Araştırma Makalesi **BAUN Fen Bil. Enst. Dergisi, 22(1), 1-12, (2020)**

DOI:10.25092/baunfbed.673243 **J. BAUN Inst. Sci. Technol., 22(1), 1-12, (2020)**

Numerical analysis for coupled systems of two-dimensional time-space fractional Schrödinger equations with trapping potentials

Betül HİÇDURMAZ*

Istanbul Medeniyet University, Faculty of Engineering and Natural Sciences, Department of Mathematics, Uskudar, Istanbul

Geliş Tarihi (Received Date): 01.03.2019
Kabul Tarihi (Accepted Date): 19.12.2019

Abstract

In this study general and classical coupled systems of nonlinear time-space fractional Schrödinger equations (TSFSDE) with trapping potentials are investigated with a numerical approach. Theorems on stability of the finite difference schemes for such problems are established and presented with their proofs. Numerical solutions are investigated for one and two-dimensional cases. Convergence rates are proved by numerical experiments. Effect of a trapping potential on such systems is searched throughout the paper.

Keywords: *Coupled system, trapping potential, finite difference method.*

Tuzaklama potansiyelli iki-boyutlu zaman-yer kesirli türevli Schrödinger denklemlerinin bağlı sistemlerinin sayısal analizi

Öz

Bu çalışmada zaman ve yer boyutlarında kesirli türevli Schrödinger diferansiyel denklemlerinin bağıl sistemlerinin genel ve klasik formları tuzaklama potansiyeli altında sayısal bir yaklaşımla ele alınmıştır. Bu tip problemlerin fark şemalarının kararlılıkları üzerine teoremler kurulmuş ve ispatlarıyla sunulmuştur. Sayısal sonuçlar tek ve iki boyutlu durumlar için incelenmiştir. Yaklaşım mertebeleri sayısal deneylerle ispatlanmıştır. Çalışma boyunca tuzaklama potansiyelinin bu tip sistemler üzerine etkisi araştırılmıştır.

Anahtar kelimeler: *Bağlı sistem, tuzaklama potansiyeli, sonlu farklar metodu.*

* Betül HİÇDURMAZ, betul.hicdurmaz@medeniyet.edu.tr, https://orcid.org/0000-0001-6797-246X

1. Introduction

Fractional Schrödinger differential equations in general form are analyzed with different techniques throughout the literature (see [1]-[6]). Presented problems differ according to the variable where fractional order derivative appears. In the present paper a coupled system of time fractional Schrödinger equations will be pointed out. The problem will be considered in both classical and general forms. General form of the model is able to include terms representing the four wave mixing effect as [7]

$$\begin{aligned} &iD_{0,t}^{\alpha}u+\frac{\partial^{\beta_1}u}{\partial|x|^{\beta_1}}+\frac{\partial^{\beta_2}u}{\partial|y|^{\beta_2}}+2(a|u|^2+c|v|^2+bu\bar{v}+\bar{b}\bar{u}v)u+V(x,y)u=f(t,x),\\ &iD_{0,t}^{\alpha}v+\frac{\partial^{\beta_1}v}{\partial|x|^{\beta_1}}+\frac{\partial^{\beta_2}v}{\partial|y|^{\beta_2}}+2(a|u|^2+c|v|^2+bu\bar{v}+\bar{b}\bar{u}v)v+V(x,y)v=g(t,x). \end{aligned} \tag{1}$$

Here, *a* and *c* are real constants scaling self-phase modulation and cross-phase modulation respectively, whereas complex constant b denotes four wave mixing effect [8] and *V(x,y)* is a real valued trapping potential. The four-wave mixing effect can be defined as an inter modulation phenomenon in nonlinear optics, whereby the interactions between the two or three wavelengths produce one or two new wavelengths [9].

Here, it may be useful to remind that the classical form is presented like that:

$$\begin{aligned} &iD_{0,t}^{\alpha}u+\frac{\partial^{\beta_1}u}{\partial|x|^{\beta_1}}+\frac{\partial^{\beta_2}u}{\partial|y|^{\beta_2}}+2(a|u|^2+c|v|^2)u+V(x,y)u=p(t,x),\\ &iD_{0,t}^{\alpha}v+\frac{\partial^{\beta_1}v}{\partial|x|^{\beta_1}}+\frac{\partial^{\beta_2}v}{\partial|y|^{\beta_2}}+2(a|u|^2+c|v|^2)v+V(x,y)v=q(t,x). \end{aligned}$$

For both cases, we consider the trapping potential *V* as a time-independent potential. Physical properties for a trapping potential can be seen in [10] with more detail.

These equations are studied by Fractional Reduced Differential Transformation, Homotopy Analysis Method, Reduced Differential Transform Method in [11-13]. In [14], general coupled systems of time fractional Schrödinger equations are considered with both theoretical and numerical approaches.

Numerical methods which are listed above lack the stability properties. Classical methods are more advantageous at this point with providing stable difference schemes for many problems. Finite difference method is not studied extensively in these type of problems. This paper will fill a gap by presenting stable difference schemes for Coupled Systems of Two-Dimensional Time-Space Fractional Schrödinger Equations with Trapping Potentials.

Furthermore, [15-17] search and present the results on the dynamics and decoherence of fractional Schrödinger equations with trapping potential.

To remind some important terms, we will give definition for Riemann-Liouville fractional derivative with order $\alpha \in (0,1)$ as

$$ {}_C D_{0,t}^{\alpha} u(t,x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u_s(s,x) ds. $$

whereas the spatial fractional derivatives are Riesz type as

$$ \frac{\partial^{\beta} u(t,x)}{\partial |x|^{\beta}} = -\frac{1}{2\cos(\pi\beta/2)} \left({}_{RL} D_{a,x}^{\beta} u(t,x) + {}_{RL} D_{x,b}^{\beta} u(t,x) \right), $$

where ${}_{RL}D^{\beta}_{a,x}u(t,x)$ and ${}_{RL}D^{\beta}_{x,b}u(t,x)$ are left and right Riemann-Liouville derivatives. We can use the term Caputo fractional derivative instead of Riemann-Louville time fractional derivative in the case of homogeneous initial condition.

Paper is organized as follows: First section is introduction. Second section is on theoretical findings for the most general case which is general coupled system of TSFSDEs with a trapping potential in two dimensional case. Third section is on numerical analysis for four different cases of the general problem which is considered in second section. Fourth section is conclusion.

2. Theoretical findings

In the present section, we will present the difference scheme for a two dimensional mixed problem for general coupled systems of time-space fractional Schrödinger equations with the required theoretical analysis. Due to the fact that the theoretical findings can easily be extended to *m*-dimensional case. A two dimensional problem for general coupled systems of time-space fractional Schrödinger equations can be given with mixed conditions as:

$$ \begin{aligned} & i\frac{\partial^{\alpha} u(t,x,y)}{\partial t^{\alpha}} = -\frac{\partial^{\beta_1} u(t,x,y)}{\partial |x|^{\beta_1}} - \frac{\partial^{\beta_2} u(t,x,y)}{\partial |y|^{\beta_2}} + f(t,x,y) - 2\Big(a|u(t,x,y)|^2 + c|v(t,x,y)|^2 \\ & + bu(t,x,y)\overline{v(t,x,y)} + \bar{b}v(t,x,y)\overline{u(t,x,y)} \Big) u(t,x,y) - V(x,y)u(t,x,y), \\ & i\frac{\partial^{\alpha} v(t,x,y)}{\partial t^{\alpha}} = -\frac{\partial^{\beta_1} v(t,x,y)}{\partial |x|^{\beta_1}} - \frac{\partial^{\beta_2} v(t,x,y)}{\partial |y|^{\beta_2}} + g(t,x,y) - 2\Big(a|u(t,x,y)|^2 + c|v(t,x,y)|^2 \\ & + bu(t,x,y)\overline{v(t,x,y)} + \bar{b}v(t,x,y)\overline{u(t,x,y)} \Big) v(t,x,y) - V(x,y)v(t,x,y), \end{aligned} \tag{1} $$

$$ x \in (0,1), y \in (0,1), t \in (0,1), u(0,x,y) = u_0(x,y), v(0,x) = v_0(x,y),\ x \in (0,1), $$

$$ u(t,0,y) = u(t,1,y) = v(t,0,y) = v(t,1,y) = 0, t \in (0,1), y \in (0,1), $$

$$ u(t,x,0) = u(t,x,1) = v(t,x,0) = v(t,x,1) = 0, t \in (0,1), x \in (0,1). $$

Then, problem (1) has a unique *{u, v}* solution in space of all continuous functions [18]: $C([0,T],L^2(\Omega))$.

To carry out the time discretization for problems above, the grid set
$[0,T]_\tau=\{t=t_k=k\tau,\ k=0,1,...,N,\tau=T/N\}$

is considered. Space discretization is provided by introduction of the grid set:

$$\overline{\Omega}_h=\{x=x_j=(h_1j_1,...,h_mj_m),\ j=(j_1,j_2,...,j_m),\ 0\le j_r\le M_r,\ \ h_rM_r=1, r=1,...,m\}$$

where $\Omega_h=\overline{\Omega}_h\cap\Omega,\ S_h=\overline{\Omega}_h\cap S$.

Throughout the paper, step size quantities M_r are considered equal for every dimension as $M_r=M$ for $r\in\{1,2,...m\}$ So, h_r step sizes are also equal as $h_r=h=1/M$ for every $r\in\{1,2,...m\}$. Here, $m=2$ is the dimension of space variable $\vec{x}$.
Prior to the discretization of problem (1), we introduce the $L_h^2(\Omega)=L^2(\overline{\Omega})$ space for the grid functions $\varphi_h(x)\in\overline{\Omega}_h$ with the norm

$$\|\phi_h(x)\|_{L_h^2(\Omega)}=\left(\sum_{x\in\overline{\Omega}_h}|\phi_h(x)|^2h_1...h_m\right)^{1/2}.$$

Here, we implement discretization formulas in [14] to obtain a $O(\tau+h^2)$ order of accurate difference scheme.

Using following formula from [19]

$$_CD_{0,t}^\alpha u(t,x)=\frac{1}{\tau^\alpha}\sum_{k=0}^{k}\eta_{k-m}(u^m-u^{m-1})+O(\tau)$$

where $u_0=0$, $\eta_m=\dfrac{\Gamma(m-\alpha+1)}{m!\Gamma(1-\alpha)}, m=0,1,2...,k+1$ and

$$\eta_i=\frac{i-\alpha}{i}\eta_{i-1},\ i\ge1, 1=\eta_0>\eta_1>\eta_2>...>\eta_k>0 \text{ and } \eta_k\to0 \text{ as } k\to0$$

with the approximation which is presented for fractional spacial derivative in [20] as

$$\frac{\partial^\beta u(t,x)}{\partial|x|^\beta}=\frac{1}{h^\beta}\sum_{l=1}^{M-1}c_{j-l}u_l^{k+1}+O(h^2)$$

where

$$w_0^\beta=(\beta/2)g_0,\ w_k^\beta=(\beta/2)g_k+((2-\beta)/2)g_{k-1},\ k\ge1,$$

$$g_0^\beta=1,\ g_k^\beta=(1-(\beta+1)/k)g_{k-1}^\beta,\ k\ge1,$$

we can construct the implicit difference scheme as

$$
\begin{aligned}
& i\tau^{-\alpha}\sum_{m=0}^{k+1}\eta_{k+1-m}(u_{j,q}^{m}-u_{j,q}^{m-1}) = \sum_{l=1}^{M-1}\frac{c_{j-l}^{1}}{h^{\beta_1}}u_{j,q}^{k+1}+\sum_{l=1}^{M-1}\frac{c_{q-l}^{2}}{h^{\beta_1}}u_{j,q}^{k+1} \\
& -2\left(\left|v_{j,q}^{k+1}\right|^2+\left|u_{j,q}^{k+1}\right|^2+u_{j,q}^{k+1}\overline{v}_{j,q}^{k+1}+v_{j,q}^{k+1}\overline{u}_{j,q}^{k+1}\right)u_{j,q}^{k+1}-V_{j,q}^{k+1}u_{j,q}^{k+1} \\
& i\tau^{-\alpha}\sum_{m=0}^{k+1}\eta_{k+1-m}(v_{j,q}^{m}-v_{j,q}^{m-1}) = \sum_{l=1}^{M-1}\frac{c_{j-l}^{1}}{h^{\beta_1}}v_{j,q}^{k+1}+\sum_{l=1}^{M-1}\frac{c_{q-l}^{2}}{h^{\beta_1}}v_{j,q}^{k+1} \\
& \quad -2\left(\left|v_{j,q}^{k+1}\right|^2+\left|u_{j,q}^{k+1}\right|^2+u_{j,q}^{k+1}\overline{v}_{j,q}^{k+1}+v_{j,q}^{k+1}\overline{u}_{j,q}^{k+1}\right)v_{j,q}^{k+1}-V_{j,q}^{k+1}u_{j,q}^{k+1} \qquad (2) \\
& 1\le j,q\le M-1, 0\le k\le N-1, \\
& u_{j,k}^{0}=0,\ v_{j,k}^{0}=0,\ 1\le j,q\le M-1, \\
& u_{j,0}^{k}=u_{j,M}^{k}=0,\ v_{j,0}^{k}=v_{j,M}^{k}=0, 1\le j\le M-1,\ 0\le k\le N, \\
& u_{0,q}^{k}=u_{M,q}^{k}=0,\ v_{0,q}^{k}=v_{M,q}^{k}=0, 1\le q\le M-1,\ 0\le k\le N.
\end{aligned}
$$

Theorem 1. The finite difference scheme (2) is convergent with the order of accuracy $O(\tau+h^2)$.

Proof. If the truncation error is z_k, then

$$
\begin{aligned}
& z_k = i\tau^{-\alpha}\sum_{m=0}^{k+1}\eta_{k+1-m}(u_{j,q}^{m}-u_{j,q}^{m-1})-i\frac{\partial^{\alpha}u(t_k,x_j,y_q)}{\partial t^{\alpha}}+\sum_{l=1}^{M-1}\frac{c_{j-l}^{1}}{h^{\beta_1}}u_{j,q}^{k+1}-\frac{\partial^{\beta_1}u(t_k,x_j,y_q)}{\partial|x|^{\beta_1}} \\
& +\sum_{l=1}^{M-1}\frac{c_{q-l}^{2}}{h^{\beta_2}}u_{j,q}^{k+1}-\frac{\partial^{\beta_2}u(t_k,x_j,y_q)}{\partial|x|^{\beta_2}}+2\left(\left|v_{j,q}^{k+1}\right|^2+\left|u_{j,q}^{k+1}\right|^2+u_{j,q}^{k+1}\overline{v}_{j,q}^{k+1}+v_{j,q}^{k+1}\overline{u}_{j,q}^{k+1}\right)u_{j,q}^{k+1} \\
& -2\left(\left|u(t_k,x_j,y_q)\right|^2+\left|v(t_k,x_j,y_q)\right|^2+u(t_k,x_j,y_q)\overline{v(t_k,x_j,y_q)}+v(t_k,x_j,y_q)\overline{u(t_k,x_j,y_q)}\right) \\
& +V(t_k,x_j,y_q)\left(u_{j,q}^{k+1}-u(t_k,x_j,y_q)\right) \\
& =O(\tau+h^2).
\end{aligned}
$$

This will be repeated for the second part of the coupled system (2). So, this is the end of the proof.

Theorem 2. The finite difference scheme (2) is unconditionally stable.

Proof. Using a similar procedure with [14], we get

$$
\operatorname{Im}\left\langle \sum_{l=1}^{M-1}\frac{c_{j-l}^{1}}{h^{\beta_1}}u_{j,q}^{k+1}+\sum_{l=1}^{M-1}\frac{c_{q-l}^{2}}{h^{\beta_2}}u_{j,q}^{k+1},u_{j,q}^{k+1}\right\rangle =0.
$$

Then, we get inner product of equations in (2) with $u^{k+1}_{j,q}$. When we consider imaginary part of the established equation and induction method, we get

$$\left\|u_{j,q}^{k+1}\right\|^2 \le \left\|u_{j,q}^{1}\right\|^2 \quad \text{and} \quad \left\|v_{j,q}^{k+1}\right\|^2 \le \left\|v_{j,q}^{1}\right\|^2 .$$

Then, these formulas constitute the proof of the stability for difference scheme (2). Following the same procedure one can construct the difference schemes and prove the stability and convergence theorems for *m*-dimensional case of problem (1) for any positive integer *m*.

3. Numerical Analysis

In the present part, one/two dimensional general/classical coupled systems of time-space fractional Schrödinger equations with trapping potentials will be investigated numerically.

3.1. Two-dimensional general TSFSDE

Two-dimensional problem for general coupled systems of time-space fractional Schrödinger equations can be stated as

$$\begin{aligned}
&i\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = -\frac{\partial^{\beta_1}u}{\partial|x|^{\beta_1}} - \frac{\partial^{\beta_2}u}{\partial|y|^{\beta_2}} - \left(|u|^2+|v|^2+bu\bar{v}+\bar{b}v\bar{u}\right)u + g_1(t,x,y) - (1-\sin^2 x\sin^2 x)u,\\
&i\frac{\partial^{\alpha}v}{\partial t^{\alpha}} = -\frac{\partial^{\beta_1}v}{\partial|x|^{\beta_1}} - \frac{\partial^{\beta_2}v}{\partial|y|^{\beta_2}} - \left(|u|^2+|v|^2+bu\bar{v}+\bar{b}v\bar{u}\right)v + g_2(t,x,y) - (1-\sin^2 x\sin^2 x)v,\\
&0<x<L, 0<t<1, \qquad\qquad (3)\\
&u(0,x)=0, v(0,x)=0, 0<x<L,\\
&u(t,0,y)=u(t,1,y)=0,\\
&v(t,0,y)=v(t,1,y)=0, 0<t<1, 0<y<1,\\
&u(t,x,0)=u(t,x,1)=0,\\
&v(t,x,0)=v(t,x,1)=0, 0<t<1, 0<x<1.
\end{aligned}$$

Here,

$$u=(1+i)(t^3)x^2(1-x)^2y^2(1-y)^2, \quad v=i(t^4)x^2(1-x)^2y^2(1-y)^2.$$

Moreover, we define

$$\begin{aligned}
&g_1(t,x,y) = (i-1)\frac{\Gamma(4)}{\Gamma(4-\alpha)}t^{3-\alpha}x^2(1-x)^2y^2(1-y)^2\\
&+\left((1+i)t^3x^2(1-x)^2\right)^3 y^2(1-y)^2 - \frac{(1+i)t^3x^{-\beta_1}y^2(1-y)^2 12x^2}{\Gamma(5-\beta_1)}\\
&+\frac{(1-x)^2x^{\beta_1}\sec\left(\frac{\pi\beta_1}{2}\right)}{(1-x)^{\beta_1}}(-6x\beta_1-\beta_1+\beta_1^2)+x^2(12(1-x)^2-7\beta_1+6x\beta_1+\beta_1^2))
\end{aligned}$$

$$-\frac{(1+i)t^3y^{-\beta_2}x^2(1-x)^2}{\Gamma(5-\beta_2)}\left(\frac{(1-y)^2y^{\beta_2}12y^2}{(1-y)^{\beta_2}}\right.$$
$$+\frac{(1-y)^2y^{\beta_2}\sec\left(\frac{\pi\beta_2}{2}\right)}{(1-y)^{\beta_2}}(-6y\beta_2-\beta_2+\beta_2^2)+y^2(12(1-y)^2-7\beta_2+6y\beta_2+\beta_2^2\Big))$$
$$-(1-\sin^2 x\sin^2 x)(1+i)(t^3)x^2(1-x)^2y^2(1-y)^2$$

and

$$g_2(t,x,y)=-\frac{\Gamma(5)}{\Gamma(5-\alpha)}t^{4-\alpha}x^2(1-x)^2y^2(1-y)^2+\left(it^3x^2(1-x)^2\right)^3y^2(1-y)^2$$
$$-\frac{it^4x^{-\beta_1}y^2(1-y)^2}{\Gamma(5-\beta_1)}\left(\frac{(1-x)^2x^{\beta_1}12x^2}{(1-x)^{\beta_1}}+\frac{(1-x)^2x^{\beta_1}\sec\left(\frac{\pi\beta_1}{2}\right)}{(1-x)^{\beta_1}}(-6x\beta_1-\beta_1+\beta_1^2)\right.$$
$$+x^2(12(1-x)^2-7\beta_1+6x\beta_1+\beta_1^2\Big))-\frac{it^4y^{-\beta_2}x^2(1-x)^2\sec\left(\frac{\pi\beta_2}{2}\right)}{\Gamma(5-\beta_2)}\left(\frac{(1-y)^2y^{\beta_2}12y^2}{(1-y)^{\beta_2}}\right.$$
$$+\frac{(1-y)^2y^{\beta_2}}{(1-y)^{\beta_2}}(-6y\beta_2-\beta_2+\beta_2^2)+y^2(12(1-y)^2-7\beta_2+6y\beta_2+\beta_2^2\Big))$$
$$-(1-\sin^2 x\sin^2 x)i(t^4)x^2(1-x)^2y^2(1-y)^2.$$

Lastly, difference scheme (2) is implemented on problem (3) and the errors of numerical experiments are presented in Tables 1 and 2. Throughout the paper, errors are computed by the following maximum norm formula which can be given for two dimensional case as:

$$E=\max_{0\le k\le N}\left(\max_{0\le j,q\le M}\left|u_{j,q}^k\right|\right)$$

Convergence rates (C) are also computed and presented. Throughout the paper, h denotes the equal step length for all dimensions of the m-dimensional spatial variables. To get the solutions of problem, we convert the problem into a system of matrices and we use MATLAB. Throughout the experiments for implicit difference schemes, iterations start with $u_{j,q}^0=0$ and terminate when the error between each iteration becomes less than 10^{-7} in the given norm. Numerical analysis supports that theoretical findings are applicable and the convergence rates are valid when difference scheme (2) is applied on two dimensional general coupled systems of time-space fractional Schrödinger problem with a trapping potential (3).

Table 1. Errors of difference scheme (2) for problem (3) when *N=100*.

α	β_1	β_2	M	E	C
0.20	1.70	1.50	4	1.4972×10^{-3}	1.97
			8	3.8157×10^{-4}	1.77
			16	1.1158×10^{-4}	-
α	β_1	β_2	N	E	C
0.70	1.20	1.70	4	1.4586×10^{-3}	1.96

			8	3.47618×10^{-4}	1.72
			16	1.1400×10^{-4}	-
α	β_1	β_2	N	E	C
0.50	1.50	1.50	4	1.4944×10^{-3}	1.97
			8	3.8266×10^{-4}	1.74
			16	1.1491×10^{-4}	-

Table 2. Errors of difference scheme (2) for problem (3) when M=80.

α	β_1	β_2	N	E	C
0.50	1.25	1.50	4	6.2474×10^{-5}	0.94
			8	3.2574×10^{-5}	0.95
			16	1.6837×10^{-5}	0.92
			32	8.8981×10^{-6}	-

3.2. Two-dimensional classical TSFSDE

Two dimensional problem for classical coupled systems of time-space fractional Schrödinger equations can be given as

$$\begin{aligned}
&i\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=-\frac{\partial^{\beta_1}u}{\partial|x|^{\beta_1}}-\frac{\partial^{\beta_2}u}{\partial|y|^{\beta_2}}-\left(|u|^2+|v|^2\right)u+g_1(t,x,y)-(1-\sin^2 x\sin^2 x)u,\\
&i\frac{\partial^{\alpha}v}{\partial t^{\alpha}}=-\frac{\partial^{\beta_1}v}{\partial|x|^{\beta_1}}-\frac{\partial^{\beta_2}v}{\partial|y|^{\beta_2}}-\left(|u|^2+|v|^2\right)v+g_2(t,x,y)-(1-\sin^2 x\sin^2 x)v,\\
&0<x<L,0<t<1, \qquad (4)\\
&u(0,x)=0,v(0,x)=0,0<x<L,u(t,0,y)=u(t,1,y)=0,\\
&v(t,0,y)=v(t,1,y)=0,0<t<1,0<y<1,\\
&u(t,x,0)=u(t,x,1)=0,0<t<1,0<x<1,\\
&v(t,x,0)=v(t,x,1)=0,0<t<1,0<x<1.
\end{aligned}$$

Here,

$$u=(1+i)(t^3)x^2(1-x)^2y^2(1-y)^2,\;\; v=i(t^4)x^2(1-x)^2y^2(1-y)^2.$$

Lastly, difference scheme (2) is implemented on problem (4). Total maximum errors and convergence rates are presented in Tables 3.

Table 3. Errors of difference scheme (2) for problem (4) when N=100.

α	β_1	β_2	M	E	C
0.50	1.25	1.50	4	1.4643×10^{-3}	1.95
			8	3.7923×10^{-4}	1.70
			16	1.1882×10^{-4}	-

3.3. One-dimensional general TSFSDE

In the present section, we present the following general coupled system of one dimensional TSFSDE with a trapping potential. Here, we consider the exact solution as

$$u=(1+i)(t^3)x^2(1-x)^2,\ v=i(t^4)x^2(1-x)^2.$$

A one dimensional mixed problem for general coupled systems of time-space fractional Schrödinger equations can be stated as

$$\begin{aligned}
&i\frac{\partial^\alpha u}{\partial t^\alpha}=-\frac{\partial^\beta u}{\partial|x|^\beta}-\left(|u|^2+|v|^2+bu\bar{v}+\bar{b}\bar{v}\bar{u}\right)u-\cos^2 xu+f_1(t,x),\\
&i\frac{\partial^\alpha v}{\partial t^\alpha}=-\frac{\partial^\beta v}{\partial|x|^\beta}-\left(|u|^2+|v|^2+bu\bar{v}+\bar{b}\bar{v}\bar{u}\right)v-\cos^2 xv+f_2(t,x)\\
&0<x<L, 0<t<1, \qquad (5)\\
&u(0,x)=0, v(0,x)=0,\ 0<x<L,\\
&u(t,0)=u(t,1)=0,\ v(t,0)=v(t,1)=0,\ 0<t<1
\end{aligned}$$

where

$$\begin{aligned}
&f_1(t,x)=(i-1)\frac{\Gamma(4)}{\Gamma(4-\alpha)}t^{3-\alpha}x^2(1-x)^2+\left((1+i)t^3x^2(1-x)^2\right)^3\\
&-\frac{(1+i)t^3x^{-\beta}\sec\left(\frac{\pi\beta}{2}\right)}{\Gamma(5-\beta)}\left(\frac{(1-x)^2x^\beta 12x^2}{(1-x)^\beta}+\frac{(1-x)^2x^\beta}{(1-x)^\beta}(-6x\beta-\beta+\beta^2)\right.\\
&\left.+x^2(12(1-x)^2-7\beta+6x\beta+\beta^2)\right)-\cos^2 x(1+i)(t^3)x^2(1-x)^2
\end{aligned}$$

and

$$\begin{aligned}
&f_2(t,x)=-\frac{\Gamma(5)}{\Gamma(5-\alpha)}t^{4-\alpha}x^2(1-x)^2+\left(it^4x^2(1-x)^2\right)^3\\
&-\frac{it^4x^{-\beta}\sec\left(\frac{\pi\beta}{2}\right)}{\Gamma(5-\beta)}\left(\frac{(1-x)^2x^\beta 12x^2}{(1-x)^\beta}+\frac{(1-x)^2x^\beta}{(1-x)^\beta}(-6x\beta-\beta+\beta^2)\right.\\
&\left.+x^2(12(1-x)^2-7\beta+6x\beta+\beta^2)\right)-i\cos^2 x(t^4)x^2(1-x)^2
\end{aligned}$$

Problem (5) is solved by difference scheme (2) and Tables 4 and 5 are constructed for the obtained errors. This experiment shows that the obtained theoretical results for difference scheme (2) are valid when *m=1*.

Table 4. Errors of difference scheme (2) for problem (5) when N=1000.

α	β	M	E	C
0.70	1.50	4	2.5122×10^{-2}	2.05
		8	6.0407×10^{-3}	2.06
		16	1.4422×10^{-3}	2.07
		32	3.4166×10^{-5}	-

Table 5. Errors of difference scheme (2) for problem (5) when M=1000.

α	β	N	E	C
0.70	1.50	10	3.9424×10^{-3}	0.96
		20	2.0324×10^{-3}	0.98
		40	1.0322×10^{-3}	0.99
		80	5.2022×10^{-4}	-

3.4. One-dimensional classical TSFSDE

In the present section, we present the following classical coupled system of one dimensional TSFSDE with a trapping potential with the following exact solution:

$$u=(1+i)(t^3)x^2(1-x)^2, v=i(t^4)x^2(1-x)^2.$$

The following mixed problem for classical coupled systems of TSFSDE will be considered:

$$\begin{aligned}
&i\frac{\partial^\alpha u}{\partial t^\alpha}=-\frac{\partial^\beta u}{\partial|x|^\beta}-\left(|u|^2+|v|^2\right)u-\cos^2 xu+f_1(t,x),\\
&i\frac{\partial^\alpha v}{\partial t^\alpha}=-\frac{\partial^\beta v}{\partial|x|^\beta}-\left(|u|^2+|v|^2\right)v-\cos^2 xv+f_2(t,x),\\
&0<x<L, 0<t<1,\\
&u(0,x)=0, v(0,x)=0,\ 0<x<L,\\
&u(t,0)=u(t,1)=0,\ v(t,0)=v(t,1)=0,\ 0<t<1
\end{aligned} \tag{6}$$

Problem (6) is solved by difference scheme (2) and Tables 6 and 7 are constructed for the obtained errors.

Table 6. Errors of difference scheme (2) for problem (5) when N=1000.

α	β	M	E	C
0.70	1.50	4	2.5059×10^{-2}	2.05
		8	6.0290×10^{-3}	2.06
		16	1.4396×10^{-3}	2.07
		32	3.4105×10^{-4}	-

Table 7. Errors of difference scheme (2) for problem (5) when M=100.

α	β	N	E	C
0.70	1.50	4	9.0170×10^{-3}	0.90
		8	4.8486×10^{-3}	0.95
		16	2.5178×10^{-3}	0.97
		32	1.2834×10^{-3}	0.99
		64	6.4800×10^{-4}	-

4. Conclusion

In the present paper, finite difference scheme is constructed for multi-dimensional general coupled systems of time-space fractional Schrödinger equations with a trapping potential. Numerical results in error analysis table prove that the constrtucted difference schemes have accurately the same convergence rate which is proved in convergence and stability theorems. Space variable discretization has second order convergence and time variable discretization has convergence rate which is approximately one.

Convergence and stability theorems are presented theoretically. Obtained theoretical results are supported by numerical experiments on one and two dimensional general and classical coupled systems of time-space fractional Schrödinger equations with trapping potentials.

References

[1] Lü, X. and Peng, M., Painlevé-integrability and explicit solutions of the general two-coupled nonlinear Schrödinger system in the optical fiber communications, **Nonlinear Dynamics**, 73, 405-410, (2013).

[2] Wang, D. S., Zhang, D. J. and Yang, J., Integrable properties of the general coupled nonlinear Schrödinger equations, **Journal of Mathematical Physics**, 51, Article ID 023510, (2010).

[3] Ding, H.F., Li, C.P. and Chen, Y.Q., High-order algorithms for Riesz derivative and their applications, **Abstract and Applied Analysis**, 653797, 19-55, (2014).

[4] Hicdurmaz B. and Ashyralyev, A., On the stability of time-fractional Schrödinger differential equations, **Numerical Functional Analysis and Optimization**, 38, 1215-1225, (2017).

[5] Wang, D., Xiao, A. and Yang, W. , A linearly implicit con-servative difference scheme for the space fractional coupled nonlinear Schrödinger equations, **Journal of Computational Physics**, 272, 644-655 (2014).

[6] Garrappa, R., Moret, I. and Popolizio, M., On the time-fractional Schrödinger equation: theoretical analysis and numerical solution by matrix Mittag-Leffler functions, **Computers and Mathematics with Applications**, 74, 977-992 (2017).

[7] Antoine, X., Besse, C, and Klein, P., Absorbing boundary conditions for general nonlinear Schrödinger equations, **SIAM Journal on Scientific Computing,** 33, 1008-1033, (2011).

[8] Yuan, Y. Q., Tian, B., Liu, L. and Sun, Y., Bright-dark solitons for a set of the general coupled nonlinear Schrödinger equations in a birefringent fiber, **Europhysics Letters,** 120, 30001, 1-5, (2017).

[9] Ashyralyev A. and Sirma, A., A note on the numerical solu-tion of the semilinear Schrödinger equation, **Nonlinear Analysis: Theory, Methods and Applications,** 71, 12, 2507-2516, (2009).

[10] Al-Hashimi, N.H.N. and Ghalib, S. K., Theoretical analysis of different external trapping potential used in experimental of BEC, **IOSR Journal of Engineering**, 2, 11, 1-5, (2012).

[11] Hamed, S. H. M., Yousif, E. A. and Arbab, A. I., Analytic and approximate solutions of the space-time fractional Schrödinger equations by homotopy

perturbation Sumudu transform method, **Abstract and Applied Analysis**, 2014, 863015, 1-13, (2014).

[12] Khan, N. A., Jamil M., and Ara, A., Approximate solutions to time-fractional Schrödinger equation via homotopy analysis method, **ISRN Mathematical Physics**, 2012, 197068, 1-11, (2012).

[13] Taghizadeh, N., Noori S. R. M., Exact solutions of the cubic nonlinear Schrödinger equation with a trapping potential by reduced differential transform method, **Mathematical Scientific Letters**, 5, 3, 297-302, (2016).

[14] Hicdurmaz, B., Multidimensional problems for general coupled systems of time-space fractional Schrödinger equations, **Journal of Coupled Systems and Multiscale Dynamics,** 6, 147-153 (2018)

[15] Antoine, X., Tang, Q., and Zhang, Y., On the ground states and dynamics of space fractional nonlinear Schrödinger/Gross–Pitaevskii equations with rotation term and nonlocal nonlinear interactions, **Journal of Computational Physics**, 325, 74-97, (2016).

[16] Kirkpatrick, K., Zhang, Y., Fractional Schrödinger dynamics and decoherence, **Physica D: Nonlinear Phenomena**, 332, 41-54, (2016).

[17] Wang, M., Shan, W. R., Lü, X., Xue, Y.S., Lin, Z.Q. and Tian B., Soliton collision in a general coupled nonlinear Schrödinger system via symbolic computation, **Applied Mathematics and Computation** 219, 11258-11264, (2013).

[18] Liu, Q., Zeng, F. and Li, C., Finite difference method for time-space-fractional Schrödinger equation, **International Journal of Computer Mathematics**, 92, 7, 1439-1451, (2015).

[19] Ashyralyev A., A note on fractional derivatives and fractional powers of operators, **Journal of Mathematical Analysis and Applications**, 357, 232-236, (2009).

[20] Tian, W.Y., Zhou, H. and Deng W.H., A class of second order difference approximation for solving space fractional diffusion equations, **Mathematics of Computation**,84,1703-1727, (2015).