &= u^{11}(t, A_1, x_2, \ldots, x_n) + (x_1 - A_1) u^{11}_{x_1}(t, A_1, x_2, \ldots, x_n)
\end{aligned}$$

the sets

$$\begin{aligned}
K_{12} &= \Big\{ u \in \mathcal{C}^1([0,m],\mathcal{C}^2(B_2)) : \max\nolimits_{t\in[0,m], x\in B_2} |u(t,x)| \le P, \\
&\quad \max\nolimits_{t\in[0,m], x\in B_2} |u_{x_i}(t,x)| \le P, \quad i = 0, 1, \ldots, n, \\
&\quad \max\nolimits_{t\in[0,m], x\in B_2} |u_{x_i x_i}(t,x)| \le P, \quad i = 1, \ldots, n \Big\}, \\
\tilde{K}_{12} &= \Big\{ u \in \mathcal{C}^1([0,m],\mathcal{C}^2(B_2)) : \max\nolimits_{t\in[0,m], x\in B_2} |u(t,x)| \le (1+\epsilon)P, \\
&\quad \max\nolimits_{t\in[0,m], x\in B_2} |u_{x_i}(t,x)| \le (1+\epsilon)P, \quad i = 0, 1, \ldots, n, \\
&\quad \max\nolimits_{t\in[0,m], x\in B_2} |u_{x_i x_i}(t,x)| \le (1+\epsilon)P, \quad i = 1, \ldots, n \Big\},
\end{aligned}$$

which are completely normed spaces with the norm

$$\begin{aligned} \|u\| &= \sup\Big\{\max_{t\in[0,m],x\in B_2}|u(t,x)|, \quad \max_{t\in[0,m],x\in B_2}|u_t(t,x)|, \\ & \max_{t\in[0,m],x\in B_2}|u_{x_i}(t,x)|, \quad \max_{t\in[0,m],x\in B_2}|u_{x_ix_i}(t,x)|, \\ & i\in\{1,\dots,n\}\Big\}. \end{aligned}$$

Let K_{12}^1 and $\tilde{K}_{12}^1$ denote the sets of all equi-continuous families in K_{12} and $\tilde{K}_{12}$, respectively. Let also, $D_{12}=\overline{K_{12}^1}$ and $\tilde{D}_{12}=\overline{\tilde{K}_{12}^1}$, where $\overline{K_{12}^1}$ and $\overline{\tilde{K}_{12}^1}$are the closures of K_{12}^1 and $\tilde{K}_{12}^1$, respectively. Note that that D_{12} and $\tilde{D}_{12}$ compact sets in $\mathcal{C}^1([0,m],\mathcal{C}^2(B_2))$. As in the previous section we prove that the IBVP

$$\begin{aligned} & u_t-\triangle u=f(t,x,u,Du), \quad t\in[0,m], \quad x\in B_2, \\ & u(0,x)=u_0(x), \quad x\in B_2, \\ & u(t,A_1,x_2,\dots,x_n)=u^{11}(t,A_1,x_2,\dots,x_n), \quad t\in[0,m], \\ & 0\le x_i\le A_i, \quad i=2,\dots,n, \end{aligned}$$

has a solution $u^{12}\in\mathcal{C}^1([0,m],\mathcal{C}^2(B_2))$. For it we have

$$\begin{aligned} & \int_A^x\int_A^s u^{12}(t,\sigma)d\sigma ds-\int_A^x\int_A^s u_0(\sigma)d\sigma ds \\ & -\int_0^t\int_A^{\overline{x}_1}\int_A^{\overline{s}_1}\left(u^{12}(\tau,\tilde{\sigma}_1)-\tilde{u}^{11}(\tau,\tilde{\sigma}_1)\right)d\overline{\sigma}_1d\overline{s}_1d\tau \\ & -\sum_{i=2}^n\int_0^t\int_A^{\overline{x}_i}\int_A^{\overline{s}_i}u^{12}(\tau,\tilde{\sigma}_id\overline{\sigma}_id\overline{s}_id\tau \\ & -\int_0^t\int_A^x\int_A^s f(\tau,\sigma,u^{12},Du^{12})d\sigma dsd\tau=0. \end{aligned} \tag{3.1}$$

We put $x_1=A_1$ in the last equality and we obtain

$$\int_0^t\int_A^{\overline{x}_1}\int_A^{\overline{s}_1}\left(u^{12}(\tau,A_1,\sigma_2,\dots,\sigma_n)-u^{11}(\tau,A_1,\sigma_2,\dots,\sigma_n)\right)d\overline{\sigma}_1d\overline{s}_1d\tau=0.$$

We differentiate the last equality once in t, twice in x_2, and etc., twice in x_n, we obtain

$$u^{12}(t,A_1,x_2,\dots,x_n)=u^{11}(t,A_1,x_2,\dots,x_n), \quad t\in[0,m], \quad 0\le x_i\le A_i,$$

$i\in\{2,\dots,n\}$, from here,

$$u_t^{12}(t,A_1,x_2,\dots,x_n)=u_t^{11}(t,A_1,x_2,\dots,x_n), \quad t\in[0,m], \quad 0\le x_i\le A_i, \tag{3.1}$$

$i\in\{2,\dots,n\}$.

Now we differentiate the equality (3.1) in x_1, then we put $x_1 = A_1$, we get

$$\int_0^t \int_A^{\overline{x}_1} \int_A^{\overline{s}_1} \left(u^{12}_{x_1}(\tau, \sigma_2, \dots, \sigma_n) - u^{11}_{x_1}(\tau, A_1, \sigma_2, \dots, \sigma_n)\right) d\overline{\sigma}_1 d\overline{s}_1 d\tau = 0.$$

The last equality we differentiate once in t, twice in x_2, and etc., twice in x_n, we get

$$u^{12}_{x_1}(t, A_1, x_2, \dots, x_n) = u^{11}_{x_1}(t, A_1, x_2, \dots, x_n), \quad t \in [0, m], \quad 0 \le x_i \le A_i, \tag{3.2}$$

$i \in \{2, \dots, n\}$.
Now we differentiate (3.1) twice in x_i, $i \in \{2, \dots, n\}$, then we put $x_1 = A_1$, we get

$$\int_0^t \int_0^{x_2} \cdots \int_0^{x_{i-1}} \int_0^{x_{i+1}} \cdots \int_0^{x_n} \int_0^{s_2} \cdots \int_0^{s_{i-1}} \int_0^{s_{i+1}} \cdots \int_0^{s_n}$$

$$\Big(u^{12}(\tau, A_1, \sigma_2, \dots, \sigma_{i-1}, x_i, \sigma_{i+1}, \dots, \sigma_n)$$

$$-u^{11}(\tau, A_1, \sigma_2, \dots, \sigma_{i-1}, x_i, \sigma_{i+1}, \dots, \sigma_n)\Big)$$

$$d\sigma_n \dots d\sigma_{i+1} d\sigma_{i-1} \dots d\sigma_2$$

$$ds_n \dots ds_{i+1} ds_{i-1} \dots ds_2 = 0,$$

which we differentiate once in t, twice in x_2, and etc., twice in x_{i-1}, twice in x_{i+1}, and etc, twice in x_n, we get

$$u^{12}(t, A_1, x_2, \dots, x_n) = u^{11}(t, A_1, x_2, \dots, x_n), \quad t \in [0, m],$$

$0 \le x_i \le A_i$, $i \in \{2, \dots, n\}$. Hence,

$$u^{12}_{x_i}(t, A_1, x_2, \dots, x_n) = u^{11}_{x_i}(t, A_1, x_2, \dots, x_n), \quad t \in [0, m], \tag{3.3}$$

$0 \le x_i \le A_i$, $i \in \{2, \dots, n\}$, and

$$u^{12}_{x_i x_i}(t, A_1, x_2, \dots, x_n) = u^{11}_{x_i x_i}(t, A_1, x_2, \dots, x_n), \quad t \in [0, m], \tag{3.4}$$

$0 \le x_i \le A_i$, $i \in \{2, \dots, n\}$.
From (3.2) and (3.3) we get

$$\begin{aligned} & f\left(t, A_1, x_2, \dots, x_n, u^{11}(t, A_1, x_2, \dots, x_n), Du^{11}(t, A_1, x_2, \dots, x_n)\right) \\ = \; & f\left(t, A_1, x_2, \dots, x_n, u^{12}(t, A_1, x_2, \dots, x_n), Du^{12}(t, A_1, x_2, \dots, x_n)\right), \end{aligned}$$

$t \in [0, m]$, $0 \le x_i \le A_i$, $i \in \{2, \dots, n\}$. Hence, (3.1) and (3.4) we obtain

$$u^{12}_{x_1 x_1}(t, A_1, x_2, \dots, x_n) = u^{11}_{x_1 x_1}(t, A_1, x_2, \dots, x_n), \quad t \in [0, m],$$

$0 \le x_i \le A_i$, $i \in \{2, \dots, n\}$.
In this way we obtain that the function

$$u = \begin{cases} u^{11} & t \in [0, m], \quad x \in B_1, \\ u^{12} & t \in [0, m], \quad x \in B_2, \end{cases}$$

is a solution to the IBVP

$$
\begin{aligned}
&u_t - \triangle u = f(t, x, u, Du), \quad t \in [0, m], \quad x \in B_1 \cup B_2 \\
&u(0, x) = u_0(x), \quad x \in B_1 \cup B_2, \\
&u(t, 0, x_2, \ldots, x_n) = v(t, x_2, \ldots, x_n), \quad t \in [0, m], \quad 0 \le x_i \le A_i, i = 2, \ldots, n.
\end{aligned}
$$

Repeat the above steps in x_1, x_2, and etc., in x_n we obtain that the IBVP

$$
\begin{aligned}
&u_t - \triangle u = f(t, x, u, Du), \quad t \in [0, m], \quad x \in \mathbb{R}^n_{1+}, \\
&u(0, x) = u_0(x), \quad x \in \mathbb{R}^n_{1+}, \\
&u(t, 0, x_2, \ldots, x_n) = v(t, x_2, \ldots, x_n), \quad t \in [0, m], \quad (x_2, \ldots, x_n) \in \mathbb{R}^{n-1},
\end{aligned}
$$

has a solution $u^1 \in \mathcal{C}^1([0, m], \mathcal{C}^2(\mathbb{R}^n_{1+}))$.

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