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On the Matrices with Harmonic Numbers

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ABSTRACT

In this study, firstly we define nxn matrices P and Q associated with harmonic numbers such that $P=(p_{ij})=[H_i]_{i,j=1}^{n}$ and $Q=(q_{ij})=[H_{i+j}]_{i,j=1}^{n}$, where H_k is denote kth harmonic number. After we study the spectral norms, Euclidean norms and determinants of these matrices.

Key Words: *Harmonic number, Norm, Determinant.*

1. INTRODUCTION

The harmonic numbers are defined by

$$H_0=0 \text{ and } H_n=\sum_{k=1}^{n}\frac{1}{k} \quad \text{for} \quad n=1,2,\ldots,.$$

A generating function for the harmonic numbers is $\frac{-\ln(1-x)}{1-x}$. The first few harmonic numbers are $1,\ \frac{3}{2},\ \frac{11}{6},\ \frac{25}{12},\ \frac{137}{12},\ \frac{49}{20},\ldots$. Harmonic numbers have many interesting properties [3,6]. For $n\geq 1$, some of them are the following

$$\sum_{k=1}^{n-1}H_k=nH_n-n\ ,$$

$$\sum_{k=m}^{n-1}\binom{k}{m}H_k=\binom{n}{m+1}\left(H_n-\frac{1}{m+1}\right),$$

$$\sum_{k=0}^{n}\binom{n}{k}H_k=2^n\left(H_n-\sum_{k=1}^{n}\frac{1}{k2^k}\right),$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k H_k=-\frac{1}{n}\ ,$$

$$H_{n+1}^2-H_n^2=\left(\frac{1}{n+1}\right)^2+\frac{2}{n+1}H_n\ .$$

The harmonic numbers have been generalized by many authors in recent works [1,2,3,5] such that

$$H_0^{(r)}=0 \quad \text{and} \quad H_n^{(r)}=\sum_{k=1}^{n}\frac{1}{k^r} \quad \text{for} \quad n,r=1,2,\ldots,$$

$$H_n^{<0>}=\frac{1}{n} \quad \text{and} \quad H_n^{<r>}=\sum_{k=1}^{n}H_k^{<r-1>} \quad \text{for} \quad n,r=1,2,\ldots,$$

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$H_{n,0}=1$ and $H_{n,r}=\sum_{1\le n_1<\cdots<n_r\le n}^{n}\frac{1}{n_1n_2\ldots n_r}$ for $n,r=1,2,..,$ (1.1)

$$H(n,r)=\sum_{1\le n_0+n_1+\cdots+n_r\le n}^{n}\frac{1}{n_0n_1\ldots n_r} \quad \text{for } n\ge 1,\ r\ge 0.$$

For $r=0$ or $r=1$, these generalizations are reduced to the ordinary harmonic numbers. There are many connections between these generalizations, Stirling numbers and ordinary harmonic numbers [4,5].

For the generalized harmonic numbers $H_{n,r}$ given by (1.1), Cheon and El-Mikkawy [4] defined an nxn matrix H as follows:

$$H=\left(h_{ij}\right)_{i,j=1}^{n}\equiv\begin{cases}H_{i,j} & \text{if } i\ge j\\ 0 & \text{if } i<j\end{cases}.$$

Moreover they characterized the inverse of the matrix H.

In this paper, firstly we define the nxn matrices P and Q which entries are consist of harmonic numbers, such that these matrices are of the forms:

$$P=\left(p_{ij}\right)_{i,j=1}^{n}=\left[H_i\right]_{i=1}^{n}=\begin{bmatrix}1 & 1 & \cdots & 1\\ \frac{3}{2} & \frac{3}{2} & \cdots & \frac{3}{2}\\ \frac{11}{6} & \frac{11}{6} & \cdots & \frac{11}{6}\\ \vdots & \vdots & & \vdots\\ H_n & H_n & \cdots & H_n\end{bmatrix}, \tag{1.2}$$

$$Q=\left(q_{ij}\right)_{i,j=1}^{n}=\left[H_{i+j}\right]_{i,j=1}^{n}=\begin{bmatrix}\frac{3}{2} & \frac{11}{6} & \cdots & H_{n+1}\\ \frac{11}{6} & \frac{25}{12} & \cdots & H_{n+2}\\ \vdots & \vdots & & \vdots\\ H_{n+1} & H_{n+2} & \cdots & H_{2n}\end{bmatrix} \tag{1.3}$$

After we study these matrices.

Now, we start with some preliminaries. Let $A=\left(a_{ij}\right)$ be any mxn matrix. The ℓ_p norms of the matrix A are defined by $\|A\|_p=\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\left|a_{ij}\right|^p\right)^{\frac{1}{p}}$, $\left(1\le p<\infty\right)$. For $p=2$, the ℓ_2 norm is called Euclidean norm. Also the spectral norm of the matrix A is $\|A\|_s=\sqrt{\max_i \lambda_i\left(A^HA\right)}$, where A^H is the conjugate transpose of the matrix A. A function ψ is called a psi (or digamma) function if $\psi(x)=\frac{d}{dx}\left\{\ln\left[\Gamma(x)\right]\right\}$, where $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt$.

Throughout this paper, P and Q denote the matrices in (1.2) and in (1.3) respectively.

2. MAIN RESULTS

Theorem 2.1. The eigenvalues of the nxn matrix P are

$$\lambda_1=(n+1)\left(H_{n+1}-1\right),$$

$$\lambda_m=0,$$

where $m=2,3,\ldots,n$.

Proof. The eigenvalues of the matrix P are roots of the equation $|\lambda I-P|=0$, such that

$$|\lambda I-P|=\begin{vmatrix}\lambda-1 & -1 & -1 & \cdots & -1 & -1\\ -\frac{3}{2} & \lambda-\frac{3}{2} & -\frac{3}{2} & \cdots & -\frac{3}{2} & -\frac{3}{2}\\ -\frac{11}{6} & -\frac{11}{6} & \lambda-\frac{11}{6} & \cdots & -\frac{11}{6} & -\frac{11}{6}\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ -H_{n-1} & -H_{n-1} & -H_{n-1} & \cdots & \lambda-H_{n-1} & -H_{n-1}\\ -H_n & -H_n & -H_n & \cdots & -H_n & \lambda-H_n\end{vmatrix}$$

From the properties of the determinant, we have

$$|\lambda I-P|=\begin{vmatrix}\lambda-1 & -\lambda & -\lambda & \cdots & -\lambda & -\lambda\\ -\frac{3}{2} & \lambda & 0 & \cdots & 0 & 0\\ -\frac{11}{6} & 0 & \lambda & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ -H_{n-1} & 0 & 0 & \cdots & \lambda & 0\\ -H_n & 0 & 0 & \cdots & 0 & \lambda\end{vmatrix}.$$

If we calculate the last determinant, we obtain

$$|\lambda I-P|=\lambda^{n-1}\left[\lambda-1-\frac{3}{2}-\frac{11}{6}-\cdots-\sum_{k=1}^{n}\frac{1}{k}\right].$$

If we solve the equation

$$|\lambda I-P|=\lambda^{n-1}\left[\lambda-1-\frac{3}{2}-\frac{11}{6}-\cdots-\sum_{k=1}^{n}\frac{1}{k}\right]=0,$$

the eigenvalues of the matrix P are

$$\lambda_1=1+\frac{3}{2}+\frac{11}{6}+\cdots+\sum_{k=1}^{n}\frac{1}{k}$$

$$=\sum_{k=1}^{n}H_k=(n+1)\left(H_{n+1}-1\right),$$

$$\lambda_m=0,$$

where $m=2,3,\ldots,n$.

Corollary 2.1. The determinant of the matrix P is zero.

Lemma 2.1. Let H_k be kth harmonic number, then the following equality holds

$$\sum_{k=1}^{n} H_k^2 = (n+1)H_{n+1}^2 - (2n+3)H_{n+1} + 2n+2 .$$

Proof. We have $H_k = \psi(k+1)+\gamma$, where ψ is digamma function and γ is Euler's constant. Then,

$$\sum_{k=1}^{n} H_k^2 = \sum_{k=1}^{n}(\psi(k+1)+\gamma)^2$$

$$=(n+1)(\psi(n+2)+\gamma)^2-(2n+3)(\psi(n+2)+\gamma)+2(n+1)$$

$$=(n+1)H_{n+1}^2-(2n+3)H_{n+1}+2n+2 .$$

Theorem 2.2. The Euclidean norm of the matrix P is

$$\|P\|_2 = \sqrt{n\left[(n+1)H_{n+1}^2-(2n+3)H_{n+1}+2n+2\right]} .$$

Proof. From the definition of the Euclidean norm, we have

$$\|P\|_2^2 = n\sum_{k=1}^{n} H_k^2 . \tag{2.1}$$

From (2.1) and Lemma 2.1., we have

$$\|P\|_2 = \sqrt{n\left[(n+1)H_{n+1}^2-(2n+3)H_{n+1}+2n+2\right]} .$$

Thus, the proof is completed.

Theorem 2.3. The singular values of the nxn matrix P satisfy the following equalities

$$\sigma_1 = \sqrt{n\left[(n+1)H_{n+1}^2-(2n+3)H_{n+1}+2n+2\right]} ,$$

$$\sigma_m = 0 ,$$

where $m = 2,3,\ldots,n$.

Proof. The singular values of the matrix P are the square roots of the eigenvalues of the matrix $P^H P$, where P^H is the conjugate transpose of the matrix P. The matrix $P^H P$ is of the form:

$$P^H P = \begin{bmatrix} \alpha & \alpha & \cdots & \alpha & \alpha \\ \alpha & \alpha & \cdots & \alpha & \alpha \\ \vdots & \vdots & & \vdots & \vdots \\ \alpha & \alpha & \cdots & \alpha & \alpha \\ \alpha & \alpha & \cdots & \alpha & \alpha \end{bmatrix},$$

where $\alpha = \sum_{k=1}^{n} H_k^2$. Since $\alpha = \sum_{k=1}^{n} H_k^2$ and from Lemma 2.1., we have

$$\alpha = (n+1)H_{n+1}^2-(2n+3)H_{n+1}+2n+2 . \tag{2.2}$$

The eigenvalues of the matrix $P^H P$ are

$$\lambda_1 = n\alpha, \quad \lambda_m = 0, \tag{2.3}$$

where $m = 2,3,\ldots,n$. From (2.2) and (2.3), the singular values of the matrix P are

$$\sigma_1 = \sqrt{n\left[(n+1)H_{n+1}^2-(2n+3)H_{n+1}+2n+2\right]} ,$$

$$\sigma_m = 0$$

where $m = 2,3,\ldots,n$.

Corollary 2.2. The spectral norm of the nxn matrix P is

$$\|P\|_s = \sqrt{n\left[(n+1)H_{n+1}^2-(2n+3)H_{n+1}+2n+2\right]} .$$

Corollary 2.3. The spectral norm of the nxn matrix P is equal to the its Euclidean norm.

Lemma 2.2. Let H_k be kth harmonic number, then the following equality holds

$$\sum_{k=1}^{n} kH_{k+1}^2 = \frac{n^2+n-2}{2}H_{n+1}^2 - \frac{n^2-3n-7}{2}H_{n+1} + \frac{n^2-9n-10}{4} .$$

Proof.

$$\sum_{k=1}^{n} kH_{k+1}^2 = \sum_{k=1}^{n} k\left[\psi(k+2)+\gamma\right]^2$$

$$=\frac{n^2+n-2}{2}\left[\psi(n+2)+\gamma\right]^2 - \frac{n^2-3n-7}{2}\left[\psi(n+2)+\gamma\right] + \frac{n^2-9n-10}{4}$$

$$=\frac{n^2+n-2}{2}H_{n+1}^2 - \frac{n^2-3n-7}{2}H_{n+1} + \frac{n^2-9n-10}{4} .$$

Lemma 2.3. Let H_k be kth harmonic number, then the equality

$$\sum_{k=1}^{n-1}(n-k)H_{n+k+1}^2 = (2n^2+3n+1)H_{2n}^2 - \frac{3n^2+7n+2}{2}H_{n+1}^2 - \left(6n^2+7n+\frac{3}{2}\right)H_{2n} + \frac{7n^2+15n+7}{2}H_{n+1} + \frac{13n^2-3n-10}{4}$$

is valid.

Proof.

$$\sum_{k=1}^{n-1}(n-k)H_{n+k+1}^2 = \sum_{k=1}^{n-1}(n-k)\left[\psi(n+k+2)+\gamma\right]^2$$

$$=(2n^2+3n+1)\left[\psi(2n+1)+\gamma\right]^2 - \left(6n^2+7n+\frac{3}{2}\right)\left[\psi(2n+1)+\gamma\right]$$

$$-\frac{3n^2+7n+2}{2}\left[\psi(n+2)+\gamma\right]^2 + \frac{7n^2+15n+7}{2}\left[\psi(n+2)+\gamma\right] + \frac{13n^2-3n-10}{4}$$

$$=(2n^2+3n+1)H_{2n}^2 - \frac{3n^2+7n+2}{2}H_{n+1}^2 - \left(6n^2+7n+\frac{3}{2}\right)H_{2n} + \frac{7n^2+15n+7}{2}H_{n+1} + \frac{13n^2-3n-10}{4}.$$

Thus the proof is completed.

Theorem 2.4. For the Euclidean norm of the nxn matrix Q

$$\|Q\|_2^2 = (2n^2+3n+1)H_{2n}^2 - (n^2+3n+2)H_{n+1}^2 - \left(6n^2+7n+\frac{3}{2}\right)H_{2n} + (3n^2+9n+7)H_{n+1} + \frac{14n^2-12n-20}{4}$$

is valid.

Proof. From the definition of the Euclidean norm, we have

$$\|Q\|_E^2 = \sum_{k=1}^{n} kH_{k+1}^2 + \sum_{k=1}^{n-1}(n-k)H_{n+k+1}^2 . \qquad (2.4)$$

From (2.4), Lemma 2.2. and Lemma 2.3.,

$$\|Q\|_2^2 = \left(2n^2+3n+1\right)H_{2n}^2 - \left(n^2+3n+2\right)H_{n+1}^2 - \left(6n^2+7n+\frac{3}{2}\right)H_{2n} + \left(3n^2+9n+7\right)H_{n+1} + \frac{14n^2-12n-20}{4}$$

is valid.

Theorem 2.5. The determinant of the matrix Q is

$$|Q| = (-1)^{n+1}\frac{\prod_{s=1}^{n-2}\left[\frac{(2s+2)!s!}{(n+s+1)!}\right]^2}{2^{n-2}n(n+1)!}\left(2H_n - \frac{1}{n(n+1)}\right).$$

Proof. If we apply Gauss Elimination Method to the determinant of the matrix Q, we have

$$|Q| = \begin{vmatrix} \alpha_1\beta_1 & & & * \\ & \alpha_2\beta_2 & & \\ & & \ddots & \\ & 0 & & (-1)^{n+1}\alpha_n\beta_n \end{vmatrix},$$

where $\alpha_1 = 1$, $\beta_1 = H_2 + H_0$, $\beta_k = \frac{H_{k+1}+H_{k-1}}{H_k+H_{k-2}}$,

$\alpha_k = \frac{(k-1)\left[(k+1)!\right]^2\left[(k-2)!\right]^2}{2k(k+1)\left[(2k-1)!\right]^2}$ and $(k = 2,3,...,n)$.

Hence we write

$$|Q| = (-1)^{n+1}\prod_{i=1}^{n}\alpha_i\beta_i = (-1)^{n+1}\prod_{i=1}^{n}\alpha_i\prod_{i=1}^{n}\beta_i .$$

Since

$$\prod_{i=1}^{n}\alpha_i = \frac{\prod_{s=1}^{n-2}\left[\frac{(2s+2)!s!}{(n+s+1)!}\right]^2}{2^{n-2}n(n+1)!}$$

and

$$\prod_{i=1}^{n}\beta_i = \left(H_{n+1}+H_{n-1}\right) = \left(2H_n - \frac{1}{n(n+1)}\right),$$

we obtain

$$|Q| = (-1)^{n+1}\frac{\prod_{s=1}^{n-2}\left[\frac{(2s+2)!s!}{(n+s+1)!}\right]^2}{2^{n-2}n(n+1)!}\left(2H_n - \frac{1}{n(n+1)}\right).$$

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