

ON BRACKET-GENERATING DISTRIBUTIONS

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ABSTRACT. Let D be a distribution on M . By using the curvature of D we obtain a necessary and sufficient condition for D to be bracket-generating of step 2. This enables us to extend the Carathéodory Theorem to any smooth distribution of corank one whose curvature is nowhere zero on M .

INTRODUCTION

A bracket-generating distribution endowed with a Riemannian metric is the main object for study in sub-Riemannian geometry (cf. R. Montgomery [4] and O. Călin and D.-C. Chang [1]). First, it was C. Carathéodory [2] who proved that if an analytic corank one distribution is not integrable, then any two points of the manifold can be joined by a horizontal path. Three decades after this event, W.L. Chow [3] and P.K. Rashevskii [5] extended Carathéodory Theorem to distributions of any corank. But in this case the distribution must be completely nonintegrable which is equivalent to the condition of bracket-generating.

Thus, given a distribution on a manifold, it is very important to know if it is bracket-generating or not. In the present paper we give a necessary and sufficient condition for a distribution to be bracket-generating of step 2. The main object used for stating the theorem is the curvature of the distribution.

Now, we outline the content of the paper. In the first section we present the bracket-generating distributions and the curvature of a distribution. Also, we introduce the rank of the curvature and define a privileged coordinate system on a manifold endowed with a distribution. In the second section we prove Theorem 2.1 which states a necessary and sufficient condition for a distribution to be bracket-generating of step 2. Then we apply this theorem to distributions of corank one (cf. Theorem 2.2 and Corollary 2.1) and give two extensions of Carathéodory Theorem (cf. Theorems 2.5 and 2.6). By these extensions we replace the analyticity condition for the distribution with the condition that the curvature of D is nowhere zero on M .

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1. PRELIMINARIES

Let M be an m -dimensional manifold and D be a distribution of rank $k < m$ (corank $m - k > 0$) on M . Thus D is a vector subbundle of the tangent bundle TM of M with k -dimensional fibers. We call D the *horizontal distribution* on M , and any vector field that is tangent to D is called the *horizontal vector field*. In what follows we consider D as a subsheaf of the sheaf TM of all smooth vector fields on M . This is to say that D assigns to each open set \mathcal{U} of M the collection $D_{\mathcal{U}}$ of all smooth horizontal vector fields on \mathcal{U} . Then, by using the Lie brackets of horizontal vector fields, we construct the flag of subsheaves

$$D \subset D^2 \subset \cdots \subset D^r \subset \cdots \subset TM,$$

with

$$D^2 = D + [D, D], \dots, D^{r+1} = D^r + [D, D^r],$$

where

$$[D, D^r] = \text{span}\{[X, Y] : X \in D, Y \in D^r\},$$

and the span is taken over the algebra of smooth functions on M . If there exists an $r \geq 2$ such that $D^r = TM$, we say that D is a *bracket-generating distribution*. In this case r is called the *step* of the distribution D .

As far as we know, there are no necessary and sufficient conditions for a distribution to be bracket-generating of a certain step. Our goal is to present such conditions for bracket-generating distributions of step 2. The main ingredient of our study is the curvature of D which is defined as follows (cf. R. Montgomery [4], p. 49). The linear bundle map

$$(1.1) \quad F : D \times D \longrightarrow \frac{TM}{D}; \quad F(X, Y) = -[X, Y] \bmod D, \quad \forall X, Y \in D,$$

is called the *curvature* of the distribution D .

Throughout the paper we are concerned only with smooth manifolds and smooth mappings. Also, we use the following range of indices:

$$a, b, c, \dots \in \{1, \dots, m\}; \quad h, i, j, \dots \in \{1, \dots, k\}; \quad \alpha, \beta, \gamma, \dots \in \{k + 1, \dots, m\}.$$

Next, we consider a horizontal frame $\{X_1, \dots, X_k\}$ on \mathcal{U} and put

$$[X_i, X_j] = C_{ij}^k X_k + \Omega_{ij}^\alpha V_\alpha,$$

where $\{X_i, V_\alpha\}$ is a frame field on \mathcal{U} , and $\{C_{ij}^k, \Omega_{ij}^\alpha\}$ are smooth functions on \mathcal{U} . Then, by using (1.1), we deduce that

$$F(X_j, X_i) = \Omega_{ij}^\alpha V_\alpha \bmod D.$$

Thus we may consider Ω_{ij}^α as local components of the curvature F with respect to the frame field $\{X_i, V_\alpha\}$ on \mathcal{U} . Let $\tilde{\mathcal{U}}$ be open set of M such that $\mathcal{U} \cap \tilde{\mathcal{U}} \neq \emptyset$, and $\{\tilde{X}_j, \tilde{V}_\alpha\}$ a frame field on $\tilde{\mathcal{U}}$ such that $\{\tilde{X}_j\}$ is a horizontal frame field. Then, on $\mathcal{U} \cap \tilde{\mathcal{U}}$ we have

$$\tilde{X}_j = X_j^i X_i \quad \text{and} \quad \tilde{V}_\beta = V_\beta^i X_i + V_\beta^\alpha V_\alpha,$$

where $[X_j^i]$ and $[V_\beta^\alpha]$ are nonsingular matrices of smooth functions on $\mathcal{U} \cap \tilde{\mathcal{U}}$. If $\{\tilde{\Omega}_{hk}^\beta\}$ are the local components of F with respect to the frame field $\{\tilde{X}_j, \tilde{V}_\beta\}$ on $\tilde{\mathcal{U}}$, then on $\mathcal{U} \cap \tilde{\mathcal{U}}$ we have:

$$(1.2) \quad \tilde{\Omega}_{hk}^\beta V_\beta^\alpha = \Omega_{ij}^\alpha X_h^i X_k^j.$$

Now, take a point $x \in \mathcal{U} \cap \tilde{\mathcal{U}}$ and consider the $(m-k) \times k(k-1)/2$ matrix

$$\Omega(x) = \begin{bmatrix} \Omega_{1\ 2}^{k+1} & \cdots & \Omega_{1\ k}^{k+1} & \Omega_{2\ 3}^{k+1} & \cdots & \Omega_{2\ k}^{k+1} & \cdots & \Omega_{k-1\ k}^{k+1} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ \Omega_{1\ 2}^m & \cdots & \Omega_{1\ k}^m & \Omega_{2\ 3}^m & \cdots & \Omega_{2\ k}^m & \cdots & \Omega_{k-1\ k}^m \end{bmatrix} (x).$$

Then by (1.2) we deduce that $\text{rank } \Omega(x) = \text{rank } \tilde{\Omega}(x)$. Thus we are entitled to define the *rank* of the curvature F at the point x , as the rank of the matrix $\Omega(x)$. If F has the same rank s at any point of M , we say that F has the rank s on M .

Next, we suppose that D is locally given on \mathcal{U} by the 1-forms $\{\omega^\alpha\}$, $\alpha \in \{k+1, \dots, m\}$, that is, we have

$$D_{\mathcal{U}} = \{X \in TM_{\mathcal{U}} : \omega^\alpha(X) = 0, \alpha \in \{k+1, \dots, m\}\}.$$

If $\{x^a\}$, $a \in \{1, \dots, m\}$ are the local coordinates on \mathcal{U} , then we put $\omega^\alpha = \omega_a^\alpha dx^a$, and D is locally given by the Pfaff system

$$\omega_a^\alpha dx^a = 0, \alpha \in \{k+1, \dots, m\}.$$

As $\text{rank } [\omega_a^\alpha] = m-k$, eventually after a renumerotation of the local coordinates, we can find a *privileged coordinate system* which we still denote by $\{\mathcal{U} : (x^1, \dots, x^m)\}$, such that D is given by the Pfaff system

$$dx^\alpha + \eta_i^\alpha dx^i = 0.$$

It is easy to check that

$$(1.3) \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \eta_i^\alpha \frac{\partial}{\partial x^\alpha}, \quad i \in \{1, \dots, k\},$$

is a local horizontal frame on \mathcal{U} . Moreover, $\{\delta/\delta x^i, \partial/\partial x^\alpha\}$ is a frame field on \mathcal{U} which we call a *privileged frame field* on \mathcal{U} . Denote by F_{ij}^α the local components of F with respect to $\{\delta/\delta x^i, \partial/\partial x^\alpha\}$, that is, we have

$$(1.4) \quad F \left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i} \right) = F_{ij}^\alpha \frac{\partial}{\partial x^\alpha} \text{ mod } D.$$

Then by using (1.1), (1.4) and (1.3) we deduce that

$$(1.5) \quad F_{ij}^\alpha \frac{\partial}{\partial x^\alpha} = \left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right] = \left(\frac{\delta \eta_i^\alpha}{\delta x^j} - \frac{\delta \eta_j^\alpha}{\delta x^i} \right) \frac{\partial}{\partial x^\alpha}.$$

In particular, we suppose that D is a distribution of corank one on M . Then we consider (x^1, \dots, x^{m-1}, t) as privileged coordinate system on \mathcal{U} , that is, D is locally given by the Pfaff equation

$$dt + \eta_i dx^i = 0,$$

where the summation is for $i \in \{1, \dots, m-1\}$. In this case (1.5) becomes

$$(1.6) \quad F_{ij} \frac{\partial}{\partial t} = \left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right] = \left(\frac{\delta \eta_i}{\delta x^j} - \frac{\delta \eta_j}{\delta x^i} \right) \frac{\partial}{\partial t},$$

where F_{ij} are the local components of F with respect to the privileged frame field $\{\delta/\delta x^i, \partial/\partial t\}$. Finally, we recall that a distribution D of corank one on M , given by the Pfaff equation $\omega = 0$, is a *contact distribution*, if the restriction of $d\omega$ on each D_x , $x \in M$, is nondegenerate. In this case we have

$$(1.7) \quad d\omega \left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i} \right) = F_{ij},$$

and therefore, *the curvature of the contact distribution is of rank 1 on M .*

Remark 1.1. We believe that the privileged coordinate systems have a great role in studying the geometry of a distribution. As we shall see in the next section, the proof of the main result is using such coordinates. ■

Sometimes the distribution D is given directly with respect to such privileged coordinate systems, as we see in the next examples.

Example 1.1. (Heisenberg Distribution) The distribution D on \mathbb{R}^3 spanned by

$$\left\{ \frac{\delta}{\delta x^1} = \frac{\partial}{\partial x^1} - \frac{1}{2} x^2 \frac{\partial}{\partial t}, \frac{\delta}{\delta x^2} = \frac{\partial}{\partial x^2} + \frac{1}{2} x^1 \frac{\partial}{\partial t} \right\},$$

is known as the Heisenberg distribution. Taking into account that

$$\left[\frac{\delta}{\delta x^1}, \frac{\delta}{\delta x^2} \right] = \frac{\partial}{\partial t},$$

we deduce that $\{\delta/\delta x^1, \delta/\delta x^2, [\delta/\delta x^1, \delta/\delta x^2]\}$ is a frame on \mathbb{R}^3 . Thus the Heisenberg distribution is bracket-generating of step 2. Using the first equality in (1.6) we conclude that the curvature F of D is given by the function $F_{12}(x) = 1$, for any $x \in \mathbb{R}^3$. Hence the curvature of the Heisenberg distribution is of rank 1 on \mathbb{R}^3 . ■

Example 1.2. (Martinet Distribution) Consider on \mathbb{R}^3 the distribution D spanned by

$$\left\{ \frac{\delta}{\delta x^1} = \frac{\partial}{\partial x^1} + (x^2)^2 \frac{\partial}{\partial t}, \frac{\delta}{\delta x^2} = \frac{\partial}{\partial x^2} \right\},$$

which is known as the Martinet distribution. In this case we have

$$\left[\frac{\delta}{\delta x^1}, \frac{\delta}{\delta x^2} \right] = -2x^2 \frac{\partial}{\partial t} \quad \text{and} \quad \left[\frac{\delta}{\delta x^2}, \left[\frac{\delta}{\delta x^1}, \frac{\delta}{\delta x^2} \right] \right] = -2 \frac{\partial}{\partial t}.$$

Hence D is not bracket-generating of step 2 on the whole \mathbb{R}^3 . Note that the curvature F of D is given by the function $F_{12}(x^1, x^2, t) = -2x^2$. Thus the curvature of D is of rank 0 on the surface $x^2 = 0$, and it is of rank 1 out of this surface.

Finally, we note that the Martinet distribution is bracket-generating of step 3. ■

2. THE MAIN RESULTS

First, we prove the following theorem.

Theorem 2.1. *Let D be a distribution of rank $k < m$ on an m -dimensional manifold M such that*

$$(2.1) \quad k \geq \frac{1}{2} (\sqrt{1 + 8m} - 1).$$

Then D is a bracket-generating distribution of step 2, if and only if, the curvature of D is of constant rank $m - k$ on M .

Proof. First, we note that condition (2.1) is equivalent to the inequality

$$m - k \leq k(k - 1)/2,$$

which must be satisfied in order to state the theorem. Now, suppose that D is bracket-generating of step 2 and take a point $x \in M$. Consider a privileged frame field $\{\delta/\delta x^i, \partial/\partial x^\alpha\}$ on a coordinate neighborhood \mathcal{U} of x . We arrange the local

components $F_i^{\alpha_j}$ of the curvature F of D with respect to this frame field in an $(m-k) \times k(k-1)/2$ matrix as follows

$$(2.2) \quad A = \begin{bmatrix} F_{1\ 2}^{k+1} & \cdots & F_{1\ k}^{k+1} & F_{2\ 3}^{k+1} & \cdots & F_{2\ k}^{k+1} & \cdots & F_{k-1\ k}^{k+1} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ F_{1\ 2}^m & \cdots & F_{1\ k}^m & F_{2\ 3}^m & \cdots & F_{2\ k}^m & \cdots & F_{k-1\ k}^m \end{bmatrix}.$$

Taking into account that there exist $m-k$ Lie brackets $[\delta/\delta x^j, \delta/\delta x^h]$ which together with $\{\delta/\delta x^i\}$, $i \in \{1, \dots, k\}$, form a frame field on \mathcal{U} , we deduce that the matrix A has $m-k$ linearly independent columns. As A has $m-k$ rows, we conclude that the rank of this matrix is $m-k$ at any point of \mathcal{U} . Thus the rank of the curvature F is $m-k$ at any point $x \in M$.

Conversely, suppose that the rank of curvature of D is $m-k$ on M . Then take a point $x \in M$ and consider a privileged frame field $\{\delta/\delta x^i, \partial/\partial x^\alpha\}$ on \mathcal{U} . As the rank of A is $m-k$ at any point of \mathcal{U} , there exist $m-k$ columns in (2.2) such that the square matrix formed by these columns is nonsingular at x . Denote by $[F_\beta^\alpha(x)]$ that matrix, and by $\{Y_\beta(x)\}$, $\beta \in \{k+1, \dots, m\}$, the $m-k$ Lie brackets $[\delta/\delta x^i, \delta/\delta x^j](x)$ whose local components form the columns of $[F_\beta^\alpha(x)]$ (see the first equality in (1.5)). Then the transition matrix from the frame $\{\delta/\delta x^i, \partial/\partial x^\alpha\}$ to the set of vectors $\{\delta/\delta x^j, Y_\beta\}$ at the point x has the form

$$B = \begin{bmatrix} \delta_j^i & 0 \\ 0 & F_\beta^\alpha(x) \end{bmatrix}.$$

As the matrix $[F_\beta^\alpha(x)]$ is nonsingular, we conclude that B is nonsingular too. Thus $T_x M$ is spanned by the vectors $\{\delta/\delta x^1, \dots, \delta/\delta x^k\}$ and the Lie brackets $\{Y_{k+1}, \dots, Y_m\}$ at the point x . Therefore, D is bracket-generating of step 2, and the proof is done. ■

By using Theorem 2.1 we can easily prove the following theorem.

Theorem 2.2. *Let D be a distribution of corank one on an m -dimensional manifold M . Then D is bracket-generating of step 2, if and only if, its curvature is nowhere zero on M .*

Proof. The local components F_{ij} of the curvature F with respect to the privileged frame field $\{\delta/\delta x^i, \partial/\partial t\}$ (see (1.6)) form a $1 \times (m-1)(m-2)/2$ matrix

$$[F_{1\ 2} \cdots F_{1\ m-1} \ F_{2\ 3} \cdots F_{2\ m-1} \cdots F_{m-2\ m-1}].$$

By Theorem 2.1 we deduce that D is bracket-generating of step 2, if and only if, the rank of this matrix is 1. This is equivalent to the fact that at each point of \mathcal{U} at least one of the functions F_{ij} is non-zero. ■

Corollary 2.1. *A contact distribution D on a manifold M is bracket-generating of step 2.*

Proof. In this case the $(m-1) \times (m-1)$ matrix $[F_{ij}]$ (see (1.7)) is nonsingular at any point of \mathcal{U} . As a consequence, the curvature of D is non-zero on M . Thus by Theorem 2.2 we conclude that D is bracket-generating of step 2. ■

Remark 2.1. With respect to Examples 1.1 and 1.2 we deduce the following. The Heisenberg distribution is bracket-generating of step 2 because its curvature is non-zero on \mathbb{R}^3 . On the other hand, the Martinet distribution is not bracket-generating of step 2 since its curvature vanishes on the surface $x^2 = 0$. ■

Next, we recall the following well known theorems.

Theorem 2.3. (C. Carathéodory [2]) *Let M be a connected manifold endowed with an analytic distribution of corank one. Then any two points of M can be joined by a horizontal path.*

Theorem 2.4. (W.L. Chow [3] and P.K. Rashevskii [5]) *Let D be a bracket-generating distribution on a connected manifold M . Then any two points of M can be joined by a horizontal path.*

By combining Theorems 2.2 and 2.4 we deduce the following.

Theorem 2.5. *Let D be a distribution of corank one on a connected manifold M . If the curvature of D is nowhere zero on M , then any two points of M can be joined by a horizontal path.*

Finally, from Theorem 2.3 and Corollary 2.1 we obtain the following.

Theorem 2.6. *Let D be a contact distribution on a connected manifold M . Then any two points of M can be joined by a horizontal path.*

Remark 2.2. Theorems 2.5 and 2.6 are extensions of the Carathéodory Theorem in the sense that we replace the analyticity condition for D with the condition that the curvature of D is nowhere zero on M .

Example 2.1. Let $M = \mathbb{R}^3$ and D be the distribution spanned by

$$\left\{ \frac{\delta}{\delta x^1} = \frac{\partial}{\partial x^1}, \frac{\delta}{\delta x^2} = \frac{\partial}{\partial x^2} + f(x^1) \frac{\partial}{\partial t} \right\},$$

where

$$f(x^1) = \begin{cases} e^{x^1} - e^{-\frac{1}{(x^1)^2}} & , \quad x^1 \neq 0, \\ 1 & , \quad x^1 = 0. \end{cases}$$

Then we have

$$\left[\frac{\delta}{\delta x^1}, \frac{\delta}{\delta x^2} \right] = f'(x^1) \frac{\partial}{\partial t}.$$

It is routine to show that $f(x^1)$ is smooth on \mathbb{R} and $f'(x^1) > 0$, for any $x^1 \in \mathbb{R}$. Thus $F_{12}(x) > 0$ for any $x \in \mathbb{R}^3$, and by Theorem 2.5 any two points of \mathbb{R}^3 can be joined by a horizontal path. However, $f(x^1)$ being not an analytic function at $x^1 = 0$, the Carathéodory Theorem cannot be applied to D . ■

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