Available online: April 10, 2019

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 68, Number 2, Pages 1733–1741 (2019)
DOI: 10.31801/cfsuasmas.551883
ISSN 1303–5991 E-ISSN 2618-6470
http://communications.science.ankara.edu.tr/index.php?series=A1

COMMUNICATIONS
SERIES A1

(p,q)-FIBONACCI AND (p,q)-LUCAS SUMS BY THE DERIVATIVES OF SOME POLYNOMIALS

B. DEMİRTÜRK BİTİM AND N. TOPAL

Abstract. The main purpose of this paper is to survey several sum formulae of $(p,q)-$Fibonacci number U_n and $(p,q)-$Lucas number V_n by using the first and the second derivatives of the equations

$$x^n = (x^2 - p\,x + q)\left(\sum_{j=0}^{n-1} U_j\, x^{n-1-j}\right) + U_n\, x - qU_{n-1}$$

and

$$2\,x^{n+1} - p\,x^n = (x^2 - p\,x + q)\left(\sum_{j=0}^{n-1} V_j\, x^{n-1-j}\right) + V_n\, x - qV_{n-1}.$$

1. Introduction

In [8], Horadam defined the sequence $\{W_n\} = \{W_n(a,b;p,q)\}$ by

$$W_n = pW_{n-1} - qW_{n-2},\ W_0 = a, W_1 = b$$

for $n \geq 2$, where $a,b,p,q \in \mathbb{Z}$ and $p > 0, q \neq 0$. The characteristic equation of this sequence is $x^2 - px + q = 0$ and indicating the distinct roots of this equation with α and β, the Binet formula for $\{W_n\}$ is given by

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}$$

where $A = b - a\beta$ and $B = b - a\alpha$, as in [13].

In this paper we consider two special cases of the Horadam sequence by determining two initial conditions. For example, the $(p,q)-$Fibonacci sequence notated by $\{U_n(p,q)\}$ is defined by

$$U_n = p\,U_{n-1} - qU_{n-2},\ U_0 = 0, U_1 = 1$$

Received by the editors: October 31, 2017; Accepted: February 16, 2019.
2010 *Mathematics Subject Classification.* Primary 11B39; Secondary 11Y55.
Key words and phrases. Fibonacci polynomial, Lucas polynomial, sum of products.

for $n \geq 2$. The $(p,q)-$Lucas sequence notated by $\{V_n(p,q)\}$ is defined by

$$V_n = p\,V_{n-1} - qV_{n-2},\ V_0 = 2,\ V_1 = p$$

for $n \geq 2$.

The Binet formulae for these sequences can be given by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n$$

where $\alpha = \dfrac{p + \sqrt{p^2 - 4q}}{2}$ and $\beta = \dfrac{p - \sqrt{p^2 - 4q}}{2}$. In this work, we will denote $\Delta = \sqrt{p^2 - 4q}$. Also, for convenience we will use $\{U_n\}$ and $\{V_n\}$ instead of $\{U_n(p,q)\}$ and $\{V_n(p,q)\}$, respectively.

The sequences $\{U_n\}$ and $\{V_n\}$ have been studied in [3],[7],[9],[11],[13], and [14]. Notably, if $p = -q = 1, \{U_n\}$ and $\{V_n\}$ are the well known Fibonacci and Lucas sequences.

Note that α and β are the roots of characteristic equation $x^2 - px + q = 0$. Thus it follows that $\alpha^2 = p\alpha - q$ and $\beta^2 = p\beta - q$. More general it can be seen that $\alpha^n = \alpha U_n - qU_{n-1}$ and $\beta^n = \beta U_n - qU_{n-1}$, for all $n \in \mathbb{Z}$. Moreover the terms of these sequences can be extended to negative subscripts as $U_{-n} = -q^{-n}U_n$ and $V_{-n} = q^{-n}\,V_n$ for $n \geq 1$.

Sums and formulae involving Fibonacci and Lucas numbers and their generalizations have always attracted the attention of many authors. For example, Long obtained sums such as $\sum\limits_{i=1}^{n} F_i\,F_{i+d}$ and their Lucas counterparts $\sum\limits_{i=1}^{n} L_i\,L_{i+d}$ and the mixed sums $\sum\limits_{i=1}^{n} F_i\,L_{i+d}$, when d is a positive integer, in [10]. Melham [12] considered two special cases of the Horadam sequence $\{W_n(a,b;p,1)\}$ by taking $a = 0, b = 1$ and $a = 2, b = 1$. He gave sums of powers of the terms of these two special cases of Horadam sequence. Moreover Belbachir and Becherif dealt with sums and formulae for products of generalized Fibonacci and Lucas numbers, in [2]. Also Čerin tackled some alternating sums of Lucas numbers in [5].

In 2008, Amini [1] considered the identity

$$x^n = \left(x^2 - x - 1\right)\left(\sum_{j=1}^{n-1} F_j\,x^{n-1-j}\right) + F_n\,x + F_{n-1}$$

where F_n is the $n-$th Fibonacci number. Then in [6], Cerin, Bitim and Keskin generalized Amini's identity and noticed another identity as follows:

$$2\,x^{n+1} - p\,x^n = \left(x^2 - p\,x - 1\right)\left(\sum_{j=0}^{n-1} V_j^*\,x^{n-1-j}\right) + V_n^*\,x + V_{n-1}^*,$$

where $V_n^* = pV_{n-1}^* + V_{n-2}^*$ for $n \geq 2$ with the initial conditions $V_0^* = 2, V_1^* = p$.

In this study we consider two more generalizations of the identities given in [6], as follows:

$$x^n = \left(x^2 - p\,x + q\right)\left(\sum_{j=0}^{n-1} U_j\, x^{n-1-j}\right) + U_n\, x - qU_{n-1}. \tag{1.1}$$

and

$$2\,x^{n+1} - p\,x^n = \left(x^2 - p\,x + q\right)\left(\sum_{j=0}^{n-1} V_j\, x^{n-1-j}\right) + V_n\, x - qV_{n-1}, \tag{1.2}$$

where U_n and V_n are the $(p, q)-$Fibonacci and (p, q)–Lucas numbers, respectively.

By derivatives of these identities we obtain some sums of products of the (p, q)–Fibonacci numbers and the (p, q)–Lucas numbers. Some similar results were investigated by Cerin in [4], Belbachir and Bencherif in [2]. In [4], Cerin gave the formulae for the sums,

$$\begin{aligned}
\Psi_1 &= \sum_{j=0}^{n} U_{a+bj}(p, q) U_{c+dj}(p, q), \\
\Psi_2 &= \sum_{j=0}^{n} U_{a+bj}(p, q) V_{c+dj}(p, q),
\end{aligned}$$

and

$$\Psi_3 = \sum_{j=0}^{n} V_{a+bj}(p, q) V_{c+dj}(p, q),$$

when $n \geq 0, a \geq 0, c \geq 0, b > 0$ and $d > 0$ are integers. In the proofs, he used the formula for the sum of the geometric series. Belbachir and Bencherif have found these formulae only in the cases when $q = \pm 1$ and $b = d = 2$, in [2]. But we obtain some formulae similar to Ψ_1, Ψ_2 and Ψ_3, by using the first and the second derivatives of the identities given in (1.1) and (1.2). The identities (2.11), (2.12) and (2.13) given in Theorem 1 are different cases of the formulae Ψ_1, Ψ_2 and Ψ_3 given in [4], since b and d are positive integers. To sum up, in this paper we extend all results given in [6].

2. Sums From The Derivatives of The Identities (1.1) and (1.2)

In this section we will use $P(x)$ and $Q(x)$ to denote polynomials $\sum\limits_{j=0}^{n-1} U_j\, x^{n-1-j}$ and $\sum\limits_{j=0}^{n-1} V_j\, x^{n-1-j}$, respectively where n is a natural number. Moreover, in proofs of our theorems we will use the following identities. For all $n, m \in \mathbb{Z}$,

$$U_{-n} = -q^{-n}U_n, \tag{2.1}$$
$$V_{-n} = q^{-n}V_n, \tag{2.2}$$
$$U_{2n} = U_nV_n, \tag{2.3}$$
$$V_n^2 = \Delta^2U_n^2 + 4q^n, \tag{2.4}$$
$$V_{2n} = V_n^2 - 2q^n, \tag{2.5}$$
$$V_n - pU_n = -2qU_{n-1} \tag{2.6}$$
$$V_{n+1} - qV_{n-1} = \Delta^2U_n, \tag{2.7}$$
$$V_n + pU_n = 2U_{n+1}, \tag{2.8}$$
$$pV_n + \Delta^2U_n = 2V_{n+1}, \tag{2.9}$$
$$q^mV_{n-2m} = V_n - \Delta^2U_mU_{n-m}. \tag{2.10}$$

These identities can be found easily by using Binet formulae.

Theorem 1. *If n is a natural number, then*

$$\Delta^2\sum_{j=0}^{n} U_j\, U_{n-j} = nV_n - pU_n, \tag{2.11}$$

$$\sum_{j=0}^{n} U_j\, V_{n-j} = \sum_{j=0}^{n} V_j\, U_{n-j} = (n+1)\, U_n \tag{2.12}$$

and

$$\sum_{j=0}^{n} V_j\, V_{n-j} = (n+1)\, V_n + 2U_{n+1}. \tag{2.13}$$

Proof. By the first derivative of (1.1) we get

$$nx^{n-1} = (2\,x - p)P(x) + (x^2 - p\,x + q)P'(x) + U_n. \tag{2.14}$$

Let $f(x) = n\,x^{n-1} - U_n$. Thus, for $x = \alpha$ and $x = \beta$, it follows that

$$f(\alpha) = \Delta P(\alpha) \text{ and } f(\beta) = -\Delta P(\beta).$$

Their sum gives

$$\Delta^2\sum_{j=0}^{n-1} U_j\, U_{n-1-j} = nV_{n-1} - 2U_n.$$

If we replace n with $n+1$ and use the identity

$$(n+1)V_n - 2U_{n+1} = nV_n - pU_n,$$

then we obtain the formula (2.11).

On the other hand, the equality $f(\alpha) - f(\beta) = n\,U_{n-1}$ is another form of the formula (2.12).

Now assume that $u(x) = 2(n+1)x^n - npx^{n-1} - V_n$. Taking the derivative of (1.2), we get

$$u(x) = (2x-p)Q(x) + (x^2 - px + q)Q'(x).$$

If we take $x = \alpha$ and $x = \beta$ in this equation, then we obtain

$$u(\alpha) = \Delta Q(\alpha) \text{ and } u(\beta) = -\Delta Q(\beta).$$

The equality $u(\alpha) + u(\beta)$ implies the formula (2.12) again. Moreover the equality $u(\alpha) - u(\beta)$ gives the formula (2.13). □

As a consequence of Theorem 1, we can give Corollary 1, 2 and 3.

Corollary 1. *Let* $g(x) = \dfrac{(n+1)x^n - U_{n+1}}{\Delta x^n}$. *Then for every natural number* n, *the following sums hold:*

$$\sum_{j=0}^{n} U_j \alpha^{-j} = g(\alpha) \tag{2.15}$$

and

$$\sum_{j=0}^{n} U_j \beta^{-j} = -g(\beta). \tag{2.16}$$

Proof. Taking $x = \alpha$ in (2.14) we can write

$$n\alpha^{n-1} - U_n = \Delta P(\alpha) = \Delta \alpha^{n-1} \sum_{j=0}^{n-1} U_j \alpha^{-j}.$$

Thus we have the formula

$$\sum_{j=0}^{n-1} U_j \alpha^{-j} = \frac{n\alpha^{n-1} - U_n}{\Delta \alpha^{n-1}},$$

which is precisely the formula (2.15). Similarly, the formula (2.16) follows from (2.14) by taking $x = \beta$. □

Corollary 2. *For every natural number* n, *the following sums hold:*

$$\sum_{j=0}^{n} q^{-j} U_{2j} = q^{-n} U_n U_{n+1}, \tag{2.17}$$

$$\Delta^2 \sum_{j=0}^{n} q^{-j} U_j^2 = q^{-n} V_n U_{n+1} - 2(n+1), \tag{2.18}$$

$$\sum_{j=0}^{n} q^{-j} V_j^2 = q^{-n} V_n U_{n+1} + 2(n+1), \tag{2.19}$$

and

$$\sum_{j=0}^{n} q^{-j} V_{2j} = q^{-n} V_n U_{n+1}.$$

Proof. By taking the sum of (2.15) and (2.16), we get $\sum_{j=0}^{n} U_j\, V_{-j} = q^{-n} U_n U_{n+1}$. Using (2.2) and (2.3), we obtain (2.17).

On the other hand, (2.15)−(2.16) gives the formula

$$\Delta \sum_{j=0}^{n} U_j\, U_{-j} = \frac{2(n+1) - q^{-n} V_n U_{n+1}}{\Delta}.$$

Using (2.1) in this equation, we get the formula

$$\Delta^2 \sum_{j=0}^{n} q^{-j} U_j^2 = q^{-n} V_n U_{n+1} - 2(n+1).$$

Moreover, if we use (2.4) and (2.18), then we can write

$$\sum_{j=0}^{n} q^{-j} V_j^2 = \Delta^2 \sum_{j=0}^{n} q^{-j} U_j^2 + 4(n+1).$$

Thus it follows that

$$\sum_{j=0}^{n} q^{-j} V_j^2 = q^{-n} V_n U_{n+1} + 2(n+1).$$

Indeed, using (2.5), the equation

$$\sum_{j=0}^{n} q^{-j} V_{2j} = q^{-n} V_n U_{n+1}$$

follows from (2.19). □

Corollary 3. *For all $n \in \mathbb{N}$, the following sums hold:*

$$\sum_{j=0}^{n} q^{j} V_{n+1-2j} = \sum_{j=0}^{n} q^{j+1} V_{n-1-2j} = V_{n+1} + 2qU_n \tag{2.20}$$

Proof. By (2.11) we can write

$$\sum_{j=0}^{n} U_j\, U_{n-j} = \frac{nV_n - pU_n}{\Delta^2}. \tag{2.21}$$

Then taking the sum of p(2.21) and (2.12), we get

$$2 \sum_{j=0}^{n} U_j\, U_{n+1-j} = \frac{n(p\, V_n + \Delta^2\, U_n) - 4qU_n}{\Delta^2}$$

by using (2.8). Thus using (2.9) in this equation, it follows that

$$\Delta^2 \sum_{j=0}^{n} U_j\, U_{n+1-j} = nV_{n+1} - 2qU_n. \tag{2.22}$$

If we consider (2.10) and (2.22), then we have the expected formula (2.20).

On the other hand using (2.12)$-p$(2.21) and (2.6), we get

$$\Delta^2 \sum_{j=0}^{n} U_j\, U_{n-1-j} = nV_{n-1} - \frac{(p^2-2q)U_n}{q}. \tag{2.23}$$

Also using (2.10), (2.23) and (2.7), we see that $\sum\limits_{j=0}^{n} q^j V_{n-1-2j}$ is equal to the right hand side of equation (2.20). □

Theorem 2 and Corollary 4 are related to results that come from the second derivatives of (1.1) and (1.2).

Theorem 2. *For every natural number n,*

$$2\,\Delta^2 \sum_{j=0}^{n} j\, U_j\, U_{n-j} = n(nV_n - pU_n), \tag{2.24}$$

$$2\,\Delta^2 \sum_{j=0}^{n} j\, U_j\, V_{n-j} = 2\,n\,V_{n+1} + \left[\Delta^2\, n(n+1) - 4q\right] U_n, \tag{2.25}$$

$$2\Delta^2 \sum_{j=0}^{n} j\, V_j\, U_{n-j} = \left(n^2\Delta^2 + 4q\right) U_n - npV_n, \tag{2.26}$$

and

$$2 \sum_{j=0}^{n} j\, V_j\, V_{n-j} = n\left[(n+1)\, V_n + 2pU_n\right]. \tag{2.27}$$

Proof. Let $h(x) = n(n-1)x^{n-2}$. If we take the second derivative of (1.1), then we get

$$h(x) = 2\,P(x) + 2(2\,x-p)P'(x) + (x^2 - p\,x + q)P''(x).$$

Taking the sum of $h(\alpha)$ and $h(\beta)$, we obtain

$$n(n-1)V_{n-2} - 2\sum_{j=0}^{n-1} U_j\, V_{n-1-j} = 2\,\Delta^2 \sum_{j=0}^{n-2} (n-1-j)U_j\, U_{n-2-j}.$$

Replacing n with $n+1$ and using (2.11) and (2.12), we get the formula (2.24).

On the other hand, the equality $h(\alpha) - h(\beta)$ gives,

$$\Delta n(n-1)U_{n-2} = 2\Delta\sum_{j=0}^{n-1} U_j\, U_{n-1-j} + 2\Delta\sum_{j=0}^{n-2}(n-1-j)U_j\, V_{n-2-j}.$$

Finally replace n with $n+1$ and use (2.11) and (2.12), we get the formula (2.25).

Taking the derivative of $u(x)$, we get

$$u'(x) = 2\,Q(x) + 2(2\,x - p)Q'(x) + (x^2 - p\,x + q)Q''(x).$$

The sum of $u'(\alpha)$ and $u'(\beta)$ implies the formula

$$2n(n+1)V_{n-1} - pn(n-1)V_{n-2} = 2\sum_{j=0}^{n-1} V_j\,V_{n-1-j} + 2\,\Delta^2\sum_{j=0}^{n-2}(n-1-j)V_j\,U_{n-2-j}.$$

Replacing n with $n+1$ and using (2.12)and (2.13), we get (2.26).

Similarly $u'(\alpha) - u'(\beta)$ gives

$$2n(n+1)U_{n-1} - pn(n-1)U_{n-2} = 2\sum_{j=0}^{n-1} V_j\,U_{n-1-j} + 2\sum_{j=0}^{n-2}(n-1-j)V_j\,V_{n-2-j}.$$

If we take n instead of $n+1$ and use (2.13) and (2.12), then we obtain the expected result for (2.27). □

Consequently, we can give the following corollary.

Corollary 4. *For every natural number n,*

$$\Delta^2\sum_{j=0}^{n} jq^{-j}\,U_{2j} = \frac{\Delta^2(n+1)U_nU_{n+1} - V_nU_{n+2} + (n+2)pq^n}{q^n}, \tag{2.28}$$

$$\Delta^2\sum_{j=0}^{n} jq^{-j}\,U_j^2 = \frac{(n+1)V_nU_{n+1} - U_nU_{n+2} - (n^2+2n+2)q^n}{q^n} \tag{2.29}$$

$$\sum_{j=0}^{n} jq^{-j}\,V_j^2 = \frac{(n+1)V_nU_{n+1} - U_nU_{n+2} + (n^2-2)q^n}{q^n} \tag{2.30}$$

and

$$\sum_{j=0}^{n} jq^{-j}\,V_{2j} = \frac{(n+1)V_nU_{n+1} - U_nU_{n+2} - (n+2)q^n}{q^n}. \tag{2.31}$$

Proof. If we multiply $h(\alpha)$ by β^{n-1} and $h(\beta)$ by α^{n-1}, then we get

$$\sum_{j=0}^{n-2}(n-1-j)U_j\alpha^{-j} = \frac{n(n-1)\Delta - 2n\alpha + 2\alpha^{-n+2}U_n}{2\Delta^2} \tag{2.32}$$

and

$$\sum_{j=0}^{n-2}(n-1-j)U_j\beta^{-j} = \frac{-n(n-1)\Delta - 2n\beta + 2\beta^{-n+2}U_n}{2\Delta^2}, \tag{2.33}$$

respectively. Now taking the sum of (2.32) and (2.33), and replacing n with $n+2$ and using (2.17) we obtain the expected formula (2.28).

On the other hand, if we consider the difference of (2.32)−(2.33), and use (2.1) with (2.18), then we obtain the formula (2.29). Moreover if we use (2.4) with (2.29), then we obtain (2.30). In addition to this, if we use (2.5) with (2.30), then we get the formula (2.31). □

References

[1] Amini, A.R., Fibonacci numbers from a long division formula, *Mathematical Spectrum*, 40 (2008), 59-61.

[2] Belbachir, H. and Bencherif, F., Sums of products of generalized Fibonacci and Lucas numbers, arXiv: 0708.2347v1, 17.08.2007.

[3] Benjamin, A. T. and Quinn, J. J., Proofs that really count, the art of combinatorial proofs, Mathematical Association of America, Providence, RI, 2003.

[4] Čerin, Z., Sums of products of generalized Fibonacci and Lucas numbers, *Demonstratio Math.*, 42(2), (2009), 247-258.

[5] Čerin, Z., Some alternating sums of Lucas numbers, *Cent. Eur. J. Math.*, 3(1), (2015), 1-13.

[6] Čerin, Z., Bitim, B. D. and Keskin, R., Sum formulae of generalized Fibonacci and Lucas numbers, *Honam Math. J.*, 40(1), (2018), 199-210.

[7] Chong, C. Y., Cheah, C. L. and Ho, C. K., Some identities of generalized Fibonacci sequence, *AIP Conf. Proc.*, 1605 (2014), 661-665 .

[8] Horadam, A. F., Basic properties of certain generalized sequences of numbers, *Fibonacci Quart.*, 3(5), (1965), 161-176.

[9] Koshy, T., Fibonacci and Lucas numbers with applications, Wiley and Sons, Canada, 2001.

[10] Long, C. T., Discovering Fibonacci identities, *Fibonacci Quart.*, 24(2), (1986), 160-167.

[11] Lucas, E., Théorie des fonctions numériques simplement périodiques, *Amer. J. Math.*, 1 (1878), 184-240.

[12] Melham, R. S., On sums of powers of terms in a linear recurrence, *Port. Math.*, 56(4), (1999), 501-508.

[13] Shannon, A. G. and Horadam, A. F., Special recurrence relations associated with the sequence $\{w_n(a, b; p, q)\}$, *Fibonacci Quart.*, 17(4), (1979), 294-299.

[14] Vajda, S., Fibonacci and Lucas numbers and the golden section, Ellis Horwood Ltd. Publ., England, 1989.

Current address: B. Demirtürk Bitim: Izmir Bakırçay University, Department of Business, Seyrek, Menemen 35665 izmir Turkey

E-mail address: `bahar.demirturk@bakircay.edu.tr`

ORCID Address: `http://orcid.org/0000-0002-5911-5190`

Current address: N. Topal: Sakarya University, Department of Mathematics, Esentepe Campus, 54187 Sakarya, Turkey

E-mail address: `nazimtopal@gmail.com`

ORCID Address: `http://orcid.org/0000-0002-2061-3432`