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POINTWISE BI-SLANT SUBMERSIONS FROM COSYMPLECTIC MANIFOLDS

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Abstract. We introduce pointwise bi-slant submersions from cosymplectic manifolds onto Riemannian manifolds as a generalization of anti-invariant, semi-invariant, semi-slant, hemi-slant, pointwise semi-slant, pointwise hemi-slant and pointwise slant Riemannian submersions. We give an example for pointwise bi-slant submersions and investigate integrability and totally geodesicness of the distributions which are mentioned in the definition of pointwise bi-slant submersions admitting vertical Reeb vector field. Also we obtain necessary and sufficient conditions for such submersions to be totally geodesic maps.

1. Introduction

The geometry of slant submanifolds was initiated by B.Y. Chen [9]. Later many geometers obtained some interesting results on this subject. As an extension of slant submanifolds, pointwise slant submanifolds were considered by F. Etayo [11] under the name of quasi-slant submanifolds. He showed that a complete totally geodesic quasi-slant submanifold of Kaehlerian manifold is a slant submanifold.

As a generalization of contact CR-manifolds, slant and semi-slant submanifolds, the geometry of bi-slant submanifolds in contact metric manifolds was studied by Carriazo [8]. A bi-slant submanifold of Kaehlerian manifold was defined by Uddin and et al. (see [27]). They investigated warped product bi-slant submanifold. Furthermore, Alqahtani and the other authors studied warped product bi-slant submanifolds of cosymplectic manifolds [4].

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The theory of submersions especially the theory of Riemannian submersions is one of the important research fields in Riemannian geometry. Riemannian submersions between Riemannian manifolds were introduced by O'Neill [18] and Gray [13]. Watson investigated the Riemannian submersions between almost Hermitian manifolds, (see [28]). Several types of Riemannian submersions have been studying in different kinds of structures, (see [1–3, 5, 10, 14–16, 19–25]).

In purpose of the present article is to investigate pointwise bi-slant submersions from cosymplectic manifolds onto Riemannian manifolds. In section 2, we review some basic properties about cosymplectic manifolds and Riemannian submersions. In section 3 we define pointwise bi-slant submersions from cosymplectic manifolds and study the geometry of leaves of distributions. Also, we obtain necessary and sufficient conditions for such submersions to be totally geodesic maps.

2. Preliminaries

In the section, we remember the basic concepts about cosymplectic manifolds and Riemannian submersions for later use.

2.1. **Cosymplectic manifolds.** Let M be $(2n+1)$-dimensional smooth manifold with an endomorphism ϕ, a vector field ξ and a 1-form η which satisfy

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1.$$

Then M is said to be an almost contact manifold. There always exist a compatible metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \ \eta(X) = g(X, \xi)$$

for $X, Y \in \Gamma(TM)$. The condition for normality in terms of ϕ, ξ and η on M is $[\phi, \phi] + 2d\eta \otimes \xi = 0$ where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. The fundamental 2-form Φ of M is defined as $\Phi(X, Y) = g(X, \phi Y)$.

An almost contact metric structure (ϕ, ξ, η, g) is said to be cosymplectic if it is normal and both $d\Phi = 0$ and $d\eta = 0$. Then considering the covariant derivative of ϕ, the structure equation of a cosymplectic manifold is characterized by the relation

$$(\nabla_X \phi) Y = 0 \ \text{ and } \ \nabla_X \xi = 0$$

for any $X, Y \in \Gamma(TM)$ [7, 17].

2.2. **Riemannian submersions.** A smooth map $\pi : M \to N$ between Riemannian manifolds M and N with dimension m and n, respectively, is called a Riemannian submersion if π_* is onto and satisfies [12]

i) π has maximal rank,
ii) π_* preserves the lengths of vectors normal to fibers.

For each $q \in N$, $\pi^{-1}(q)$ is a submanifold of M with dimension $m - n$. The submanifold $\pi^{-1}(q)$ are called fibers and a vector field X on M is called vertical (resp. horizontal) if it is always tangent (resp. orthogonal). If X is horizontal and π-related to a vector field X_* on N then X is called basic. The projection morphisms on the distributions $\ker \pi_*$ and $(\ker \pi_*)^\perp$ are denoted by $\mathcal{V}$ and $\mathcal{H}$, respectively.

The type of $(1,2)$ tensor fields $\mathcal{T}$ and $\mathcal{A}$ on M are given by

$$\mathcal{T}(X,Y) = \mathcal{T}_X Y = \mathcal{H}\nabla_{\mathcal{V}X}\mathcal{V}Y + \mathcal{V}\nabla_{\mathcal{V}X}\mathcal{H}Y \tag{1}$$

$$\mathcal{A}(X,Y) = \mathcal{A}_X Y = \mathcal{V}\nabla_{\mathcal{H}X}\mathcal{H}Y + \mathcal{H}\nabla_{\mathcal{H}X}\mathcal{V}Y \tag{2}$$

for $X, Y \in \Gamma(TM)$ where ∇ denotes the Levi-Civita connection of (M, g). On the other hand for $U, V \in \Gamma(\ker \pi_*)$ and $X, Y \in \Gamma\left((\ker \pi_*)^\perp\right)$ the tensor fields satisfy the following equations

$$\mathcal{T}_U V = \mathcal{T}_V U \tag{3}$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X,Y]. \tag{4}$$

Note that a Riemannian submersion $\pi : M \longrightarrow N$ has totally geodesic fibers if and only if $\mathcal{T}$ vanishes identically. Considering the equations (1) and (2), one can write

$$\nabla_U V = \mathcal{T}_U V + \bar{\nabla}_U V \tag{5}$$

$$\nabla_U X = \mathcal{H}\nabla_U X + \mathcal{T}_U X \tag{6}$$

$$\nabla_X U = \mathcal{A}_X U + \mathcal{V}\nabla_X U \tag{7}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y \tag{8}$$

for $X, Y \in \Gamma\left((\ker \pi_*)^\perp\right)$ and $U, V \in \Gamma(\ker \pi_*)$, where $\bar{\nabla}_U V = \mathcal{V}\nabla_U V$. Moreover, if X is basic then $\mathcal{H}\nabla_U X = \mathcal{A}_X U$.

Lemma 1. *([18]) Let $\pi : M \longrightarrow N$ be a Riemannian submersion between Riemannian manifolds and suppose that X and Y are basic vector fields of M π-related to X_* and Y_* on N. Then*

i) $\mathcal{H}[X,Y]$ *is a basic vector field i.e.* $\pi_*(\mathcal{H}[X,Y]) = [X_*, Y_*] \circ \pi$,

ii) $[U,X]$ *is vertical for any vector field U of* $(\ker \pi_*)$,

iii) $\mathcal{H}\nabla_X Y$ *is the basic vector field i.e.* $\pi_*(\mathcal{H}\nabla_X Y) = \overline{\nabla}_{X_*} Y_*$,

where ∇ and $\overline{\nabla}$ are the Levi-Civita connection on M and N, respectively.

Let (M, g) and (N, g') be Riemannian manifolds and $\Psi : M \longrightarrow N$ is a smooth mapping between them. The second fundamental form of Ψ is given by

$$\nabla \Psi_*(X,Y) = \nabla^{\Psi}_X \Psi_*(Y) - \Psi_*(\nabla_X Y) \tag{9}$$

for $X, Y \in \Gamma(TM)$, where ∇^{Ψ} is the pullback connection. The smooth map Ψ is said to be harmonic if $trace \nabla \Psi_* = 0$ and ψ is called a totally geodesic map if $(\nabla \Psi_*)(X,Y) = 0$ for $X, Y \in \Gamma(TM)$ [6].

Remark 2. *Throughout this article, we consider that the characteristic vector field ξ is a vertical vector field.*

3. Pointwise Bi-Slant Submersions

In the present section of the paper we define pointwise bi-slant submersions from cosymplectic manifolds and obtain necessary and sufficient conditions for integrability and total geodesicness of the distributions.

Definition 3. *Let (M, ϕ, ξ, η, g) be a cosymplectic manifold and (N, g') a Riemannian manifold. A Riemannian submersion $\pi : M \longrightarrow N$ is called a pointwise bi-slant submersion if*

i) *for nonzero any $U \in \Gamma(D_1)_p$ and $p \in M$, the angle θ_1 between ϕU and the space $(D_1)_p$ is independent of the choice of the nonzero vector $U \in \Gamma(D_1)$,*

ii) *for nonzero any $V \in \Gamma(D_2)_q$ and $q \in M$, the angle θ_2 between ϕV and the space $(D_2)_q$ are independent of the choice of the nonzero vector $V \in \Gamma(D_2)$*

such that $\ker \pi_* = D_1 \oplus D_2 \oplus \xi$. Then the angle $\theta_{i,}$ is called the slant function of the pointwise bi-slant submersion. π is called proper if its slant functions satisfy $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$.

We can give the following example using cosymplectic structure (ϕ, ξ, η, g) as in Example 2.1 of [26].

Example 4. *Define $\pi : \mathbb{R}^9 \to \mathbb{R}^4$ as follows:*

$$\pi\left(x_1, ..., x_8, z\right) = \left(x_1, (\cos\alpha)x_2 + (sin\alpha)x_4, (-\cos\beta)x_5 + (\sin\beta)x_7, x_6\right),$$

where $(x_1, ..., x_8, z)$ are natural coordinates of $\mathbb{R}^9$. Then we obtain

$$D_1 = \{V_1 = \frac{\partial}{\partial x_3}, V_2 = \sin\beta\frac{\partial}{\partial x_5} + \cos\beta\frac{\partial}{\partial x_7}\} \text{ and}$$
$$D_2 = \{V_3 = \frac{\partial}{\partial x_8}, V_4 = \sin\alpha\frac{\partial}{\partial x_2} - \cos\alpha\frac{\partial}{\partial x_4}\}.$$

Thus π is a pointwise bi-slant submersion with slant functions $\theta_1 = \beta$ and $\theta_2 = \alpha$.

Suppose that π is a pointwise bi-slant submersion from a cosymplectic manifold (M, ϕ, ξ, η, g) onto a Riemannian manifold (N, g'). For $U \in \Gamma\left(\ker \pi_*\right)$, we have

$$U = PU + QU + \eta(U)\xi \tag{10}$$

where $PU \in \Gamma\left(D_1\right)$ and $QU \in \Gamma\left(D_2\right)$.
Also, for $U \in \Gamma\left(\ker \pi_*\right)$, we write

$$\phi U = \psi U + \omega U \tag{11}$$

where $\psi U \in \Gamma\left(\ker \pi_*\right)$ and $\omega U \in \Gamma\left(\ker \pi_*\right)^{\perp}$.
For $X \in \Gamma\left(\ker \pi_*\right)^{\perp}$, we have

$$\phi X = \mathcal{B}X + \mathcal{C}X \tag{12}$$

where $\mathcal{B}X \in \Gamma\left(\ker \pi_*\right)$ and $\mathcal{C}X \in \Gamma\left(\ker \pi_*\right)^{\perp}$.
The horizontal distribution $\left(\ker \pi_*\right)^{\perp}$ is decomposed as

$$\left(\ker \pi_*\right)^{\perp} = \omega D_1 \oplus \omega D_2 \oplus \mu$$

where μ is the complementary distribution to $\omega D_1 \oplus \omega D_2$ in $(\ker \pi_*)^\perp$.
By using (3.2) and (3.3) we obtain

$$\psi D_1 = D_1, \quad \psi D_2 = D_2, \quad \mathcal{B}\omega D_1 = D_1, \quad \mathcal{B}\omega D_2 = D_2.$$

Considering Definition 3 we can give the following result.

Theorem 5. *Suppose that π is a Riemannian submersion from cosymplectic manifold (M, ϕ, ξ, η, g) onto a Riemannian manifold (N, g'). Then π is a pointwise bi-slant submersion if and only if there exist bi-slant function θ_i defined on D_i such that*

$$\psi^2 = -\cos^2\theta_i \left(I - \eta \otimes \xi\right), \ i = 1, 2.$$

Proof. The proof is similar to the proof of Theorem 2 of [10], so we omit it. □

Theorem 6. *Suppose that π is a pointwise bi-slant submersion from cosymplectic manifold (M, ϕ, ξ, η, g) onto a Riemannian manifold (N, g') with bi-slant functions θ_1, θ_2. Then*

i) the distribution D_1 is integrable if and only if

$$g\left(\mathcal{T}_U\omega\psi V - \mathcal{T}_V\omega\psi U, W\right) = g\left(\mathcal{T}_U\omega V - \mathcal{T}_V\omega U, \psi W\right) + g\left(\mathcal{H}\nabla_U\omega V - \mathcal{H}\nabla_V\omega U, \omega W\right)$$

ii) the distribution D_2 is integrable if and only if

$$g\left(\mathcal{T}_W\omega\psi Z - \mathcal{T}_Z\omega\psi W, U\right) = g\left(\mathcal{T}_W\omega Z - \mathcal{T}_Z\omega W, \psi U\right) + g\left(\mathcal{H}\nabla_W\omega Z - \mathcal{H}\nabla_Z\omega W, \omega U\right)$$

where $U, V \in \Gamma(D_1)$, $W, Z \in \Gamma(D_2)$.

Proof. From $U, V \in \Gamma(D_1)$ and $W \in \Gamma(D_2)$ we have

$$\begin{aligned} g\left([U,V], W\right) =& g\left(\nabla_U\phi V, \phi W\right) - g\left(\nabla_V\phi U, \phi W\right) \\ =& g\left(\nabla_U\psi V, \phi W\right) - g\left(\nabla_U\omega V, \phi W\right) + g\left(\nabla_V\psi U, \phi W\right) \\ & - g\left(\nabla_V\omega U, \phi W\right). \end{aligned}$$

Considering Theorem 5 we arrive

$$\begin{aligned} g\left([U,V], W\right) =& \cos^2\theta_1 g\left(\nabla_U V, W\right) - g\left(\nabla_U\omega\psi V, \phi W\right) - \cos^2\theta_1 g\left(\nabla_V U, W\right) \\ & + g\left(\nabla_V\omega\psi U, \phi W\right) + g\left(\nabla_U\omega V, \phi W\right) - g\left(\nabla_V\omega U, \phi W\right). \end{aligned}$$

Thus we get

$$\begin{aligned} \sin^2\theta_1 g\left([U,V], W\right) =& - g\left(\nabla_U\omega\psi V, W\right) + g\left(\nabla_V\omega\psi U, W\right) + g\left(\nabla_U\omega V, \phi W\right) \\ & - g\left(\nabla_V\omega U, \phi W\right). \end{aligned}$$

By using the equation (6) we obtain

$$\begin{aligned} \sin^2\theta_1 g\left([U,V], W\right) =& - g\left(\mathcal{T}_U\omega\psi V, W\right) + g\left(\mathcal{T}_V\omega\psi U, W\right) + g\left(\mathcal{T}_U\omega V, \psi W\right) \\ & + g\left(\mathcal{H}\nabla_U\omega V, \omega W\right) - g\left(\mathcal{T}_V\omega U, \psi W\right) - g\left(\mathcal{H}\nabla_V\omega U, \omega W\right). \end{aligned}$$

This completes the proof. □

Theorem 7. *Suppose that π is a pointwise bi-slant submersion from cosymplectic manifold (M, ϕ, ξ, η, g) onto a Riemannian manifold (N, g') with bi-slant functions θ_1, θ_2. Then the distribution D_1 defines a totally geodesic foliation if and only if*

$$\begin{aligned}\sin^2\theta_1 g\left([U,X],V\right) = &\sin 2\theta_1 X\left[\theta_1\right] g\left(\phi U, \phi V\right) - g\left(\mathcal{A}_X \omega\psi U, V\right)\\ &+ g\left(\mathcal{A}_X \omega U, \psi V\right) + g\left(\mathcal{H}\nabla_X \omega U, \omega V\right)\end{aligned}$$

and

$$g\left(\mathcal{H}\nabla_U \omega V, \omega W\right) = g\left(\mathcal{T}_U \omega\psi V, W\right) - g\left(\mathcal{T}_U \omega V, \psi W\right)$$

where $U, V \in \Gamma\left(D_1\right)$, $W \in \Gamma\left(D_2\right)$ and $X \in \Gamma\left(\left(\ker\pi_*\right)^{\perp}\right)$.

Proof. For any $U, V \in D_1$ and $X \in \Gamma\left(\left(\ker\pi_*\right)^{\perp}\right)$ we write

$$\begin{aligned}g\left(\nabla_U V, X\right) &= -\,g\left([U,X],V\right) - g\left(\nabla_X U, V\right)\\ &= -\,g\left([U,X],V\right) + g\left(\nabla_X \phi\psi U, V\right) - g\left(\nabla_X \omega U, \phi V\right).\end{aligned}$$

From Theorem 5, the above equation is obtained as follows

$$\begin{aligned}g\left(\nabla_U V, X\right) = &-\,g\left([U,X],V\right) + \sin 2\theta_1 X[\theta_1] g\left(\phi U, \phi V\right) - \cos^2\theta_1 g\left(\nabla_X U, V\right)\\ &+ g\left(\nabla_X \omega\psi U, V\right) - g\left(\nabla_X \omega U, \phi V\right)\end{aligned}$$

Using the equation (8) we have

$$\begin{aligned}\sin^2\theta_1 g\left(\nabla_U V, X\right) = &-\,sin^2\theta_1 g\left([U,X],V\right) + \sin 2\theta_1 X[\theta_1] g\left(\phi U, \phi V\right)\\ &+ g\left(\mathcal{A}_X \omega\psi U, V\right) - g\left(\mathcal{A}_X \omega U, \psi V\right) - g\left(\mathcal{H}\nabla_X \omega U, \omega V\right)\end{aligned}$$

Similarly for $W \in \Gamma\left(D_2\right)$ we have

$$g\left(\nabla_U V, W\right) = -g\left(\nabla_U \psi^2 V, W\right) - g\left(\nabla_U \omega\psi V, W\right) + g\left(\nabla_U \omega V, \phi W\right).$$

Thus we write

$$\begin{aligned}\sin^2\theta_1 g\left(\nabla_U V, W\right) = &-\,g\left(\mathcal{T}\omega\psi V, W\right) + g\left(\mathcal{T}_U \omega V, \psi W\right)\\ &+ g\left(\mathcal{H}\nabla_U \omega V, \omega W\right).\end{aligned}$$

This completes the proof. □

Theorem 8. *Suppose that π is a pointwise bi-slant submersion from cosymplectic manifold (M, ϕ, ξ, η, g) onto a Riemannian manifold (N, g') with bi-slant functions θ_1, θ_2. Then the distribution D_2 defines a totally geodesic foliation if and only if*

$$\begin{aligned}\sin^2\theta_2 g\left([W,X],Z\right) = &\sin 2\theta_2 X\left[\theta_2\right] g\left(\phi W, \phi Z\right) - g\left(\mathcal{A}_X \omega\psi W, Z\right)\\ &+ g\left(\mathcal{A}_X \omega W, \psi Z\right) + g\left(\mathcal{H}\nabla_X \omega W, \omega Z\right)\end{aligned}$$

and

$$g\left(\mathcal{H}\nabla_W \omega Z, \omega U\right) = g\left(\mathcal{T}_W \omega\psi Z, U\right) - g\left(\mathcal{T}_W \omega Z, \psi U\right)$$

where $U \in D_1$, $W, Z \in D_2$ and $X \in \Gamma\left(\left(\ker\pi_*\right)^{\perp}\right)$.

Proof. The proof of this theorem is similar to the proof of Theorem 5. □

Theorem 9. *Suppose that π is a pointwise bi-slant submersion from cosymplectic manifold (M,ϕ,ξ,η,g) onto a Riemannian manifold (N,g') with bi-slant functions θ_1,θ_2. Then the distribution $(\ker\pi_*)^{\perp}$ defines a totally geodesic foliation if and only if*

$$\begin{aligned}\sin^2\theta_1 g\left(\nabla_X Y,U\right)=&\left(\cos^2\theta_2-\cos^2\theta_1\right)g\left(A_X Y,QU\right)-g\left(\mathcal{H}\nabla_X Y,\omega\phi U\right)\\&+g\left(\omega\mathcal{A}_X Y,\omega U\right)+g\left(C\mathcal{H}\nabla_X Y,\omega U\right).\end{aligned}$$

where $X,Y\in\Gamma(\ker\pi_*)^{\perp}$ and $U\in\Gamma(\ker\pi_*)$.

Proof. For $X,Y\in\Gamma(\ker\pi_*)^{\perp}$ and $U\in\Gamma(\ker\pi_*)$ we write

$$g\left(\nabla_X Y,U\right)=g\left(\phi\nabla_X Y,\psi PU\right)+g\left(\phi\nabla_X Y,\psi QU\right)+g\left(\phi\nabla_X Y,\omega U\right)$$

From Theorem 5 we have

$$\begin{aligned}g\left(\nabla_X Y,U\right)=&-g\left(\nabla_X Y,\psi^2 PU\right)-g\left(\nabla_X Y,\psi^2 QU\right)-g\left(\nabla_X Y,\omega\psi U\right)\\&+g\left(\phi\nabla_X Y,\omega U\right)\end{aligned}$$

By using the equation (8) we arrive

$$\begin{aligned}\sin^2\theta_1 g\left(\nabla_X Y,U\right)=&\left(\cos^2\theta_2-\cos^2\theta_1\right)g\left(\mathcal{A}_X Y,QU\right)-g\left(\mathcal{H}\nabla_X Y,\omega\psi U\right)\\&+g\left(\omega\mathcal{A}_X Y,\omega U\right)+g\left(C\mathcal{H}\nabla_X Y,\omega U\right).\end{aligned}$$

Thus we have the desired equation. □

Theorem 10. *Suppose that π is a pointwise bi-slant submersion from cosymplectic manifold (M,ϕ,ξ,η,g) onto a Riemannian manifold (N,g') with bi-slant functions θ_1,θ_2. Then the distribution $(\ker\pi_*)$ defines a totally geodesic foliation on M if and only if*

$$\begin{aligned}\sin^2\theta_1 g\left([U,X],V\right)=&\left(\cos^2\theta_1-\cos^2\theta_2\right)g\left(\phi\nabla_X QU,\phi V\right)+\sin 2\theta_1 X[\theta_1]\,g\left(\phi U,\phi V\right)\\&-\left(\sin 2\theta_1 X[\theta_1]-\sin 2\theta_2 X[\theta_2]\right)g\left(\phi QU,QV\right)+g\left(\mathcal{A}_X\omega\psi U,V\right)\\&-g\left(\mathcal{A}_X\omega U,\psi V\right)-g\left(\mathcal{H}\nabla_X\omega U,\omega V\right)-\sin^2\theta_1\eta\left(\nabla_X U\right)\eta(V)\end{aligned}$$

where $X\in\Gamma(\ker\pi_*)^{\perp}$ and $U,V\in\Gamma(\ker\pi_*)$.

Proof. Given $X\in\Gamma(\ker\pi_*)^{\perp}$ and $U,V\in\Gamma(\ker\pi_*)$. Then we derive

$$\begin{aligned}g\left(\nabla_U V,X\right)&=-g\left([U,X],V\right)-g\left(\nabla_X U,V\right)\\&=-g\left([U,X],V\right)-g\left(\nabla_X\phi U,\phi V\right)-\eta(\nabla_X U)\eta(V)\end{aligned}$$

By using the equations (10) and (11), we have

$$\begin{aligned}g\left(\nabla_U V,X\right)=&-g\left([U,X],V\right)-g\left(\nabla_X\psi PU,\phi V\right)-g\left(\nabla_X\psi QU,\phi V\right)\\&-g\left(\nabla_X\omega U,\phi V\right)-\eta\left(\nabla_X U\right)\eta(V).\end{aligned}$$

Thus, we obtain

$$g\left(\nabla_U V,X\right)=-g\left([U,X],V\right)+g\left(\nabla_X\psi^2 PU,V\right)+g\left(\nabla_X\psi^2 QU,V\right)$$

$$+ g\left(\nabla_X \omega\psi U, V\right) - g\left(\nabla_X \omega U, \phi V\right) - \eta\left(\nabla_X U\right)\eta(V)$$

Using Theorem 5 we arrive

$$\begin{aligned} g\left(\nabla_U V, X\right) = & - g\left(\left[U, X\right], V\right) + \sin 2\theta_1 X\left[\theta_1\right] g\left(PU, V\right) \\ & - \sin 2\theta_1 X[\theta_1]\eta(PU)\eta(V) + \sin 2\theta_2 X\left[\theta_2\right] g\left(QU, V\right) \\ & - \sin 2\theta_2 X[\theta_2]\eta(QU)\eta(V) - \cos^2\theta_1 g\left(\nabla_X PU, V\right) \\ & + \cos^2\theta_1 \eta\left(\nabla_X PU\right)\eta(V) - \cos^2\theta_2 g\left(\phi\nabla_X QU, \phi V\right) \\ & + g\left(\nabla_X \omega\psi U, V\right) - g\left(\nabla_X \omega U, \phi V\right) - \eta\left(\nabla_X U\right)\eta(V) \end{aligned}$$

From the equation (8) we obtain

$$\begin{aligned} \sin^2\theta_1 g\left(\nabla_U V, X\right) = & - \sin^2\theta_1 g\left(\left[U, X\right], V\right) + \sin 2\theta_1 X\left[\theta_1\right] g\left(\phi U, \phi V\right) \\ & + \left(\sin 2\theta_2 X[\theta_2] - \sin 2\theta_1 X[\theta_1]\right) g\left(\phi QU, \phi V\right) \\ & + \left(\cos^2\theta_1 - \cos^2\theta_2\right) g\left(\phi\nabla_X QU, \phi V\right) + g\left(\mathcal{A}_X \omega\psi U, V\right) \\ & - g\left(\mathcal{A}_X \omega U, \psi V\right) - g\left(\mathcal{H}\nabla_X \omega U, \omega V\right) - \sin^2\theta_1 \eta\left(\nabla_X U\right)\eta(V) \end{aligned}$$

Using above equation the proof is completed. □

Theorem 11. *Suppose that π be a pointwise bi-slant submersion from cosymplectic manifold (M, ϕ, ξ, η, g) onto a Riemannian manifold (N, g') with bi-slant functions θ_1, θ_2. Then π is totally geodesic if and only if*

$$-\cos^2\theta_1 \mathcal{T}_U PV - \cos^2\theta_2 \mathcal{T}_U QV + \mathcal{H}\nabla_U \omega\psi V + C\mathcal{H}\nabla_U \omega V + \omega\mathcal{T}_U \omega V = 0$$

and

$$-\cos^2\theta_1 \mathcal{A}_X PU - \cos^2\theta_2 \mathcal{A}_X QU + \mathcal{H}\nabla_X \omega\psi U + C\mathcal{H}\nabla_X \omega U + \omega\mathcal{A}_X \omega U = 0$$

where $X \in \Gamma\left(\ker\pi_*\right)^\perp$ and $U, V \in \left(\ker\pi_*\right)$.

Proof. Since π is a Riemannian submersion for $X, Y \in \Gamma\left(\ker\pi_*\right)^\perp$ we have

$$\left(\nabla\pi_*\right)(X, Y) = 0.$$

Thus for $U, V \in \Gamma\left(\ker\pi_*\right)$ it is enough to show that $\left(\nabla\pi_*\right)(U, V) = 0$ and $\left(\nabla\pi_*\right)(X, U) = 0$. Then we can write

$$\left(\nabla\pi_*\right)(U, V) = -\pi_*\left(\nabla_U V\right).$$

Thus from the equation (9), we obtain

$$\begin{aligned} \left(\nabla\pi_*\right)(U, V) = & - \pi_*\left(\nabla_U V\right) = \pi_*\left(\phi\nabla_U \psi V + \phi\nabla_U \omega V\right) \\ = & \pi_*\left(\nabla_U \psi^2 PV + \nabla_U \psi^2 QV + + \nabla_U \omega\psi V + \phi\nabla_U \omega V\right). \end{aligned}$$

Considering Theorem 5 we find

$$\left(\nabla\pi_*\right)(U, V) = \pi_*\left(-\cos^2\theta_1 \nabla_U PV - \cos^2\theta_2 \nabla_U QV + \nabla_U \omega\psi V + \phi\nabla_U \omega V\right).$$

Therefore we obtain the first equation of Theorem 11.
On the other hand, we can write

$$(\nabla\pi_*)(X,U) = -\pi_*(\nabla_X U).$$

Using the equation (7) and (8), we arrive

$$(\nabla\pi_*)(X,U) = \pi_*\left(-\cos^2\theta_1\mathcal{A}_X PU - \cos^2\theta_2\mathcal{A}_X QU + \mathcal{H}\nabla_X\omega\psi U + C\mathcal{H}\nabla_X\omega U\right).$$

This concludes the proof. □

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