ESKİŞEHİR TEKNİK ÜNİVERSİTESİ BİLİM VE TEKNOLOJİ DERGİSİ
B- TEORİK BİLİMLER

Eskişehir Technical University Journal of Science and Technology B- Theoritical Sciences

2018, 6(2), pp. 185 - 192, DOI: 10.20290/aubtdb.375635

EXPLICIT DETERMINATION OF GROMOV-PRODUCT TYPES OF FIVE-POINT METRIC SPACES

Derya ÇELİK *

[1] Mathematics, Science Faculty, Eskişehir Technical University, Eskişehir, Turkey

ABSTRACT

We determine the Gromov-product types of five-point metric spaces in terms of optimal realization types.

Keywords: Finite metric spaces, Gromov-products, Optimal realizations

1. INTRODUCTION

Finite metric spaces can be classified from different points of view. An important classification is based on optimal realization types ([1], [2]) and recently another classification based on Gromov-products was proposed in [3]. For $n = 3$ and $n = 4$ both classifications coincide and in [3] it was remarked that they coincide also in the first non-trivial case $n = 5$. In this note we give an explicit verification of this fact. By the way, it is studied the Gromov-product decomposition of seven-point metric spaces in [4].

Let (X, d) be a finite metric space with n elements x_i, $i = 0, \dots, n-1$ $(n \geq 3)$ and let the triple (x_i, x_j, x_k) be a "triangle" with vertices x_i, x_j and x_k. If some of the indices i, j, k coincide, then we view it as a degenerate triangle. Then the Gromov-product of the triangle (x_i, x_j, x_k) at the vertex x_i is defined in [3] as

$$\Delta_{ijk} = \frac{1}{2}(d_{ij} + d_{ik} - d_{jk})$$

where $d_{ij} = d(x_i, x_j)$. For a generic five-point metric space, it is proved in [3] there are three Gromov-product equivalence classes. There are also three optimal realization types for a five-point metric space ([1], [2]). We will give below the explicit correspondence between these equivalence classes.

We recall that a split $S = A \mid B$ is defined to be a partition of X into disjoint, nonempty subsets A and B. The split (pseudo-)metric $\delta_{A|B}$ associated with any such split is defined by

$$\delta_{A|B}(x, y) = \begin{cases} 0 & \text{if } x, y \in A \text{ or } x, y \in B \\ 1 & \text{else.} \end{cases}$$

For a split $S = A \mid B$, the isolation index $\alpha_{A|B}$ is defined as

$$\alpha_{A|B} = \frac{1}{2}\min\left\{\max\left\{\begin{matrix} d(a,b) + d(a',b'), \\ d(a',b) + d(a,b'), \\ d(a,a') + d(b,b') \end{matrix}\right\} - d(a,a') - d(b,b') \mid a, a' \in A, b, b' \in B\right\}.$$

*Corresponding Author: deryacelik@anadolu.edu.tr
Received: 05 January 2018 Accepted: 02.07.2018

There are three optimal realization types of a generic five-point space called Type-I, Type-II and Type-III as given below ([1], [2]):

Definition:

Let $(X,d) = (\{x_0, x_1, x_2, x_3, x_4\}, d)$ be a generic five-point metric space.

1) If the metric d can be expressed as

$$d = \alpha_{x_0}\delta_{x_0} + \alpha_{x_1}\delta_{x_1} + \alpha_{x_2}\delta_{x_2} + \alpha_{x_3}\delta_{x_3} + \alpha_{x_4}\delta_{x_4} + \alpha_{x_0,x_1}\delta_{x_0,x_1} + \alpha_{x_1,x_2}\delta_{x_1,x_2} + \alpha_{x_2,x_3}\delta_{x_2,x_3} + \alpha_{x_3,x_4}\delta_{x_3,x_4} + \alpha_{x_4,x_0}\delta_{x_4,x_0},$$

then the metric d is called to be of Type-I.

2) If the metric d can be expressed as

$$d = \alpha_{x_0}\delta_{x_0} + \alpha_{x_1}\delta_{x_1} + \alpha_{x_2}\delta_{x_2} + \alpha_{x_3}\delta_{x_3} + \alpha_{x_4}\delta_{x_4} + \alpha_{x_0,x_2}\delta_{x_0,x_2} + \alpha_{x_0,x_3}\delta_{x_0,x_3} + \alpha_{x_1,x_2}\delta_{x_1,x_2} + \alpha_{x_1,x_3}\delta_{x_1,x_3} + cd'$$

where

$$d'(a,b) = \begin{cases} 0 & , \text{ if } a = b \\ 2 & , \text{ if } \{a,b\} \in \big\{\{x_0,x_1\},\{x_2,x_3\},\{x_2,x_4\},\{x_3,x_4\}\big\} \\ 1 & , \text{ else} \end{cases}$$

then d belongs to Type-II class.

3) If the metric d can be expressed as

$$d = \alpha_{x_0}\delta_{x_0} + \alpha_{x_1}\delta_{x_1} + \alpha_{x_2}\delta_{x_2} + \alpha_{x_3}\delta_{x_3} + \alpha_{x_4}\delta_{x_4} + \alpha_{x_0,x_2}\delta_{x_0,x_2} + \alpha_{x_0,x_3}\delta_{x_0,x_3} + \alpha_{x_1,x_4}\delta_{x_1,x_4} + \alpha_{x_1,x_3}\delta_{x_1,x_3} + cd'$$

then the metric d is of Type-III.

The coefficients $\alpha_{x_i}, \alpha_{x_i,x_j}$ above are isolation indices in short-notation:

$$\alpha_{x_i} = \alpha_{\{x_i\} \mid X-\{x_i\}} \text{ and } \alpha_{x_i,x_j} = \alpha_{\{x_i,x_j\} \mid X-\{x_i,x_j\}}.$$

And $\delta_{x_i}, \delta_{x_i,x_j}$ are split metrics in short notation:

$$\delta_{x_i} = \delta_{\{x_i\}|X-\{x_i\}} \; and \; \delta_{x_i,x_j} = \delta_{\{x_i,x_j\} \mid X-\{x_i,x_j\}}.$$

On the other hand, there are three Gromov-product types called A, B, and C for a five-point metric space (see [3]):

$$\text{A}= \{\Delta_{014}, \Delta_{102}, \Delta_{213}, \Delta_{324}, \Delta_{403}\}$$

$$\text{B}= \{\Delta_{023}, \Delta_{123}, \Delta_{201}, \Delta_{301}, \Delta_{401}\}$$

$$\text{C}= \{\Delta_{023}, \Delta_{134}, \Delta_{201}, \Delta_{301}, \Delta_{401}\}.$$

This means for example for A that the least Gromov-product at the vertex x_0 is Δ_{014}; at the vertex x_1 is Δ_{102}; at the vertex x_2 is Δ_{213}; at the vertex x_3 is Δ_{324} and at the vertex x_4 is Δ_{403}.

We now give the following proposition regarding the correspondence between the two classifications:

Proposition: Let $X = \{x_0, x_1, x_2, x_3, x_4\}$ be a generic five-point metric space. Then, a Type-I metric has Gromov-product type A, a Type-II metric has Gromov-product type B and a Type-III metric has Gromov-product type C.

Proof:

First we assume that d is of Type-I. Then d is written as follows:

$$d = \alpha_{x_0}\delta_{x_0} + \alpha_{x_1}\delta_{x_1} + \alpha_{x_2}\delta_{x_2} + \alpha_{x_3}\delta_{x_3} + \alpha_{x_4}\delta_{x_4} + \alpha_{x_0,x_1}\delta_{x_0,x_1} + \alpha_{x_1,x_2}\delta_{x_1,x_2} + \alpha_{x_2,x_3}\delta_{x_2,x_3} + \alpha_{x_3,x_4}\delta_{x_3,x_4} + \alpha_{x_4,x_0}\delta_{x_4,x_0} \tag{1}$$

Using the equality (1) we can obtain the distances between all points as below.

$d_{01} = \alpha_{x_0} + \alpha_{x_1} + \alpha_{x_1,x_2} + \alpha_{x_4,x_0}$

$d_{02} = \alpha_{x_0} + \alpha_{x_2} + \alpha_{x_0,x_1} + \alpha_{x_1,x_2} + \alpha_{x_2,x_3} + \alpha_{x_4,x_0}$

$d_{03} = \alpha_{x_0} + \alpha_{x_3} + \alpha_{x_0,x_1} + \alpha_{x_2,x_3} + \alpha_{x_3,x_4} + \alpha_{x_4,x_0}$

$d_{04} = \alpha_{x_0} + \alpha_{x_4} + \alpha_{x_0,x_1} + \alpha_{x_3,x_4}$

$d_{12} = \alpha_{x_1} + \alpha_{x_2} + \alpha_{x_0,x_1} + \alpha_{x_2,x_3}$

$d_{13} = \alpha_{x_1} + \alpha_{x_3} + \alpha_{x_0,x_1} + \alpha_{x_1,x_2} + \alpha_{x_2,x_3} + \alpha_{x_3,x_4}$

$d_{14} = \alpha_{x_1} + \alpha_{x_4} + \alpha_{x_0,x_1} + \alpha_{x_1,x_2} + \alpha_{x_3,x_4} + \alpha_{x_4,x_0}$

$d_{23} = \alpha_{x_2} + \alpha_{x_3} + \alpha_{x_1,x_2} + \alpha_{x_3,x_4}$

$d_{24} = \alpha_{x_2} + \alpha_{x_4} + \alpha_{x_1,x_2} + \alpha_{x_2,x_3} + \alpha_{x_3,x_4} + \alpha_{x_4,x_0}$

$d_{34} = \alpha_{x_3} + \alpha_{x_4} + \alpha_{x_2,x_3} + \alpha_{x_4,x_0}$

Denoting these numbers as

$$\alpha = \alpha_{x_0}, \beta = \alpha_{x_1}, \gamma = \alpha_{x_2}, \delta = \alpha_{x_3}, \varepsilon = \alpha_{x_4}, \vartheta = \alpha_{x_0,x_1}, \zeta = \alpha_{x_1,x_2}, \iota = \alpha_{x_2,x_3}, \kappa = \alpha_{x_3,x_4}, \eta = \alpha_{x_4,x_0},$$

the following table given in the article [1] is obtained.

Table 1. Distances that belong to the Type-I metric

Type I	x_0	x_1	x_2	x_3	x_4
x_0	0	$\alpha+\eta+\zeta+\beta$	$\alpha+\eta+\zeta+\vartheta+\iota+\gamma$	$\alpha+\eta+\vartheta+\iota+\kappa+\delta$	$\alpha+\vartheta+\kappa+\varepsilon$
x_1	$\alpha+\eta+\zeta+\beta$	0	$\beta+\vartheta+\iota+\gamma$	$\beta+\zeta+\vartheta+\iota+\kappa+\delta$	$\beta+\eta+\zeta+\vartheta+\kappa+\varepsilon$
x_2	$\alpha+\eta+\zeta+\vartheta+\iota+\gamma$	$\beta+\vartheta+\iota+\gamma$	0	$\gamma+\zeta+\kappa+\delta$	$\gamma+\eta+\zeta+\iota+\kappa+\varepsilon$
x_3	$\alpha+\eta+\vartheta+\iota+\kappa+\delta$	$\beta+\zeta+\vartheta+\iota+\kappa+\delta$	$\gamma+\zeta+\kappa+\delta$	0	$\delta+\eta+\iota+\varepsilon$
x_4	$\alpha+\vartheta+\kappa+\varepsilon$	$\beta+\eta+\zeta+\vartheta+\kappa+\varepsilon$	$\gamma+\eta+\zeta+\iota+\kappa+\varepsilon$	$\delta+\eta+\iota+\varepsilon$	0

We now evaluate the Gromov-products at each vertex as below.

Gromov-products at the vertex x_0	Gromov-products at the vertex x_1
$\Delta_{012} = \alpha_{x_0} + \alpha_{x_1,x_2} + \alpha_{x_0,x_4}$ $\Delta_{013} = \alpha_{x_0} + \alpha_{x_0,x_4}$ $\boldsymbol{\Delta_{014} = \alpha_{x_0}}$ $\Delta_{023} = \alpha_{x_0} + \alpha_{x_0,x_1} + \alpha_{x_2,x_3} + \alpha_{x_0,x_4}$ $\Delta_{024} = \alpha_{x_0} + \alpha_{x_0,x_1}$ $\Delta_{034} = \alpha_{x_0} + \alpha_{x_0,x_1} + \alpha_{x_3,x_4}$	$\boldsymbol{\Delta_{102} = \alpha_{x_1}}$ $\Delta_{103} = \alpha_{x_1} + \alpha_{x_1,x_2}$ $\Delta_{104} = \alpha_{x_1} + \alpha_{x_1,x_2} + \alpha_{x_0,x_4}$ $\Delta_{123} = \alpha_{x_1} + \alpha_{x_0,x_1} + \alpha_{x_2,x_3}$ $\Delta_{124} = \alpha_{x_1} + \alpha_{x_0,x_1}$ $\Delta_{134} = \alpha_{x_1} + \alpha_{x_0,x_1} + \alpha_{x_1,x_2} + \alpha_{x_3,x_4}$
The minimal Gromov-product is Δ_{014}.	The minimal Gromov-product is Δ_{102}.

Gromov-products at the vertex x_2	Gromov-products at the vertex x_3
$\Delta_{201} = \alpha_{x_2} + \alpha_{x_0,x_1} + \alpha_{x_2,x_3}$ $\Delta_{203} = \alpha_{x_2} + \alpha_{x_1,x_2}$ $\Delta_{204} = \alpha_{x_2} + \alpha_{x_1,x_2} + \alpha_{x_2,x_3} + \alpha_{x_0,x_4}$ $\boldsymbol{\Delta_{213} = \alpha_{x_2}}$ $\Delta_{214} = \alpha_{x_2} + \alpha_{x_2,x_3}$ $\Delta_{234} = \alpha_{x_2} + \alpha_{x_1,x_2} + \alpha_{x_3,x_4}$	$\Delta_{301} = \alpha_{x_3} + \alpha_{x_0,x_1} + \alpha_{x_2,x_3} + \alpha_{x_3,x_4}$ $\Delta_{302} = \alpha_{x_3} + \alpha_{x_3,x_4}$ $\Delta_{304} = \alpha_{x_3} + \alpha_{x_2,x_3} + \alpha_{x_0,x_4}$ $\Delta_{312} = \alpha_{x_3} + \alpha_{x_1,x_2} + \alpha_{x_3,x_4}$ $\Delta_{314} = \alpha_{x_3} + \alpha_{x_2,x_3}$ $\boldsymbol{\Delta_{324} = \alpha_{x_3}}$
The minimal Gromov-product is Δ_{213}.	The minimal Gromov-product is Δ_{324}.

Gromov-products at the vertex x_4
$\Delta_{401} = \alpha_{x_4} + \alpha_{x_0,x_1} + \alpha_{x_3,x_4}$ $\Delta_{402} = \alpha_{x_4} + \alpha_{x_3,x_4}$ $\boldsymbol{\Delta_{403} = \alpha_{x_4}}$ $\Delta_{412} = \alpha_{x_4} + \alpha_{x_1,x_2} + \alpha_{x_0,x_4} + \alpha_{x_3,x_4}$ $\Delta_{413} = \alpha_{x_4} + \alpha_{x_0,x_4}$ $\Delta_{423} = \alpha_{x_4} + \alpha_{x_2,x_3} + \alpha_{x_0,x_4}$
The minimal Gromov-product is Δ_{403}.

Thus we obtain the Gromov-product structure equivalent to type A as given in [3]:

$$\{\Delta_{014}, \Delta_{102}, \Delta_{213}, \Delta_{324}, \Delta_{403}\}.$$

If d is of Type-II then

$$d = \alpha_{x_0}\delta_{x_0} + \alpha_{x_1}\delta_{x_1} + \alpha_{x_2}\delta_{x_2} + \alpha_{x_3}\delta_{x_3} + \alpha_{x_4}\delta_{x_4} + \alpha_{x_0,x_2}\delta_{x_0,x_2} + \alpha_{x_0,x_3}\delta_{x_0,x_3} + \alpha_{x_1,x_2}\delta_{x_1,x_2} + \alpha_{x_1,x_3}\delta_{x_1,x_3} + cd'$$

where

$$d'(a,b) = \begin{cases} 0 & , \quad \text{if } a = b \\ 2 & , \quad \text{if } \{a,b\} \in \{\{x_0,x_1\},\{x_2,x_3\},\{x_2,x_4\},\{x_3,x_4\}\}. \\ 1 & , \quad \text{else} \end{cases}$$

Now, we give all d_{ij}'s to be used in Gromov-products:

$d_{01} = \alpha_{x_0} + \alpha_{x_1} + \alpha_{x_0,x_2} + \alpha_{x_0,x_3} + \alpha_{x_1,x_2} + \alpha_{x_1,x_3} + 2c$

$d_{02} = \alpha_{x_0} + \alpha_{x_2} + \alpha_{x_0,x_3} + \alpha_{x_1,x_2} + c$

$d_{03} = \alpha_{x_0} + \alpha_{x_3} + \alpha_{x_0,x_2} + \alpha_{x_1,x_3} + c$

$d_{04} = \alpha_{x_0} + \alpha_{x_4} + \alpha_{x_0,x_2} + \alpha_{x_0,x_3} + c$

$d_{12} = \alpha_{x_1} + \alpha_{x_2} + \alpha_{x_0,x_2} + \alpha_{x_1,x_3} + c$

$d_{13} = \alpha_{x_1} + \alpha_{x_3} + \alpha_{x_0,x_3} + \alpha_{x_1,x_2} + c$

$d_{14} = \alpha_{x_1} + \alpha_{x_4} + \alpha_{x_1,x_2} + \alpha_{x_1,x_3} + c$

$d_{23} = \alpha_{x_2} + \alpha_{x_3} + \alpha_{x_0,x_2} + \alpha_{x_0,x_3} + \alpha_{x_1,x_2} + \alpha_{x_1,x_3} + 2c$

$d_{24} = \alpha_{x_2} + \alpha_{x_4} + \alpha_{x_0,x_2} + \alpha_{x_1,x_2} + 2c$

$d_{34} = \alpha_{x_3} + \alpha_{x_4} + \alpha_{x_0,x_3} + \alpha_{x_1,x_3} + 2c$

Denoting these numbers as

$$\alpha = \alpha_{x_0}, \beta = \alpha_{x_1}, \gamma = \alpha_{x_2}, \delta = \alpha_{x_3}, \varepsilon = \alpha_{x_4}, \zeta = \alpha_{x_0,x_2}, \eta = \alpha_{x_0,x_3}, \iota = \alpha_{x_1,x_2}, \vartheta = \alpha_{x_1,x_3}, \kappa = c$$

we obtain Table 2 for Type-II metrics:

Table 2. Distances that belong to the Type-II metric

Type II	x_0	x_1	x_2	x_3	x_4
x_0	0	$\alpha+\beta+\zeta+\eta+\iota+\vartheta+2\kappa$	$\alpha+\gamma+\eta+\iota+\kappa$	$\alpha+\delta+\zeta+\vartheta+\kappa$	$\alpha+\varepsilon+\zeta+\eta+\kappa$
x_1	$\alpha+\beta+\zeta+\eta+\iota+\vartheta+2\kappa$	0	$\beta+\gamma+\zeta+\vartheta+\kappa$	$\beta+\delta+\eta+\iota+\kappa$	$\beta+\varepsilon+\iota+\vartheta+\kappa$
x_2	$\alpha+\gamma+\eta+\iota+\kappa$	$\beta+\gamma+\zeta+\vartheta+\kappa$	0	$\gamma+\delta+\zeta+\eta+\iota+\vartheta+2\kappa$	$\gamma+\varepsilon+\zeta+\iota+2\kappa$
x_3	$\alpha+\delta+\zeta+\vartheta+\kappa$	$\beta+\delta+\eta+\iota+\kappa$	$\gamma+\delta+\zeta+\eta+\iota+\vartheta+2\kappa$	0	$\delta+\varepsilon+\eta+\vartheta+2\kappa$
x_4	$\alpha+\varepsilon+\zeta+\eta+\kappa$	$\beta+\varepsilon+\iota+\vartheta+\kappa$	$\gamma+\varepsilon+\zeta+\iota+2\kappa$	$\delta+\varepsilon+\eta+\vartheta+2\kappa$	0

Let us now compute the Gromov-products at each of the vertices:

Gromov-products at the vertex x_0	Gromov-products at the vertex x_1
$\Delta_{012} = \alpha_{x_0} + \alpha_{x_0,x_3} + \alpha_{x_1,x_2} + c$ $\Delta_{013} = \alpha_{x_0} + \alpha_{x_0,x_2} + \alpha_{x_1,x_3} + c$ $\Delta_{014} = \alpha_{x_0} + \alpha_{x_0,x_2} + \alpha_{x_0,x_3} + c$ $\boldsymbol{\Delta_{023} = \alpha_{x_0}}$ $\Delta_{024} = \alpha_{x_0} + \alpha_{x_0,x_3}$ $\Delta_{034} = \alpha_{x_0} + \alpha_{x_0,x_2}$	$\Delta_{102} = \alpha_{x_1} + \alpha_{x_0,x_2} + \alpha_{x_1,x_3} + c$ $\Delta_{103} = \alpha_{x_1} + \alpha_{x_0,x_3} + \alpha_{x_1,x_2} + c$ $\Delta_{104} = \alpha_{x_1} + \alpha_{x_1,x_2} + \alpha_{x_1,x_3} + c$ $\boldsymbol{\Delta_{123} = \alpha_{x_1}}$ $\Delta_{124} = \alpha_{x_1} + \alpha_{x_1,x_3}$ $\Delta_{134} = \alpha_{x_1} + \alpha_{x_1,x_2}$
The minimal Gromov-product is Δ_{023}.	The minimal Gromov-product is Δ_{123}.

Gromov-products at the vertex x_2	Gromov-products at the vertex x_3
$\boldsymbol{\Delta_{201} = \alpha_{x_2}}$ $\Delta_{203} = \alpha_{x_2} + \alpha_{x_0,x_3} + \alpha_{x_1,x_2} + c$ $\Delta_{204} = \alpha_{x_2} + \alpha_{x_1,x_2} + c$ $\Delta_{213} = \alpha_{x_2} + \alpha_{x_0,x_2} + \alpha_{x_1,x_3} + c$ $\Delta_{214} = \alpha_{x_2} + \alpha_{x_0,x_2} + c$ $\Delta_{234} = \alpha_{x_2} + \alpha_{x_1,x_2} + \alpha_{x_0,x_2} + c$	$\boldsymbol{\Delta_{301} = \alpha_{x_3}}$ $\Delta_{302} = \alpha_{x_3} + \alpha_{x_0,x_2} + \alpha_{x_1,x_3} + c$ $\Delta_{304} = \alpha_{x_3} + \alpha_{x_1,x_3} + c$ $\Delta_{312} = \alpha_{x_3} + \alpha_{x_0,x_3} + \alpha_{x_1,x_2} + c$ $\Delta_{314} = \alpha_{x_3} + \alpha_{x_0,x_3} + c$ $\Delta_{324} = \alpha_{x_3} + \alpha_{x_0,x_3} + \alpha_{x_1,x_3} + c$
The minimal Gromov-product is Δ_{201}.	The minimal Gromov-product is Δ_{301}.

Gromov-products at the vertex x_4
$\boldsymbol{\Delta_{401} = \alpha_{x_4}}$ $\Delta_{402} = \alpha_{x_4} + \alpha_{x_0,x_2} + c$ $\Delta_{403} = \alpha_{x_4} + \alpha_{x_0,x_3} + c$ $\Delta_{412} = \alpha_{x_4} + \alpha_{x_1,x_2} + c$ $\Delta_{413} = \alpha_{x_4} + \alpha_{x_1,x_3} + c$ $\Delta_{423} = \alpha_{x_4} + c$
The minimal Gromov-product is Δ_{401}.

As a result we obtain the Gromov-product structure that is equivalent to type B in the sense of [3]:

$$\{\Delta_{023}, \Delta_{123}, \Delta_{201}, \Delta_{301}, \Delta_{401}\}.$$

For the metric of Type-III we obtain the distances using the following equality.

$$d = \alpha_{x_0}\delta_{x_0} + \alpha_{x_1}\delta_{x_1} + \alpha_{x_2}\delta_{x_2} + \alpha_{x_3}\delta_{x_3} + \alpha_{x_4}\delta_{x_4} + \alpha_{x_0,x_2}\delta_{x_0,x_2} + \alpha_{x_0,x_3}\delta_{x_0,x_3} + \alpha_{x_1,x_4}\delta_{x_1,x_4} + \alpha_{x_1,x_3}\delta_{x_1,x_3} + cd'$$

$$d_{01} = \alpha_{x_0} + \alpha_{x_1} + \alpha_{x_0,x_2} + \alpha_{x_0,x_3} + \alpha_{x_1,x_4} + \alpha_{x_1,x_3} + 2c$$

$$d_{02} = \alpha_{x_0} + \alpha_{x_2} + \alpha_{x_0,x_3} + c$$

$$d_{03} = \alpha_{x_0} + \alpha_{x_3} + \alpha_{x_0,x_2} + \alpha_{x_1,x_3} + c$$

$$d_{04} = \alpha_{x_0} + \alpha_{x_4} + \alpha_{x_0,x_2} + \alpha_{x_0,x_3} + \alpha_{x_1,x_4} + c$$

$$d_{12} = \alpha_{x_1} + \alpha_{x_2} + \alpha_{x_0,x_2} + \alpha_{x_1,x_4} + \alpha_{x_1,x_3} + c$$

$$d_{13} = \alpha_{x_1} + \alpha_{x_3} + \alpha_{x_0,x_3} + \alpha_{x_1,x_4} + c$$

$$d_{14} = \alpha_{x_1} + \alpha_{x_4} + \alpha_{x_1,x_3} + c$$

$$d_{23} = \alpha_{x_2} + \alpha_{x_3} + \alpha_{x_0,x_2} + \alpha_{x_0,x_3} + \alpha_{x_1,x_3} + 2c$$

$$d_{24} = \alpha_{x_2} + \alpha_{x_4} + \alpha_{x_0,x_2} + \alpha_{x_1,x_4} + 2c$$

$$d_{34} = \alpha_{x_3} + \alpha_{x_4} + \alpha_{x_0,x_3} + \alpha_{x_1,x_4} + 2c$$

Denoting these numbers as

$$\alpha = \alpha_{x_0}, \beta = \alpha_{x_1}, \gamma = \alpha_{x_2}, \delta = \alpha_{x_3}, \varepsilon = \alpha_{x_4}, \zeta = \alpha_{x_0,x_2}, \eta = \alpha_{x_0,x_3}, \iota = \alpha_{x_1,x_3}, \vartheta = \alpha_{x_1,x_4}, \kappa = c$$

we obtain Table 3 for Type-III metrics:

Table 3. Distances that belongs the Type-III metric

Type-III	x_0	x_1	x_2	x_3	x_4
x_0	0	$\alpha+\beta+\zeta+\eta+\vartheta+\iota+2\kappa$	$\alpha+\gamma+\eta+\kappa$	$\alpha+\delta+\zeta+\iota+\kappa$	$\alpha+\varepsilon+\zeta+\eta+\vartheta+\kappa$
x_1	$\alpha+\beta+\zeta+\eta+\vartheta+\iota+2\kappa$	0	$\beta+\gamma+\zeta+\vartheta+\iota+\kappa$	$\beta+\delta+\eta+\vartheta+\kappa$	$\beta+\varepsilon+\iota+\kappa$
x_2	$\alpha+\gamma+\eta+\kappa$	$\beta+\gamma+\zeta+\vartheta+\iota+\kappa$	0	$\gamma+\delta+\zeta+\eta+\iota+2\kappa$	$\gamma+\varepsilon+\zeta+\vartheta+2\kappa$
x_3	$\alpha+\delta+\zeta+\iota+\kappa$	$\beta+\delta+\eta+\vartheta+\kappa$	$\gamma+\delta+\zeta+\eta+\iota+2\kappa$	0	$\delta+\varepsilon+\eta+\vartheta+2\kappa$
x_4	$\alpha+\varepsilon+\zeta+\eta+\vartheta+\kappa$	$\beta+\varepsilon+\iota+\kappa$	$\gamma+\varepsilon+\zeta+\vartheta+2\kappa$	$\delta+\varepsilon+\eta+\vartheta+2\kappa$	0

We now give all Gromov-products at each vertex.

Gromov-products at the vertex x_0	Gromov-products at the vertex x_1
$\Delta_{012}=\alpha_{x_0}+\alpha_{x_0,x_3}+c$ $\Delta_{013}=\alpha_{x_0}+\alpha_{x_0,x_2}+\alpha_{x_1,x_3}+c$ $\Delta_{014}=\alpha_{x_0}+\alpha_{x_0,x_2}+\alpha_{x_0,x_3}+\alpha_{x_1,x_4}+c$ $\boldsymbol{\Delta_{023}=\alpha_{x_0}}$ $\Delta_{024}=\alpha_{x_0}+\alpha_{x_0,x_3}$ $\Delta_{034}=\alpha_{x_0}+\alpha_{x_0,x_2}$	$\Delta_{102}=\alpha_{x_1}+\alpha_{x_0,x_2}+\alpha_{x_1,x_3}+\alpha_{x_0,x_4}+c$ $\Delta_{103}=\alpha_{x_1}+\alpha_{x_0,x_3}+\alpha_{x_1,x_4}+c$ $\Delta_{104}=\alpha_{x_1}+\alpha_{x_1,x_3}+c$ $\Delta_{123}=\alpha_{x_1}+\alpha_{x_1,x_4}$ $\Delta_{124}=\alpha_{x_1}+\alpha_{x_1,x_3}$ $\boldsymbol{\Delta_{134}=\alpha_{x_1}}$
The minimal Gromov-product is Δ_{023}.	The minimal Gromov-product is Δ_{134}.

Gromov-products at the vertex x_2	Gromov-products at the vertex x_3
$\boldsymbol{\Delta_{201}=\alpha_{x_2}}$ $\Delta_{203}=\alpha_{x_2}+\alpha_{x_0,x_3}+c$ $\Delta_{204}=\alpha_{x_2}+c$ $\Delta_{213}=\alpha_{x_2}+\alpha_{x_0,x_2}+\alpha_{x_1,x_3}+c$ $\Delta_{214}=\alpha_{x_2}+\alpha_{x_0,x_2}+\alpha_{x_1,x_4}+c$ $\Delta_{234}=\alpha_{x_2}+\alpha_{x_0,x_2}+c$	$\boldsymbol{\Delta_{301}=\alpha_{x_3}}$ $\Delta_{302}=\alpha_{x_3}+\alpha_{x_0,x_2}+\alpha_{x_1,x_3}+c$ $\Delta_{304}=\alpha_{x_3}+\alpha_{x_1,x_3}+c$ $\Delta_{312}=\alpha_{x_3}+\alpha_{x_0,x_3}+c$ $\Delta_{314}=\alpha_{x_3}+\alpha_{x_0,x_3}+\alpha_{x_1,x_4}+c$ $\Delta_{324}=\alpha_{x_3}+\alpha_{x_0,x_3}+\alpha_{x_1,x_3}+c$
The minimal Gromov-Product is Δ_{201}.	The minimal Gromov-product is Δ_{301}

Gromov-products at the vertex x_4
$\boldsymbol{\Delta_{401}=\alpha_{x_4}}$ $\Delta_{402}=\alpha_{x_4}+\alpha_{x_0,x_2}+\alpha_{x_1,x_4}+c$ $\Delta_{403}=\alpha_{x_4}+\alpha_{x_0,x_3}+\alpha_{x_1,x_4}+c$ $\Delta_{412}=\alpha_{x_4}+c$ $\Delta_{413}=\alpha_{x_4}+\alpha_{x_1,x_3}+c$ $\Delta_{423}=\alpha_{x_4}+\alpha_{x_1,x_4}+c$
The minimal Gromov-product is Δ_{401}.

Hence we obtain the Gromov-product structure equivalent to type C according to the classification in [3]:

$$\{\Delta_{023}, \Delta_{134}, \Delta_{201}, \Delta_{301}, \Delta_{401}\}.$$

REFERENCES

[1] Dress WMA. Trees, tight Extensions of metric spaces, and the cohomological dimension of certain groups: A note on combinatorial properties of metric spaces. Advances in Mathematics 1984; 53: 321-402.

[2] Koolen J, Lesser A, Moulton V. Optimal realizations of generic five-point metrics. European Journal of Combinatorics 2009; 30: 1164-1171.

[3] Bilge AH., Çelik D, Koçak Ş. An equivalence class decomposition of finite metric spaces via Gromov products. Discrete Mathematics 2017; 340: 1928-1932.

[4] Bilge A H. and İncegül M. Gromov product decomposition of 7-point metric spaces. Preprint.