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APPROXIMATION PROPERTIES OF BERNSTEIN-KANTOROVICH TYPE OPERATORS OF TWO VARIABLES

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ABSTRACT. In this study, the generalized Bernstein-Kantorovich type operators are introduced and some approximation properties of these operators are studied in the space of continuous functions of two variables on a compact set. The rate of convergence of these operators are obtained by means of the modulus of continuity. The Voronovskaya type theorem is given and some differential properties of these operators are proved.

1. INTRODUCTION

In 1912, for a function f defined on the closed interval $[0,1]$, the expression

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} \tag{1.1}$$

was called the Bernstein polynomial of order n of the function f in [1]. In [2], Korovkin's theorem sometimes also called Bohman-Korovkin theorem, arose from the study of the role of Bernstein polynomials in the proof of the Weierstrass approximation theorem. Later, the various generalizations of Bernstein polynomials (1.1) were investigated in [3]-[9]. In 1930, L.V. Kantorovich constructed and studied the linear positive operators $K_m : L_1([0,1]) \to C([0,1])$ defined by

$$(K_m f)(x) = (m+1)\sum_{k=0}^{m} p_{m,k}(x) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(s)ds. \tag{1.2}$$

The operators (1.2) are known as the Kantorovich operators. These operators are obtained from the classical Bernstein operators (1.1). In [9], Kahvecibasi studied approximation properties of generalized Bernstein-Kantorovich operators on closed interval $[-1,1]$.

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There are many investigations devoted to the problem of approximating continuous functions by classical Bernstein polynomials, as well as by two dimensional Bernstein polynomials and their generalizations. We refer to papers [11]-[14].

In this note, inspired by the operators (1.1) and (1.2), we consider generalized Bernstein-Kantorovich operators for functions of two variables. To this end, let $f \in C(\mathbb{A})$, where $\mathbb{A} = [-1,1] \times [-1,1]$ and define the linear positive operators $D_{n,m}(f;x,y)$, $n,m \in \mathbb{N}$ in the following way:

$$D_{n,m}(f;x,y) = \frac{n+1}{2}\frac{m+1}{2}\sum_{k=0}^{n}\sum_{j=0}^{m}\phi_{n,m}^{k,j}(x,y)\int_{2\frac{k}{n+1}-1}^{2\frac{k+1}{n+1}-1}\int_{2\frac{j}{m+1}-1}^{2\frac{j+1}{m+1}-1} f(t,u)dtdu, \tag{1.3}$$

with

$$\phi_{n,m}^{k,j}(x,y) = \varphi_{n,k}(x)\varphi_{m,j}(y) \tag{1.4}$$

and

$$\varphi_{n,k}(x) = \frac{1}{2^n}\binom{n}{k}(1+x)^k(1-x)^{n-k}. \tag{1.5}$$

As usual, let $C(\mathbb{A})$ be the the space of all real valued continuous functions on $\mathbb{A}$ endowed with the norm

$$\|f\|_{(\mathbb{A})} = \max_{(x,y)\in\mathbb{A}} |f(x,y)|$$

Let $f \in C(\mathbb{A})$.The full modulus of continuity of f is defined as follows:

$$\omega(f,\delta) = \max_{\substack{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \leq \delta \\ (x,y)\in\mathbb{A}}} |f(x_1,y_1) - f(x_2,y_2)| \tag{1.6}$$

Partial modulus of continuity with respect to x and y is defined by

$$\omega^{(1)}(f,\delta) = \max_{\substack{(x_1,y),(x_2,y)\in\mathbb{A} \\ |x_1-x_2|\leq\delta}} |f(x_1,y) - f(x_2,y)|, \tag{1.7}$$

$$\omega^{(2)}(f,\delta) = \max_{\substack{(x,y_1),(x,y_2)\in\mathbb{A} \\ |y_1-y_2|\leq\delta}} |f(x,y_1) - f(x,y_2)|, \tag{1.8}$$

respectively. It is known that the full and partial modulus of continuity satisfy the following properties:

$$\omega(f,\delta) \leq (1+\lambda)\omega(f,\delta), \quad \lim_{\delta\to 0}\omega(f,\delta) = 0. \tag{1.9}$$

2. Main results

In this section we give some classical approximation properties of the operators $D_{n,m}$ on the set $\mathbb{A}$.

Then by simple calculations, one can obtain the following lemmas.

Lemma 2.1. *For $\forall(x,y) \in \mathbb{A}$ and $\forall n,m \in \mathbb{N}$, Bernstein-Kantorovich operators (1.3) satisfy the following equalities:*

$$D_{n,m}(1;x,y) = 1, \tag{2.1}$$

$$D_{n,m}(t;x,y) = x - \frac{x}{n+1}, \tag{2.2}$$

$$D_{n,m}(u;x,y) = y - \frac{y}{m+1}, \tag{2.3}$$

$$D_{n,m}(t^2+u^2;x,y) = x^2 - \frac{3nx^2+x^2-n-\frac{1}{3}}{(n+1)^2} + y^2 - \frac{3my^2+y^2-m-\frac{1}{3}}{(m+1)^2}, \tag{2.4}$$

$$\begin{aligned} D_{n,m}(t^3+u^3;x,y) &= x^3 - \frac{x^3+6n^2x^3+3nx^3-3n^2x+6n+7nx+6nx^2}{(n+1)^3} \\ &+ y^3 - \frac{y^3+6m^2y^3+3my^3-3m^2y+6m+7my+6my^2}{(m+1)^3}, \end{aligned} \tag{2.5}$$

$$\begin{aligned} D_{n,m}(t^4+u^4;x,y) &= x^4 - \frac{10n^3x^4-5n^2x^4+10nx^4-x^4+6n^3x^2+6nx^2}{(n+1)^4} \\ &+ \frac{10n^2x^2+4nx-3n^2-4n-\frac{1}{5}}{(n+1)^4} \\ &+ y^4 - \frac{10m^3y^4-5m^2y^4+10my^4-y^4+6m^3y^2+6my^2}{(m+1)^4} \\ &+ \frac{10m^2y^2+4my-3m^2-4m-\frac{1}{5}}{(m+1)^4} \end{aligned} \tag{2.6}$$

From Lemma 2.1, we obtained following lemmas.

Lemma 2.2. *For the operators $D_{n,m}$ in (1.3), we have*

$$D_{n,m}((t-x)^2;x,y) = \frac{3x^2-3nx^2+3n+1}{3(n+1)^2}, \tag{2.7}$$

$$D_{n,m}((u-y)^2;x,y) = \frac{3y^2-3my^2+3m+1}{3(m+1)^2}, \tag{2.8}$$

$$\begin{aligned} D_{n,m}((t-x)^4;x,y) &= \frac{n^2x^4+8nx^4+x^4+44nx^2+20n^2x^2+24n^2x+24n^2x^3}{(n+1)^4} \\ &+ \frac{24nx^3+20nx+2x^2+3n^2+4n+\frac{1}{5}}{(n+1)^4} \end{aligned} \tag{2.9}$$

$$D_{n,m}((u-y)^4;x,y)=\frac{m^2y^4+8my^4+y^4+44my^2+20m^2y^2+24m^2y+24m^2y^3}{(m+1)^4}+\frac{24my^3+20my+2y^2+3m^2+4m+\frac{1}{5}}{(m+1)^4} \tag{2.10}$$

Lemma 2.3. *For every fixed $(x_0,y)\in\mathbb{A}$ there exists a positive constant $M_1(x_0)$ such that for $n\in\mathbb{N}$, $D_{n,n}((t-x_0)^4;x_0,y)\le M_1(x_0)n^{-2}$.*

Theorem 2.1. *(See [10]) If $(T_{n,m})$ is a sequence of linear positive operators satisfying the conditions*

$$\begin{aligned}\lim_{n,m\to\infty}\left\|T_{n,m}(1;x,y)-1\right\|_{C(\mathbb{X})}&=0\\ \lim_{n,m\to\infty}\left\|T_{n,m}(t;x,y)-x\right\|_{C(\mathbb{X})}&=0\\ \lim_{n,m\to\infty}\left\|T_{n,m}(u;x,y)-y\right\|_{C(\mathbb{X})}&=0\\ \lim_{n,m\to\infty}\left\|T_{n,m}((t^2+u^2;x,y)-(x^2+y^2)\right\|_{C(\mathbb{X})}&=0\end{aligned} \tag{2.11}$$

then for any function $f\in C(\mathbb{X})$, which is bounded in $\mathbb{R}^2$,

$$\lim_{n,m\to\infty}\left\|T_{n,m}(f;x,y)-f(x,y)\right\|_{C(\mathbb{X})}=0, \tag{2.12}$$

where $\mathbb{X}$ is a compact set.

In the following theorem we show that the linear positive operator $D_{n,m}$ in (1.3) converges to f uniformly with the help of Theorem 2.1.

Theorem 2.2. *For every $f\in C(\mathbb{A})$, the the operators $D_{n,m}$ defined by (1.3) converge uniformly to f on the set $\mathbb{A}$ as $n,m\to\infty$.*

Proof. From (2.1)-(2.4), we have

$$\begin{aligned}\lim_{n,m\to\infty}\left\|D_{n,m}(1;x,y)-1\right\|_{C(\mathbb{A})}&=0\\ \lim_{n,m\to\infty}\left\|D_{n,m}(t;x,y)-x\right\|_{C(\mathbb{A})}&=0\\ \lim_{n,m\to\infty}\left\|D_{n,m}(u;x,y)-y\right\|_{C(\mathbb{A})}&=0\\ \lim_{n,m\to\infty}\left\|D_{n,m}(t^2+u^2;x,y)-(x^2+y^2)\right\|_{C(\mathbb{A})}&=0.\end{aligned} \tag{2.13}$$

□

Applying Theorem 2.1, we obtain the desired result.

Theorem 2.3. *For $f\in C(\mathbb{A})$, the the following inequalities hold:*

$$\left\|D_{n,m}(f;x,y)-f\right\|_{C(\mathbb{A})}\le 2\left(\omega^{(1)}\left(f;\frac{1}{\sqrt{n}}\right)+\omega^{(2)}\left(f;\frac{1}{\sqrt{m}}\right)\right) \tag{2.14}$$

$$\left\|D_{n,m}(f;x,y)-f\right\|_{C(\mathbb{A})}\leq 2\omega\left(f;\sqrt{\frac{1}{n}+\frac{1}{m}}\right) \tag{2.15}$$

where ω, $\omega^{(1)}$ and $\omega^{(2)}$ are given by (1.6), (1.7) and (1.8), respectively.

Proof. From (1.3) and (2.1), we can write

$$\begin{aligned}
&|D_{n,m}(f;x,y)-f(x,y)| \\
&\leq D_{n,m}(|f(t,y)-f(x,y)|;x,y)+D_{n,m}(|f(t,u)-f(t,y)|;x,y) \\
&\leq \omega^{(1)}(f;\delta_n)\left[1+\frac{1}{\delta_n}\frac{n+1}{2}\sum_{k=0}^{n}\varphi_{n,k}(x)\int_{2\frac{k}{n+1}-1}^{2\frac{k+1}{n+1}-1}|t-x|dt\right] \\
&+\omega^{(2)}(f;\delta_m)\left[1+\frac{1}{\delta_m}\frac{m+1}{2}\sum_{j=0}^{m}\varphi_{m,j}(y)\int_{2\frac{j}{m+1}-1}^{2\frac{j+1}{m+1}-1}|u-y|du\right].
\end{aligned}$$

Applying the Cauchy-Schwartz inequality once more, we get

$$\begin{aligned}
&|D_{n,m}(f;x,y)-f(x,y)| \\
&\leq \omega^{(1)}(f;\delta_n)\left[1+\frac{1}{\delta_n}\frac{n+1}{2}\sum_{k=0}^{n}\varphi_{n,k}(x)\left(\int_{2\frac{k}{n+1}-1}^{2\frac{k+1}{n+1}-1}|t-x|^2dt\right)^{\frac{1}{2}}\left(\int_{2\frac{k}{n+1}-1}^{2\frac{k+1}{n+1}-1}dt\right)^{\frac{1}{2}}\right] \\
&+\omega^{(2)}(f;\delta_m)\left[1+\frac{1}{\delta_m}\frac{m+1}{2}\sum_{j=0}^{m}\varphi_{m,j}(y)\left(\int_{2\frac{j}{m+1}-1}^{2\frac{j+1}{m+1}-1}|u-y|^2du\right)^{\frac{1}{2}}\left(\int_{2\frac{j}{m+1}-1}^{2\frac{j+1}{m+1}-1}du\right)^{\frac{1}{2}}\right].
\end{aligned}$$

Applying the Cauchy-Schwartz inequality once more, we get

$$\begin{aligned}
&|D_{n,m}(f;x,y)-f(x,y)| \\
&\leq \omega^{(1)}(f;\delta_n)\left[1+\frac{1}{\delta_n}\left(\frac{n+1}{2}\right)^{\frac{1}{2}}\left(\sum_{k=0}^{n}\varphi_{n,k}(x)\right)^{\frac{1}{2}}\left(\sum_{k=0}^{n}\varphi_{n,k}(x)\int_{2\frac{k}{n+1}-1}^{2\frac{k+1}{n+1}-1}|t-x|^2dt\right)^{\frac{1}{2}}\right] \\
&+\omega^{(2)}(f;\delta_m)\left[1+\frac{1}{\delta_m}\left(\frac{m+1}{2}\right)^{\frac{1}{2}}\left(\sum_{j=0}^{m}\varphi_{m,j}(y)\right)^{\frac{1}{2}}\left(\sum_{j=0}^{m}\varphi_{m,j}(y)\int_{2\frac{j}{m+1}-1}^{2\frac{j+1}{m+1}-1}|u-y|^2du\right)^{\frac{1}{2}}\right] \\
&=\omega^{(1)}(f;\delta_n)\left[1+\frac{1}{\delta_n}\left(D_{n,m}((t-x)^2;x,y)\right)^{\frac{1}{2}}\right] \\
&+\omega^{(2)}(f;\delta_m)\left[1+\frac{1}{\delta_m}\left(D_{n,m}((u-y)^2;x,y)\right)^{\frac{1}{2}}\right].
\end{aligned}$$

From (2.7) and (2.8), we obtain the (2.14) inequality. Now we prove the (2.15) inequality:

$$\begin{aligned}|D_{n,m}(f;x,y)-f(x,y)| &\leq D_{n,m}(|f(t,u)-f(x,y)|;x,y)\\ &\leq \omega\left(f;\delta_{n,m}\right)\Bigg[1+\frac{1}{\delta_{n,m}}\frac{n+1}{2}\frac{m+1}{2}\sum_{k=0}^{n}\sum_{j=0}^{m}\phi_{n,m}^{k,j}(x,y)\\ &\quad\times\int_{2^{\frac{k}{n+1}}-1}^{2^{\frac{k+1}{n+1}}-1}\int_{2^{\frac{j}{m+1}}-1}^{2^{\frac{j+1}{m+1}}-1}\sqrt{(t-x)^2+(u-y)^2}\,dudt\Bigg].\end{aligned}$$

Applying the Cauchy-Schwartz inequality once more, we get

$$\begin{aligned}&|D_{n,m}(f;x,y)-f(x,y)|\\ &\leq \omega\left(f;\delta_{n,m}\right)\Bigg[1+\frac{1}{\delta_{n,m}}\Bigg(\frac{n+1}{2}\frac{m+1}{2}\sum_{k=0}^{n}\sum_{j=0}^{m}\phi_{n,m}^{k,j}(x,y)\\ &\quad\times\int_{2^{\frac{k}{n+1}}-1}^{2^{\frac{k+1}{n+1}}-1}\int_{2^{\frac{j}{m+1}}-1}^{2^{\frac{j+1}{m+1}}-1}(t-x)^2+(u-y)^2\,dudt\Bigg)^{\frac{1}{2}}\Bigg]\\ &=\omega\left(f;\delta_{n,m}\right)\left[1+\frac{1}{\delta_{n,m}}\left(D_{n,m}((t-x)^2+(u-y)^2;x,y)\right)^{\frac{1}{2}}\right].\end{aligned}$$

From (2.7) and (2.8), we obtain inequality (2.15). □

Now, we obtain a Voronovskaya-type theorem and some differential properties for the positive linear operators $D_{n,m}$ in (1.3) for $n=m$.

Theorem 2.4. *For every $f\in C^2(\mathbb{A})$, we have*

$$\begin{aligned}&\lim_{n\to\infty} n\{D_{n,n}(f;x,y)-f(x,y)\}\\ &=-xf_x'(x,y)-yf_y'(x,y)+\frac{1}{2}\{(1-x^2)f_{xx}''^2)f_{yy}''(x,y)\}.\end{aligned}\tag{2.16}$$

Proof. Let $(x,y)\in\mathbb{A}$ and $f\in C^2(\mathbb{A})$. Define the function ψ by

$$\psi(t,u;x,y)$$
$$=\begin{cases}\frac{f(t,u)-f(x,y)-f_x(t-x)-f_y(u-y)-\frac{1}{2}\{f_{xx}(t-x)^2+2f_{xy}(t-x)(u-y)+f_{yy}(u-y)^2\}}{\sqrt{(t-x)^4+(u-y)^4}} & ,\quad (t,u)\neq(x,y)\\ 0 & ,\quad (t,u)=(x,y).\end{cases}$$

Then by assumption we get $\psi(.,.;x,y)=\psi(.,.)\in C(\mathbb{A})$. By the Taylor formula for $f\in C(\mathbb{A})$, we have

$$\begin{aligned}f(t,u)&=f(x,y)+f_x'(x,y)(t-x)+f_y'(x,y)(u-y)\\ &\quad+\frac{1}{2}\{f_{xx}''(x,y)(t-x)^2+2f_{xy}''(x,y)(t-x)(u-y)+f_{yy}''(x,y)(u-y)^2\}\\ &\quad+\psi(t,u)\sqrt{(t-x)^4+(u-y)^4}\end{aligned}$$

Since the operator $D_{n,n}$ is linear, we obtain

$$\begin{aligned}
&D_{n,n}(f(t,u);x,y)\\
&= f(x,y) + f'_x(x,y)D_{n,n}((t-x);x,y) + f'_y(x,y)D_{n,n}((u-y);x,y)\\
&+ \frac{1}{2}\left[f''_{xx}(x,y)D_{n,n}((t-x)^2;x,y) + 2f''_{xy}(x,y)D_{n,n}((t-x);x,y)D_{n,n}((u-y);x,y)\right.\\
&\left.+f''_{yy}(x,y)D_{n,n}((u-y)^2;x,y)\right]\\
&+ D_{n,n}\left(\psi(t,u)\sqrt{(t-x)^4+(u-y)^4};x,y\right)
\end{aligned} \tag{2.17}$$

Applying the Cauchy-Schwartz inequality for the last term on the right-hand side of (2.17), we get

$$\begin{aligned}
&|D_{n,n}\left(\psi(t,u)\sqrt{(t-x)^4+(u-y)^4};x,y\right)|\\
&\leq \{D_{n,n}\left(\psi^2(t,u);x,y\right)\}^{\frac{1}{2}}\{D_{n,n}\left((t-x)^4+(u-y)^4;x,y\right)\}^{\frac{1}{2}}\\
&= \{D_{n,n}\left(\psi^2(t,u);x,y\right)\}^{\frac{1}{2}}\{D_{n,n}\left((t-x)^4;x,y\right) + D_{n,n}\left((u-y)^4;x,y\right)\}^{\frac{1}{2}}
\end{aligned}$$

Theorem 2.2 implies

$$\lim_{n\to\infty} D_{n,n}\left(\psi^2(t,u);x,y\right) = \psi^2(x,y) = 0. \tag{2.18}$$

Using (2.18) and Lemma 2.3, we obtain

$$\lim_{n\to\infty} nD_{n,n}\left(\psi(t,u)\sqrt{(t-x)^4+(u-y)^4};x,y\right) = 0. \tag{2.19}$$

Using (2.19) and Lemma 2.1, we derive (2.16) from (2.17). □

Theorem 2.5. *For every $f \in C^1(\mathbb{A})$ such that $f_x, f_y \in C(\mathbb{A})$, we have*

$$\lim_{n\to\infty} \frac{\partial}{\partial x} D_{n,n}(f;x,y) = \frac{\partial f}{\partial x}(x,y), \quad x \neq -1,1 \tag{2.20}$$

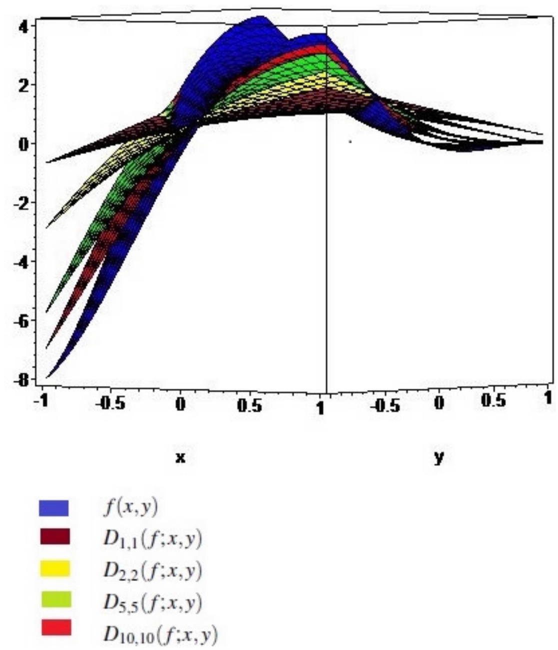
$$\lim_{n\to\infty} \frac{\partial}{\partial y} D_{n,n}(f;x,y) = \frac{\partial f}{\partial y}(x,y) \quad y \neq -1,1. \tag{2.21}$$

Proof. We shall prove only (2.20) because the proof of (2.21) is identical. Let $(x,y) \in \mathbb{A}$ for $x \neq -1,1$ be a fixed point. From (1.3) it follows that

$$\begin{aligned}
\frac{\partial}{\partial x} D_{n,n}(f(t,u);x,y) &= -\frac{n}{1-x} D_{n,n}(f(t,u);x,y)\\
&+ \frac{2}{(1+x)(1-x)} D_{n,n}(kf(t,u);x,y), \quad \forall n \in \mathbb{N}.
\end{aligned}$$

By the Taylor formula for $f \in C^1(\mathbb{A})$, we have

$$\begin{aligned}
f(t,u) = f(x,y) &+ f'_x(x,y)(t-x) + f'_y(x,y)(u-y)\\
&+ \chi(t,u;x,y)\sqrt{(t-x)^2+(u-y)^2}, \quad (t,u) \in \mathbb{A}
\end{aligned}$$



where $\chi(.,.;x,y)=\chi(.,.)\in C$ and $\chi(x,y)=0$.

From (2.1) and (2.7), we get

$$\begin{aligned}&\frac{\partial}{\partial x}D_{n,n}(f(t,u);x,y)\\&=f(x,y)\Big\{-\frac{n}{1-x}D_{n,n}(1;x,y)+\frac{2}{(1-x)(1+x)}D_{n,n}(k;x,y)\Big\}\\&+f'_x(x,y)\Big\{-\frac{n}{1-x}D_{n,n}(t-x;x,y)+\frac{2}{(1-x)(1+x)}D_{n,n}(k(t-x);x,y)\Big\}\\&+f'_y(x,y)\Big\{-\frac{n}{1-x}D_{n,n}(u-y;x,y)+\frac{2}{(1-x)(1+x)}D_{n,n}(k(u-y);x,y)\Big\}\\&+\frac{n}{(1-x)(1+x)}D_{n,n}\left(\left(2\frac{k}{n}-x-1\right)\chi(t,u)\sqrt{(t-x)^2+(u-y)^2};x,y\right)\end{aligned}$$

From Lemma 2.1, we obtain

$$\begin{aligned}\frac{\partial}{\partial x}D_{n,n}(f(t,u);x,y)=\frac{n}{n+1}f'_x(x,y)+\frac{n}{(1-x)(1+x)}D_{n,n}\Bigg(\Big(2\frac{k}{n}-x-1\Big)\\ \times\ \chi(t,u)\sqrt{(t-x)^2+(u-y)^2};x,y\Bigg).\end{aligned} \tag{2.22}$$

Using the Cauchy-Schwartz inequality, we get

$$\begin{aligned}
&n|D_{n,n}\left(\left(2\frac{k}{n}-x-1\right)\chi(t,u)\sqrt{(t-x)^2+(u-y)^2};x,y\right)|\\
&\le\left\{D_{n,n}\left(\left(2\frac{k}{n}-x-1\right)^2;x,y\right)\right\}^{\frac{1}{2}}\left\{n^2D_{n,n}\left(\chi^2(t,u)((t-x)^2+(u-y)^2);x,y\right)\right\}^{\frac{1}{2}}\\
&\le\left\{D_{n,n}\left(\left(2\frac{k}{n}-x-1\right)^2;x,y\right)\right\}^{\frac{1}{2}}\left\{n^2D_{n,n}\left(\chi^4(t,u);x,y\right)\left[D_{n,n}\left((t-x)^4\right);x,y\right)\right.\\
&\left.+\,2D_{n,n}\left((t-x)^2;x,y\right))D_{n,n}\left((u-y)^2);x,y\right)+D_{n,n}\left((u-y)^4);x,y\right)\right]\Bigg\}^{\frac{1}{4}}.
\end{aligned}$$

From (1.3), we obtain

$$\begin{aligned}
D_{n,n}\left(\left(2\frac{k}{n}-x-1\right)^2;x,y\right)&=D_{n,n}\left(4\frac{k^2}{n^2}-4\frac{k}{n}(x+1)+(x+1)^2;x,y\right)\\
&=\frac{4}{n^2}D_{n,n}(k^2;x,y)-\frac{4}{n}(x+1)D_{n,n}(k;x,y)\\
&+(x+1)^2D_{n,n}(1;x,y).
\end{aligned}$$

From (2.1) and

$$D_{n,n}(k^2;x,y)=\frac{n(n-1)(1+x)^2}{4}+\frac{n(1+x)}{2},$$

$$D_{n,n}(k;x,y)=\frac{n(1+x)}{2}$$

equalities, we get

$$D_{n,n}\left(\left(2\frac{k}{n}-x-1\right)^2;x,y\right)=\frac{1-x^2}{n}.$$

So, we obtain from Lemma 2.2 and Lemma 2.3

$$\begin{aligned}
&n\Big|D_{n,n}\left(\left(2\frac{k}{n}-x-1\right)\chi(t,u)\sqrt{(t-x)^2+(u-y)^2};x,y\right)|\\
&\le M(x,y)n^{-1}\left(D_{n,n}\left(\chi^4(t,u);x,y\right)\right)^{\frac{1}{4}}.
\end{aligned}\tag{2.23}$$

From Theorem 2.2, we get

$$\lim_{n\to\infty}D_{n,n}\left(\chi^4(t,u);x,y\right)=\chi^4(x,y)=0,\quad (x,y)\in\mathbb{A}$$

which used to (2.23) gives

$$\lim_{n\to\infty}nD_{n,n}\left(\left(2\frac{k}{n}-x-1\right)\chi(t,u)\sqrt{(t-x)^2+(u-y)^2};x,y\right)=0.$$

Consequently, we obtain from (2.22)

$$\lim_{n\to\infty} \frac{\partial}{\partial x} D_{n,n}(f;x,y) = \frac{\partial f}{\partial x}(x,y).$$

$\square$

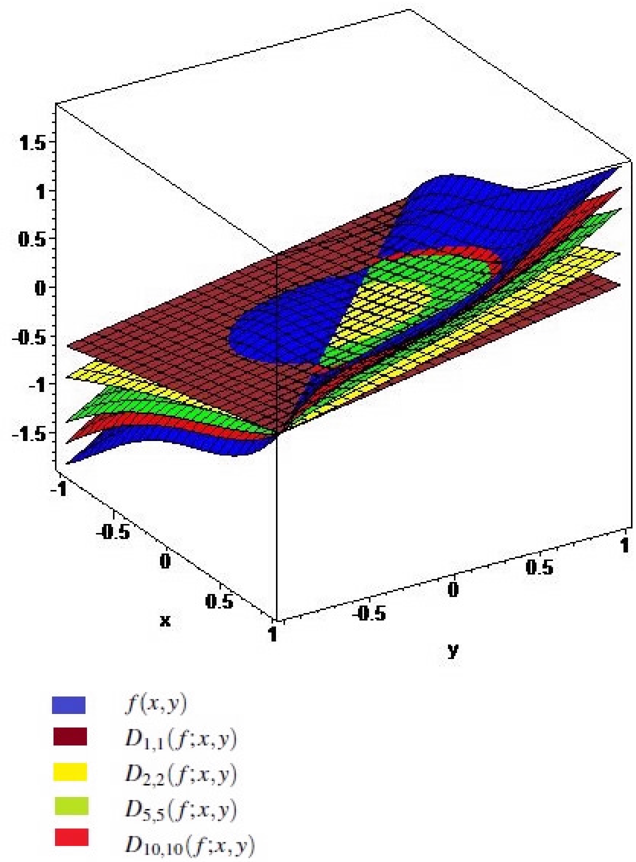


Figure 1. Approximation to the function $f(x,y) = (x^2 + y^2)\sin(x+y)$ by the generalized Bernstein-Kantorovich operators.

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