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$H_B^{\tau_1,\tau_2,\tau_3}$ Srivastava Hypergeometric Function

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Abstract
Formulas and identities involving many well known special functions (such as the Gamma and Beta functions, Gauss hypergeometric function, and so on) play important roles in themselves and their diverse applications. In this paper, we will add τ_1, τ_2, τ_3 parameters to the H_B Srivastava hypergeometric function and we introduce new $H_B^{\tau_1,\tau_2,\tau_3}$ Srivastava's triple τ-hypergeometric function. Then, we present some properties of the $H_B^{\tau_1,\tau_2,\tau_3}$ Srivastava's triple τ-hypergeometric function.

Keywords: *Srivastava hypergeometric function; Integral representations;Recurrence relations.*

AMS Subject Classification (2010): *Primary: 33C60 ; Secondary: 33C90.*

1. Introduction

In this paper, $\mathbb{N}$, $\mathbb{Z}^-$, and $\mathbb{C}$ denote the sets of positive integers, negative integers, complex numbers,respectively.Also $\mathbb{N}_0$ and $\mathbb{Z}_0^-$ represent th sets of positive integers and complex numbers byexcluding origin,respectively.($\mathbb{N}_0 := \mathbb{N} \cup \{0\}$and $\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}$.)

The classical Gamma function $\Gamma(x)$ defined by [18, 20, 22, 23],

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \quad (\Re(x) > 0). \tag{1.1}$$

The familiar Beta function $B(x, y)$ is a function of two complex variables x and y, defined by Eulerian integral of the first kind [18, 20, 22, 23],

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt. \tag{1.2}$$

Srivastava [21, 24, 25] noticed the existence of three additional complete triple hypergeometric functions of the second order; of which H_B is defined as [11–13, 22, 23, 26]:

$$\begin{aligned} &H_B[\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x_1, x_2, x_3] \\ &= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p}(\beta_1)_{m+n}(\beta_2)_{n+p}}{(\gamma_1)_m(\gamma_2)_n(\gamma_3)_p}\frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}\ , \\ &(|x_1| < r, |x_2| < s, |x_3| < t, r+s+t = 1+st)\,. \end{aligned} \tag{1.3}$$

There,$\mathbb{C}$ and $\mathbb{Z}_0^-$ denote the set of complex numbers and the set of nonpositive integers respectively.Here $(\lambda)_n$ is the Pochhammer symbol is defined by $(\lambda \in \mathbb{C})$ [18, 20, 22, 23];

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ (\lambda)(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N}) \end{cases} \tag{1.4}$$

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where Γ being well-known Gamma functions (1.1).

In 2001, Virchenko et al. [30, 31] have studied and investigated the following generilazied τ-hypergeometric function:

$$\begin{aligned} {}_2R_1^{\tau}(a,b;c;z) &= {}_2R_1(a,b;c;\tau;z) = \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a)_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!} \\ (\tau &> 0, |z|<1, \Re(c)>\Re(b)>0) \end{aligned} \tag{1.5}$$

They gave the Euler type integral representation as follows [30, 31]:

$$\begin{aligned} {}_2R_1(a,b;c;\tau;z) &= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt^{\tau})^{-a}\,dt. \\ (\tau &> 0; |\arg(1-z)|<\pi, \Re(c)>\Re(b)>0) \end{aligned} \tag{1.6}$$

The special case when $\tau=1$ in (1.5) and (1.6) give the familiar representations of Gauss's hypergeometric functions [7, 8, 18, 20, 22, 23].

Furthermore,Al-Shammery and Kalla [4] introduced and studied various properties of second τ-Appell's hypergeometric functions as:

$$\begin{aligned} F_2^{\tau_1,\tau_2}[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2] &= \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\beta_1)\Gamma(\beta_2)} \\ &\times\sum_{m_1,m_2=0}^{\infty}\frac{(\alpha)_{m_1+m_2}\Gamma(\beta_1+\tau_1 m_1)\Gamma(\beta_2+\tau_2 m_2)}{\Gamma(\gamma_1+\tau_1 m_1)\Gamma(\gamma_2+\tau_2 m_2)}\frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!} \\ &(\tau_1,\tau_2>0,\ \ |x_1|+|x_2|<1). \end{aligned} \tag{1.7}$$

The interested reader may be referred to several recent papers on the subject [10, 16, 17, 27, 30, 31]. The special case when $\tau_1,\tau_2=1$ in (1.7) give the familiar representations of Appell's hypergeometric function[4, 5, 14, 15, 17, 18, 20, 22, 23].

However, recently there has been an increasing interest in and widely extended use of differential equations and systems of fractional order (that is, of arbitrary order), as better models of phenomena of various physics, engineering, automatization, biology and biomedicine, chemistry, earth science, economics, nature and so on. The extensions of a number of well-known special functions were investigated recently by several authors [see. [1–3, 6–9, 19, 28, 29, 32]].

In a sequel to the aforementioned work by Şahin et al. [27], we aim here is to introduce $H_B^{\tau_1,\tau_2,\tau_3}$ Srivastava's triple τ-hypergeometric function. After, we will present some properties of this function such as integral representations, partial differential equation, derivative formula and recurrence relations.

2. The Srivastava's $H_B^{\tau_1,\tau_2,\tau_3}$ Triple τ-Hypergeometric Function

By adding parameters τ_1,τ_2,τ_3 to a known $H_B(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3)$ Srivastava hypergeometric function, new $H_B^{\tau_1,\tau_2,\tau_3}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3)$ Srivastava's triple τ-hypergeometric function defined as:

$$\begin{aligned} H_B^{\tau_1,\tau_2,\tau_3}[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3] &= \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\gamma_3)}{\Gamma(\beta_1)\Gamma(\beta_2)} \\ \times\sum_{m,n,p=0}^{\infty}&\frac{(\alpha)_{m+p}\Gamma(\beta_1+\tau_1 m+n)\Gamma(\beta_2+\tau_2 n+\tau_3 p)}{\Gamma(\gamma_1+\tau_1 m)\Gamma(\gamma_2+\tau_2 n)\Gamma(\gamma_3+\tau_3 p)}\frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}, \\ &\left(\tau_1,\tau_2,\tau_3>0; |x_1|<r, |x_2|<s, |x_3|<t, r+s+t+2\sqrt{rst}=1\right). \end{aligned} \tag{2.1}$$

The special case when $\tau_1,\tau_2,\tau_3=1$ in (2.1) give the familiar representations of Srivastava's H_B triple hypergeometric function[4, 5, 13, 15].

2.1 Integral Representation of $H_B^{\tau_1,\tau_2,\tau_3}$ Srivastava 's τ-Hypergeometric Function

In this section we give some integral representations of $H_B^{\tau_1,\tau_2,\tau_3}$ Srivastava's triple τ-hypergeometric function.

Theorem 2.1. *The following integral representations hold true :*

$$H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right] = \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)} \times \int_0^\infty \int_0^\infty t^{\alpha-1}s^{\beta_1-1}e^{-t-s}\, {}_0F_1^{\tau_1}\left(-;\gamma_1;ts^{\tau_1}x_1\right)\ \Phi_2^{\tau_2,\tau_3}\left(\beta_2;\gamma_2,\gamma_3;sx_2,tx_3\right)dtds, \tag{2.2}$$

$$\left(\tau_1,\tau_2,\tau_3>0,\ \Re(x_2)<1,\Re(x_3)<1,\Re(\alpha)>0,\Re(\beta_1)>0\ \right),$$

where $\Phi_2^{\tau_2,\tau_3}$ is one of the confluent forms of $\tau-$Appell series in two variables defined by [4],

$$\Phi_2^{\tau_2,\tau_3}\left(\beta_2;\gamma_2,\gamma_3;sx_2,tx_3\right) = \sum_{n,p=0}^{\infty}\frac{(\beta_2)_{\tau_2 n+\tau_3 p}}{(\gamma_2)_{\tau_2 n}(\gamma_3)_{\tau_3 p}}\frac{(sx_2)^n}{n!}\frac{(tx_3)^p}{p!}, \tag{2.3}$$

$$\left(\tau_1,\tau_2>0,\ \Re(a)>\Re(b)>\Re(c)>0\right),$$

and

$$H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right] = \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)\Gamma(\beta_2)} \times \int_0^\infty \int_0^\infty \int_0^\infty t^{\alpha-1}s^{\beta_1-1}v^{\beta_2-1}e^{-t-s} \times {}_0F_1^{\tau_1}\left(-;\gamma_1;ts^{\tau_1}x_1\right)\ {}_0F_1^{\tau_2}\left(-;\gamma_2;sv^{\tau_2}x_1\right)\ {}_0F_1^{\tau_3}\left(-;\gamma_1;tv^{\tau_3}x_1\right)dtdsdv \tag{2.4}$$

$$\left(\tau_1,\tau_2,\tau_3>0,\ \Re(x_2)<1,\Re(x_3)<1,\Re(\alpha)>0,\Re(\beta_1)>0\right).$$

Proof. Using the equation(1.4) in (2.1) we get,

$$H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right] = \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)} \times \sum_{m,n,p=0}^{\infty}\frac{\Gamma(\alpha+m+p)\Gamma\beta_1+\tau_1 m+n)(\beta_2)_{\tau_2 n+\tau_3 p}}{(\gamma_1)_{\tau_1 m}(\gamma_2)_{\tau_2 n}(\gamma_3)_{\tau_3 p}}\frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}.$$

Replacing the values $\Gamma(\alpha+m+p)$ and $\Gamma(\beta_1+\tau_1 m+n)$ from the equation(1.1),the desired result can be obtained.Then, the similar way we can easily get (2.4). □

The special case when $\tau_{1,}\tau_{2,}\tau_3=1$ in (2.2) and (2.4) give the familiar integral representations of Srivastava's H_B triple hypergeometric function [21–23]. Also, the special case of (2.3) when $\tau_2,\tau_3=1$ is seen to yield the confluent hypergeometric function [18, 20, 22, 23].

Theorem 2.2. *The following integral representations hold true :*

$$H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right] = \frac{1}{B(\alpha,s-\alpha)} \times \int_0^1 t^{\alpha-1}(1-t)^{s-\alpha-1}H_B^{\tau_1,\tau_2,\tau_3}\left[s,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right]dt\ , \tag{2.5}$$

$$\left(\tau_1,\tau_2,\tau_3>0,\ \Re(x_2)<1,\Re(x_3)<1,\Re(\alpha)>0\ \right),$$

$$H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right] = \frac{1}{B(\beta_1,s-\beta_1)} \times \int_0^1 t^{\beta_1-1}(1-t)^{s-\beta_1-1}H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,s,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right]dt\ , \tag{2.6}$$

$$\left(\tau_1,\tau_2,\tau_3>0,\ \Re(x_2)<1,\Re(x_3)<1,\Re(\beta_1)>0\ \right),$$

$$H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right]=\frac{1}{B(\beta_2,s-\beta_2)} \times \int_0^1 t^{\beta_2-1}(1-t)^{s-\beta_2-1}H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,s;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right]dt\ , \quad (\tau_1,\tau_2,\tau_3>0,\ \Re(x_2)<1,\Re(x_3)<1,\Re(\beta_2)>0\ ), \tag{2.7}$$

$$H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right]=\frac{1}{B(s,\gamma_1-s)} \times \int_0^1 t^{s-1}(1-t)^{\gamma_1-s-1}H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;s,\gamma_2,\gamma_3;x_1,x_2,x_3\right]dt\ , \quad (\tau_1,\tau_2,\tau_3>0,\ \Re(x_2)<1,\Re(x_3)<1,\Re(\gamma_1)>0\ ), \tag{2.8}$$

$$H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right]=\frac{1}{B(s,\gamma_2-s)} \times \int_0^1 t^{s-1}(1-t)^{\gamma_2-s-1}H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1,s;x_1,t^{\tau_2}x_2,x_3\right]dt\ , \quad (\tau_1,\tau_2,\tau_3>0,\ \Re(x_2)<1,\Re(x_3)<1,\Re(\gamma_2)>0\ ), \tag{2.9}$$

$$H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right]=\frac{1}{B(s,\gamma_3-s)} \times \int_0^1 t^{s-1}(1-t)^{\gamma_3-s-1}H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,s;x_1,x_2,t^{\tau_3}x_3\right]dt\ , \quad (\tau_1,\tau_2,\tau_3>0,\ \Re(x_2)<1,\Re(x_3)<1,\Re(\gamma_3)>0\ ), \tag{2.10}$$

$$\begin{aligned}
&H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right]\\
&=\frac{1}{B(\beta_2,\gamma_3-\beta_2)B(\beta_2+\tau_3 p,\gamma_2-\beta_2-\tau_3 p)}\\
&\times\int_0^1\int_0^1 t^{\beta_2-1}u^{\beta_2-1}(1-t)^{\gamma_3-\beta_2-1}(1-u)^{\gamma_2-\beta_2-1}\\
&\times(1-u^{\tau_2}x_2)^{-\beta_1}(1-\left(\frac{tu}{1-u}\right)^{\tau_3}x_3)^{-\alpha}\\
&\times {}_2F_1^{\tau_1}\left[\begin{array}{c}\alpha,\beta_1;\\ \gamma_1;\end{array}\ \frac{x_1}{(1-u^{\tau_2}x_2)^{\tau_1}(1-\left(\frac{tu}{1-u}\right)^{\tau_3}x_3)}\right]dtdu\ ,\\
&(\tau_1,\tau_2,\tau_3>0,\ \Re(x_2)<1,\Re(x_3)<1,\Re(\beta_2)>0,\Re(\gamma_3-\beta_2)>0\ ),
\end{aligned} \tag{2.11}$$

$$
\begin{aligned}
&H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right] \\
&\quad= \frac{1}{B(\beta_2,\gamma_3-\beta_2)B(\beta_2,\gamma_2-\beta_2)B(\beta_1,\gamma_1-\beta_1)} \\
&\times \int_0^1\int_0^1\int_0^1 t^{\beta_2-1}u^{\beta_2-1}v^{\beta_2-1} \\
&\times(1-t)^{\gamma_3-\beta_2-1}(1-u)^{\gamma_2-\beta_2-1}(1-v)^{\gamma_1-\beta_2-1} \\
&\times(1-u^{\tau_2}x_2)^{-\beta_1}(1-\left(\frac{tu}{1-u}\right)^{\tau_3}x_3)^{-\alpha} \\
&\times\left(1-\frac{x_1v^{\tau_1}}{(1-u^{\tau_2}x_2)^{\tau_1}(1-\left(\frac{tu}{1-u}\right)^{\tau_3}x_3)}\right)^{-\alpha} dtdudv\ .
\end{aligned}
\tag{2.12}
$$

$$(\tau_1,\tau_2,\tau_3>0,\ \Re(x_2)<1,\Re(x_3)<1,\Re(\beta_1)>0,\Re(\beta_2)>0,\Re(\gamma_3-\beta_2)>0\ ),$$

Proof. From the equations (1.4) and (2.1), we have

$$
\begin{aligned}
&H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right] \\
&\quad=\sum_{m,n,p=0}^{\infty}\frac{(s)_{m+p}(\alpha)_{m+p}(\beta_1)_{\tau_1m+n}(\beta_2)_{\tau_2n+\tau_3p}}{(s)_{m+p}(\gamma_1)_{\tau_1m}(\gamma_2)_{\tau_2n}(\gamma_3)_{\tau_3p}}\frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}\ .
\end{aligned}
\tag{2.13}
$$

If we consider the following equality and using definition of Beta function in equation (1.3),

$$\frac{(\alpha)_{m+p}}{(s)_{m+p}}=\frac{B(\alpha+m+p,s-\alpha)}{B(\alpha,s-\alpha)},\tag{2.14}$$

the (2.5) is obtained.Then, we easily can obtain the equations (2.6) $-$ (2.11) in similar ways. Substituting the following equality (1.6),

$${}_2F_1^{\tau}\left[\begin{array}{c}\alpha,\beta;\\ \gamma;\end{array}x\right]=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-u^{\tau}x)\,du$$

into (2.11), we get the desired result (2.12). □

The special case when $\tau_{1,}\tau_{2,}\tau_3=1$ in (2.5) -(2.12) give the familiar integral representations of Srivastava's H_B triple hypergeometric function [12, 13].

Theorem 2.3. *The following Euler integral representation holds true:*

$$
\begin{aligned}
&H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right] \\
&\quad=\frac{1}{B(\beta_1,\gamma_1-\beta_1)B(1-\gamma_1+\beta_1,\gamma_1+\gamma_2-\beta_1-1)B(\beta_2,\gamma_3-\beta_2)} \\
&\quad\times\int_0^1\int_0^1\int_0^1 u^{\beta_1-1}v^{\beta_1-\gamma_1}w^{\beta_2-1}(1-u)^{\gamma_1-\gamma_3-\beta_1+\beta_2}(1-v)^{\gamma_1+\gamma_2-\beta_1+2} \\
&\quad\times(1-x_1u^{\tau_1}-x_3w^{\tau_3})^{-\alpha}(1-x_2v^{\tau_2})^{-\beta_1}dudvdw
\end{aligned}
\tag{2.15}
$$

$$(\tau_1,\tau_2,\tau_3>0,\ \Re(\gamma_1-\beta_1)<1,\Re(\beta_1)>1,\Re(\gamma_1+\gamma_2-\beta_1)>1,\Re(\beta_2)>0\ ).$$

Proof. From the definitions of $H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right]$ Srivastava's triple τ-hypergeometric function (2.1) and the second τ-Appell's hypergeometric functions (1.7), we get,

$$
\begin{aligned}
H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right]&=\sum_{n=0}^{\infty}\frac{(\beta_1)_n(\beta_2)_{\tau_2n}}{(\gamma_2)_{\tau_2n}} \\
&\times F_2^{\tau_1,,\tau_3}\left[\alpha,\beta_1+n,\beta_2+\tau_2n;\gamma_1,\gamma_3;x_1,x_3\right]\frac{x_2^n}{n!}.
\end{aligned}
\tag{2.16}
$$

In [4], we have the following equation :

$$F_2^{\tau_1,,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_3\right]=\frac{1}{B\left(\beta_1,\gamma_1-\beta_1\right)B\left(\beta_2,\gamma_3-\beta_2\right)}$$
$$\times\int_0^1\int_0^1 t^{\beta_1-1}s^{\beta_2-1}\left(1-t\right)^{\gamma_1-\beta_1-1}\left(1-s\right)^{\gamma_3-\beta_2-1}$$
$$\times\left(1-x_1t^{\tau_1}-x_3s^{\tau_3}\right)^{-\alpha}dtds. \tag{2.17}$$

Using (2.17) and making necessary arrangements in (2.16), we get,

$$H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right]=\frac{1}{B\left(\beta_1,\gamma_1-\beta_1\right)B\left(\beta_2,\gamma_3-\beta_2\right)}$$
$$\times\int_0^1\int_0^1 u^{\beta_1-1}w^{\beta_2-1}(1-u)^{\gamma_1-\beta_1-1}(1-w)^{\gamma_3-\beta_2-1}$$
$$\times(1-x_1u^{\tau_1}-x_3v^{\tau_3})^{-\alpha}\;{}_2R_1^{\tau_2}\left[\beta_1,\beta_2;\gamma_2;x_2uw\left(1-u\right)\right]dudw. \tag{2.18}$$

Afterwards, using the integral representation of the ${}_2R_1$ from (1.6) and putting into (2.18), we can be obtained (2.15). □

The special case when $\tau_{1,}\tau_{2,}\tau_3=1$ in (2.15) give the familiar integral representations of Srivastava's H_B triple hypergeometric function [12, 13].

Theorem 2.4. *The following integral representations hold true :*

$$H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right]$$
$$=\frac{\left(1-x_3\right)^{-\alpha}\left(1-x_2\right)^{-\beta_1}}{\Gamma\left(\alpha\right)\Gamma\left(\beta_1\right)}$$
$$\times\int_0^\infty\int_0^\infty e^{-u-v}u^{\alpha-1}v^{\beta_1-1}\;{}_0F_1^{\tau_1}\left(-;\gamma_1;\frac{x_2x_3uv^{\tau_1}}{\left(1-x_3\right)^{\tau_1}\left(1-x_2\right)}\right)$$
$$\times{}_0F_1^{\tau_2}\left(-;\gamma_2;\frac{x_2x_3u^{\tau_2}v}{\left(1-x_3\right)^{\tau_2}\left(1-x_2\right)}\right)dudv.$$
$$\left(\tau_1,\tau_2,\tau_3>0,\;\Re(x_2)<1,\Re(x_3)<1,\Re(\alpha)>0,\Re(\beta_1)>0\;\right), \tag{2.19}$$

Proof. If we put $\beta_2=\gamma_2$, $x_1=x_2x_3$ and using $\Phi_1^{\tau_2,\tau_3}\left(\gamma_2;\gamma_{2,}\gamma_3;sx_2,tx_3\right)=e^{x_2s+x_3t}\;{}_0F_1^{\tau_2}\left(-;\gamma_2;x_2x_3\right)$ in (2.2), we get,

$$H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right]=\frac{1}{\Gamma\left(\alpha\right)\Gamma\left(\beta_1\right)}$$
$$\times\int_0^\infty\int_0^\infty e^{-s(1-x_2)-t(1-x_3)}t^{\alpha-1}s^{\beta_1-1}$$
$$\times{}_0F_1^{\tau_1}\left(-;\gamma_1;ts^{\tau_1}x_1\right)\;{}_0F_1^{\tau_2}\left(-;\gamma_2;x_2x_3t^{\tau_2}s\right)dtds. \tag{2.20}$$

Setting $t\left(1-x_3\right)=u$ and $s\left(1-x_2\right)=v$ in (2.20), we are led to the desired integral representation of (2.19). □

The special case when $\tau_{1,}\tau_{2,}\tau_3=1$ in (2.19) give the familiar integral representations of Srivastava's H_B triple hypergeometric function[12, 13].

2.2 Partial Differential Equations

Here, we will give partial differential equations for $H_B^{\tau_1,\tau_2,\tau_3}$ Srivastava's triple τ-hypergeometric function.

Theorem 2.5. *The following system of partial differential equations hold true:*

$$\left\{\begin{array}{c}\left[\theta\left(\tau_1\theta+\gamma_1-1\right)_{\tau_1}-x_1\left(\theta+\phi+\alpha\right)\left(\tau_1\theta+\varphi+\beta_1\right)_{\tau_1}\right]H_B^{\tau_1,\tau_2,\tau_3}=0\\ \left[\varphi\left(\tau_2\varphi+\phi+\gamma_2-1\right)_{\tau_2}-x_2\left(\tau_1\theta+\varphi+\beta_1\right)\left(\tau_2\varphi+\phi+\beta_2\right)_{\tau_2}\right]H_B^{\tau_1,\tau_2,\tau_3}=0\\ \left[\phi\left(\varphi+\tau_3\phi+\gamma_3-1\right)_{\tau_3}-x_3\left(\theta+\phi+\alpha\right)\left(\varphi+\tau_3\phi+\beta_2\right)\right]H_B^{\tau_1,\tau_2,\tau_3}=0\end{array}\right. \tag{2.21}$$

where, $\theta=x_1\frac{d}{dx_1}$, $\varphi=x_2\frac{d}{dx_2}$ *and* $\phi=x_3\frac{d}{dx_3}$.

Proof. Using the $H_B^{\tau_1,\tau_2,\tau_3}[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3]$ Srivastava's triple τ-hypergeometric function in equation (2.1) and multiplying this equation $\theta\left(\theta+\gamma_1-1\right)_{\tau_1}$, we have

$$\begin{aligned}&\theta\left(\tau_1\theta+\gamma_1-1\right)_{\tau_1}H_B^{\tau_1,\tau_2,\tau_3}[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3]\\&=\sum_{m,n,p=0}^{\infty}\frac{m\left(\tau_1m+\gamma_1-1\right)_{\tau_1}(\alpha)_{m+p}(\beta_1)_{\tau_1m+n}(\beta_2)_{\tau_2n+\tau_3p}}{(\gamma_1)_{\tau_1m}(\gamma_2)_{\tau_2n}(\gamma_3)_{\tau_3p}}\frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}\qquad(m\to m+1)\\&=\sum_{m,n,p=0}^{\infty}\frac{\left(\tau_1m+\tau_1+\gamma_1-1\right)_{\tau_1}(\alpha)_{m+1+p}(\beta_1)_{\tau_1m+\tau_1+n}(\beta_2)_{\tau_2n+\tau_3p}}{(\gamma_1)_{\tau_1m+\tau_1}(\gamma_2)_{\tau_2n}(\gamma_3)_{\tau_3p}}\frac{x_1^{m+1}}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}.\end{aligned} \tag{2.22}$$

Taking advantage of the following property of Pochhammer symbol [30, 31] in (2.22)

$$(\alpha)_{\tau_1m+\tau_2n}=(\alpha+\tau_1m)_{\tau_2n}(\alpha)_{\tau_1m}$$

and making some useful arragement in the equation (2.22), we have the desired result (2.21). □

The special case when τ_1,τ_2,$\tau_3=1$ in (2.21) gives the partial differential equations of Srivastava's H_A triple hypergeometric function [21–23]

2.3 Derivative Formula

In this section, we derive derivative formula for $H_B^{\tau_1,\tau_2,\tau_3}$ Srivastava's $\tau-$hypergeometric function.

Theorem 2.6. *The following derivative formula for* $H_B^{\tau_1,\tau_2,\tau_3}$ *holds true:*

$$\begin{aligned}&\frac{d^{r+s+t}}{dx_1^rdx_2^sdx_3^t}H_B^{\tau_1,\tau_2,\tau_3}[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3]\\&=\frac{(\alpha)_{r+t}(\beta_1)_{\tau_1r+s}(\beta_2)_{\tau_2s+\tau_3t}}{(\gamma_1)_{\tau_1r}(\gamma_2)_{\tau_2s}(\gamma_3)_{\tau_3t}}\\&\times H_B^{\tau_1,\tau_2,\tau_3}\left[(\alpha+r+t),(\beta_1+\tau_1r+s),(\beta_2+\tau_2s+\tau_3t);(\gamma_1+\tau_1r),(\gamma_2+\tau_2s)(\gamma_3+\tau_3t);x_1,x_2,x_3\right].\end{aligned} \tag{2.23}$$

Proof. By using the definition of $H_B^{\tau_1,\tau_2,\tau_3}[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3]$ Srivastava's $\tau-$hypergeometric function in(2.1) and takes derivative this equation r times, we get

$$\begin{aligned}&\frac{d^r}{dx_1^r}H_B^{\tau_1,\tau_2,\tau_3}[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3]\\&=\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{m+r+p}(\beta_1)_{\tau_1m+\tau_1r+s}(\beta_2)_{\tau_2n+\tau_3p}}{(\gamma_1)_{\tau_1m+\tau_1r}(\gamma_2)_{\tau_2n}(\gamma_3)_{\tau_3t}}\frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}.\end{aligned} \tag{2.24}$$

Then, taking derivative s times of (2.24), we have

$$\begin{aligned}&\frac{d^{r+s}}{dx_1^rdx_2^s}H_B^{\tau_1,\tau_2,\tau_3}[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3]\\&=\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{m+r+p}(\beta_1)_{\tau_1m+\tau_1r+n+s}(\beta_2)_{\tau_2s+\tau_2n+\tau_3p}}{(\gamma_1)_{\tau_1m+\tau_1r}(\gamma_2)_{\tau_2s+\tau_2n}(\gamma_3)_{\tau_3t}}\frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}.\end{aligned} \tag{2.25}$$

Thereafter, we take derivative t times of (2.25), we get

$$\begin{aligned}&\frac{d^{r+s+t}}{dx_1^rdx_2^sdx_3^t}H_B^{\tau_1,\tau_2,\tau_3}[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3]\\&=\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{m+r+p+t}(\beta_1)_{\tau_1m+\tau_1r+n+s}(\beta_2)_{\tau_2s+\tau_2n+\tau_3p+\tau_3t}}{(\gamma_1)_{\tau_1m+\tau_1r}(\gamma_2)_{\tau_2s+\tau_2n}(\gamma_3)_{\tau_3p+\tau_3t}}\frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}.\end{aligned} \tag{2.26}$$

Using following property of Pochhammer symbol [18, 20] in (2.26),

$$(\alpha)_{m+n} = (\alpha+m)_n\,(\alpha)_m\,, \tag{2.27}$$

we led to desired result (2.23). □

The special case when $\tau_1, \tau_2, \tau_3 = 1$ in (2.23) give the familiar derivative of Srivastava's H_B triple hypergeometric function [21].

2.4 Recurrence Relations

In this section, we present certain reccurence relation for $H_B^{\tau_1,\tau_2,\tau_3}$ Srivastava's $\tau-$hypergoemetric function (2.1).

Theorem 2.7. *The following recurrence relation for $H_B^{\tau_1,\tau_2,\tau_3}$ holds true:*

$$\begin{aligned} &H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right] \\ &\quad= H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1-1,\gamma_2,\gamma_3;x_1,x_2,x_3\right] \\ &\quad+\frac{\alpha\beta_1 x_1}{\gamma_1(1-\gamma_1)} H_B^{\tau_1,\tau_2,\tau_3}\left[(\alpha+1),\beta_1+1,\beta_2;\gamma_1+1,\gamma_2,\gamma_3;x_1,x_2,x_3\right]. \end{aligned} \tag{2.28}$$

Proof. Using integral representation of $H_B^{\tau_1,\tau_2,\tau_3}\left[(\alpha),\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right]$ Srivastava's $\tau-$hypergoemetric function in (2.2) and following contigoous relation for the function ${}_0F_1^{\tau_1}$ (2.29),

$${}_0F_1^{\tau_1}\left(-;\gamma-1;x\right) - {}_0F_1^{\tau_1}\left(-;\gamma;x\right) - \frac{\alpha\beta_1 x_1}{\gamma_1(1-\gamma_1)}\; {}_0F_1^{\tau_1}\left(-;\gamma+1;x\right) = 0, \tag{2.29}$$

we can easily obtain the recurrence relation(2.28). □

Theorem 2.8. *The following recurrence relation for $H_B^{\tau_1,\tau_2,\tau_3}$ holds true:*

$$\begin{aligned} &(\beta_2-\gamma_2+1) H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right] = \\ &\qquad \beta_2 H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2+1;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right] \\ &\qquad -(\gamma_2-1) H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2-1,\gamma_3;x_1,x_2,x_3\right]. \end{aligned} \tag{2.30}$$

Proof. The series on the right side of(2.30) are :

$$\begin{aligned} &\beta_2 H_B^{\tau_1,\tau_2,\tau_3}\left[(\alpha),\beta_1,\beta_2+1;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right] = \frac{\beta_2\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\beta_1)\Gamma(\beta_2+1)} \\ &\quad\times \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p}\Gamma(\beta_1+\tau_1 m+n)\Gamma(\beta_2+1+\tau_2 n+\tau_3 p)}{\Gamma(\gamma_1+\tau_1 m+n)\Gamma(\gamma_2+\tau_2 n+\tau_3 p)} \frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}, \\ &\quad= \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\beta_1)\Gamma(\beta_2)} \\ &\quad\times \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p}\Gamma(\beta_1+\tau_1 m+n)\Gamma(\beta_2+\tau_2 n+\tau_3 p)}{\Gamma(\gamma_1+\tau_1 m+n)\Gamma(\gamma_2+\tau_2 n+\tau_3 p)} \frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}\frac{(\beta_2+\tau_2 n+\tau_3 p)}{(\beta_2)}, \end{aligned} \tag{2.31}$$

and

$$\begin{aligned} &(\gamma_2-1) H_B^{\tau_1,\tau_2,\tau_3}\left[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2-1,\gamma_3;x_1,x_2,x_3\right] = \frac{(\gamma_2-1)\Gamma(\gamma_1)\Gamma(\gamma_2-1)}{\Gamma(\beta_1)\Gamma(\beta_2)} \\ &\quad\times \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p}\Gamma(\beta_1+\tau_1 m+n)\Gamma(\beta_2+\tau_2 n+\tau_3 p)}{\Gamma(\gamma_1+\tau_1 m+n)\Gamma(\gamma_2-1+\tau_2 n+\tau_3 p)} \frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}, \\ &\quad= \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\beta_1)\Gamma(\beta_2)} \\ &\quad\times \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p}\Gamma(\beta_1+\tau_1 m+n)\Gamma(\beta_2+\tau_2 n+\tau_3 p)}{\Gamma(\gamma_1+\tau_1 m+n)\Gamma(\gamma_2+\tau_2 n+\tau_3 p)} \frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}\frac{(\gamma_2-1+\tau_2 n+\tau_3 p)}{(\gamma_2-1)}. \end{aligned} \tag{2.32}$$

Substuting(2.31) and (2.32) in the series expression into the right hand side of (2.30), we get the desired result of $(\beta_2-\gamma_2+1) H_B^{\tau_1,\tau_2,\tau_3}\left[(\alpha),\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right]$. □

The special case when τ_1,τ_2,$\tau_3 = 1$ in (2.28) and (2.30) give the recurrence relations for Srivastava's H_B triple hypergeometric function [25, 26].

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