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Homotopy of Lie-Rinehart Crossed Module Morphisms

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Abstract

In this paper our aim is to give the concept homotopy of morphisms of Lie-Rinehart crossed modules. We show that the homotopy relation gives rise to an equivalence relation. Additionally a groupoid structure of Lie- Rinehart crossed module morphisms and their homotopies.

Keywords: Lie-Rinehart algebra, Crossed modules, Derivations, Homotopy.

Lie-Rinehart Çapraz Modül Morfizimlerinin Homotopisi

Özet

Bu makalede amacımız Lie-Rinehart çapraz modüllerin morfizmlerinin homotopi kavramını vermektir. Homotopi bağıntısı bir denklik bağıntısı oluşturduğunu, buna ek olarak Lie- Rinehart çapraz modül morfizmleri ve homotopileri bir groupoid yapısı olduğunu gösterdik.

Anahtar Kelimeler: Lie-Rinehart Cebir, Çapraz Modüller, Türevler, Homotopi.

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1. Introduction

Crossed modules were introduced by Whitehead [10,11,12] in his investigations into the structure of the second relative homotopy group $\pi_2(X, X_0)$ where X is a pathwise connected topological space and X_0 a pathwise connected subspace.

Lie-Rinehart algebra is described by Herz (1953) [7] as "Pseudo-algebras de Lie". Let K to be a field and A to be a unital commutative algebra over K. Lie-Rinehart algebra is $K-$ algebra and $A-$ module. (1990) by J. Huebschmann [8] associated these two frame.

Lie-Rinehart algebras appear in various areas of Mathematics, for example; In diferantial geometry, Lie-Rinehart algebras appear as algebraic counterpart of Lie-algebroids.[9]

Crossed modules of Lie-Rinehart algebras were introduced by Casas, Ladra and Pirashvili, (2004) [4]. In [8] by Huebschmann, the third cohomology consept of Lie-Rinehart algebras is developed, in [4] by Casas, Ladra and Pirashvili this consept were classified. Subsequently, (2005) by Casas, Ladra and Pirashvili [5] were defined triple cohomology of Lie-Rinehart algebras. (by [4],[8])

Lie-Rinehart algebras are not in the modified category of interest, since this category does not contain anchor map. However, in this article it is observed that crossed module morphisms of Lie-Rinehart algebras fall within this category [6].

In this paper, we have described the homotopy of the crossed modules morphisms of the Lie-Rinehart modules, using crossed modules and morphsims in [4].

2. Preliminaries

In this section we recollect some notion from [4,5]. We start by recalling the definition of Lie-Rinehart algebras. We assume that K be a field and A be a unital commutative algebra over K, in this paper. We let $Der(A)$ be the set of all $K-$ derivations of A. Thus elements of $Der(A)$ are $K-$ linear maps $d: A \to A$ such that $d(ab) = ad(b) + d(a)b$ holds. It is well-known that Der(A) is a Lie K-algebra under the bracket $[d, d'] = dd' - d'd$.

For $a \in A$ and $d \in Der(A)$ one has $ad \in Der(A)$, here ad is defined by

$(ad)(b) = ad(b), b \in A$. Thus $Der(A)$ is also an $A-$ module. It is well-known and it is easy to check that the following holds

$$[d, ad'] = a[d, d'] - d(a)d',\ d, d' \in Der(A).$$

In particular, $Der(A)$ is not a Lie $A-$ algebra. Following Huebschmann [8], a Lie-Rinehart algebra over A consists of a Lie $K-$ algebra L together with an $A-$ module structure on L and a map $\alpha : \mathcal{L} \to Der(A)$ which is simultaneously a Lie algebra and an A-module homomorphism such that $[l, al'] = a[l, l'] + l(a)l'$ holds. Here $l, l' \in L, a \in A$ and we write $l(a)$ for $\alpha(l)(a)$. It is clear that the Lie-Rinehart algebras with $\alpha = 0$ are exactly the Lie $A-$algebras. On the other hand, any commutative $K-$ algebra A defines a Lie-Rinehart algebra with $\mathcal{L} = Der(A)$. If $\mathcal{L}$ and $\mathcal{L}'$ are Lie-Rinehart algebras, then a Lie-Rinehart homomorphism $f : \mathcal{L} \to \mathcal{L}'$ is a map, which is simultaneously a Lie $K-$ algebra homomorphism and a homomorphism of $A-$ modules. Furthermore one requires that the diagram

$$\begin{tikzcd} \mathcal{L} \arrow[rr, "\alpha"] \arrow[d, "f"'] & & Der(A) \\ \mathcal{L}' \arrow[urr, "\alpha'"'] & & \end{tikzcd}$$

commutes. We denote by $\mathcal{LR}(A)$ the category of Lie-Rinehart algebras. As we said one has the full inclusion $\mathcal{L}(A) \subset \mathcal{LR}(A)$ where $\mathcal{L}(A)$ denotes the category of Lie $A-$ algebras. Let us observe that the kernel of any Lie-Rinehart algebra homomorphism is a Lie $A-$ algebra.

Definition 1 *Let $\mathcal{L}$ be a Lie-Rinehart algebra and let R be a Lie $A-$ algebra. We will say that $\mathcal{L}$ acts on R if $K-$ linear map,*

$$\mathcal{L} \otimes R \to R,\quad (l, r) \mapsto l \triangleright r, l \in \mathcal{L}, r \in R,$$

is given such that we refer [4] to recall these action conditions.

Definition 2 *A crossed module $\beta : \mathcal{R} \to \mathcal{L}$ of Lie-Rinehart algebras over A*

consists of a Lie-Rinehart algebra $\mathcal{L}$ *and a Lie* $A-$ *algebra* R *together with the action of* $\mathcal{L}$ *on* R *and the Lie* $K-$ *algebra homomorphism* β *such that the following identities hold:*

$$(1)\ \beta(l \triangleright r) = [l, \beta(r)]\ (2)\ \beta(r') \triangleright r = [r', r]\ (3)\ \beta(ar) = a\beta(r)\ (4)\ \beta(r)(a) = 0$$

for all $r \in \mathcal{R}, l \in \mathcal{L}, a \in A$. *The first two conditions say that* $\beta : \mathcal{R} \to \mathcal{L}$ *is a crossed module of Lie* $K-$ *algebras (see [4]), the condition (3) says that* β *is a map of* $A-$ *modules and the condition (4) says that the composition of the following maps is zero*

$$\mathcal{R} \xrightarrow{\beta} \mathcal{L} \xrightarrow{\alpha} Der(A).$$

Definition 3 *A morphism between two crossed modules* $(R, \mathcal{L}, \beta) \to (R', \mathcal{L}', \beta')$ *is a pair* (m_1, m_0) *of morphisms* $m_1 : R \to R'$, $m_0 : \mathcal{L} \to \mathcal{L}'$, *such that the diagram:*

$$\begin{tikzcd} R \arrow[r, "m_1"] \arrow[d, "\beta"'] & R' \arrow[d, "\beta'"] \\ \mathcal{L} \arrow[r, "m_0"'] & \mathcal{L}' \end{tikzcd}$$

commutes and:

$$m_1(l \triangleright r) = m_0(l) \triangleright m_1(r)$$

for all $r \in R$, $l \in \mathcal{L}$.

3. Homotopy of Crossed Modules for Lie-Rinehart Algebras

In the rest of the paper, we fix two arbitrary crossed modules for Lie-Rinehart algebra $(R, \mathcal{L}, \beta)$ and $(R', \mathcal{L}', \beta')$.

3.1 Derivation and Homotopy

Definition 4 *Let* $m_0 : \mathcal{L} \to \mathcal{L}'$ *be a Lie-Rinehart algebra homomorphism. An* m_0-

derivation $t:\mathcal{L}\to R'$ *is a map satisfying:*

$$t[al,l'] = m_0(al)\rhd t(l') - m_0(l')\rhd t(al) + [t(al),t(l')]$$

for all $a\in A, l, l'\in\mathcal{L}$.

$$\begin{tikzcd} R \arrow[r, "m_1"] \arrow[d, "\beta"'] & R' \arrow[d, "\beta'"] \\ \mathcal{L} \arrow[r, "m_0"'] \arrow[ur, "t"] \arrow[d, "\alpha"'] & \mathcal{L}' \arrow[dl, "\alpha'"] \\ Der(A) & \end{tikzcd}$$

Theorem 5 $(R,\mathcal{L},\beta)\to(R',\mathcal{L}',\beta')$ *be a crossed module homomorphism for Lie-Rinehart algebra. If* t *is an* m_0- *derivation and if we define* $n=(n_0,n_1)$ *as (where* $l\in\mathcal{L}$ *and* $r\in R$ *)*

$$n_0(l) = m_0(l) + (\beta' t)(l),\ n_1(r) = m_1(r) + (t\beta)(r)$$

then n *is also defines a crossed module morphism for Lie-Rinehart algebra* $(R,\mathcal{L},\beta)\to(R',\mathcal{L}',\beta')$.

Proof. For all $l,l'\in\mathcal{L}$,

$$n_0(l+l') = m_0(l+l') + \beta' t(l+l')$$
$$= m_0(l) + m_0(l') + \beta'(t(l)+t(l'))$$
$$= m_0(l) + m_0(l') + \beta' t(l) + \beta' t(l')$$
$$= n_0(l) + n_0(l'),$$

$$n_0[l,l'] = m_0[l,l'] + \beta' t[l,l']$$
$$= [m_0(l), m_0(l')] + \beta'(m_0(l)t(l') - m_0(l')t(l) + [t(l),t(l')])$$

$$=[m_0(l),m_0(l')]+\beta'(m_0(l)t(l'))-\beta'(m_0(l')t(l))+\beta'[t(l),t(l')]$$

$$=[m_0(l),m_0(l')]+[m_0(l),\beta't(l')]-[m_0(l'),\beta't(l)]+[\beta't(l),\beta't(l')]$$

$$=[m_0(l),m_0(l')]+[m_0(l),\beta't(l')]+[\beta't(l),m_0(l')]+[\beta't(l),\beta't(l')]$$

$$=[m_0(l)+\beta't(l),m_0(l')+\beta't(l')]$$

$$=[n_0(l),n_0(l')].$$

Thus n_0 is a Lie $K-$ algebra morphism; and

for $a \in A$,

$$n_0(al)=m_0(al)+\beta't(al)$$

$$=am_0(l)+\beta't(al)$$

$$=am_0(l)+a\beta't(l)$$

$$=a(m_0(l)+\beta't(l))$$

$$=an_0(l),$$

$$\alpha'n_0(l)=\alpha'(m_0(l)+\beta't(l))$$

$$=\alpha'(m_0(l))+\alpha'(\beta't(l))$$

$$=\alpha(l)+0$$

$$=\alpha(l).$$

Thus n_0 is $A-$ modul morphism. Consequently n_0 is a Lie-Rinehart algebra homomorphism.

For all $l,l' \in R$,

$$n_1(l+l')=m_1(l+l')+(t\beta)(l+l')$$

$$=m_1(l)+m_1(l')+t(\beta(l)+\beta(l'))$$

$$=m_1(l)+m_1(l')+t\beta(l)+t\beta(l')$$

$$=n_1(l)+n_1(l'),$$

$$n_1[l,l']=m_1[l,l']+(t\beta)[l,l']$$

$$=[m_1(l),m_1(l')]+t[\beta(l),\beta(l')]$$

$$=[m_1(l),m_1(l')]+m_0(\beta(l))t\beta(l')-m_0(\beta(l')t(\beta l)+[t(\beta l),t(\beta l')]$$

$$= [m_1(l), m_1(l')] + \beta' m_1(l) t(\beta(l')) - \beta' m_1(l') t(\beta(l)) + [t\beta(l), t(\beta(l'))]$$

$$= [m_1(l), m_1(l')] + [m_1(l), t(\beta(l'))] - [m_1(l'), t(\beta(l))] + [t(\beta(l)), t(\beta(l'))]$$

$$= [m_1(l), m_1(l')] + [m_1(l), t(\beta(l'))] + [t(\beta(l)), m_1(l')] + [t(\beta(l)), t(\beta(l'))]$$

$$= [m_1(l) + t\beta(l), m_1(l') + t\beta(l')]$$

$$= [n_1(l), n_1(l')].$$

Thus n_1 is a Lie algebra morphism.

For $l \in R$,

$$(n_0\beta)(l) = n_0(\beta(l))$$

$$= m_0(\beta(l)) + \beta' t(\beta(l))$$

$$= \beta'(m_1(l) + t\beta(l))$$

$$= \beta' n_1(l)$$

$$= (\beta' n_1)(l).$$

So

$$n_0\beta = \beta' n_1.$$

Finally; n_1 preserves the action of $\mathcal{L}$ on R. Indeed:

For $a \in A, l \in \mathcal{L}, r \in R$

$$n_1(al \triangleright r) = m_1(al \triangleright r) + t(\beta(al \triangleright r))$$

$$= m_0(al) \triangleright m_1(r)) + t[a(l), \beta(r)]$$

$$= [am_0(l), m_1(r)] + t([al, \beta(r)])$$

$$= a[m_0(l), m_1(r)] + [am_0(l), t\beta(r)] - [m_0(\beta r), t(al)] + [t(al), t\beta(r)]$$

$$= a[m_0(l), m_1(r)] + a[m_0(l), t\beta(r)] - [\beta' m_1, t(a(l))] + [ta(l), t\beta(r)]$$

$$= a(m_0(l) \triangleright m_1(r)) + a(m_0(l) \triangleright t\beta(r)) + t(al) \triangleright (m_1(r) + t\beta(r))$$

$$= a(m_0(l) \triangleright (m_1(r) + t\beta(r)) + a\beta' t(l) \triangleright (m_1(r) + t\beta(r))$$

$$= a(m_0(l) + \beta' t(l)) \triangleright (m_1(r) + t\beta(r))$$

$$= an_0(l) \triangleright n_1(r),$$

$$
\begin{aligned}
n_1(l \triangleright ar) &= m_1(l \triangleright ar) + t\beta(l \triangleright ar) \\
&= m_0(l) \triangleright m_1(ar) + t(\beta(l \triangleright ar)) \\
&= m_0(l) \triangleright m_1(ar) + t[l, \beta(ar)] \\
&= m_0(l) \triangleright m_1(ar) + t[l, a\beta(r)] \\
&= m_0(l) \triangleright m_1(ar) - t[a\beta(r), l] \\
&= m_0(l) \triangleright m_1(ar) - (m_0(a\beta(r)) \triangleright t(l) - m_0(l) \triangleright t(a\beta(r)) + [t(a\beta(r)), t(l)]) \\
&= m_0(l) \triangleright m_1(ar) - m_0(\beta(ar)) \triangleright t(l) + m_0(l) \triangleright t(\beta(ar)) - [t(\beta(ar)), t(l)] \\
&= m_0(l) \triangleright m_1(ar) - \beta' m_1(ar) \triangleright t(l) + m_0(l) \triangleright t\beta(ar)) - [t\beta(ar), t(l)] \\
&= m_0(l) \triangleright m_1(ar) - [m_1(ar), t(l)] + m_0(l) \triangleright t\beta(ar)) - [t\beta(ar), t(l)] \\
&= m_0(l) \triangleright ((m_1(ar) + t\beta(ar)) + t(l) \triangleright (m_1(ar) + t\beta(ar)) \\
&= (m_0(l) + \beta' t(l)) \triangleright (m_1(ar) + t\beta(ar)) \\
&= n_0(l) \triangleright n_1(ar)\,.
\end{aligned}
$$

Therefore $n = (n_1, n_0)$ is a crossed module morphism between $(R, \mathcal{L}, \beta) \to (R', \mathcal{L}', \beta')$.

Remark 6 *Specially we give rise to* $A = \{\mathrm{id}\}$ *for the above definition and theorem, we obtain Lie crossed module morphism homotopy* [1].

Definition 7 *In the condition of the previous theorem, we write* $f \xrightarrow{(m_0,t)} g$ *or shortly* $f \simeq g$, *and say that* (m_0, t) *is a homotopy (or derivation) connecting* f *to* g .

4. A Groupoid

Now we construct a groupoid structure which is induced from homotopy of Lie-Rinehart crossed module morphisms.

Lemma 8 *(Identity) Let* $m = (m_1, m_0)$ *be a Lie-Rinehart crossed module morphism* $(R, \mathcal{L}, \beta) \to (R', \mathcal{L}', \beta')$. *The null function* $0 : \mathcal{L} \to R', 0(l) = 0_{R'}$ *defines an* m_0- *derivation connecting* f *to* f .

Proof. Easy calculations.

Lemma 9 *(Inverse) Let* $m=(m_1,m_0)$ *and* $n=(n_1,n_0)$ *be crossed module morphisms for Lie-Rinehart algebras.* $(R,\mathcal{L},\beta)\to(R',\mathcal{L}',\beta')$ *and* t *be an* m_0- *derivation connecting* f *to* g*. Then, the map* $\bar{t}=-t:\mathcal{L}\to R',$ *with* $\bar{t}(r)=-t(r),$ *where* $r\in R,$ *is a* n_0- *derivation connecting* g *to* f*.*

Proof. Since t is an m_0- derivation connecting f to g, we have (for all $r\in\mathcal{L}),r'\in R: m_0(r)=n_0(r)+(\beta'\circ\bar{t})(r),$ and $m_1(r')=n_1(r')+(\bar{t}\circ\beta)(r').$ Morever $\bar{t}$ is a n_0 - derivation, since:

For $a\in A, l\in\mathcal{L}, r\in R$

$$
\begin{aligned}
&\bar{t}[al,l']=-t[al,l']\\
&=-(m_0(al)\rhd t(l')-m_0(l')\rhd t(al)+[t(al),t(l')])\\
&=-m_0(al)\rhd t(l')+m_0(l')\rhd t(al)-[t(al),t(l')]\\
&\quad -[t(al),t(l')]+[t(al),t(l')]\\
&=-m_0(al)\rhd t(l')+m_0(l')\rhd t(al)+[t(al),t(l')]\\
&\quad -\beta't(al)\rhd t(l')+\beta't(l')\rhd t(al)\\
&=(-m_0(al)+\beta't(al))\rhd t(l')+(m_0(l')+\beta't(l'))\\
&\quad \rhd t(al)-[t(al),t(l')]\\
&=-(m_0+\beta't)(al)\rhd t(l')+(m_0\\
&\quad +\beta't)(l')\rhd(t(al))+[t(al),t(l')]\\
&=-n_0(al)\rhd t(l')+n_0(l')\rhd(t(al))+[t(al),-t(l')]\\
&=n_0(al)\rhd(-t(l'))-n_0(l')\rhd(-t(al))+[-t(al),-t(l')]\\
&=n_0(al)\rhd\bar{t}(l')-n_0(l')\rhd\bar{t}(al)+[\bar{t}(al),\bar{t}(l')].
\end{aligned}
$$

Lemma 10 *(Concatenation) Let* $m=(m_1,m_0), n=(n_1,n_0)$ *and* $k=(k_1,k_0)$ *be crossed module morphisms for Lie- Rinehart algebra.* $(R,\mathcal{L},\beta)\to(R',\mathcal{L}',\beta')$ t *be an* m_0- *derivation connecting* f *to* g*, and* t' *be a* n_0- *derivation connecting* g *to* k*. Then the linear map* $(t+t'):\mathcal{L}\to R'$*, such that* $(t+t')(l)=t(l)+t'(l),$ *defines an* m_0-

derivation (therefore a homotopy) connecting m *to* k.

Proof. We know that $m \overset{(m_0,t)}{\to} n$ and $n \overset{(n_0,t')}{\to} k$. Therefore by definition

$$k_0(r) = m_0(r) + (\beta'(t+t'))(r), \quad k_1(r') = m_1(r') + ((t+t')\beta)(r').$$

Let us see that $t+t'$ satisfies the condition for it to be an m_0 derivation:

$$\begin{aligned}
&(t+t')[al,l'] = t[al,l'] + t'[al,l'] \\
&= m_0(al) \triangleright t(l') - m_0(l') \triangleright t(al) + [t(al), t(l')] \\
&\quad + n_0(al) \triangleright t'(l') - n_0(l') \triangleright t'(al) + [t'(al), t'(l')] \\
&= m_0(al) \triangleright t(l') - m_0(l') \triangleright t(al) + [t(al), t(l')] \\
&\quad + (m_0(al) + \beta' t(al)) \triangleright t'(l') - (m_0(l') + \beta' t(l')) \triangleright t'(al) + [t'(al), t'(l')] \\
&= m_0(al) \triangleright t(l') - m_0(l') \triangleright t(al) + [t(al), t(l')] \\
&\quad + (m_0(al) \triangleright t'(l')) + \beta' t(al) \triangleright t'(l') - m_0(l') \triangleright t'(al) \\
&\quad - \beta' t(l') \triangleright t'(al) + [t'(al), t'(l')] \\
&= m_0(al) \triangleright t(l') - m_0(l') \triangleright t(al) + [t(al), t(l')] \\
&\quad + (m_0(al) \triangleright t'(l')) + [t(al), t'(l')] - m_0(l') \triangleright t'(al) \\
&\quad - [t(l'), t'(al)] + [t'(al), t'(l')] \\
&= m_0(al) \triangleright (t+t')(l') - m_0(l') \triangleright (t+t')(al) + [(t+t')(al), (t+t')(l')] \\
&= [(t+t')(al), (t+t')(l')].
\end{aligned}$$

Corollary 11 *Let* $(R, \mathcal{L}, \beta)$ *and* $(R', \mathcal{L}', \beta')$ *be two arbitrary Lie-Rinehart crossed modules. We have a groupoid* $HOM((R, \mathcal{L}, \beta), (R', \mathcal{L}', \beta'))$*, whose objects are the Lie-Rinehart crossed module morphisms* $(R, \mathcal{L}, \beta) \to (R', \mathcal{L}', \beta')$*, the morphisms being their homotopies. In particular the relation below, for Lie-Rinehart crossed module morphisms* $(R, \mathcal{L}, \beta) \to (R', \mathcal{L}', \beta')$*, is an equivalence relation:*

" $m \simeq n \Leftrightarrow$ there exists an m_0 – derivation t connecting m with n ".

Proof. Follows from previous three lemmas.

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