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IMPROVED HERMITE HADAMARD TYPE INEQUALITIES FOR HARMONICALLY CONVEX FUNCTIONS VIA KATUGAMPOLA FRACTIONAL INTEGRALS

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Abstract. In this paper, we prove three new Katugampola fractional Hermite-Hadamard type inequalities for harmonically convex functions by using the left and the right fractional integrals independently. One of our Katugampola fractional Hermite-Hadamard type inequalities is better than given in [17]. Also, we give two new Katugampola fractional identities for differentiable functions. By using these identities, we obtain some new trapezoidal type inequalities for harmonically convex functions. Our results generalize many results from earlier papers.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

is well known in the literature as Hermite-Hadamard's inequality. There are so many generalizations and extensions of inequalities (1) for various classes of functions. One of this classes of functions is harmonically convex functions defined by İşcan.

In [7], İşcan gave the definition of harmonically convex functions as follows:

Definition 1. [7] *Let $I \subset \mathbb{R}\backslash\{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x) \tag{2}$$

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for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (2) is reversed, then f is said to be harmonically concave.

For some similar studies with this work about harmonically convex functions, readers can see [1, 2, 3, 5, 6, 7, 8, 9, 13, 14, 15, 16, 17, 20] and references therein.

In [7], İşcan gave Hermite-Hadamard type inequalities for harmonically convex functions as follows:

Theorem 2. [7] *Let $f : I \subset \mathbb{R}\backslash \{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$, then the following inequalities hold:*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}. \tag{3}$$

Following definitions of the left and right side Riemann-Liouville fractional integrals are well known in the literature.

Definition 3. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f \in L[a,b]$. The left and right Riemann-Liouville fractional integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \ x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \ x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int\limits_0^{\infty} e^{-t}t^{\alpha-1}dt$ (see [12, page 69]*).*

In [9], İşcan and Wu presented Hermite-Hadamard type inequalities for harmonically convex functions in fractional integral forms as follows:

Theorem 4. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a function such that $f \in L[a,b]$, where $a, b \in I$ with $a < b$. If f is a harmonically convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leq \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left[\begin{array}{c} J^{\alpha}_{1/a-}(f\circ h)(1/b) \\ +J^{\alpha}_{1/b+}(f\circ h)(1/a) \end{array}\right] \\ &\leq \frac{f(a)+f(b)}{2} \end{aligned} \tag{4}$$

with $\alpha > 0$ and $h(x) = 1/x$.

In [20], Şanlı et al. proved the following three Riemann-Liouville fractional Hermite-Hadamard type inequalities for harmonically convex functions by using the left and the right fractional integrals seperately as follows:

Theorem 5. *Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a, b]$, then the following inequality for the left Riemann-Liouville fractional integral holds:*

$$f\left(\frac{(\alpha+1)\,ab}{\alpha a+b}\right) \leq \Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha} J_{\frac{1}{a}+}^{\alpha}(f\circ h)\left(\frac{1}{b}\right) \leq \frac{f(a)+\alpha f(b)}{\alpha+1} \tag{5}$$

where $h(x) = \frac{1}{x}$ and $\alpha > 0$.

Theorem 6. *Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a, b]$, then the following inequality for the right Riemann-Liouville fractional integral holds:*

$$f\left(\frac{(\alpha+1)\,ab}{a+\alpha b}\right) \leq \Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha} J_{\frac{1}{b}-}^{\alpha}(f\circ h)\left(\frac{1}{a}\right) \leq \frac{\alpha f(a)+f(b)}{\alpha+1} \tag{6}$$

where $h(x) = \frac{1}{x}$ and $\alpha > 0$.

Theorem 7. *Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a, b]$, then the following inequality for the Riemann-Liouville fractional integral holds:*

$$\begin{aligned} \frac{f\left(\frac{(\alpha+1)ab}{a+\alpha b}\right)+f\left(\frac{(\alpha+1)ab}{\alpha a+b}\right)}{2} &\leq \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left[\begin{array}{c} J_{\frac{1}{b}-}^{\alpha}(f\circ h)\left(\frac{1}{a}\right) \\ +J_{\frac{1}{a}+}^{\alpha}(f\circ h)\left(\frac{1}{b}\right) \end{array}\right] \\ &\leq \frac{f(a)+f(b)}{2} \end{aligned} \tag{7}$$

where $h(x) = \frac{1}{x}$ and $\alpha > 0$.

The following definitions of Katugampola fractional integrals could be found in [4, 11].

Definition 8. *Let $[a, b] \subset \mathbb{R}$ be a finite interval. Then the left and right-side Katugampola fractional integrals of order $\alpha > 0$ of $f \in X_c^p(a, b)$ are defined by*

$${}^{\rho}I_{a+}^{\alpha}f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x \frac{t^{\rho-1}}{(x^{\rho}-t^{\rho})^{1-\alpha}}f(t)dt,$$

and

$${}^{\rho}I_{b-}^{\alpha}f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_x^b \frac{t^{\rho-1}}{(t^{\rho}-x^{\rho})^{1-\alpha}}f(t)dt,$$

with $a < x < b$ and $\rho > 0$, respectively.

(See [10], for the definition of the set $X_c^p(a, b)$)

It is easily seen that if one takes $\rho \to 1$ in the Definition 8, one has the Definition 3.

Lemma 9. *For $0 < \alpha \leq 1$ and $0 \leq a < b$ we have*

$$\left|a^{\alpha} - b^{\alpha}\right| \leq (b-a)^{\alpha}.$$

(see [18]).

In [17, Theorem 2.1], Mumcu et al. presented Hermite-Hadamard type inequalities for harmoncally convex functions in Katugampola fractional integral forms as follows:

Theorem 10. *Let $\alpha > 0$ and $\rho > 0$. Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a function such that $f \in X_c^p(a^\rho, b^\rho)$, where $a^\rho, b^\rho \in I$ with $a < b$. If f is a harmonically convex function on $[a,b]$, then the following inequalities hold:*

$$\begin{aligned} f\left(\frac{2\alpha^\rho b^\rho}{a^\rho + b^\rho}\right) &\leq \frac{\rho^\alpha \Gamma(\alpha+1)}{2}\left(\frac{a^\rho b^\rho}{b^\rho - a^\rho}\right)^\alpha \left[\begin{array}{c} {}^\rho I^\alpha_{\frac{1}{a}-}(f\circ h)\left(\frac{1}{b}\right) \\ + {}^\rho I^\alpha_{\frac{1}{b}+}(f\circ h)\left(\frac{1}{a}\right) \end{array}\right] \\ &\leq \frac{f(a^\rho)+f(b^\rho)}{2} \end{aligned}$$

where $h(x) = 1/x$.

For the Theorem 10 , the correct inequality should be expressed as follows:

$$\begin{aligned} f\left(\frac{2\alpha^\rho b^\rho}{a^\rho + b^\rho}\right) &\leq \frac{\rho^\alpha \Gamma(\alpha+1)}{2}\left(\frac{a^\rho b^\rho}{b^\rho - a^\rho}\right)^\alpha \left[\begin{array}{c} {}^\rho I^\alpha_{\frac{1}{a}-}(f\circ h)\left(\frac{1}{b^\rho}\right) \\ + {}^\rho I^\alpha_{\frac{1}{b}+}(f\circ h)\left(\frac{1}{a^\rho}\right) \end{array}\right] \\ &\leq \frac{f(a^\rho)+f(b^\rho)}{2}. \end{aligned} \tag{8}$$

In (8), if one takes $\rho \to 1$, one obtaines the inequality (4) in the Theorem 4.

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals. The papers [3, 8, 9, 13, 14, 15, 17, 20] are based on Hermite-Hadamard type inequalities involving several fractional integrals.

Definition 11. [19, page 12] *A function f defined on I has a support at $x_0 \in I$ if there exists an affine functions $A(x) = f(x_0) + m(x - x_0)$ such that $A(x) \leq f(x)$ for all $x \in I$. The graph of the support function A is called a line of support for f at x_0.*

Theorem 12. [19, page 12] *$f : (a,b) \to \mathbb{R}$ is a convex function if and only if there is at least one line of support for f at each $x_0 \in (a,b)$.*

Remark 13. [6] *Let $[a,b] \subset I \subseteq (0,\infty)$, if the function $g : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$ defined $g(x) = f\left(\frac{1}{x}\right)$, then f is harmonically convex on $[a,b]$ if and only if g is convex on $\left[\frac{1}{b}, \frac{1}{a}\right]$.*

In literature, there are so many studies for Hermite-Hadamard type inequalities by using the left and right fractional integrals (such as Riemann-Liouville fractional integrals, Hadamard fractional integrals, Katugampola fractional integrals etc.). In all of them, the left and right fractional integrals are used together. As much as we know, the studies [20] are the first two works by using only the right fractional integrals or the left fractional integrals.

In this paper, our aim is to obtain new Katugampola fractional Hermite-Hadamard type inequalities by using only the right or the left fractional integrals separately for harmonically convex functions.

2. Katugampola Fractional Hermite Hadamard Type Inequalities for Harmonically Convex Functions

Theorem 14. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a function such that $f \in X_c^p(a^\rho, b^\rho)$, where $a^\rho, b^\rho \in I$ with $a < b$. If f is a harmonically convex function on $[a^\rho, b^\rho]$, then the following inequality for the left katugampola fractional integral holds:*

$$\begin{aligned} f\left(\frac{(\alpha+1)\,a^\rho b^\rho}{\alpha a^\rho + b^\rho}\right) &\leq \left(\frac{a^\rho b^\rho}{b^\rho - a^\rho}\right)^\alpha \rho^\alpha \Gamma(\alpha+1)\ \ {}^\rho I^\alpha_{\frac{1}{b}+}(f\circ h)\left(\tfrac{1}{a^\rho}\right) \\ &\leq \frac{f(a^\rho) + \alpha f(b^\rho)}{\alpha+1} \end{aligned} \tag{9}$$

where $\alpha > 0$, $\rho > 0$ and $h(x) = \frac{1}{x}$

Proof. Let $\alpha > 0$. Since f is harmonically convex on $[a^\rho, b^\rho]$, then by using Remark 13 the function $g(x) = f\left(\frac{1}{x}\right)$ is convex on $\left[\frac{1}{b^\rho}, \frac{1}{a^\rho}\right]$. Hence using Theorem 12, there is at least one line of support

$$A(x) = g\left(\frac{\alpha a^\rho + b^\rho}{(\alpha+1)\,a^\rho b^\rho}\right) + m\left(x - \frac{\alpha a^\rho + b^\rho}{(\alpha+1)\,a^\rho b^\rho}\right) \leq g(x) \tag{10}$$

for all $x \in \left[\frac{1}{b^\rho}, \frac{1}{a^\rho}\right]$ and $m \in \left[g'_-\left(\frac{\alpha a^\rho + b^\rho}{(\alpha+1)a^\rho b^\rho}\right), g'_+\left(\frac{\alpha a^\rho + b^\rho}{(\alpha+1)a^\rho b^\rho}\right)\right]$. From (10) and harmonically convexity of f, we have

$$\begin{aligned} A\left(\frac{t^\rho a^\rho + (1-t^\rho)b^\rho}{a^\rho b^\rho}\right) &= f\left(\frac{(\alpha+1)\,a^\rho b^\rho}{\alpha a^\rho + b^\rho}\right) + m\left(\begin{array}{c} \frac{t^\rho a^\rho + (1-t^\rho)b^\rho}{a^\rho b^\rho} \\ -\frac{\alpha a^\rho + b^\rho}{(\alpha+1)a^\rho b^\rho} \end{array}\right) \\ &\leq g\left(\frac{t^\rho a^\rho + (1-t^\rho)b^\rho}{a^\rho b^\rho}\right) = f\left(\frac{a^\rho b^\rho}{t^\rho a^\rho + (1-t^\rho)b^\rho}\right) \\ &\leq t^\rho f(b^\rho) + (1-t^\rho)\, f(a^\rho) \end{aligned} \tag{11}$$

for all $t \in [0,1]$. Multiplying all sides of (11) with $t^{\alpha\rho-1}$ and integrating over $[0,1]$ respect to t, we have

$$\begin{aligned} &\int_0^1 t^{\alpha\rho-1}\left[f\left(\frac{(\alpha+1)\,a^\rho b^\rho}{\alpha a^\rho + b^\rho}\right) + m\left(\frac{t^\rho a^\rho + (1-t^\rho)b^\rho}{a^\rho b^\rho} - \frac{\alpha a^\rho + b^\rho}{(\alpha+1)\,a^\rho b^\rho}\right)\right] dt \\ &= f\left(\frac{(\alpha+1)\,a^\rho b^\rho}{\alpha a^\rho + b^\rho}\right)\int_0^1 t^{\alpha\rho-1}dt + m\left[\begin{array}{c} \int_0^1 t^{\alpha\rho-1}\frac{t^\rho a^\rho + (1-t^\rho)b^\rho}{a^\rho b^\rho}dt \\ -\frac{\alpha a^\rho + b^\rho}{(\alpha+1)a^\rho b^\rho}\int_0^1 t^{\alpha\rho-1}dt \end{array}\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\alpha\rho} f\left(\frac{(\alpha+1)\,a^{\rho}b^{\rho}}{\alpha a^{\rho}+b^{\rho}}\right) + m\left[\frac{\alpha a^{\rho}+b^{\rho}}{\rho\alpha\,(\alpha+1)\,a^{\rho}b^{\rho}} - \frac{\alpha a^{\rho}+b^{\rho}}{\rho\alpha\,(\alpha+1)\,a^{\rho}b^{\rho}}\right] \\
&= \frac{1}{\alpha\rho} f\left(\frac{(\alpha+1)\,a^{\rho}b^{\rho}}{\alpha a^{\rho}+b^{\rho}}\right) \\
&\le \int_0^1 t^{\alpha\rho-1} f\left(\frac{a^{\rho}b^{\rho}}{t^{\rho}a^{\rho}+(1-t^{\rho})b^{\rho}}\right) dt \\
&= \left(\frac{a^{\rho}b^{\rho}}{b^{\rho}-a^{\rho}}\right)^{\alpha} \int_{\frac{1}{b}}^{\frac{1}{a}} \frac{t^{\rho-1}}{\left(\frac{1}{a^{\rho}}-t^{\rho}\right)^{1-\alpha}} f\left(\frac{1}{t^{\rho}}\right) dt \\
&= \left(\frac{a^{\rho}b^{\rho}}{b^{\rho}-a^{\rho}}\right)^{\alpha} \rho^{\alpha-1}\Gamma(\alpha)\ \ {}^{\rho}I^{\alpha}_{\frac{1}{b}+}\,(f\circ h)\left(\tfrac{1}{a^{\rho}}\right) \\
&\le f(b^{\rho})\int_0^1 t^{\alpha\rho-1+\rho}dt + f(a^{\rho})\int_0^1 \left(t^{\alpha\rho-1}-t^{\alpha\rho-1+\rho}\right) dt \\
&= \frac{1}{\alpha\rho}\,\frac{f(a^{\rho})+\alpha f(b^{\rho})}{\alpha+1}.
\end{aligned}
$$

It means that

$$
\begin{aligned}
f\left(\frac{(\alpha+1)\,a^{\rho}b^{\rho}}{\alpha a^{\rho}+b^{\rho}}\right) &\le \left(\frac{a^{\rho}b^{\rho}}{b^{\rho}-a^{\rho}}\right)^{\alpha} \rho^{\alpha}\Gamma(\alpha+1)\ \ {}^{\rho}I^{\alpha}_{\frac{1}{b}+}\,(f\circ h)\left(\tfrac{1}{a^{\rho}}\right) \\
&\le \frac{f(a^{\rho})+\alpha f(b^{\rho})}{\alpha+1}.
\end{aligned}
$$

This completes the proof. □

Remark 15. *In Theorem 14,*

(1) *if one takes $\rho\to 1$, one has the inequality* (6).
(2) *if one takes $\rho\to 1$, and after that if one takes $\alpha=1$, one has the inequality* (3).

Theorem 16. *Let $f : I\subset(0,\infty)\to\mathbb{R}$ be a function such that $f\in X^p_c(a^{\rho},b^{\rho})$, where $a^{\rho},b^{\rho}\in I$ with $a<b$. If f is a harmonically convex function on $[a^{\rho},b^{\rho}]$, then the following inequality for the right katugampola fractional integral holds:*

$$
\begin{aligned}
f\left(\frac{(\alpha+1)\,a^{\rho}b^{\rho}}{a^{\rho}+\alpha b^{\rho}}\right) &\le \left(\frac{a^{\rho}b^{\rho}}{b^{\rho}-a^{\rho}}\right)^{\alpha} \rho^{\alpha}\Gamma(\alpha+1)\ \ {}^{\rho}I^{\alpha}_{\frac{1}{a}-}\,(f\circ h)\left(\tfrac{1}{b^{\rho}}\right) \\
&\le \frac{\alpha f(a^{\rho})+f(b^{\rho})}{\alpha+1} \qquad (12)
\end{aligned}
$$

where $\alpha>0$, $\rho>0$ and $h(x)=\frac{1}{x}$.

Proof. Let $\alpha>0$. Since f is harmonically convex on $[a^{\rho},b^{\rho}]$, then by using Remark 13 the function $g(x)=f\left(\frac{1}{x}\right)$ is convex on $\left[\frac{1}{b^{\rho}},\frac{1}{a^{\rho}}\right]$. Hence using Theorem 12, there is at least one line of support

$$
A(x) = g\left(\frac{a^{\rho}+\alpha b^{\rho}}{(\alpha+1)\,a^{\rho}b^{\rho}}\right) + m\left(x-\frac{a^{\rho}+\alpha b^{\rho}}{(\alpha+1)\,a^{\rho}b^{\rho}}\right) \le g(x) \qquad (13)
$$

for all $x \in \left[\frac{1}{b^\rho}, \frac{1}{a^\rho}\right]$ and $m \in \left[g'_{-}\left(\frac{a^\rho+\alpha b^\rho}{(\alpha+1)a^\rho b^\rho}\right), g'_{+}\left(\frac{a^\rho+\alpha b^\rho}{(\alpha+1)a^\rho b^\rho}\right)\right]$. From (10) and harmonically convexity of f, we have

$$\begin{aligned}
A\left(\frac{t^\rho b^\rho+(1-t^\rho)a^\rho}{a^\rho b^\rho}\right) &= f\left(\frac{(\alpha+1)\,a^\rho b^\rho}{a^\rho+\alpha b^\rho}\right)+m\left(\begin{array}{c}\frac{t^\rho b^\rho+(1-t^\rho)a^\rho}{a^\rho b^\rho}\\ -\frac{a^\rho+\alpha b^\rho}{(\alpha+1)a^\rho b^\rho}\end{array}\right)\\
&\le g\left(\frac{t^\rho b^\rho+(1-t^\rho)a^\rho}{a^\rho b^\rho}\right)=f\left(\frac{a^\rho b^\rho}{t^\rho b^\rho+(1-t^\rho)a^\rho}\right)\\
&\le t^\rho f(a^\rho)+(1-t^\rho)f(b^\rho) \qquad (14)
\end{aligned}$$

for all $t \in [0,1]$. Multiplying all sides of (14) with $t^{\alpha\rho-1}$ and integrating over $[0,1]$ respect to t, we have

$$\begin{aligned}
&\int_0^1 t^{\alpha\rho-1}\left[f\left(\frac{(\alpha+1)\,a^\rho b^\rho}{a^\rho+\alpha b^\rho}\right)+m\left(\frac{t^\rho b^\rho+(1-t^\rho)a^\rho}{a^\rho b^\rho}-\frac{a^\rho+\alpha b^\rho}{(\alpha+1)\,a^\rho b^\rho}\right)\right]dt\\
&= f\left(\frac{(\alpha+1)\,a^\rho b^\rho}{a^\rho+\alpha b^\rho}\right)\int_0^1 t^{\alpha\rho-1}dt+m\left[\begin{array}{c}\int_0^1 t^{\alpha\rho-1}\frac{t^\rho b^\rho+(1-t^\rho)a^\rho}{a^\rho b^\rho}dt\\ -\frac{a^\rho+\alpha b^\rho}{(\alpha+1)a^\rho b^\rho}\int_0^1 t^{\alpha\rho-1}dt\end{array}\right]\\
&= \frac{1}{\alpha\rho}f\left(\frac{(\alpha+1)\,a^\rho b^\rho}{a^\rho+\alpha b^\rho}\right)+m\left[\frac{a^\rho+\alpha b^\rho}{\rho\alpha\,(\alpha+1)\,a^\rho b^\rho}-\frac{a^\rho+\alpha b^\rho}{\rho\alpha\,(\alpha+1)\,a^\rho b^\rho}\right]\\
&= \frac{1}{\alpha\rho}f\left(\frac{(\alpha+1)\,a^\rho b^\rho}{a^\rho+\alpha b^\rho}\right)\\
&\le \int_0^1 t^{\alpha\rho-1}f\left(\frac{a^\rho b^\rho}{t^\rho b^\rho+(1-t^\rho)a^\rho}\right)dt\\
&= \left(\frac{a^\rho b^\rho}{b^\rho-a^\rho}\right)^\alpha\int_{\frac{1}{b}}^{\frac{1}{a}}\frac{t^{\rho-1}}{\left(t^\rho-\frac{1}{b^\rho}\right)^{1-\alpha}}f\left(\frac{1}{t^\rho}\right)dt\\
&= \left(\frac{a^\rho b^\rho}{b^\rho-a^\rho}\right)^\alpha\rho^{\alpha-1}\Gamma(\alpha)\ {}^\rho I^\alpha_{\frac{1}{a}-}(f\circ h)\left(\tfrac{1}{b^\rho}\right)\\
&\le f(a^\rho)\int_0^1 t^{\alpha\rho-1+\rho}dt+f(b^\rho)\int_0^1\left(t^{\alpha\rho-1}-t^{\alpha\rho-1+\rho}\right)dt\\
&= \frac{1}{\alpha\rho}\frac{\alpha f(a^\rho)+f(b^\rho)}{\alpha+1}.
\end{aligned}$$

It means that

$$\begin{aligned}
f\left(\frac{(\alpha+1)\,a^\rho b^\rho}{a^\rho+\alpha b^\rho}\right) &\le \left(\frac{a^\rho b^\rho}{b^\rho-a^\rho}\right)^\alpha\rho^\alpha\Gamma(\alpha+1)\ {}^\rho I^\alpha_{\frac{1}{a}-}(f\circ h)\left(\tfrac{1}{b^\rho}\right)\\
&\le \frac{\alpha f(a^\rho)+f(b^\rho)}{\alpha+1}.
\end{aligned}$$

This completes the proof. $\square$

Remark 17. *In Theorem 16,*

(1) *If one takes* $\rho \to 1$*, one has the inequality* (5).

(2) *If one takes* $\rho \to 1$, *and after that if one takes* $\alpha = 1$, *one has the inequality* (3).

Theorem 18. *Let* $f : I \subset (0,\infty) \to \mathbb{R}$ *be a function such that* $f \in X_c^p(a^\rho, b^\rho)$, *where* $a^\rho, b^\rho \in I$ *with* $a < b$. *If* f *is a harmonically convex function on* $[a^\rho, b^\rho]$, *then the following inequality for katugampola fractional integrals hold:*

$$\begin{aligned}
&\frac{f\left(\frac{(\alpha+1)a^\rho b^\rho}{\alpha a^\rho + b^\rho}\right) + f\left(\frac{(\alpha+1)a^\rho b^\rho}{a^\rho + \alpha b^\rho}\right)}{2} \\
\leq\ & \frac{\left(\frac{a^\rho b^\rho}{b^\rho - a^\rho}\right)^\alpha \rho^\alpha \Gamma(\alpha+1)}{2} \left[{}^\rho I^\alpha_{\frac{1}{b}+} (f\circ h)\left(\tfrac{1}{a^\rho}\right) + {}^\rho I^\alpha_{\frac{1}{a}-} (f\circ h)\left(\tfrac{1}{b^\rho}\right) \right] \\
\leq\ & \frac{f(a^\rho) + f(b^\rho)}{2} \qquad (15)
\end{aligned}$$

where $\alpha > 0$, $\rho > 0$ *and* $h(x) = \frac{1}{x}$

Proof. Adding the inequalities (9) and (12) side by side, then multiplying the resulting inequalities by $\frac{1}{2}$, we have the inequalities (15). □

Remark 19. *In Theorem 18;*

(1) *if one takes* $\rho \to 1$, *one has the inequality* (7).
(2) *if one takes* $\rho \to 1$, *and after that if one takes* $\alpha = 1$, *one has the inequality* (3).

Corollary 20. *The left hand side of* (15) *is better than the left hand side of* (8).

Proof. Since f is harmonically convex on $[a^\rho, b^\rho]$, it is clear from

$$f\left(\frac{2a^\rho b^\rho}{a^\rho + b^\rho}\right) = f\left(\frac{1}{\frac{a^\rho + b^\rho}{2a^\rho b^\rho}}\right) = f\left(\frac{1}{\frac{(\alpha+1)(a^\rho+b^\rho)}{2a^\rho b^\rho(\alpha+1)}}\right)$$

$$= f\left(\frac{1}{\frac{a^\rho + \alpha b^\rho}{2a^\rho b^\rho(\alpha+1)} + \frac{\alpha a^\rho + b^\rho}{2a^\rho b^\rho(\alpha+1)}}\right) = f\left(\frac{\frac{a^\rho b^\rho(\alpha+1)}{a^\rho+\alpha b^\rho}\frac{a^\rho b^\rho(\alpha+1)}{\alpha a^\rho + b^\rho}}{\frac{a^\rho b^\rho(\alpha+1)}{\alpha a^\rho + b^\rho}\frac{1}{2} + \frac{a^\rho b^\rho(\alpha+1)}{a^\rho + \alpha b^\rho}\frac{1}{2}}\right)$$

$$\leq \frac{f\left(\frac{(\alpha+1)a^\rho b^\rho}{a^\rho + \alpha b^\rho}\right) + f\left(\frac{(\alpha+1)a^\rho b^\rho}{\alpha a^\rho + b^\rho}\right)}{2}.$$

□

3. Lemmas

In this section we will prove two new identities used in forward results.

Lemma 21. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a^\rho, b^\rho \in I^\circ$ with $a < b$. If the fractional integrals exist and $f' \in L[a^\rho, b^\rho]$, then the following equality for the left katugampola fractional integral holds:*

$$\frac{\alpha f(a^\rho)+f(b^\rho)}{\alpha+1}-\rho^\alpha\left(\frac{a^\rho b^\rho}{b^\rho-a^\rho}\right)^\alpha\Gamma(\alpha+1)\ \ {}^\rho I^\alpha_{\frac{1}{b}+}(f\circ h)\left(\tfrac{1}{a^\rho}\right)$$

$$=\frac{\rho a^\rho b^\rho\left(b^\rho-a^\rho\right)}{\alpha+1}\int_0^1\frac{\left(1-(\alpha+1)\,t^{\rho\alpha}\right)}{\left(t^\rho b^\rho+(1-t^\rho)a^\rho\right)^2}t^{\rho-1}f'\left(\frac{a^\rho b^\rho}{t^\rho b^\rho+(1-t^\rho)a^\rho}\right)dt \tag{16}$$

where $\alpha > 0$ and $\rho > 0$.

Proof. It could be prove directly by applying the partial integration to the right hand side of the equation (16) as follows:

$$\frac{\rho a^\rho b^\rho\left(b^\rho-a^\rho\right)}{\alpha+1}\int_0^1\frac{\left(1-(\alpha+1)\,t^{\rho\alpha}\right)}{\left(t^\rho b^\rho+(1-t^\rho)a^\rho\right)^2}t^{\rho-1}f'\left(\frac{a^\rho b^\rho}{t^\rho b^\rho+(1-t^\rho)a^\rho}\right)dt$$

$$=\frac{\rho a^\rho b^\rho\left(b^\rho-a^\rho\right)}{\alpha+1}\left[\begin{array}{c}\int_0^1\frac{t^{\rho-1}}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2}f'\left(\frac{a^\rho b^\rho}{t^\rho b^\rho+(1-t^\rho)a^\rho}\right)dt\\ -(\alpha+1)\int_0^1\frac{t^{\rho\alpha}}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2}t^{\rho-1}f'\left(\frac{a^\rho b^\rho}{t^\rho b^\rho+(1-t^\rho)a^\rho}\right)dt\end{array}\right]$$

$$=\frac{\rho a^\rho b^\rho\left(b^\rho-a^\rho\right)}{\alpha+1}\left[\begin{array}{c} f\left(\frac{a^\rho b^\rho}{t^\rho b^\rho+(1-t^\rho)a^\rho}\right)\frac{1}{\rho a^\rho b^\rho(a^\rho-b^\rho)}\Big|_0^1\\ -\frac{(\alpha+1)}{\rho}\left[\begin{array}{c}t^{\alpha\rho}f\left(\frac{a^\rho b^\rho}{t^\rho b^\rho+(1-t^\rho)a^\rho}\right)\frac{1}{a^\rho b^\rho(a^\rho-b^\rho)}\Big|_0^1\\ -\alpha\rho\int_0^1\frac{t^{\alpha\rho-1}}{a^\rho b^\rho(a^\rho-b^\rho)}f\left(\frac{a^\rho b^\rho}{t^\rho b^\rho+(1-t^\rho)a^\rho}\right)dt\end{array}\right]\end{array}\right]$$

$$\begin{aligned}=\ &\frac{1}{\alpha+1}\left(f\left(b^\rho\right)-f\left(a^\rho\right)\right)+f\left(a^\rho\right)\\ &-\alpha\rho\int_0^1\frac{t^{\alpha\rho-1}}{a^\rho b^\rho\left(a^\rho-b^\rho\right)}f\left(\frac{a^\rho b^\rho}{t^\rho b^\rho+(1-t^\rho)a^\rho}\right)dt\end{aligned}$$

$$=\frac{\alpha f(a^\rho)+f(b^\rho)}{\alpha+1}-\alpha\rho\left(\frac{a^\rho b^\rho}{b^\rho-a^\rho}\right)^\alpha\int_{\frac{1}{b}}^{\frac{1}{a}}\frac{t^{\rho-1}}{\left(\frac{1}{a^\rho}-t^\rho\right)^{1-\alpha}}f\left(\frac{1}{t^\rho}\right)dt$$

$$=\frac{\alpha f(a^\rho)+f(b^\rho)}{\alpha+1}-\rho^\alpha\left(\frac{a^\rho b^\rho}{b^\rho-a^\rho}\right)^\alpha\Gamma(\alpha+1)\ \ {}^\rho I^\alpha_{\frac{1}{b}+}(f\circ h)\left(\tfrac{1}{a^\rho}\right)\ .$$

This completes the proof. □

Remark 22. *In Lemma 21,*

(1) *if one takes $\rho \to 1$, one has the inequality* [20, Lemma 3].
(2) *if one takes $\rho \to 1$, and after that if one takes $\alpha = 1$, one has the inequality* [7, 2.5. Lemma].

Lemma 23. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a^\rho, b^\rho \in I^\circ$ with $a^\rho < b^\rho$. If the fractional integrals exist and $f' \in L[a^\rho, b^\rho]$, then the following equality for the right katugampola fractional integral holds:*

$$\begin{aligned} & \frac{f(a^\rho)+\alpha f(b^\rho)}{\alpha+1} - \rho^\alpha \left(\frac{a^\rho b^\rho}{b^\rho - a^\rho}\right)^\alpha \Gamma(\alpha+1) \ {}^\rho I^\alpha_{\frac{1}{a}-} (f \circ h)(\tfrac{1}{b^\rho}) \\ = \ & \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{\alpha+1} \\ & \times \int_0^1 \frac{\left[(\alpha+1)(1-t^\rho)^\alpha - 1\right]}{(t^\rho b^\rho + (1-t^\rho)a^\rho)^2} t^{\rho-1} f'\left(\frac{a^\rho b^\rho}{t^\rho b^\rho + (1-t^\rho)a^\rho}\right) dt \end{aligned} \tag{17}$$

where $\alpha > 0$ and $\rho > 0$.

Proof. It could be prove directly by applying the partial integration to the right hand side of the equation (17) as follows:

$$\begin{aligned} & \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{\alpha+1} \int_0^1 \frac{\left[(\alpha+1)(1-t^\rho)^\alpha - 1\right]}{(t^\rho b^\rho + (1-t^\rho)a^\rho)^2} t^{\rho-1} f'\left(\frac{a^\rho b^\rho}{t^\rho b^\rho + (1-t^\rho)a^\rho}\right) dt \\ = & \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{\alpha+1} \left[\begin{array}{c} (\alpha+1) \int_0^1 \frac{(1-t^\rho)^\alpha}{(t^\rho b^\rho + (1-t^\rho)a^\rho)^2} t^{\rho-1} f'\left(\frac{a^\rho b^\rho}{t^\rho b^\rho + (1-t^\rho)a^\rho}\right) dt \\ - \frac{t^{\rho-1}}{(t^\rho b^\rho + (1-t^\rho)a^\rho)^2} f'\left(\frac{a^\rho b^\rho}{t^\rho b^\rho + (1-t^\rho)a^\rho}\right) dt \end{array} \right] \\ = & \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{\alpha+1} \left[\begin{array}{c} \frac{(\alpha+1)}{\rho} \left[\begin{array}{c} t^{\alpha\rho} f\left(\frac{a^\rho b^\rho}{t^\rho a^\rho + (1-t^\rho)b^\rho}\right) \frac{1}{a^\rho b^\rho (b^\rho - a^\rho)} \Big|_0^1 \\ -\alpha\rho \int_0^1 \frac{t^{\alpha\rho-1}}{a^\rho b^\rho (b^\rho - a^\rho)} f\left(\frac{a^\rho b^\rho}{t^\rho a^\rho + (1-t^\rho)b^\rho}\right) dt \end{array} \right] \\ - f\left(\frac{a^\rho b^\rho}{t^\rho b^\rho + (1-t^\rho)a^\rho}\right) \frac{1}{\rho a^\rho b^\rho (a^\rho - b^\rho)} \Big|_0^1 \end{array} \right] \\ = & \ f(b^\rho) - \alpha\rho \int_0^1 \frac{t^{\alpha\rho-1}}{a^\rho b^\rho (a^\rho - b^\rho)} f\left(\frac{a^\rho b^\rho}{t^\rho b^\rho + (1-t^\rho)a^\rho}\right) dt \\ & + \frac{1}{\alpha+1} \left(f(a^\rho) - f(b^\rho)\right) \\ = & \frac{f(a^\rho)+\alpha f(b^\rho)}{\alpha+1} - \alpha\rho \left(\frac{a^\rho b^\rho}{b^\rho - a^\rho}\right)^\alpha \int_{\frac{1}{b}}^{\frac{1}{a}} \frac{t^{\rho-1}}{\left(t^\rho - \frac{1}{b^\rho}\right)^{1-\alpha}} f\left(\frac{1}{t^\rho}\right) dt \\ = & \frac{f(a^\rho)+\alpha f(b^\rho)}{\alpha+1} - \rho^\alpha \left(\frac{a^\rho b^\rho}{b^\rho - a^\rho}\right)^\alpha \Gamma(\alpha+1) \ {}^\rho I^\alpha_{\frac{1}{a}-} (f \circ h)(\tfrac{1}{b^\rho}) \ . \end{aligned}$$

This completes the proof. □

Remark 24. *In Lemma 23,*

(1) *if one takes $\rho \to 1$, one has the inequality* [20, Lemma 2].
(2) *if one takes $\rho \to 1$, and after that if one takes $\alpha = 1$, one has the inequality* [7, 2.5. Lemma].

4. Some new conformable fractional trapezoid type inequalities for harmonically convex functions

In this section, we will prove some new conformable fractional trapezoid type inequalities for harmonically convex functions by using Lemma 21 and Lemma 23.

Theorem 25. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a^{\rho}, b^{\rho} \in I^{\circ}$ with $a < b$. If $f' \in L[a^{\rho}, b^{\rho}]$ and $|f'|^q$ is harmonically convex on $[a^{\rho}, b^{\rho}]$ for $q \geq 1$, then the following inequality for the left katugampola fractional integral holds:*

$$\left|\frac{\alpha f(a^{\rho}) + f(b^{\rho})}{\alpha + 1} - \rho^{\alpha}\left(\frac{a^{\rho}b^{\rho}}{b^{\rho} - a^{\rho}}\right)^{\alpha}\Gamma(\alpha+1)\ {}^{\rho}I^{\alpha}_{\frac{1}{b}+}(f\circ h)\left(\tfrac{1}{a^{\rho}}\right)\right|$$

$$\begin{aligned}\leq\ & \frac{\rho a^{\rho}b^{\rho}(b^{\rho} - a^{\rho})}{\alpha+1} Z_1(a,b,\alpha,\rho)^{1-\frac{1}{q}} \\ & \left[\left|f'(a^{\rho})\right|^q Z_2(a,b,\alpha,\rho) + \left|f'(b^{\rho})\right|^q Z_3(a,b,\alpha,\rho)\right]^{\frac{1}{q}} \end{aligned} \tag{18}$$

where

$$Z_1(a,b,\alpha,\rho) = \left[\begin{array}{c} \frac{\frac{2}{\rho}\sqrt[\alpha]{\frac{1}{\alpha+1}}}{\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}(b^{\rho}-a^{\rho})+a^{\rho}\right]^2}\ {}_2F_1\left(2,1;2;1-\frac{a^{\rho}}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b^{\rho}-a^{\rho})+a^{\rho}}\right) \\ -\frac{b^{-2\rho}}{\rho}\ {}_2F_1\left(2,1;2;1-\frac{a^{\rho}}{b^{\rho}}\right) \\ -\frac{\frac{2}{\rho(\alpha+1)}\sqrt[\alpha]{\frac{1}{\alpha+1}}}{\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}(b^{\rho}-a^{\rho})+a^{\rho}\right]^2}\ {}_2F_1\left(2,1;\alpha+2;1-\frac{a^{\rho}}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b^{\rho}-a^{\rho})+a^{\rho}}\right) \\ +\frac{1}{\rho}b^{-2}\ {}_2F_1\left(2,1;\alpha+2;1-\frac{a^{\rho}}{b^{\rho}}\right) \end{array}\right],$$

$$Z_2(a,b,\alpha,\rho) = \left[\begin{array}{c} \frac{\frac{2}{\rho}\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}}}{\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}(b^{\rho}-a^{\rho})+a^{\rho}\right]^2}\ {}_2F_1\left(2,1;3;1-\frac{a^{\rho}}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b^{\rho}-a^{\rho})+a^{\rho}}\right) \\ -\frac{1}{2\rho}b^{-2\rho}\ {}_2F_1\left(2,1;3;1-\frac{a^{\rho}}{b^{\rho}}\right) \\ -\frac{\frac{2}{\rho(\alpha+2)}\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}}}{\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}(b^{\rho}-a^{\rho})+a^{\rho}\right]^2}\ {}_2F_1\left(2,1;\alpha+3;1-\frac{a^{\rho}}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b^{\rho}-a^{\rho})+a^{\rho}}\right) \\ +\frac{\alpha+1}{\rho(\alpha+2)}b^{-2}\ {}_2F_1\left(2,1;\alpha+3;1-\frac{a^{\rho}}{b^{\rho}}\right) \end{array}\right],$$

$$Z_3(a,b,\alpha,\rho) = Z_1(a,b,\alpha,\rho) - Z_2(a,b,\alpha,\rho),$$

with $\alpha > 0$ and $\rho > 0$.

Proof. By using Lemma 21, power mean inequality and harmonically convexity of $\left|f'\right|^q$, we have

$$\left|\frac{f(a^{\rho}) + \alpha f(b^{\rho})}{\alpha + 1} - \rho^{\alpha}\left(\frac{a^{\rho}b^{\rho}}{b^{\rho} - a^{\rho}}\right)^{\alpha}\Gamma(\alpha+1)\ {}^{\rho}I^{\alpha}_{\frac{1}{b}+}(f\circ h)\left(\tfrac{1}{a^{\rho}}\right)\right|$$

$$\leq \frac{\rho a^{\rho}b^{\rho}(b^{\rho} - a^{\rho})}{\alpha+1}\int_0^1 \frac{\left|1-(\alpha+1)t^{\rho\alpha}\right|}{\left(t^{\rho}b^{\rho} + (1-t^{\rho})a^{\rho}\right)^2} t^{\rho-1}\left|f'\left(\frac{a^{\rho}b^{\rho}}{t^{\rho}b^{\rho} + (1-t^{\rho})a^{\rho}}\right)\right| dt$$

$$\leq \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{\alpha+1}\left[\begin{array}{c}\left(\int_0^1 \frac{|1-(\alpha+1)t^{\rho\alpha}|}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2} t^{\rho-1}\right)^{1-\frac{1}{q}} \\ \times\left(\int_0^1 \frac{|1-(\alpha+1)t^{\rho\alpha}|}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2} t^{\rho-1}\left|f'\left(\frac{a^\rho b^\rho}{t^\rho b^\rho+(1-t^\rho)a^\rho}\right)\right| dt\right)^{\frac{1}{q}}\end{array}\right]$$

$$\leq \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{\alpha+1}\left[\begin{array}{c}\left(\int_0^1 \frac{|1-(\alpha+1)t^{\rho\alpha}|}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2} t^{\rho-1}\right)^{1-\frac{1}{q}} \\ \times\left(\int_0^1 \frac{|1-(\alpha+1)t^{\rho\alpha}|}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2} t^{\rho-1}\left[\begin{array}{c} t^\rho |f(a^\rho)|^q \\ +(1-t^\rho)|f(b^\rho)|^q\end{array}\right] dt\right)^{\frac{1}{q}}\end{array}\right]$$

$$\leq \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{\alpha+1}\left[\begin{array}{c}\left(\int_0^1 \frac{|1-(\alpha+1)t^{\rho\alpha}|}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2} t^{\rho-1}\right)^{1-\frac{1}{q}} \\ \times\left(\left[\begin{array}{c} |f(a^\rho)|^q \int_0^1 \frac{|1-(\alpha+1)t^{\rho\alpha}|}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2} t^{2\rho-1} dt \\ +|f(b^\rho)|^q \int_0^1 \frac{|1-(\alpha+1)t^{\rho\alpha}|}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2} t^{\rho-1}(1-t^\rho)\, dt\end{array}\right]\right)^{\frac{1}{q}}\end{array}\right] \tag{19}$$

Calculating the appearing integrals in (19) we have,

$$\int_0^1 \frac{|1-(\alpha+1)t^{\rho\alpha}|}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2} t^{\rho-1} dt = \left[\begin{array}{c}\int_0^{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}} \frac{1-(\alpha+1)t^{\rho\alpha}}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2} t^{\rho-1} dt \\ +\int_{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}}^1 \frac{(\alpha+1)t^{\rho\alpha}-1}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2} t^{\rho-1} dt\end{array}\right]$$

$$= \int_0^{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}} \frac{t^{\rho-1}}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2} dt - \int_{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}}^1 \frac{t^{\rho-1}}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2} dt$$

$$-(\alpha+1)\left[\begin{array}{c}\int_0^{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}} \frac{t^{\rho\alpha}}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2} t^{\rho-1} dt \\ -\int_{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}}^1 \frac{t^{\rho\alpha}}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2} t^{\rho-1} dt\end{array}\right]$$

$$= 2\int_0^{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}} \frac{t^{\rho-1}}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2} dt - \int_0^1 \frac{t^{\rho-1}}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2} dt$$

$$-(\alpha+1)\left[\begin{array}{c}2\int_0^{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}} \frac{t^{\rho\alpha}}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2} t^{\rho-1} dt \\ -\int_0^1 \frac{t^{\rho\alpha}}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2} t^{\rho-1} dt\end{array}\right]$$

$$= \frac{2}{\rho}\sqrt[\alpha]{\frac{1}{\alpha+1}} \int_0^1 \frac{1}{\left(\sqrt[\alpha]{\frac{1}{\alpha+1}} u b^\rho + \left(1-\sqrt[\alpha]{\frac{1}{\alpha+1}} u\right) a^\rho\right)^2} du$$

$$-\frac{1}{\rho}\int_0^1 \frac{1}{(u^\rho a^\rho + (1-u^\rho) b^\rho)^2} du$$

$$-\left[\begin{array}{c}\frac{2}{\rho}\sqrt[\alpha]{\frac{1}{\alpha+1}} \int_0^1 \frac{u^\alpha}{\left(\sqrt[\alpha]{\frac{1}{\alpha+1}} u b^\rho + \left(1-\sqrt[\alpha]{\frac{1}{\alpha+1}} u\right) a^\rho\right)^2} du \\ -\frac{(\alpha+1)}{\rho}\int_0^1 \frac{(1-u)^\alpha}{(u^\rho a^\rho + (1-u^\rho) b^\rho)^2} du\end{array}\right]$$

$$
\begin{aligned}
&= \frac{2}{\rho}\sqrt[\alpha]{\frac{1}{\alpha+1}}\int_0^1 \frac{1}{\left(\sqrt[\alpha]{\frac{1}{\alpha+1}}\left(1-v\right)b^\rho+\left(1-\sqrt[\alpha]{\frac{1}{\alpha+1}}\left(1-v\right)\right)a^\rho\right)^2}dv \\
&\quad -\frac{1}{\rho}\int_0^1 \frac{1}{\left(u^\rho a^\rho+\left(1-u^\rho\right)b^\rho\right)^2}du \\
&\quad -\left[\begin{array}{c} \frac{2}{\rho}\sqrt[\alpha]{\frac{1}{\alpha+1}}\int_0^1 \frac{(1-v)^\alpha}{\left(\sqrt[\alpha]{\frac{1}{\alpha+1}}(1-v)b^\rho+\left(1-\sqrt[\alpha]{\frac{1}{\alpha+1}}(1-v)\right)a^\rho\right)^2}dv \\ -\frac{(\alpha+1)}{\rho}\int_0^1 \frac{(1-u)^\alpha}{\left(u^\rho a^\rho+(1-u^\rho)b^\rho\right)^2}du \end{array}\right] \\
&= \frac{2}{\rho}\sqrt[\alpha]{\frac{1}{\alpha+1}}\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}\left(b^\rho-a^\rho\right)+a^\rho\right]^{-2} \\
&\quad \times\int_0^1\left(1-v\left[1-\frac{a^\rho}{\sqrt[\alpha]{\frac{1}{\alpha+1}}\left(b^\rho-a^\rho\right)+a^\rho}\right]\right)^{-2}dv \\
&\quad -\frac{b^{-2\rho}}{\rho}\int_0^1\left(1-u\left(1-\frac{a^\rho}{b^\rho}\right)\right)^{-2}du \\
&\quad -\left[\begin{array}{c} \frac{2}{\rho}\sqrt[\alpha]{\frac{1}{\alpha+1}}\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}\left(b^\rho-a^\rho\right)+a^\rho\right]^{-2} \\ \times\int_0^1(1-v)^\alpha\left(1-v\left[1-\frac{a^\rho}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b^\rho-a^\rho)+a^\rho}\right]\right)^{-2}dv \\ -\frac{(\alpha+1)}{\rho}b^{-2\rho}\int_0^1(1-u)^\alpha\left(1-u\left(1-\frac{a^\rho}{b^\rho}\right)\right)^{-2}du \end{array}\right] \\
&= \left[\begin{array}{c} \frac{2}{\rho}\sqrt[\alpha]{\frac{1}{\alpha+1}}\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}\left(b^\rho-a^\rho\right)+a^\rho\right]^{-2} \\ \times{}_2F_1\left(2,1;2;1-\frac{a^\rho}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b^\rho-a^\rho)+a^\rho}\right) \\ -\frac{b^{-2\rho}}{\rho}\ {}_2F_1\left(2,1;2;1-\frac{a^\rho}{b^\rho}\right) \\ -\frac{2}{\rho(\alpha+1)}\sqrt[\alpha]{\frac{1}{\alpha+1}}\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}\left(b^\rho-a^\rho\right)+a^\rho\right]^{-2} \\ \times{}_2F_1\left(2,1;\alpha+2;1-\frac{a^\rho}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b^\rho-a^\rho)+a^\rho}\right) \\ +\frac{1}{\rho}b^{-2\rho}\ {}_2F_1\left(2,1;\alpha+2;1-\frac{a^\rho}{b^\rho}\right) \end{array}\right] \\
&= Z_1\left(a,b,\alpha,\rho\right),
\end{aligned} \tag{20}
$$

and

$$
\int_0^1 \frac{\left|1-(\alpha+1)\,t^{\rho\alpha}\right|}{\left(t^\rho b^\rho+(1-t^\rho)a^\rho\right)^2}t^{2\rho-1}dt = \left[\begin{array}{c} \int_0^{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}}\frac{1-(\alpha+1)t^{\rho\alpha}}{\left(t^\rho b^\rho+(1-t^\rho)a^\rho\right)^2}t^{2\rho-1}dt \\ +\int_{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}}^1\frac{(\alpha+1)t^{\rho\alpha}-1}{\left(t^\rho b^\rho+(1-t^\rho)a^\rho\right)^2}t^{2\rho-1}dt \end{array}\right]
$$

$$
=\int_0^{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}}\frac{t^{2\rho-1}}{\left(t^\rho b^\rho+(1-t^\rho)a^\rho\right)^2}dt-\int_{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}}^1\frac{t^{2\rho-1}}{\left(t^\rho b^\rho+(1-t^\rho)a^\rho\right)^2}dt
$$

$$-(\alpha+1)\left[\begin{array}{c}\int_0^{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}}\frac{t^{\rho\alpha}}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2}t^{2\rho-1}dt\\ -\int_{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}}^1\frac{t^{\rho\alpha}}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2}t^{2\rho-1}dt\end{array}\right]$$

$$= 2\int_0^{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}}\frac{t^{2\rho-1}}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2}dt$$

$$-\int_0^1\frac{t^{2\rho-1}}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2}dt$$

$$-(\alpha+1)\left[\begin{array}{c}2\int_0^{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}}\frac{t^{\rho\alpha}}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2}t^{2\rho-1}dt\\ -\int_0^1\frac{t^{\rho\alpha}}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^2}t^{2\rho-1}dt\end{array}\right]$$

$$= \frac{2}{\rho}\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}}\int_0^1\frac{u}{\left(\sqrt[\alpha]{\frac{1}{\alpha+1}}ub^\rho+\left(1-\sqrt[\alpha]{\frac{1}{\alpha+1}}u\right)a^\rho\right)^2}du$$

$$-\frac{1}{\rho}\int_0^1\frac{1-u}{(u^\rho a^\rho+(1-u^\rho)b^\rho)^2}du$$

$$-\left[\begin{array}{c}\frac{2}{\rho}\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}}\int_0^1\frac{u^{\alpha+1}}{\left(\sqrt[\alpha]{\frac{1}{\alpha+1}}ub^\rho+\left(1-\sqrt[\alpha]{\frac{1}{\alpha+1}}u\right)a^\rho\right)^2}du\\ -\frac{(\alpha+1)}{\rho}\int_0^1\frac{(1-u)^{\alpha+1}}{(u^\rho a^\rho+(1-u^\rho)b^\rho)^2}du\end{array}\right]$$

$$= \frac{2}{\rho}\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}}\int_0^1\frac{1-v}{\left(\sqrt[\alpha]{\frac{1}{\alpha+1}}(1-v)b^\rho+\left(1-\sqrt[\alpha]{\frac{1}{\alpha+1}}(1-v)\right)a^\rho\right)^2}dv$$

$$-\frac{1}{\rho}\int_0^1\frac{1-u}{(u^\rho a^\rho+(1-u^\rho)b^\rho)^2}du$$

$$-\left[\begin{array}{c}\frac{2}{\rho}\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}}\int_0^1\frac{(1-v)^{\alpha+1}}{\left(\sqrt[\alpha]{\frac{1}{\alpha+1}}(1-v)b^\rho+\left(1-\sqrt[\alpha]{\frac{1}{\alpha+1}}(1-v)\right)a^\rho\right)^2}dv\\ -\frac{(\alpha+1)}{\rho}\int_0^1\frac{(1-u)^{\alpha}}{(u^\rho a^\rho+(1-u^\rho)b^\rho)^2}du\end{array}\right]$$

$$= \frac{2}{\rho}\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}}\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}(b^\rho-a^\rho)+a^\rho\right]^{-2}$$

$$\times\int_0^1\left(1-v\left[1-\frac{a^\rho}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b^\rho-a^\rho)+a^\rho}\right]\right)^{-2}(1-v)\,dv$$

$$-\frac{b^{-2\rho}}{\rho}\int_0^1\left(1-u\left(1-\frac{a^\rho}{b^\rho}\right)\right)^{-2}(1-u)\,du$$

$$-\left[\begin{array}{c} \frac{2}{\rho}\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}}\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}\left(b^{\rho}-a^{\rho}\right)+a^{\rho}\right]^{-2} \\ \times\int_0^1(1-v)^{\alpha+1}\left(1-v\left[1-\frac{a^{\rho}}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b^{\rho}-a^{\rho})+a^{\rho}}\right]\right)^{-2}dv \\ -\frac{(\alpha+1)}{\rho}b^{-2\rho}\int_0^1(1-u)^{\alpha}\left(1-u\left(1-\frac{a^{\rho}}{b^{\rho}}\right)\right)^{-2}du \end{array}\right]$$

$$=\left[\begin{array}{c} \frac{2}{\rho}\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}}\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}\left(b^{\rho}-a^{\rho}\right)+a^{\rho}\right]^{-2} \\ {}_2F_1\left(2,1;3;1-\frac{a^{\rho}}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b^{\rho}-a^{\rho})+a^{\rho}}\right) \\ -\frac{1}{2\rho}b^{-2\rho}\ {}_2F_1\left(2,1;3;1-\frac{a^{\rho}}{b^{\rho}}\right) \\ -\frac{2}{\rho(\alpha+2)}\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}}\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}\left(b^{\rho}-a^{\rho}\right)+a^{\rho}\right]^{-2} \\ \times{}_2F_1\left(2,1;\alpha+3;1-\frac{a^{\rho}}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b^{\rho}-a^{\rho})+a^{\rho}}\right) \\ +\frac{\alpha+1}{\rho(\alpha+2)}b^{-2}\ {}_2F_1\left(2,1;\alpha+3;1-\frac{a^{\rho}}{b^{\rho}}\right) \end{array}\right]$$

$$= Z_2\left(a,b,\alpha,\rho\right), \tag{21}$$

and

$$\int_0^1 \frac{\left|1-(\alpha+1)\,t^{\rho\alpha}\right|}{\left(t^{\rho}b^{\rho}+(1-t^{\rho})a^{\rho}\right)^2}t^{\rho-1}\left(1-t^{\rho}\right)dt = \left[\begin{array}{c} \int_0^1 \frac{|1-(\alpha+1)t^{\rho\alpha}|}{(t^{\rho}b^{\rho}+(1-t^{\rho})a^{\rho})^2}t^{\rho-1} \\ -\int_0^1 \frac{|1-(\alpha+1)t^{\rho\alpha}|}{(t^{\rho}b^{\rho}+(1-t^{\rho})a^{\rho})^2}t^{2\rho-1} \end{array}\right]$$

$$Z_3\left(a,b,\alpha,\rho\right) = Z_1\left(a,b,\alpha,\rho\right) - Z_2\left(a,b,\alpha,\rho\right). \tag{22}$$

If we use (20) − (22) in (19), we have (18). This completes the proof. □

Remark 26. *In Theorem 25,*

(1) *if one takes $\rho \to 1$, one has the inequality* [20, Theorem 7].
(2) *if one takes $\rho \to 1$, and after that if one takes $\alpha = 1$, one has the inequality* [7, 2.6. Theorem].

Theorem 27. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a^{\rho}, b^{\rho} \in I^{\circ}$ with $a^{\rho} < b^{\rho}$. If $f' \in L\left[a^{\rho}, b^{\rho}\right]$ and $|f'|^q$ is harmonically convex on $\left[a^{\rho}, b^{\rho}\right]$ for $q > 1$ and $\frac{1}{q} + \frac{1}{p} = 1$, then the following inequality for the left katugampola fractional integral holds:*

$$\begin{aligned} &\left|\frac{\alpha f(a^{\rho}) + f(b^{\rho})}{\alpha+1} - \rho^{\alpha}\left(\frac{a^{\rho}b^{\rho}}{b^{\rho}-a^{\rho}}\right)^{\alpha}\Gamma\left(\alpha+1\right)\ {}^{\rho}I^{\alpha}_{\frac{1}{b}+}\left(f\circ h\right)\left(\tfrac{1}{a^{\rho}}\right)\right| \\ &\quad\leq \frac{\rho a^{\rho}b^{\rho}\left(b^{\rho}-a^{\rho}\right)}{\alpha+1}Z_4\left(a,b,\alpha,\rho\right)^{\frac{1}{p}} \\ &\quad\quad \left(\left|f'\left(a^{\rho}\right)\right|^q Z_5\left(a,b,\alpha,\rho\right) + \left|f'\left(b^{\rho}\right)\right|^q Z6\left(\alpha,\rho,p\right)\right)^{\frac{1}{q}} \end{aligned} \tag{23}$$

where

$$Z_4\left(\alpha,\rho,p\right) = \frac{1}{\sqrt[\alpha\rho]{\alpha+1}\left(\alpha\rho p+1\right)} + \frac{\left(\sqrt[\alpha\rho]{\alpha+1}-1\right)^{\alpha\rho p+1}}{\sqrt[\alpha\rho]{\alpha+1}\left(\alpha\rho p+1\right)},$$

$$Z_5\left(a,b,q,\rho\right)=\frac{b^{-2q\rho}}{2\rho}\ {}_2F_1\left(2q,1;3;1-\tfrac{a^\rho}{b^\rho}\right)\ ,$$
$$Z_6\left(a,b,q,\rho\right)==\frac{b^{-2q\rho}}{2\rho}\ {}_2F_1\left(2q,2;3;1-\tfrac{a^\rho}{b^\rho}\right)\ ,$$

with $\alpha>0$ and $\rho>0$.

Proof. By using Lemma 21, Hölder inequality and harmonically convexity of $\left|f'\right|^q$, we have

$$\begin{aligned}
&\left|\frac{\alpha f(a^\rho)+f(b^\rho)}{\alpha+1}-\rho^\alpha\left(\frac{a^\rho b^\rho}{b^\rho-a^\rho}\right)^\alpha\Gamma\left(\alpha+1\right)\ {}^\rho I^\alpha_{\frac{1}{b}+}\left(f\circ h\right)\left(\tfrac{1}{a^\rho}\right)\ \right|\\
&\le\frac{\rho a^\rho b^\rho\left(b^\rho-a^\rho\right)}{\alpha+1}\int_0^1\frac{\left|1-\left(\alpha+1\right)t^{\rho\alpha}\right|}{\left(t^\rho b^\rho+(1-t^\rho)a^\rho\right)^2}t^{\rho-1}\left|f'\left(\frac{a^\rho b^\rho}{t^\rho b^\rho+(1-t^\rho)a^\rho}\right)\right|dt\\
&\le\frac{\rho a^\rho b^\rho\left(b^\rho-a^\rho\right)}{\alpha+1}\left[\begin{array}{c}\left(\int_0^1\left|1-\left(\alpha+1\right)t^{\rho\alpha}\right|^p\right)^{\frac{1}{p}}\\ \times\left(\int_0^1\frac{t^{\rho-1}}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^{2q}}\left|f'\left(\frac{a^\rho b^\rho}{t^\rho b^\rho+(1-t^\rho)a^\rho}\right)\right|^q dt\right)^{\frac{1}{q}}\end{array}\right]\\
&\le\frac{\rho a^\rho b^\rho\left(b^\rho-a^\rho\right)}{\alpha+1}\left[\begin{array}{c}\left(\int_0^1\left|1-\left(\alpha+1\right)t^{\rho\alpha}\right|^p\right)^{\frac{1}{p}}\\ \times\left(\int_0^1\frac{t^{\rho-1}}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^{2q}}\left[t^\rho\left|f\left(a^\rho\right)\right|^q+\left(1-t^\rho\right)\left|f\left(b^\rho\right)\right|^q\right]dt\right)^{\frac{1}{q}}\end{array}\right]\\
&\le\quad\frac{\rho a^\rho b^\rho\left(b^\rho-a^\rho\right)}{\alpha+1}\left(\int_0^1\left|1-\left(\alpha+1\right)t^{\rho\alpha}\right|^p\right)^{\frac{1}{p}}\\
&\quad\times\left(\left[\begin{array}{c}\left|f\left(a^\rho\right)\right|^q\int_0^1\frac{t^{2\rho-1}}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^{2q}}dt\\ +\left|f\left(b^\rho\right)\right|^q\int_0^1\frac{t^{\rho-1}}{(t^\rho b^\rho+(1-t^\rho)a^\rho)^{2q}}\left(1-t^\rho\right)dt\end{array}\right]\right)^{\frac{1}{q}} \qquad (24)
\end{aligned}$$

Calculating the appearing integrals in (20), we have

$$\begin{aligned}
\int_0^1\frac{t^{2\rho-1}}{\left(t^\rho b^\rho+(1-t^\rho)a^\rho\right)^2}dt&=b^{-2q\rho}\int_0^1 t\left(1-t\left(1-\frac{a^\rho}{b^\rho}\right)\right)^{-2q}dt\\
&=\frac{b^{-2q\rho}}{2\rho}\ {}_2F_1\left(2q,1;3;1-\tfrac{a^\rho}{b^\rho}\right)\\
&=Z_5\left(a,b,q,\rho\right) \qquad (25)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1\frac{t^{\rho-1}}{\left(t^\rho b^\rho+(1-t^\rho)a^\rho\right)^{2q}}\left(1-t^\rho\right)dt&=b^{-2q\rho}\int_0^1\left(1-t\right)\left(1-t\left(1-\frac{a^\rho}{b^\rho}\right)\right)^{-2q}dt\\
&=\frac{b^{-2q\rho}}{2\rho}\ {}_2F_1\left(2q,2;3;1-\tfrac{a^\rho}{b^\rho}\right)\\
&=Z_6\left(a,b,q,\rho\right), \qquad (26)
\end{aligned}$$

and if we use the Lemma 9 for the following integrals, we have

$$\int_0^1 |1-(\alpha+1)t^{\rho\alpha}|^p\,dt = \int_0^{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}} (1-(\alpha+1)t^{\rho\alpha})^p\,dt + \int_{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}}^1 ((\alpha+1)t^{\rho\alpha}-1)^p\,dt$$

$$\leq \int_0^{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}} \left(1-\sqrt[\alpha\rho]{\alpha+1}t\right)^{\alpha\rho p}dt + \int_{\sqrt[\alpha]{\frac{1}{\alpha+1}}}^1 \left(\sqrt[\alpha\rho]{\alpha+1}t-1\right)^{\alpha\rho p}dt$$

$$= \left.\frac{\left(1-\sqrt[\alpha\rho]{\alpha+1}t\right)^{\alpha\rho p+1}}{-\sqrt[\alpha\rho]{\alpha+1}\,(\alpha\rho p+1)}\right|_0^{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}} + \left.\frac{\left(\sqrt[\alpha\rho]{\alpha+1}t-1\right)^{\alpha\rho p+1}}{\sqrt[\alpha\rho]{\alpha+1}\,(\alpha\rho p+1)}\right|_{\sqrt[\alpha\rho]{\frac{1}{\alpha+1}}}^1$$

$$= \frac{1}{\sqrt[\alpha\rho]{\alpha+1}\,(\alpha\rho p+1)} + \frac{\left(\sqrt[\alpha\rho]{\alpha+1}-1\right)^{\alpha\rho p+1}}{\sqrt[\alpha\rho]{\alpha+1}\,(\alpha\rho p+1)}$$

$$= Z_4(\alpha,\rho,p). \tag{27}$$

If we use (25) – (27) in (24), we have (23). This completes the proof. □

Remark 28. *In Theorem 27,*

(1) *if one takes $\rho\to 1$, one has the inequality* [20, Theorem 8].
(2) *if one takes $\rho\to 1$, and after that if one takes $\alpha=1$, one has the inequality* [7, 2.7. Theorem].

Theorem 29. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, $a^\rho, b^\rho\in I^\circ$ with $a^\rho<b^\rho$. If $f'\in L[a^\rho,b^\rho]$ and $|f'|^q$ is harmonically convex on $[a^\rho,b^\rho]$ for $q\geq 1$, then the following inequality for the right katugampola fractional integral holds:*

$$\left|\frac{f(a^\rho)+\alpha f(b^\rho)}{\alpha+1} - \rho^\alpha\left(\frac{a^\rho b^\rho}{b^\rho-a^\rho}\right)^\alpha \Gamma(\alpha+1)\ {}^\rho I^\alpha_{\frac{1}{a}-}(f\circ h)\left(\tfrac{1}{b^\rho}\right)\right|$$

$$\leq \frac{\rho a^\rho b^\rho (b^\rho-a^\rho)}{\alpha+1} Z_7(a,b,\alpha,\rho)^{1-\frac{1}{q}}$$
$$\left(|f'(a^\rho)|^q Z_8(a,b,\alpha,\rho) + |f'(b^\rho)|^q Z_9(a,b,\alpha,\rho)\right)^{\frac{1}{q}} \tag{28}$$

where

$$Z_7(a,b,\alpha,\rho) = \begin{bmatrix} \frac{\frac{2}{\rho}\sqrt[\alpha]{\frac{1}{\alpha+1}}}{\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}(b^\rho-a^\rho)+a^\rho\right]^2} \\ \times {}_2F_1\left(2,1;2;1-\frac{a^\rho}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b^\rho-a^\rho)+a^\rho}\right) \\ -\frac{b^{-2\rho}}{\rho}\ {}_2F_1\left(2,1;2;1-\frac{a^\rho}{b^\rho}\right) \\ -\frac{\frac{2}{\rho(\alpha+1)}\sqrt[\alpha]{\frac{1}{\alpha+1}}}{\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}(b^\rho-a^\rho)+a^\rho\right]^2} \\ \times {}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a^\rho}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b^\rho-a^\rho)+a^\rho}\right) \\ +\frac{1}{\rho}b^{-2}\ {}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a^\rho}{b^\rho}\right) \end{bmatrix},$$

$$Z_8(a,b,\alpha,\rho)=\left[\begin{array}{c}\frac{\frac{2}{\rho}\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}}}{\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}(b^\rho-a^\rho)+a^\rho\right]^2}\\ \times{}_2F_1\left(2,2;3;1-\frac{a^\rho}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b^\rho-a^\rho)+a^\rho}\right)\\ -\frac{1}{2\rho}b^{-2\rho}\ {}_2F_1\left(2,2;3;1-\frac{a^\rho}{b^\rho}\right)\\ -\frac{\frac{2}{\rho(\alpha+2)}\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}}}{\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}(b^\rho-a^\rho)+a^\rho\right]^2}\\ \times{}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{a^\rho}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b^\rho-a^\rho)+a^\rho}\right)\\ +\frac{\alpha+1}{\rho(\alpha+2)}b^{-2}\ {}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{a^\rho}{b^\rho}\right)\end{array}\right],$$

$$Z_9(a,b,\alpha,\rho)=Z_7(a,b,\alpha,\rho)-Z_8(a,b,\alpha,\rho),$$

with $\alpha>0$ *and* $\rho>0$.

Proof. Similarly the proof of the Theorem 25, by using Lemma 23, power mean inequality and harmonically convexity of $\left|f'\right|^q$, we have (28). □

Remark 30. *In Theorem 29,*

(1) *if one takes* $\rho\to 1$, *one has the inequality* [20, Theorem 9].
(2) *if one takes* $\rho\to 1$, *and after that if one takes* $\alpha=1$, *one has the inequality* [7, 2.6. Theorem].

Theorem 31. *Let* $f:I\subseteq\mathbb{R}\to\mathbb{R}$ *be a differentiable function on* I°, $a^\rho,b^\rho\in I^\circ$ *with* $a^\rho<b^\rho$. *If* $f'\in L[a^\rho,b^\rho]$ *and* $|f'|^q$ *is harmonically convex on* $[a^\rho,b^\rho]$ *for* $q>1$ *and* $\frac{1}{q}+\frac{1}{p}=1$, *then the following inequality for the right katugampola fractional integral holds:*

$$\begin{aligned}&\left|\frac{f(a^\rho)+\alpha f(b^\rho)}{\alpha+1}-\rho^\alpha\left(\frac{a^\rho b^\rho}{b^\rho-a^\rho}\right)^\alpha\Gamma(\alpha+1)\ {}^\rho I^\alpha_{\frac{1}{a}-}(f\circ h)\left(\tfrac{1}{b^\rho}\right)\right|\\ &\quad\le\frac{\rho a^\rho b^\rho(b^\rho-a^\rho)}{\alpha+1}Z_4(a,b,\alpha,\rho)^{\frac{1}{p}}\\ &\qquad\left(|f'(a^\rho)|^q Z_5(a,b,\alpha,\rho)+|f'(b^\rho)|^q Z_6(\alpha,\rho,p)\right)^{\frac{1}{q}}\end{aligned}\tag{29}$$

where $Z_4(\alpha,\rho,p)$, $Z_5(a,b,q,\rho)$, $Z_6(a,b,q,\rho)$ *are same as in Theorem 27 and* $\alpha>0$, $\rho>0$.

Proof. Similarly the proof of the Theorem 27, by using Lemma 23, Hölder inequality and harmonically convexity of $\left|f'\right|^q$, we have (29). □

Remark 32. *In Theorem 31,*

(1) *if one takes* $\rho\to 1$, *one has the inequality* [20, Theorem 10].
(2) *if one takes* $\rho\to 1$, *and after that if one takes* $\alpha=1$, *one has the inequality* [7, 2.7. Theorem].

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