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Exact Solutions for Forced Vibration of Non-Uniform Rods by Laplace Transformation

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ABSTRACT

Longitudinal forced vibration behavior of non-uniform rods subjected to dynamic axial load is studied. Exact displacement solutions are obtained using the Laplace transformation method. Free vibration behavior is readily obtained in the analysis. Natural frequencies available in the literature for the cases considered are fully recovered. Inverse transformation into the time domain is performed using calculus of residues. Closed-form displacement expressions are tractable and efficiently implemented. Their efficiency is demonstrated by comparing the results with those obtained using Mode Superposition Method.

Key Words: *Longitudinal vibrations; Forced vibrations; Natural frequency; Non-uniform rod; Laplace transformation; Residue theorem.*

1. INTRODUCTION

Longitudinal vibrations of non-uniform bars have attracted considerable scientific and practical attention in the study of composite structures subjected to high velocity impact and the study of foundations [1]. The use of variable cross-section members can help the designer reduce the weight; improve strength and stability of structures [2]. Free vibration analysis and presentation of fundamental frequencies along with mode shapes constitute most of the archival works. The researchers are expected to subsequently obtain the forced vibration response through methods such as Mode Superposition Method. Recent works on free vibration analysis of non-uniform rods include works by Abrate [3], Li, *et al.* [4], Li [5], Qiusheng, *et al.*[6] , Raj and Sujith [7], Nachum and Altus [8], Horgan and Chan [9] where exact solutions are obtained either in closed-form or by using methods such as lumped parameters and finite elements.

In the present paper forced vibration analysis of non-uniform rods subjected at the end point to various time-dependent axial forces will be presented in closed-form equations. The need for exact solutions is obvious: they give adequate insight into the physics of the problem as well as establishing the accuracy of the approximate or numerical solutions. In optimization problems using closed-form solutions will greatly reduce the solution

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time. Laplace transformation will be employed in the analysis. The inversion into the time domain is performed exactly using calculus of residues. Free vibration behavior is readily obtained since substituting the complex Laplace parameter in the governing equation directly gives natural frequencies [10]. Uniform mass and stiffness are assumed along the rod. The cross section is assumed to vary along the non-dimensional axial coordinate η in the forms $A(\eta)=A_o\sin^2[a\eta+b]$, $A(\eta)=A_o(1+a\eta)^2$ and $A(\eta)=A_o e^{-a\eta}$. Natural frequencies for these cross sections are given in tabular form and good agreement with the benchmark results presented by Kumar and Sujith [1], Abrate [3] and Li, *et al.* [4], is displayed. The first ten fundamental frequencies are listed. Using only the first ten frequencies provided six-digit accuracy in the forced vibration analysis. The results are compared to those obtained via Mode Superposition Method (MSM). Among other advantages, the efficiency of analytical results is obvious: for some cases, up to a hundred frequencies were needed in MSM to achieve the same accuracy.

2. THEORY

The longitudinal motion of a rod with varying cross-section $A(x)$, uniform density and Young's modulus is governed by the differential equation

$$\frac{\partial^2 u(x,t)}{\partial x^2}+\frac{1}{A(x)}\frac{\partial A(x)}{\partial x}\frac{\partial u(x,t)}{\partial x}=\frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2},$$

$$0<x<L,\quad t>0 \tag{1}$$

where $c^2=E/\rho$.

Using the dimensionless variables

$$v=\frac{u}{L},\qquad \eta=\frac{x}{L},\qquad \tau=\frac{ct}{L} \tag{2}$$

renders Eq. (1) in the form

$$\frac{\partial^2 v(\eta,\tau)}{\partial\eta^2}+\frac{1}{A(\eta)}\frac{\partial A(\eta)}{\partial\eta}\frac{\partial v(\eta,\tau)}{\partial\eta}=\frac{\partial^2 v(\eta,\tau)}{\partial\tau^2},$$

$$0<\eta<1,\quad \tau>0 \tag{3}$$

The above equation will be solved for rods with cross-sections varying as $A(\eta)=A_o\sin^2[a\eta+b]$, $A(\eta)=A_o(1+a\eta)^2$ and $A(\eta)=A_o e^{-a\eta}$. Axial end forces to be considered are: $P_1(\tau)=P_0(1-\cos[\gamma\tau])$, $P_2(\tau)=P_0$, $P_3(\tau)=P_0(1-e^{-\gamma\tau})$.

2.1 Solutions for the Cross Section Varying as $A(\eta)=A_o\sin^2[a\eta+b]$

A detailed analysis for the load type $P_1(\tau)=P_0(1-\cos[\gamma\tau])$ will be presented, and subsequently, the outline of analyses and results for other cases will be listed. A fixed-free rod with the axial force applied at the free end $\eta=1.0$ will be considered. Initial and boundary conditions accompanying the governing differential equation (3) are

$$v(\eta,0)=0,\quad \frac{\partial v(\eta,0)}{\partial\tau}=0,$$

$$v(0,\tau)=0,\quad \left.\frac{\partial v}{\partial\eta}\right|_{\eta=1}=\frac{P(\tau)}{A(1)\,E} \tag{4}$$

where E is Young's modulus.

Substituting $A(\eta)=A_o\sin^2[a\eta+b]$ and taking the Laplace transform of Eqs. (3-4) yields

$$y''(\eta,p)+2a\cot[a\eta+b]\,y'(\eta,p)-p^2\,y(\eta,p)=0 \tag{5}$$

$$y(0,p)=0,\quad \frac{\partial y(1,p)}{\partial\eta}=\frac{1}{E\,A(1)}L\{P_1(\tau)\} \tag{6}$$

where $y(\eta,p)=L\{v(\eta,\tau)\}$, p being the complex Laplace parameter. Introducing a new variable *z* defined as [1]

$$y(\eta,p)=\frac{z}{\sin[a\eta+b]} \tag{7}$$

into Eq. (5) results in

$$z''+(a^2-p^2)z=0 \tag{8}$$

At this stage free vibration analysis can easily be performed and determination of fundamental frequencies would be in order. The Laplace parameter *p* in Eq. (8) is now replaced by $i\alpha$ to get

$$z''+(a^2+\alpha^2)z=0 \tag{9}$$

Note that α now corresponds to the natural frequency whose determination will not require inverse transformation [10]. The general solution of Eq. (9) is

$$z=c_1\sin[k\eta]+c_2\cos[k\eta] \tag{10}$$

where

$$k^2=a^2+\alpha^2 \tag{11}$$

Therefore,

$$y=\frac{1}{\sin[a\eta+b]}(c_1\sin[k\eta]+c_2\cos[k\eta]) \tag{12}$$

Imposing boundary conditions $y(0)=0$ and $(\partial y/\partial\eta)_{|\eta=1}=0$ gives the following transcendental equation for natural frequencies

$$k=\frac{a}{\tan[a+b]}\tan[k] \tag{13}$$

For arbitrary c_1, the mode shape corresponding to each α_n found from Eq. (13) is

$$y_n = \frac{\sin[k_n \eta]}{\sin[a\,\eta + b]} \tag{14}$$

First ten frequencies are listed in Table 1. for a=0, 1, 2 and b=1 which match the results in reference [1].

Table1. First ten natural frequencies for $A(\eta) = A_o \sin^2[a\eta + b]$

Mode	a=0	a=1	a=2
1	1.57079	1.51764	2.14856
2	4.71239	4.70214	5.53576
3	7.85398	7.84831	8.63281
4	10.9956	10.9916	11.6946
5	14.1372	14.1341	14.7579
6	17.2788	17.2763	17.8306
7	20.4204	20.4183	20.9137
8	23.5619	23.5601	24.0062
9	26.7035	26.7019	27.1064
10	29.8451	29.8437	30.2129

To determine the dynamic response in the Laplace space due to $P_1(\tau) = P_0(1 - \cos[\,\gamma\,\tau])$ with $L\{P_1(\tau)\} = \mathrm{P}_0\,\gamma^2 / p\,(p^2 + \gamma^2)$, boundary conditions given by Eq. (6) are applied to Eq. (12) yielding

$$y = \frac{P_0\gamma^2}{EA_0 \sin[a\eta + b]} \frac{F(p)}{p(p^2 + \gamma^2)G(p)} \tag{15}$$

where

$$F(p) = \sin[\sqrt{a^2 - p^2}\,\eta] \tag{16}$$

$$G(p) = [\sqrt{a^2 - p^2}\,\sin[a+b]\cos[\sqrt{a^2 - p^2}\,] - a\sin[\sqrt{a^2 - p^2}\,]\cos[a+b]] \tag{17}$$

and the complex Laplace parameter p has been maintained in the formulation.

Using the inversion theorem we may write the displacement function v in the real time space as

$$v(\eta,\tau) = \frac{P_0\gamma^2}{E A_0 \sin[a\eta + b] 2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{\xi\tau}\;F(\xi)\;d\xi}{(\xi^2 + \gamma^2)G(\xi)\,\xi} \tag{18}$$

Applying the residue theorem yields

$$v(\eta,\tau) = 2\,\pi\,i \sum \text{Residues at the poles of } v \tag{19}$$

The singular points of Eq. (18) are $\xi = \pm i\gamma$ and the zeros of the equation $G(\xi) = 0$ and $\xi = 0$.

The residues at the pole $i\gamma$ and $-i\gamma$ can be obtained readily. They are

$$-\frac{e^{i\gamma\tau}F(i\gamma)}{2\gamma^2\,G(i\gamma)} \quad \text{and} \quad -\frac{e^{-i\gamma\tau}F(-i\gamma)}{2\gamma^2\,G(-i\gamma)} \tag{20}$$

Noting that $F(-i\gamma) = F(i\gamma)$ and $G(-i\gamma) = G(i\gamma)$ sum of the residues becomes

$$R_1 = -\frac{F(i\gamma)\,\cos[\gamma\,\tau]}{\gamma^2\,G(i\gamma)} \tag{21}$$

As for the roots of $G(\xi) = 0$; in view of Eq. (17), replacing ξ with $i\alpha$ and setting $G(i\alpha) = G(-i\alpha)$ equal to zero gives

$$[\sqrt{a^2 + \alpha_s^2}\,\sin[a+b]\,\cos[\sqrt{a^2 + \alpha_s^2}\,] - a\sin[\sqrt{a^2 + \alpha_s^2}\,]\cos[a+b]] = 0 \tag{22}$$

The roots α_s, $s = 1,2,...$ of Eq.(22) correspond to natural frequencies given by Eq. (13). Without resorting to lengthy discussions, based on this observation, it can be concluded that these are all real and simple.

The residues of the integrand at the simple poles $\xi = \pm i\alpha_s$ are

$$\left.\frac{e^{i\alpha_s\tau}F(i\alpha_s)}{(\gamma^2 - \alpha_s^{\,2})\alpha_s \dfrac{dG}{d\xi}}\right|_{\xi = \alpha_s} \quad \text{and}$$

$$\left.\frac{e^{-i\alpha_s\tau}F(-i\alpha_s)}{(\gamma^2 - \alpha_s^{\,2})\alpha_s \dfrac{dG}{d\xi}}\right|_{\xi = -\alpha_s} \tag{23}$$

The residue sum is expressed explicitly as

$$R_s = \frac{2\cos[\alpha_s\tau]\sin[\sqrt{a^2 + \alpha_s^{\,2}}\,]\sqrt{a^2 + \alpha_s^{\,2}}}{(\gamma^2 - \alpha_s^{\,2})(\alpha_s^{\,2}(a\cos[a+b]\cos[\sqrt{a^2 + \alpha_s^{\,2}}\,] + \sin[a+b]\,(-\cos[\sqrt{a^2 + \alpha_s^{\,2}}\,] + \sqrt{a^2 + \alpha_s^{\,2}}\,\sin[\sqrt{a^2 + \alpha_s^{\,2}}\,]\,)))} \tag{24}$$

The sum of the residues at the poles of the zeros of $G(\xi)$ is

$$R_2 = \sum_{s=1}^{\infty} R_s \tag{25}$$

For $\xi = 0$ the residue is as follows:

$$R_3 = \frac{\sin[a\;]}{\gamma^2 a \sin[b]} \tag{26}$$

Final form of the displacement can now be written as

$$v(1,\tau) = \frac{P_0\gamma^2}{E\,A_0(\sin[a+b])}(R_1 + R_2 + R_3) \tag{27}$$

Table 2. lists these results along with displacement due to $P_2(\tau) = P_0$ and $P_3(\tau) = P_0(1 - e^{-\gamma\tau})$.

Table 2. End displacements due to $P_1(\tau), P_2(\tau)$ and $P_3(\tau)$ for $A(\eta) = A_o \sin^2 [a\eta + b]$

$P_1(\tau)$	$v(1,\tau) = \dfrac{P_0 \gamma^2}{E_0\ A_0\ \sin[a+b]}(R_1 + R_2 + R_3)$	
	R_1	$-\dfrac{F(i\gamma)\cos[\gamma\,\tau]}{\gamma^2 G(i\gamma)}$
	R_2	$\sum\limits_{s=1}^{\infty} \dfrac{2\cos[\alpha_s \tau]\sin[\sqrt{a^2+\alpha_s^{\ 2}}](\sqrt{a^2+\alpha_s^{\ 2}})}{(\gamma^2-\alpha_s^{\ 2})(\alpha_s^2(a\cos[a+b]\cos[\sqrt{a^2+\alpha_s^{\ 2}}] + \sin[a+b](-\cos[\sqrt{a^2+\alpha_s^{\ 2}}] + \sqrt{a^2+\alpha_s^{\ 2}}\sin[\sqrt{a^2+\alpha_s^{\ 2}}])))}$
	R_3	$\dfrac{\sin[a]}{\gamma^2 (a)\sin[b]}$
$P_2(\tau)$	$v(1,\tau) = \dfrac{P_0}{E_0\ A_0\ \sin[a+b]}(R_1 + R_2 + R_3)$	
	R_1	0
	R_2	$\sum\limits_{s=1}^{\infty} \dfrac{2\cos[\alpha_s \tau]\sin[\sqrt{a^2+\alpha_s^{\ 2}}](\sqrt{a^2+\alpha_s^{\ 2}})}{\alpha_s^2(a\cos[a+b]\cos[\sqrt{a^2+\alpha_s^{\ 2}}] + \sin[a+b](-\cos[\sqrt{a^2+\alpha_s^{\ 2}}] + \sqrt{a^2+\alpha_s^{\ 2}}\sin[\sqrt{a^2+\alpha_s^{\ 2}}]))}$
	R_3	$\dfrac{\sin[a]}{a\sin[b]}$
$P_3(\tau)$	$v(1,\tau) = \dfrac{P_0 \gamma}{E_0\ A_0\ \sin[a+b]}(R_1 + R_2 + R_3)$	
	R_1	$-\dfrac{F(-\gamma)e^{-\gamma\tau}}{\gamma\, G(-\gamma)}$
	R_2	$\sum\limits_{s=1}^{\infty} \dfrac{(2\gamma\cos[\alpha_s \tau] + 2\alpha_s\sin[\alpha_s\,\tau])\sin[\sqrt{a^2+\alpha_s^{\ 2}}](\sqrt{a^2+\alpha_s^{\ 2}})}{(\gamma^2+\alpha_s^2)(\alpha_s^2(a\cos[a+b]\cos[\sqrt{a^2+\alpha_s^{\ 2}}] + \sin[a+b](-\cos[\sqrt{a^2+\alpha_s^{\ 2}}] + \sqrt{a^2+\alpha_s^{\ 2}}\sin[\sqrt{a^2+\alpha_s^{\ 2}}])))}$
	R_3	$\dfrac{\sin[a]}{\gamma(a)\sin[b]}$

2.2. Solutions for the Cross Section Varying as $A(\eta) = A_o(1 + a\,\eta)^2$ and $A(\eta) = A_o\, e^{-a\,\eta}$

Following the same steps as in the preceding section the natural frequencies can be found for $A(\eta) = A_o(1 + a\,\eta)^2$ and $A(\eta) = A_o\, e^{-a\,\eta}$.

For $A(\eta) = A_o(1 + a\,\eta)^2$, natural frequencies and the mode shape are given by Abrate [3] as

$$y_n = \frac{\sin[k_n \eta]}{(1 + a\eta)} \quad , \quad \alpha_n = k_n \tag{28}$$

where a new variable z is introduced as

$$y = \frac{z}{(1 + a\,\eta)} \tag{29}$$

For $A(\eta) = A_o\, e^{-a\,\eta}$, natural frequencies and the mode shape are given by Li, *et al.* [4] as

$$y_n = e^{a\eta/2}\sin[k_n \eta] \quad , \quad k_n^2 = \alpha_n^2 - \frac{a^2}{4} \tag{30}$$

which is valid for $\Delta = a^2 - 4\alpha_n^2 < 0$

The displacement solutions can be obtained following the same steps as in the preceding section. Tables 3. and 4. show the axial displacement expressions of the end point ($\eta = 1$) for $A(\eta) = A_o(1 + a\,\eta)^2$ and $A(\eta) = A_o\, e^{-a\,\eta}$, respectively.

3. COMPARISION WITH MODE SUPERPOSITION METHOD (MSM)

In this section, efficient implementation of the presented results will be demonstrated by comparing them with those from MSM. It is common knowledge that MSM is comprised of three main steps [11]:

1) Determining natural frequencies (α_n) and mode shapes (y_n) as discussed in preceding sections

2) Computing generalized mass (M_n) and loading (P_n) by

$$M_n = c^2 \int_0^1 y_n^2\, m(\eta)\, d\eta \tag{31}$$

$$P_n(\tau) = P(\tau)\, y_n(1) \tag{32}$$

3) Evaluating displacement by

$$v(\eta,\tau) = \sum_{n=1}^{\infty} y_n\, R_n(\tau) \tag{33}$$

where $R_n(\tau)$ is the normal-coordinate response for an undamped single-degree-of-freedom system subjected to any form of dynamic loading. It can readily be calculated through the Duhamel Integral as

$$R_n(\tau) = \frac{1}{M_n\, \alpha_n} \int_0^{\tau} P_n(\tau^*) \sin[\alpha_n(\tau - \tau^*)] d\tau^* \tag{34}$$

First ten frequencies have been taken in the calculations. Figs. 1-3 show the results for $A(\eta) = A_o \sin^2 [a\eta + b]$ subjected to dynamic loads $P_1(\tau) = P_0(1-\cos[\gamma\,\tau])$, $P_2(\tau) = P_0$ and $P_3(\tau) = P_0(1-e^{-\gamma\tau})$, respectively. A γ value of 0.6 has been used throughout. Increasing a causes an increase in the displacement. In all loading cases, for $a = 2$, a noticeable difference was present between the exact and MSM results. To achieve the accuracy of the exact results, up to a hundred frequencies have been needed in MSM.

Table 3. End displacements due to $P_1(\tau), P_2(\tau)$ and $P_3(\tau)$ for $A(\eta) = A_o\, e^{-a\eta}$

$P_1(\tau)$	$v(1,\tau) = \frac{2e^a P_0 \gamma^2}{E_0 A_0}(R_1 + R_2 + R_3)$	
	R_1	$-\frac{F(i\gamma)\cos[\gamma\,\tau]}{\gamma^2 G(i\gamma)}$
	R_2	$\sum_{s=1}^{\infty} \frac{2\cos[\alpha_s\tau]\sinh[\frac{1}{2}\sqrt{a^2-4\alpha_s^2}]\sqrt{a^2-4\alpha_s^2}}{(\gamma^2-\alpha_s^2)(\alpha_s(-2\alpha_s((2+a)\cosh[\frac{1}{2}\sqrt{a^2-4\alpha_s^2}]+\sqrt{a^2-4\alpha_s^2})\sinh[\frac{1}{2}\sqrt{a^2-4\alpha_s^2}])))}$
	R_3	$\frac{\sinh[a/2]}{a(\cosh[a/2]+\sinh[a/2])\,\gamma^2}$
$P_2(\tau)$	$v(1,\tau) = \frac{2e^a P_0}{E_0 A_0}(R_1 + R_2 + R_3)$	
	R_1	0
	R_2	$\sum_{s=1}^{\infty} \frac{2\cos[\alpha_s\tau]\sinh[\frac{1}{2}\sqrt{a^2-4\alpha_s^2}]\sqrt{a^2-4\alpha_s^2}}{(\alpha_s(-2\alpha_s((2+a)\cosh[\frac{1}{2}\sqrt{a^2-4\alpha_s^2}]+\sqrt{a^2-4\alpha_s^2})\sinh[\frac{1}{2}\sqrt{a^2-4\alpha_s^2}])))}$
	R_3	$\frac{\sinh[a/2]}{a(\cosh[a/2]+\sinh[a/2])}$
$P_3(\tau)$	$v(1,\tau) = \frac{2e^a P_0 \gamma}{E_0 A_0}(R_1 + R_2 + R_3)$	
	R_1	$-\frac{F(-\gamma)e^{-\gamma\tau}}{\gamma\, G(-\gamma)}$
	R_2	$\sum_{s=1}^{\infty} \frac{(2\alpha_s\sin[\alpha_s\tau]+2\gamma\cos[\alpha_s\tau])\sinh[\frac{1}{2}\sqrt{a^2-4\alpha_s^2}]\sqrt{a^2-4\alpha_s^2}}{(\gamma^2+\alpha_s^2)(\alpha_s(-2\alpha_s((2+a)\cosh[\frac{1}{2}\sqrt{a^2-4\alpha_s^2}]+\sqrt{a^2-4\alpha_s^2})\sinh[\frac{1}{2}\sqrt{a^2-4\alpha_s^2}])))}$
	R_3	$\frac{\sinh[a/2]}{\gamma\, a(\cosh[a/2]+\sinh[a/2])}$

Table 4. End displacements due to $P_1(\tau), P_2(\tau)$ and $P_3(\tau)$ for $A(\eta) = A_o(1 + a\,\eta)^2$.

$P_1(\tau)$	$v(1,\tau) = \frac{P_0\gamma^2}{E_0A_0(1+a)}(R_1 + R_2 + R_3)$	
	R_1	$-\frac{F(i\gamma)\cos[\gamma\,\tau]}{\gamma^2 G(i\gamma)}$
	R_2	$\sum_{s=1}^{\infty} \frac{2F(i\alpha_s)\cos[\alpha_s\tau]}{(\gamma^2 - \alpha_s^2)\alpha_s(\cos[\alpha_s] - (1+a)\alpha_s \sin[\alpha_s])}$
	R_3	$1/\gamma^2$
$P_2(\tau)$	$v(1,\tau) = \frac{P_0}{E_0A_0(1+a)}(R_1 + R_2 + R_3)$	
	R_1	0
	R_2	$\sum_{s=1}^{\infty} \frac{2F(i\alpha_s)\cos[\alpha_s\tau]}{\alpha_s(\cos[\alpha_s] - (1+a)\alpha_s \sin[\alpha_s])}$
	R_3	1
$P_3(\tau)$	$v(1,\tau) = \frac{P_0\gamma}{E_0A_0(1+a)}(R_1 + R_2 + R_3)$	
	R_1	$-\frac{F(-\gamma)e^{-\gamma\tau}}{\gamma\, G(-\gamma)}$
	R_2	$\sum_{s=1}^{\infty} \frac{F(i\alpha_s)(2\alpha_s \sin[\alpha_s\tau] + 2\gamma\, \cos[\alpha_s\tau])}{(\gamma^2 + \alpha_s^2)\alpha_s(\cos[\alpha_s] - (1+a)\alpha_s \sin[\alpha_s])}$
	R_3	$\frac{1}{\gamma}$

4. CONCLUSIONS

Closed-form solutions for free and forced vibration analyses of a rod with varying cross-section have been performed using Laplace transform technique. The problem is solved in the Laplace domain and the inversion into the time domain is done exactly by the residue theorem. The numerical results obtained from the residue theorem are compared with those from Mode Superposition Method (MSM). It is seen that when use of first ten frequencies in the analytical solutions gives six-digit accuracy, up to a hundred frequencies are needed in MSM. In addition to their efficient use, exact solutions give adequate insight into the physics of the problem. In optimization problems using closed-form solutions will greatly reduce the solution time.

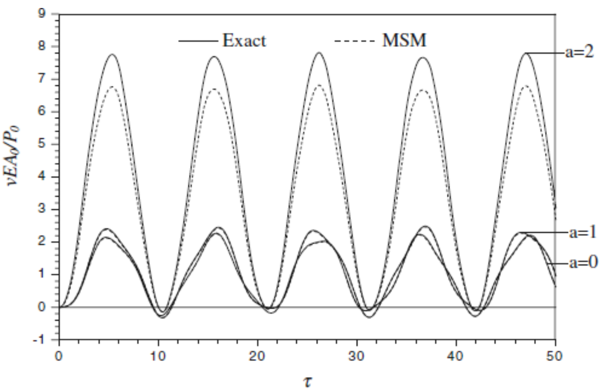

Figure 1. End displacement for $A(\eta) = A_o \sin^2 [a\eta + b]$ under $P_1(\tau) = P_0(1 - \cos(\gamma\tau))$.

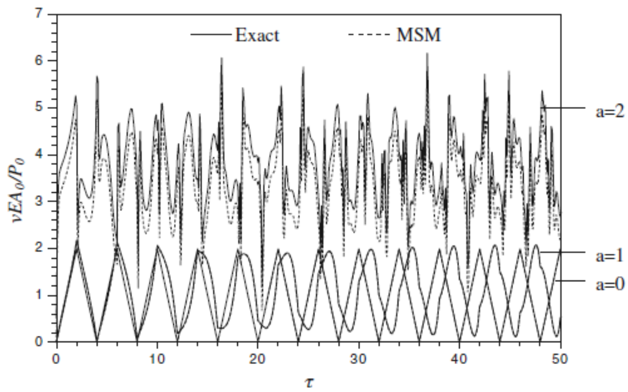

Figure 2. End displacement for $A(\eta) = A_o \sin^2 [a\eta + b]$ under $P_2(\tau) = P_0$.

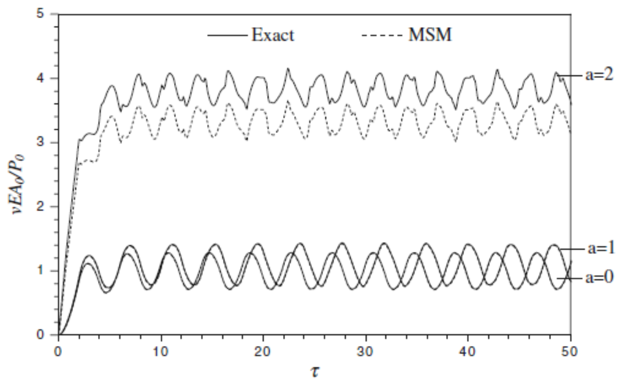

Figure 3. End displacement for $A(\eta) = A_o \sin^2 [a\eta + b]$ under $P_3(\tau) = P_0(1 - e^{-\gamma\tau})$.

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