Available online: January 24, 2020

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 69, Number 1, Pages 576–602 (2020)
DOI: 10.31801/cfsuasmas.606890
ISSN 1303–5991 E-ISSN 2618-6470
http://communications.science.ankara.edu.tr/index.php?series=A1

A STUDY ON COMPARISONS OF BAYESIAN AND CLASSICAL PARAMETER ESTIMATION METHODS FOR THE TWO-PARAMETER WEIBULL DISTRIBUTION

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Abstract. The main objective of this paper is to determine the best estimators of the shape and scale parameters of the two parameter Weibull distribution. Therefore, both classical and Bayesian approximation methods are considered. For parameter estimation of classical approximation methods maximum likelihood estimators (MLEs), modified maximum likelihood estimators-I (MMLEs-I), modified maximum likelihood estimators -II (MMLEs-II), least square estimators (LSEs), weighted least square estimators (WLSEs), percentile estimators (PEs), moment estimators (MEs), L-moment estimators (LMEs) and TL- moment estimators (TLMEs) are used. Since the Bayesian estimators don't have the explicit form. There are Bayes estimators are obtained by using Lindley's and Tierney Kadane's approximation methods in this study. In Bayesian approximation, the choice of loss function and prior distribution is very important. Hence, Bayes estimators are given based on both the non- informative and informative prior distribution. Moreover, these estimators have been calculated under different symmetric and asymmetric loss functions. The performance of classical and Bayesian estimators are compared with respect to their biases and MSEs through a simulation study. Finally, a real data set taken from Turkish State Meteorological Service is analysed for better understanding of methods presented in this paper.

1. Introduction

Weibull distribution is one of the most popular among life-time distributions. The Weibull distribution was first proposed by W. Weibull who used it to model the distribution of the breaking strength of materials. The distribution has played major role in the reliability theory, see for example, [1] and [2]. Also, the distribution has found wide applications in many areas of environmental sciences, and renewable energy [3],[4],[5]and [6] . In addition to these application areas, Weibull distribution

Received by the editors: August 20, 2019; Accepted: December 18, 2019.
2010 *Mathematics Subject Classification.* Primary 05C38, 15A15; Secondary 05A15, 15A18.
Key words and phrases. Bayes approximation, parameter estimation, new estimator, L- moment estimator, simulation study.

is now being used in a wide range of fields in medical, biological, and earth sciences. For details, see [7],[8] and [9] .

It is crucial to determine the best parameter estimation method for any probability function. There are various different estimation methods in the literature for estimating the parameters of the Weibull distribution. Notable among them are given as follows: In terms of classical parameter estimation methods, Trustrum and Jayatilaka [10] investigated the moment estimator, maximum likelihood estimator and least squares method based on the Monte Carlo simulation.Hung [11] and Lu et al.[12] discussed the properties of the weighted least square estimators and showed that weighted least squares estimators performed better than least squares estimators. Pobocikova and Sedliackova [13] compared the maximum likelihood estimators, moment estimators, least squares estimators and weighted least square estimators. Teimouri et al. [14] presented the maximum likelihood estimators, method of logarithm moment, percentile estimator, L- moment estimator, method of moment. Alizadeh et al. [15] considered estimation of the probability density function and cumulative density function.

In terms of Bayesian parameter estimation methods, Al Omari and Ibrahim [16] conducted a study on Bayesian survival estimator for Weibull distribution with censored data. Also, Guure et al. [17] provided the Bayesian estimation of two parameter Weibull distribution under three loss functions using extension of Jeffey's prior information. Pandey et. al [18] compared Bayesion estimator and maximum likelihood estimation of the scale parameter of the Weilbull distribution under linex loss function, with the assumption that the shape parameter is kwown. Similar work can be seen in [19],[20] .

The maximum likelihood estimators (MLEs) and the moment estimators (MEs) are the most well-known among parameter estimation methods. In this article, the least square error estimators (LSEs) and the weighted square error estimators (WLSEs), the percentile estimators (PEs),the L-moment estimators (LMEs),the TL-moment estimators (TLMEs), modified maximum likelihood estimators (MMLE-I) are considered besides these methods. Moreover, we propose the modified maximum likelihood estimators-II (MMLE-II). Further, we compute Bayes estimators of the unknown parameters with informative prior and non-informative prior under squared error loss function (SELF), general entropy loss function (GELF), weighted square loss function (WSELF) and precautionary loss function (PLF). It is clear that Bayesian estimators cannot be found in explicit form. Therefore, in this paper, we consider the Lindley's and Tierney Kadane's procedures.

There are numerous studies for Weibull distribution in literature. But, as far as we know this, this is the first study which compares all these aforementioned estimation methods for choosing the best estimation method for the two- parameter Weibull distribution. The objective of this study is to estimate the parameters of the model from both classical and Bayesian viewpoint. Finally, a better estimation method is given for the distribution parameters. In the recent past, many researchers have compared various parameter estimation methods for estimating the parameters of the different distribution. See, for example, [21] for the generalized Rayleigh distribution, [22] for the Fréchet distribution, [23] for two parameter Maxwell distribution,[24] for generalized logistic distribution.

The rest of the paper is organized as follows: Weibull distribution is described in section 2. In section 3, some classical estimation methods are given to estimate the unknown parameters. In section 4, Bayes estimators of the unknown parameters are obtained by using Lindley's and Tierney Kadane's approximations. In Section 5, a simulation study is presented to evaluate the performances of the estimators with respect to their biases and mean square errors (MSE). Finally, a real life example taken from Turkish State Meteorological Service is given.

2. Weibull Distribution

The popularity of the Weibull distribution is attributable to the fact that it is commonly used to model different data types, such as wind speed, geothermal energy and finance.

The probability density function (PDF) and the cumulative density function (CDF) of the two-parameter Weibull distribution with the shape parameter α and the scale parameter β are given by:

$$F(x;\alpha,\beta)=1-exp\left\{-\left(\frac{x}{\beta}\right)^{\alpha}\right\}\ 0<x<\infty;\ \alpha>0,\ \beta>0 \tag{1}$$

and

$$f(x;\alpha,\beta)=\frac{\alpha}{\beta^{\alpha}}x^{\alpha-1}exp\left\{-\left(\frac{x}{\beta}\right)^{\alpha}\right\},\ 0<x<\infty. \tag{2}$$

The mean and variance of the Weibull distribution are defined as follows:

$$E(x)=\beta\Gamma\left(1+\frac{1}{\alpha}\right)\ and\ V(x)=\beta^{2}\left[\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma^{2}\left(1+\frac{1}{\alpha}\right)\right]$$

respectively. Here, Γ is the gamma function.

3. The Methods for Parameter Estimation

In this section, we presented the methods of classical estimation for the Weibull distribution used in this study.

3.1 Moment Estimators. The MEs are found by equating theoretical moments to corresponding sample moments as shown below:

$$\beta\Gamma\left(1+\frac{1}{\alpha}\right)=\bar{X}\ and\ \beta^{2}\Gamma\left(1+\frac{2}{\alpha}\right)=\frac{\sum_{i=1}^{n}X_i^2}{n}. \tag{3}$$

Then, by solving equation 3 the MEs of α and β are found as

$$\hat{\beta} = \frac{\bar{X}}{\Gamma\left(1 + \frac{1}{\alpha}\right)} \; and \; \frac{\Gamma\left(1 + \frac{2}{\hat{\alpha}}\right)}{\Gamma^2\left(1 + \frac{2}{\hat{\alpha}}\right)} = \frac{\sum_{i=1}^{n} X_i^2}{n\bar{X}^2} \tag{4}$$

respectively.

3.2 Maximum Likelihood Estimators. Let $X_1, X_2, ..., X_n$ be a random sample from Weibull distribution. The log-likelihood function is given by:

$$lnL = nln\alpha - n\alpha ln\beta + (\alpha - 1)\sum_{i=1}^{n} lnx_i - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha}. \tag{5}$$

By taking the partial derivative of 5 with respect to α and β, and equating them to zero, we obtain the following log-likelihood equations:

$$\frac{\partial lnL}{\alpha} = \frac{n}{\alpha} - nln\beta + \sum_{i=1}^{n} lnx_i - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{a} ln\frac{x_i}{\beta} = 0 \tag{6}$$

and

$$\frac{\partial lnL}{\partial\beta} = \frac{n\alpha}{\beta} + \frac{\alpha\sum_{i=1}^{n} x_i^{\alpha}}{\beta^{\alpha+1}} = 0. \tag{7}$$

Solutions of these likelihood equations are called as the MLEs of shape parameter α and scale parameter β , see for example [25] , [26] . However, they do not give closed form expressions since they include nonlinear terms $g_1(x) = lnx$ and $g_2(x) = x_i^{\alpha}$ in 6 and 7. Therefore, numerical methods are applied to solve the required equations. In this study, we apply the well-known Newton Rapson method to solve these equations.

3.3 Least Squares and Weighted Least Squares Estimators. The LSEs and WLSEs were originally suggested by Swain et al.[27] to estimate the parameters of beta distributions. See, for example, Kundu and Ragab [21] and Alkasabeh and Ragab [24] .

Let $X_1, ..., X_n$ is a random sample of size n from a distribution function $G(.)$ and $X_{i:n}; i = 1, 2, ..., n$ denotes the ordered sample. The expected value and variance of $G(X_{i:n})$ are easily obtained from the relation between the Beta and uniform distribution as

$$E(G(X_{i:n})) = \frac{i}{n+1} \; and \; Var(G(X_{i:n})) = \frac{i(n-i+1)}{(n+1)^2(n+2)}.$$

Since $E(G(X_{i:n})) = \frac{i}{n+1}$, $i = 1, 2, ..., n$, a regression model can be written as follows:

$$G(X_{i:n}) = \frac{i}{n+1} + \varepsilon_i, i = 1, 2, ..., n.$$

Then the LSEs of the unknown parameters can be obtained by minimizing the sum of squares of errors

$$\sum_{i=1}^{n}(G(X_{i:n}) - \frac{i}{n+1})^2 \tag{8}$$

with respect to unknown parameters. Therefore, the LSEs of the unknown parameters of Weibull distribution are found by minimizing

$$\sum_{i=1}^{n}(1 - \exp(-(x_{i:n}/\beta)^{\alpha}))^2 \tag{9}$$

with respect to α and β . Since the variances of errors depend on i, the heteroscedasticity problem arises. This problem adversely affects the performance of the estimators. To overcome this problem, we use the method of weighted least squares. The weighted least squares estimators of the unknown parameters can be obtained by minimizing

$$\sum_{i=1}^{n}W_i(G(X_{i:n}) - \frac{i}{n+1})^2 \tag{10}$$

with respect to the unknown parameters. Therefore, the WLSEs of the unknown parameters of the two-parameter Weibull distribution are obtained by minimizing

$$\sum_{i=1}^{n}w_i(1 - \exp\{-(x_{i:n}/\beta)^{\alpha}\})^2 \tag{11}$$

with respect to α and β. Where $W_i = \frac{(n+1)^2(n+2)}{i(n-i+1)}$.

3.4 The Percentile Estimators. The Percentile estimators (PEs) of α and β are obtained by minimizing the function given below:

$$\sum_{i=1}^{n}\left\{X_{i:n} - F^{-1}(\frac{i}{n+1})\right\}^2 \tag{12}$$

with respect to unknown parameters [28], [29] . Here, F^{-1} is the inverse distribution function and $X_{i:n}$ is ordered observations i.e. $X_{1:n} < X_{2:n} < ... < X_{n:n}$.

Then the PEs of the shape and scale parameters of the Weibull distribution are obtained by minimizing function

$$\sum_{i=1}^{n}\left(X_{i:n} - \beta ln(\frac{n+1}{n+1-i})^{\frac{1}{\alpha}}\right)^2 \tag{13}$$

with respect to α and β.

3.5 L- Moment Estimators. The L- moment estimators (LMEs) was introduced by Hosking [30]. These estimators have an estimation method based on linear combination of order statistics. The LMEs have lower sample variances and they are more robust outliers in data. In recently, a few authors have studied the L-moment estimator for the Weibull distribution [14]-[31] .

Let $X_1, X_2, ..., X_n$ be a random sample of size n and $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ be the order random variables. Then the population L-moments and sample L-moments are given as follows:

$$L_k = k^{-1} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} E(X_{k-j:k}), k = 1, 2, 3, ..., \tag{14}$$

$$l_k = \frac{1}{k\binom{n}{k}} \sum_{i=1}^{n} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \binom{i-1}{k-j-1} \binom{n-i}{j} X_{i:n}, k = 1, 2, 3, ... \tag{15}$$

respectively. Here, k is the number of the unknown parameters, $E(X_{i:n})$ are the expected values of the order statistics and n is sample size.
By using equations 14, the population L-moments of two-parameter Weibull distribution derived as

$$L_1 = \beta\Gamma(1 + \frac{1}{\alpha}) \ and \ L_2 = \beta\Gamma(1 + \frac{1}{\alpha}) - \frac{\beta\Gamma(1 + \frac{1}{\alpha})}{2^{\frac{1}{\alpha}}}. \tag{16}$$

The idea lying under L moment estimators are the same as in the moment estimators. In other words, on equating the first two population moments to corresponding sample moments, the estimating equations are

$$\beta\Gamma(1 + \frac{1}{\alpha}) = l_1 \ and \ \beta\Gamma(1 + \frac{1}{\alpha}) - \frac{\beta\Gamma(1 + \frac{1}{\alpha})}{2^{\frac{1}{\alpha}}} = l_2. \tag{17}$$

Then the LMEs of the parameters follow from 17 as

$$\hat{\alpha} = \frac{ln2}{ln(\frac{l_1}{l_1 - l_2})} \ and \ \hat{\beta} = \frac{l_1}{\Gamma(1 + \frac{1}{\hat{\alpha}})}, \tag{18}$$

respectively, where, $l_1 = \bar{x}$ and $l_2 = \frac{1}{n(n-1)} \sum (2j - n - 1) X_{j:n}$.

3.6 Trimmed L-Moments Estimators. Elamir and Seheult [32] proposed TL-moments as a robust generalization of L-moments. The TL-moments always exist even if the mean of the distribution does not exist, for example, the TL-moments exist for Cauchy distribution.

Let $X_1, X_2, ..., X_n$ be a random sample of size n and $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ denote the corresponding order statistics. Elamir and Seheult [32] defined the kth the population and sample TL-moments

$$\lambda_k^{(s,t)} = k^{-1} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} E(X_{k+s-j:k+s+t}), k = 1, 2, 3, ..., s, t = 0, 1, 2, ... \tag{19}$$

and

$$l_k^{(s,t)} = \frac{1}{k\binom{n}{k+s+t}} \sum_{j=s}^{n-t} \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \binom{j-1}{k+s-i-1} \binom{n-j}{t+i} X_{j:n} \ k = 1, 2, 3... \tag{20}$$

respectively. It should be noted that TL-moments reduce to the L-moments when $s = t = 0$. In this study, we focus on asymmetric cases where $s = 0, t = 1$. By putting $s = 0$ and $t = 1$ in equations 19 and 20 , we have

$$\lambda_k^{(0,1)} = k^{-1} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} E(X_{k-j:k+1}) \tag{21}$$

and

$$l_k^{(0,1)} = \frac{1}{k\binom{n}{k+t}} \sum_{j=0}^{n-1} \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \binom{j-1}{k-i-1} \binom{n-j}{i+1} X_{j:n}. \tag{22}$$

The population TL-moments of the two-parameter Weibull distribution can be obtained from 21 as

$$\lambda_1^{(0,1)} = \frac{\beta\Gamma(1+\frac{1}{\alpha})}{2^{\frac{1}{\alpha}}} \ and \ \lambda_2^{(0,1)} = \frac{3\beta\Gamma(1+\frac{1}{\alpha})}{2^{\frac{1}{\alpha}}} - \frac{2\beta\Gamma(1+\frac{1}{\alpha})}{3^{\frac{1}{\alpha}}} \tag{23}$$

The TLMEs are obtained by equating the first two sample TL-moments to the corresponding population TL-moments. Hence, the estimating equations are

$$\frac{\beta\Gamma(1+\frac{1}{\alpha})}{2^{\frac{1}{\alpha}}} = l_1^{(0,1)} \ and \ \frac{3\beta\Gamma(1+\frac{1}{\alpha})}{2^{\frac{1}{\alpha}}} - \frac{2\beta\Gamma(1+\frac{1}{\alpha})}{3^{\frac{1}{\alpha}}} = l_2^{(0,1)}. \tag{24}$$

The solutions of these equations are the following TLMEs:

$$\hat{\alpha} = \frac{log(\frac{2}{3})}{log(\frac{3l_1^{(0,1)} - 2l_2^{(0,1)}}{3l_1^{(0,1)}})} \ and \ \hat{\beta} = \frac{2^{1/\alpha} l_1^{(0,1)}}{\Gamma(1+\frac{1}{\hat{\alpha}})}$$

where

$$l_1^{(0,1)} = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} (n-j) X_{j:n} \ and \ l_2^{(0,1)} = \frac{3}{2n(n-1)(n-2)} \sum_{j=1}^{n} (n-j)(3j-n-1)) X_{j:n}.$$

3.7 Modified Maximum Likelihood Estimators-I. Cohen and Whitten [33] recommend modifications of the MLEs for estimating the shape and scale parameters of the Weibull distribution. The MMLE-I of the shape parameter α and scale parameter β , say $\hat{\alpha}_{MMLE-I}$ and $\hat{\beta}_{MMLE-I}$ respectively, of the Weibull distribution is obtained by solving the following equations:

$$\frac{-nX_{1:n}^{\alpha}}{ln(\frac{n}{n+1})} = \sum_{i=1}^{n} X_{i:n}^{\alpha} \ and \ \hat{\beta}_{MMLE-I} = \frac{1}{n} (\sum_{i=1}^{n} X_i^{\hat{\alpha}_{MMLE}})^{\frac{1}{\hat{\alpha}_{MMLE-I}}}. \tag{25}$$

3.8 Modified Maximum Likelihood Estimators-II. We proposed modifications of the MLEs for estimating the unknown parameters of the Weibull distribution. Then, MMLE of the shape parameter α, say $\hat{\alpha}_{MMLE-II}$, is estimated by solving the following equation:

$$\frac{\gamma + \frac{ln \sum_{i=1}^{n} x_i^{\alpha}}{n}}{\alpha} = \frac{\sum_{i=1}^{n} lnx_i}{n}, \tag{26}$$

where $\gamma \cong 0.57722$ is Euler constant.
Here, by inserting $\hat{\alpha}_{MMLE-II}$ instead of $\hat{\alpha}$ into equation 7 , MMLE of the scale parameter β , say $\hat{\beta}_{MMLE-II}$ is obtained as

$$\hat{\beta}_{MMLE-II} = (\frac{1}{n}\sum_{i=1}^{n} x_i^{\hat{\alpha}_{MMLE-II}})^{\frac{1}{\hat{\alpha}_{MMLE-II}}}. \tag{27}$$

4. Bayesian Analysis

In this section, we consider the Bayesian estimation by using Lindley's and Tierney-Kadane's approximations under different loss function for estimating the unknown parameters of Weibull distribution. Bayesian analysis has many applications in statistical theory and analysis[34]. In Bayesian analysis the role of two factors are crucial. These are (i) the choice of the loss function (LF) and (ii) the choice of the prior distribution. For more details about the priors and loss functions, see [35],[36].
In this study, GELF, PLF,WSELF and SELF are considered and described as follows:
The SELF was proposed by Legendre [37] and Gauss [38] to developed least square theory. This loss function is commonly used and defined as

$$L_{SELF} = (\hat{\theta} - \theta)^2, \tag{28}$$

where θ is the parameter to be estimated by an estimator $\hat{\theta}$. The Bayes estimator under equation 28 is the posterior mean given by

$$\hat{\theta}_{SELF} = E(\theta|x). \tag{29}$$

This loss function is symmetrical in nature. It gives equal weight to both underestimation and over estimation. However, from a practical point of view, this is not always appropriate and realistic, see for example [39]. Hence, asymmetric loss functions would be more useful to develop Bayesian procedures.
Calabria and Pulcini [40] proposed general entropy loss function. It is one of the most popular asymmetrical loss functions.
The GELF is given by

$$L_{GELF} = (\frac{\hat{\theta}}{\theta})^k - klog(\frac{\hat{\theta}}{\theta}) - 1, k \neq 0, \tag{30}$$

where $\hat{\theta}$ is the estimator of θ. k reflects the magnitude and degree of symmetry. The Bayes estimator under equation 30 is given by

$$\hat{\theta}_{GELF} = [E(\theta^{-k}|x)]^{-\frac{1}{k}}, \tag{31}$$

provided $E_\theta(\theta^{-k}|x)$.
The PLF, which is proposed by Norstrom [41] , is one of the asymmetric loss functions. This loss function approach is useful to derive conservative estimators since it approaches infinity near the origin and prevents underestimation. It is

very useful when underestimation may lead to significant results [42]. The PLF is defined as

$$L_{PLF} = \frac{(\theta - \hat{\theta})^2}{\hat{\theta}}, \tag{32}$$

where $\hat{\theta}$ is the estimator of θ. The Bayes estimator of under equation 32 is given by

$$\hat{\theta}_{PLF} = \sqrt{E(\theta^2|x)}, \tag{33}$$

provided $\sqrt{E(\theta^2|x)}$ exists and is finite.
WSELF is another useful asymmetric loss function. This function is a weighted version of SELF. More detail about this loss function can be found in [35]and [43]. The WSELF is defined as:

$$L_{WSELF}(\hat{\theta}, \theta) = \frac{(\theta - \hat{\theta})^2}{\hat{\theta}}. \tag{34}$$

The Bayes estimator under WSELF is given by

$$\hat{\theta}_{WSELF} = [E(\theta^{-1}|x)]^{-1}. \tag{35}$$

provided $E(\theta^{-1}|x)^{-1}$ exists and is finite.
The prior distribution summarizes the information about unknown parameter before the data is available. The prior distribution is then synthesized with the information in the data procedure the posterior distribution. In other words, analytically, combining the prior distribution and likelihood function results in the posterior distribution. The posterior distribution expresses what is known after seeing data. In the Bayesian analysis, all inferences are made from the posterior distribution [44].
The prior distribution has two forms: these are (i) "non-informative prior" and (ii) "informative prior" [45].
Here we assume that α and β have two independent gamma prior distributions i.e. $\alpha \sim gamma(a, b)$ and $\beta \sim gamma(c, d)$ respectively. The gamma prior is very flexible and suitable. Thus, this paper considers two special cases of the gamma prior corresponding to $a = b = c = d = 0$ and $a, b, c, d \geq 0$ (a, b, c, d are the hyper-parameters of the prior distribution). It should be mentioned that for $a = b = c = d = 0$ the prior distribution is non-informative prior (NP) distribution. For $a, b, c, d \geq 0$, the prior distribution is referred to as the gamma prior (GP) distribution. Thus, the proposed prior for α and β may be considered as

$$v_1(\alpha) \propto \alpha^{a-1} e^{-b\alpha}, \alpha > 0 \text{ and } v_2(\beta) \propto \beta^{c-1} e^{-d\beta}, \beta > 0. \tag{36}$$

The joint prior distribution α and β is given as

$$v(\alpha, \beta) \propto \alpha^{a-1} \beta^{c-1} e^{-d\beta - b\alpha},\ \alpha, \beta, a, b, c, d \geq 0. \tag{37}$$

Based on the observations, the likelihood function becomes

$$L(\alpha,\beta) = \alpha^n \beta^{-n\alpha} \prod_{i=1}^{n} X_i^{(\alpha-1)} e^{-\sum(\frac{X_i}{\beta})^\alpha}. \tag{38}$$

Combining 37 with 38 and using Bayes theorem, the joint posterior density of α and β is

$$p(\alpha,\beta|x) = K^{-1}\alpha^{n+a-1}\beta^{-n\alpha+c-1} exp(-d\beta - b\alpha) \prod_{i=1}^{n} x_i^{\alpha-1} e^{\sum_{i=1}^{n}(\frac{x_i}{\beta})^\alpha}. \tag{39}$$

Here $K = \int_0^\infty \int_0^\infty \alpha^{n+a-1}\beta^{-n\alpha+c-1} exp(-d\beta - b\alpha) \prod_{i=1}^{n} x_i^{\alpha-1} e^{\sum_{i=1}^{n}(\frac{x_i}{\beta})^\alpha} d\alpha d\beta$.
It can be seen that the analytical solution of the Bayes estimators are not obtained. Hence, we use the Lindley's and Tierney-Kadane's approximation. These methods are described below.

4.1 Lindley's procedure. Lindley's [46] introduced an approximation method for the evaluation of the ratio of the two integrals. This procedure can be applied to compute the posterior expectation of the arbitrary function $u(\theta)$ as given by

$$E(u(\theta)|x) = \frac{\int u(\theta) e^{L(\theta)+G(\theta)} d\theta}{\int e^{L(\theta)+G(\theta)} d\theta},$$

where
$u(\theta)$ = a function of θ only,
$L(\theta)$ = Log-likelihood function,
$G(\theta)$ = Log of joint prior density function.
According to Lindley's approximation, the ratio of integral $E\{u(\theta)|x\}$ can be approximated asymptotically given below:

$$E(u(\theta)|x) \approx [u + \frac{1}{2}\sum_i\sum_j (u_{ij} + 2u_i\rho_i)\sigma_{ij} + \frac{1}{2}\sum_i\sum_j\sum_k\sum_l L_{ijk}\sigma_{ij}\sigma_{kl}u_l] + O(1/n^2). \tag{40}$$

Here, $i;j;k;l = 1,2,...,n; \theta = (\theta_1,\theta_2,...,\theta_m)$, $u_i = \frac{\partial u(\theta)}{\partial\theta_i}$, $u_{ij} = \frac{\partial u(\theta)}{\partial\theta_i\partial\theta_j}$, $L_{ijk} = \frac{\partial L(\theta)}{\partial\theta_i\partial\theta_j\partial\theta_k}$, $\rho_j = \frac{\partial G(\theta)}{\partial\theta_j}$ and σ_{ij} are elements of the covariance matrix.
For the two-parameter Weibull distribution, equation 40 reduces to

$$\begin{aligned} E(u(\alpha,\beta)|x) = u &+ \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22}) + u_{12}\sigma_{12} + u_1(\sigma_{11}\rho_1 + \sigma_{21}\rho_2) + u_2(\sigma_{12}\rho_1 + \sigma_{22}\rho_2) \\ &+ 0.5[L_{111}(u_1\sigma_{11}^2 + u_2\sigma_{11}\sigma_{12}) + L_{112}(3u_1\sigma_{11}\sigma_{12} + u_2(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2)) \\ &+ L_{122}(u_1(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2) + 3u_2\sigma_{12}\sigma_{22}) + L_{222}(u_1\sigma_{12}\sigma_{22} + u_2\sigma_{22}^2)]_{\hat{\alpha},\hat{\beta},}. \end{aligned} \tag{41}$$

Here, the $\hat{\alpha}$ and $\hat{\beta}$ are the MLEs of α and β, respectively.
All other quantities appearing in the above expression of $E(u(\alpha,\beta)|x)$ for Weibull

distribution is given by

$$\hat{\rho}_\alpha = \frac{a-1}{\hat{\alpha}} - b, \hat{\rho}_\beta = \frac{c-1}{\hat{\beta}} - d,$$

$$\hat{L}_{111} = \frac{2n}{\hat{\alpha}^3} - \sum_{i=1}^{n}((ln(\frac{x_i}{\hat{\beta}}))^3(\frac{x_i}{\hat{\beta}})^{\hat{\alpha}}),$$

$$\hat{L}_{112} = \sum_{i=1}^{n} log(\frac{x_i}{\hat{\beta}})(\frac{x_i}{\hat{\beta}})^{\hat{\alpha}}[\frac{1}{\hat{\beta}}(2+\hat{\alpha}.log(\frac{x_i}{\hat{\beta}}))],$$

$$\hat{L}_{122} = \sum_{i=1}^{n}((\frac{x_i}{\hat{\beta}})^{\hat{\alpha}}(\frac{1}{\hat{\beta}^2}(\hat{\alpha}+1)(\hat{\alpha}+ln(\frac{x_i}{\hat{\beta}})+1)+\hat{\alpha})) + \frac{n}{\hat{\beta}}^2$$

$$\hat{L}_{222} = \frac{-2n\hat{\alpha}}{\hat{\beta}^3} + \frac{\hat{\alpha}(\hat{\alpha}+1)(\hat{\alpha}+2)}{\hat{\beta}^3}\sum_{i=1}^{n}(\frac{x_i}{\hat{\beta}})^{\hat{\alpha}},$$

and

$$\sigma_{ij} = \begin{bmatrix} Var\{\hat{\alpha}\} & Cov\{\hat{\alpha},\hat{\beta}\} \\ Cov\{\hat{\alpha},\hat{\beta}\} & Var\{\hat{\beta}\} \end{bmatrix} = \frac{1}{n}\begin{bmatrix} 0.6080\alpha^2 & 0.2570\beta \\ 0.2570\beta & 1.1087\frac{\beta^2}{\alpha^2} \end{bmatrix}.$$

All constant are evaluated at $(\hat{\alpha}, \hat{\beta})$.
Then, by using Lindley's method the Bayesion estimators of the parameter α under SELF is obtained as
If $u(\alpha,\beta) = \alpha, u_1 = 1, u_2 = u_{22} = u_{12} = u_{21} = u_{11} = 0$, then

$$\begin{aligned}\hat{\alpha}_{SELF} =& \hat{\alpha} + (\hat{\sigma}_{11}\hat{\rho}_1 + \hat{\sigma}_{21}\hat{\rho}_2) \\ &+ 0.5[\hat{L}_{111}\sigma_{11}^2 + 3\hat{L}_{112}\hat{\sigma}_{11}\hat{\sigma}_{12} + \hat{L}_{122}(\hat{\sigma}_{11}\hat{\sigma}_{22} + 2\hat{\sigma}_{12}^2) + \hat{L}_{222}\hat{\sigma}_{12}\hat{\sigma}_{22}].\end{aligned}$$

So, the Bayes estimator of β under SELF is obtained as,
If $u(\alpha,\beta) = \beta$, $u_2 = 1, u_{22} = u_{12} = u_{21} = u_{11} = 0$, then

$$\begin{aligned}\hat{\beta}_{SELF} =& \hat{\beta} + (\hat{\sigma}_{12}\hat{\rho}_1 + \hat{\sigma}_{22}\hat{\rho}_2) \\ &+ 0.5[\hat{L}_{111}\sigma_{11}\sigma_{12} + \hat{L}_{112}(\hat{\sigma}_{11}\hat{\sigma}_{22} + 2\hat{\sigma}_{12}^2) + 3\hat{L}_{122}\hat{\sigma}_{12}\hat{\sigma}_{22} + \hat{L}_{222}\hat{\sigma}_{22}^2].\end{aligned}$$

Bayes estimator of α under the GELF is defined as
If $u(\alpha,\beta) = \alpha^{-k}, u_1 = -k\alpha^{-(k+1)}, u_{11} = k(k+1)\alpha^{-(k+2)}, u_2 = u_{22} = u_{12} = u_{21} = 0$, then

$$\begin{aligned}&E(\alpha^{-k}|x) = \hat{\alpha}^{-k} + 0.5(\hat{u}_{11}\hat{\sigma}_{11}) + \hat{u}_1(\hat{\sigma}_{11}\hat{\rho}_1 + \hat{\sigma}_{21}\hat{\rho}_2) + \\ &0.5[\hat{L}_{111}\hat{u}_1\hat{\sigma}_{11}^2 + 3\hat{L}_{112}\hat{u}_1\hat{\sigma}_{11}\hat{\sigma}_{12} + \hat{L}_{122}\hat{u}_1(\hat{\sigma}_{11}\hat{\sigma}_{22} + 2\hat{\sigma}_{12}^2) + \hat{L}_{222}\hat{u}_1\hat{\sigma}_{12}\hat{\sigma}_{22}].\end{aligned}$$

Therefore, $\hat{\alpha}_{GELF} = E[\alpha^{-k}|x]^{-1/k}$.
Bayes estimator of β under the general entropy loss function is given by
If $u(\alpha,\beta) = \beta^{-k}, u_2 = -k\beta^{-(k+1)}$, $u_{22} = k(k+1)\beta^{-(k+2)}$, $u_1 = u_{11} = u_{12} = u_{21} = 0$, then

$$E(\beta^{-k}|x) = \hat{\beta}^{-k} + 0.5(\hat{u}_{22}\hat{\sigma}_{22}) + \hat{u}_2(\hat{\sigma}_{12}\hat{\rho}_1 + \hat{\sigma}_{22}\hat{\rho}_2) +$$

$$0.5[\hat{L}_{111}\hat{u}_2\hat{\sigma}_{11}\hat{\sigma}_{12} + \hat{L}_{112}\hat{u}_2(\hat{\sigma}_{11}\hat{\sigma}_{22} + 2\sigma_{12}^2) + 3\hat{L}_{122}\hat{\sigma}_{12}\hat{\sigma}_{22} + \hat{L}_{222}\hat{u}_2\hat{\sigma}_{22}^2].$$

Hence, $\hat{\beta}_{GELF} = E[\beta^{-k}|x]^{-1/k}$.
Bayes estimator of α under the WSELF is as follows
If $u(\alpha,\beta) = \alpha^{-1}, u_1 = -\alpha^{-2}, u_{11} = 2\alpha^{-3}, u_2 = u_{22} = u_{12} = 0$, then

$$E(\alpha^{-1}|x) = \hat{\alpha}^{-1} + 0.5(\hat{u}_{11}\hat{\sigma}_{11}) + \hat{u}_1(\hat{\sigma}_{11}\hat{\rho}_1 + \hat{\sigma}_{21}\hat{\rho}_2) +$$
$$0.5[\hat{L}_{111}\hat{u}_1\hat{\sigma}_{11}^2 + 3\hat{L}_{112}\hat{u}_1(\hat{\sigma}_{11}\hat{\sigma}_{12} + \hat{L}_{122}\hat{u}_1(\hat{\sigma}_{11}\hat{\sigma}_{22} + 2\hat{\sigma}_{12}^2) + \hat{L}_{222}\hat{u}_1\hat{\sigma}_{12}\hat{\sigma}_{22}].$$

So, $\hat{\alpha}_{WSELF} = [E(\alpha^{-1}|x)]^{-1}$.
The Bayes estimator of β under the WSELF is given in following form
If $u(\alpha,\beta) = \beta^{-1}, u_2 = -\beta^{-2}, u_{22} = 2\beta^{-3}, u_1 = u_{11} = u_{12} = 0$, then

$$E(\beta^{-1}|x) = \hat{\beta}^{-1} + 0.5(\hat{u}_{22}\hat{\sigma}_{22}) + \hat{u}_2(\hat{\sigma}_{12}\hat{\rho}_1 + \hat{\sigma}_{22}\hat{\rho}_2) +$$
$$0.5[\hat{L}_{111}\hat{u}_2\hat{\sigma}_{11}\hat{\sigma}_{12} + \hat{L}_{112}\hat{u}_2(\hat{\sigma}_{11}\hat{\sigma}_{22} + 2\hat{\sigma}_{12}^2) + 3\hat{L}_{122}\hat{\sigma}_{12}\hat{\sigma}_{22} + \hat{L}_{222}\hat{u}_2\hat{\sigma}_{22}^2].$$

So, the Bayes estimator of β is $\hat{\beta}_{WSELF} = [E(\beta^{-1}|x)]^{-1}$.
Finally, the Bayes estimator of α under PLF is
If $u(\alpha,\beta) = \alpha^2, u_1 = 2\alpha, u_{11} = 2, u_2 = u_{22} = u_{12} = 0$, then

$$E(\alpha^2|x) = \hat{\alpha}^2 + 0.5(\hat{u}_{11}\hat{\sigma}_{11}) + \hat{u}_1(\hat{\sigma}_{11}\hat{\rho}_1 + \hat{\sigma}_{21}\hat{\rho}_2) +$$
$$0.5[\hat{L}_{111}\hat{u}_1\hat{\sigma}_{11}^2 + 3\hat{L}_{112}\hat{u}_1\hat{\sigma}_{11}\hat{\sigma}_{12} + \hat{L}_{122}\hat{u}_1(\hat{\sigma}_{11}\hat{\sigma}_{22} + 2\hat{\sigma}_{12}^2) + \hat{L}_{222}\hat{u}_1\hat{\sigma}_{12}\hat{\sigma}_{22}].$$

Hence,the Bayes estimator of α is as follows
$\hat{\alpha}_{PLF} = \sqrt{E(\alpha^2|x)}$.
Bayes estimator of β under PLF is given by
If $u(\alpha,\beta) = \beta^2, u_2 = 2\beta, u_{22} = 2, u_1 = u_{11} = u_{12} = 0$, then

$$E(\beta|x) = \hat{\beta} + 0.5(\hat{u}_{22}\hat{\sigma}_{22}) + \hat{u}_2(\hat{\sigma}_{12}\hat{\rho}_1 + \hat{\sigma}_{22}\hat{\rho}_2) +$$
$$0.5[\hat{L}_{111}\hat{u}_2\hat{\sigma}_{11}\hat{\sigma}_{12} + \hat{L}_{112}\hat{u}_2(\hat{\sigma}_{11}\hat{\sigma}_{22} + 2\hat{\sigma}_{12}^2) + 3\hat{L}_{122}\hat{\sigma}_{12}\hat{\sigma}_{22} + \hat{L}_{222}\hat{u}_2\hat{\sigma}_{22}^2].$$

So, $\hat{\beta}_{PLF} = \sqrt{E(\beta^2|x)}$.

4.2 Tierney Kadane's Procedure. Lindley's procedure seems to be become more and more complex in p- parameter case ($p > 2$). Therefore, in multi-parameter case, Tierney Kadane's (T-K) procedure is used as an alternative to Lindley's procedure [47],[48].
According to this procedure, posterior expectation for multi-parameter case can be approximated by:

$$E(u(\theta)|x) = \sqrt{\frac{|\Sigma^*|}{|\Sigma|}}exp[n(L_1^*(\hat{\theta}^*) - L_1(\hat{\theta}))]. \tag{42}$$

Here, $\hat{\theta}^*$ and $\hat{\theta}$ maximize L_1^* and L_1,respectively,

$$L_1 = \frac{[L(\theta) + log(v(\theta))]}{n}, L_1^* = L_1 + \frac{[log(u(\theta))]}{n},$$

where
$v(\theta) =$ joint prior distribution of θ,
$L(\theta) =$ Log-likelihood function of θ,
$u(\theta) =$ loss function of θ.
In equation 42, $\sum^*$ and $\sum$ are elements of the negative of the inverse of the matrices of the second derivatives of L_1^* and L_1 at the point $\hat{\theta}^*$ and θ, respectively.
For the two parameter case, $\theta = (\alpha, \beta)$, equation 42 becomes:

$$E(u(\alpha,\beta)|x) = \sqrt{\frac{|\sum^*|}{|\sum|}} exp[n(L_1^*(\hat{\alpha}^*, \hat{\beta}^*) - L_1(\hat{\alpha}, \hat{\beta}))]. \tag{43}$$

Here, $(\hat{\beta}, \hat{\alpha})$ and $(\hat{\beta}^*, \hat{\alpha}^*)$ maximize $L_1(\alpha, \beta)$ and $L_1^*(\alpha, \beta)$, respectively. $\sum$ and $\sum^*$ are given below:

$$\sum^* = \begin{bmatrix} -\frac{\partial^2 L_1^*}{\partial\alpha^2} & -\frac{\partial^2 L_1^*}{\partial\alpha\partial\beta} \\ -\frac{\partial^2 L_1^*}{\partial\alpha\partial\beta} & -\frac{\partial^2 L_1^*}{\partial\beta} \end{bmatrix}^{-1}_{(\hat{\alpha}^*, \hat{\beta}^*)} \quad and \quad \sum = \begin{bmatrix} -\frac{\partial^2 L_1}{\partial\alpha^2} & -\frac{\partial^2 L_1}{\partial\alpha\partial\beta} \\ -\frac{\partial^2 L_1}{\partial\alpha\partial\alpha} & -\frac{\partial^2 L_1}{\partial\beta^2} \end{bmatrix}^{-1}_{(\hat{\alpha}, \hat{\beta})}. \tag{44}$$

All other quantities appearing in the above expression of $E(u(\alpha, \beta|x))$ for Weibull distribution can be obtained as

$$L_1(\alpha,\beta) = \frac{1}{n}[nln\alpha - n\alpha ln\beta + (\alpha-1)\sum_{i=1}^{n} x_i - \sum_{i=1}^{n}(\frac{x_i}{\beta})^\alpha + (\alpha-1)ln\alpha + (c-1)ln\beta - (b\alpha + d\beta)]. \tag{45}$$

Thus the Bayes estimator of α under SELF is given in the following form:
If $u(\alpha, \beta) = \alpha$ and $L_1^* = \frac{1}{n} log\alpha + L_1(\alpha, \beta)$, then

$$\hat{\alpha}_{SELF} = [\sqrt{\frac{|\sum^*|}{|\sum|}} exp[n(\frac{1}{n} log\hat{\alpha} + L_1(\hat{\alpha}^*, \hat{\beta}^*) - L_1(\hat{\alpha}, \hat{\beta}))]].$$

Also, the Bayes estimator of β under SELF using this procedure is defined as:
If $u(\alpha, \beta) = \beta$ and $L_1^* = \frac{1}{n} log\beta + L_1(\alpha, \beta)$, then

$$\hat{\beta}_{SELF} = [\sqrt{\frac{|\sum^*|}{|\sum|}} exp[n(\frac{1}{n} log\hat{\beta} + L_1(\hat{\alpha}^*, \hat{\beta}^*) - L_1(\hat{\alpha}, \hat{\beta}))]].$$

Bayes estimator of α under GELF is given by:
If $u(\alpha, \beta) = \alpha^{-k}$ and $L_1^* = \frac{1}{n} log(\alpha^{-k}) + L_1(\alpha, \beta)$, then

$$\hat{\alpha}_{GELF} = [\sqrt{\frac{|\sum^*|}{|\sum|}} exp[n(\frac{1}{n} log(\hat{\alpha}^{-k}) + L_1^*(\hat{\alpha}^*, \hat{\beta}^*) - L_1(\hat{\alpha}, \hat{\beta}))]]^{-1/k}.$$

Bayes estimator of β under GELF is given by:
If $u(\alpha,\beta)=\beta^{-k}$ and $L_1^*=\frac{1}{n}log(\beta^{-k})+L_1(\alpha,\beta)$, then

$$\hat{\beta}_{GELF}=[\sqrt{\frac{|\sum^*|}{|\sum|}}exp[n(\frac{1}{n}log(\hat{\beta}^{-k})+L_1^*(\hat{\alpha}^*,\hat{\beta}^*)-L_1(\hat{\alpha},\hat{\beta}))]]^{-1/k}.$$

Bayes estimator of α under WSELF is as follows:
If $u(\alpha,\beta)=\alpha^{-1}$ and $L_1^*=\frac{1}{n}log(\alpha^{-1})+L_1(\alpha,\beta)$, then

$$\hat{\alpha}_{WSELF}=[\sqrt{\frac{|\sum^*|}{|\sum|}}exp[n(\frac{1}{n}log(\hat{\alpha^{-1}})+L_1^*(\hat{\alpha}^*,\hat{\beta}^*)-L_1(\hat{\alpha},\hat{\beta}))]]^{-1}.$$

Bayes estimator of β under WSELF is as follows:
If $u(\alpha,\beta)=\beta^{-1}$ and $L_1^*=\frac{1}{n}log(\beta)^{-1}+L_1(\alpha,\beta)$, then

$$\hat{\beta}_{WSELF}=[\sqrt{\frac{|\sum^*|}{|\sum|}}exp[n(\frac{1}{n}log\hat{(}\beta^{-1})+L_1^*(\hat{\alpha}^*,\hat{\beta}^*)-L_1(\hat{\alpha},\hat{\beta}))]]^{-1}.$$

Bayes estimator of α under PLF is
If $u(\alpha,\beta)=\alpha^2$ and $L_1^*=\frac{1}{n}log(\alpha)^2+L_1(\alpha,\beta)$, then

$$\hat{\alpha}_{PLF}=\sqrt{\sqrt{\frac{|\sum^*|}{|\sum|}}exp[n(\frac{1}{n}log(\hat{\alpha}^2)+L_1^*(\hat{\alpha}^*,\hat{\beta}^*)-L_1(\hat{\alpha},\hat{\beta}))]}.$$

Bayes estimator of β under PLF is
If $u(\alpha,\beta)=\beta^2$ and $L_1^*=\frac{1}{n}log(\beta)^2+L_1(\alpha,\beta)$, then

$$\hat{\beta}_{PLF}=\sqrt{\sqrt{\frac{|\sum^*|}{|\sum|}}exp[n(\frac{1}{n}log(\hat{\beta}^2)+L_1^*(\hat{\alpha}^*,\hat{\beta}^*)-L_1(\hat{\alpha},\hat{\beta}))]}.$$

5. Simulation study

In this section, an extensive Monte Carlo simulation study was carried out to compare the performances of the Bayesian and classical estimators with respect to the biases and mean squared errors (MSEs) for different sample sizes and parameter values. All The computations were performed in Matlab R. 2013. over 10.000 replications for different cases. We consider the sample sizes $n=10(10)100$, the shape parameter values $\alpha=0.5,1.5$ and the scale parameter β was taken to be 1 throughout the study. The bias and MSE values of the classical estimators are given in Table 1.

For Bayesian estimators, we know that the Gamma prior provides flexible approach in both informative and non-informative cases [48]. In case of the non-informative prior (NP), we chose hyper-parameter values as $a=b=c=d=0$. In case of the GP, we chose hyper-parameter values as $a=0.4,1,1.5,3$, $b=0.2,1$, $c=0.4,1,1.5,3$ and $d=0.2,1$. In both cases i.e. informative and non-informative, we considered as $k = \pm 1.5$ for GELF. Because of the large number of tables and results, only results for $a = c = 0.4$, $b = d = 0.2$ and $k = 1.5$ were reported. Moreover, Lindley's and T-K methods were used to obtain the Bayes estimator of the unknown parameters. The results of simulation for these approximation methods were summarized in Table 2-3.

Table 1. The simulated, means and MSEs values for the classical different parameter estimators of α and β

			$\hat{\alpha}$		$\hat{\beta}$	
n	α	Estimator	Mean	MSE	Mean	MSE
		MLE	0.5382	0.0121	1.0704	0.2607
		LME	0.5290	0.0152	0.9837	0.2598
		TLME	0.5079	0.0138	0.9579	0.2395
		MMLE-II	0.5330	0.0123	1.0687	0.2610
20	0.5	MMLE-I	0.4739	0.0220	0.9874	0.2616
		ME	0.6511	0.0403	1.3510	0.5322
		LSE	0.5008	0.0147	1.1271	0.3244
		WLSE	0.5052	0.0131	1.1114	0.2973
		PE	0.4887	0.0336	1.1229	0.2395
		MLE	0.5243	0.0068	1.0544	0.1694
		LME	0.5210	0.0101	0.9960	0.1752
		TLME	0.5039	0.0081	0.9807	0.1622
		MMLE-II	0.5226	0.0069	1.0541	0.1700
30	0.5	MMLE-I	0.4702	0.0172	0.9816	0.1779
		ME	0.6151	0.0263	1.2863	0.3425
		LSE	0.4984	0.0087	1.0932	0.2037
		WLSE	0.5034	0.0076	1.0795	0.1878
		PE	0.4800	0.0276	1.0577	0.2268
		MLE	0.5147	0.0037	1.0276	0.0951
		LME	0.5125	0.0059	0.9891	0.1019
		TLME	0.5032	0.0045	0.9829	0.0945
		MMLE-II	0.5139	0.0038	1.0280	0.0959
50	0.5	MMLE-I	0.4689	0.0139	0.9654	0.1154
		ME	0.5782	0.0149	1.1955	0.1874
		LSE	0.5000	0.0048	1.0490	0.1094
		WLSE	0.5039	0.0041	1.0401	0.1014
		PE	0.4664	0.0210	0.9622	0.3263

Table 1. Continued

			$\hat{\alpha}$		$\hat{\beta}$	
n	α	Estimator	Mean	MSE	Mean	MSE
		MLE	0.5068	0.0016	1.0112	0.0458
		LME	0.5068	0.0029	0.9931	0.0506
		TLME	0.5012	0.0022	0.9894	0.0472
		MMLE-II	0.5063	0.0017	1.0111	0.0461
100	0.5	MMLE-I	0.4697	0.0106	0.9586	0.0661
		ME	0.5487	0.0076	1.1264	0.0959
		LSE	0.4994	0.0023	1.0223	0.0516
		WLSE	0.5020	0.0019	1.0170	0.0481
		PE	0.4653	0.0142	0.9208	0.1755
		MLE	1.6090	0.1041	1.0010	0.0245
		LME	1.5292	0.0878	0.9943	0.0247
		TLME	1.5333	0.1304	0.9968	0.0270
		MMLE-II	1.5989	0.1055	1.0008	0.0246
20	1.5	MMLE-I	1.4225	0.1959	0.9712	0.0281
		ME	1.6194	0.1051	1.0016	0.0245
		LSE	1.4949	0.1262	1.0165	0.0276
		WLSE	1.5087	0.1116	1.0129	0.0262
		PE	1.4362	0.0968	1.0172	0.0734
		MLE	1.5740	0.0625	0.9999	0.0166
		LME	1.5225	0.0563	0.9956	0.0167
		TLME	1.5218	0.0793	0.9977	0.0179
		MMLE-II	1.5689	0.0634	0.9998	0.0167
30	1.5	MMLE-I	1.4112	0.1556	0.9732	0.0202
		ME	1.5826	0.0651	1.0003	0.0167
		LSE	1.4981	0.0787	1.0104	0.0183
		WLSE	1.5125	0.0683	1.0071	0.0174
		PE	1.4471	0.0667	1.0107	0.0021
		MLE	1.5438	0.0331	0.9995	0.0099
		LME	1.5137	0.0319	0.9969	0.0100
		TLME	1.5133	0.0445	0.9982	0.0108
		MMLE-II	1.5413	0.0344	0.9994	0.0100
50	1.5	MMLE-I	1.4103	0.1258	0.9763	0.0133
		ME	1.5496	0.0353	0.9997	0.0100
		LSE	1.4989	0.0450	1.0058	0.0110
		WLSE	1.5107	0.0381	1.0034	0.0104
		PE	1.4549	0.0398	1.0057	0.1723
		MLE	1.5213	0.0151	1.0000	0.0050
		LME	1.5065	0.0152	0.9988	0.0050
		TLME	1.5067	0.0206	0.9993	0.0054
		MMLE-II	1.5199	0.0158	1.0000	0.0050
100	1.5	MMLE-I	1.4063	0.0976	0.9790	0.0081
		ME	1.5244	0.0165	1.0001	0.0050
		LSE	1.4998	0.0213	1.003	0.0055
		WLSE	1.5075	0.0175	1.0017	0.0052
		PE	1.4658	0.0205	1.0025	0.0735

Table 2.The simulated, means and MSEs values under different loss function for the Lindley approximation of α and β

			Lindley's approximation							
			$\hat{\alpha}$				$\hat{\beta}$			
			NP		GP		NP		GP	
n	α	LF	Mean	MSE	Mean	MSE	Mean	MSE	Mean	MSE
20	0.5	SELF	0.5178	0.0102	0.5240	0.0100	1.1895	0.3391	1.2313	0.3487
		GELF	0.5008	0.0092	0.5060	0.0089	0.9441	0.2152	0.9746	0.2060
		WSELF	0.5038	0.0094	0.5092	0.0090	0.9825	0.2258	1.0182	0.2177
		PLF	0.5258	0.0108	0.5321	0.0108	1.2758	0.4192	1.3101	0.4347
30	0.5	SELF	0.5110	0.0061	0.5154	0.0062	1.1349	0.2053	1.1635	0.2104
		GELF	0.4993	0.0057	0.5032	0.0057	0.9640	0.1468	0.9868	0.1429
		WSELF	0.5015	0.0057	0.5054	0.0058	0.9930	0.1524	1.0186	0.1494
		PLF	0.5163	0.0063	0.5207	0.0065	1.1976	0.2441	1.2220	0.2518
50	0.5	SELF	0.5068	0.0034	0.5097	0.0035	1.0759	0.1071	1.0974	0.1128
		GELF	0.4996	0.0033	0.5023	0.0033	0.9709	0.0875	0.9897	0.0880
		WSELF	0.5009	0.0033	0.5037	0.0033	0.9899	0.0894	1.0099	0.0906
		PLF	0.5099	0.0035	0.5129	0.0036	1.1157	0.1209	1.1356	0.1284
100	0.5	SELF	0.5029	0.0016	0.5031	0.0016	1.0354	0.0487	1.0418	0.0507
		GELF	0.4992	0.0015	0.4994	0.0016	0.9816	0.0441	0.9876	0.0453
		WSELF	0.4999	0.0015	0.5002	0.0016	0.9918	0.0445	0.9979	0.0459
		PLF	0.5045	0.0016	0.5047	0.0016	1.0564	0.0521	1.0628	0.0545
20	1.5	SELF	1.5480	0.0877	1.5585	0.0894	1.0161	0.0250	1.0177	0.0250
		GELF	1.4974	0.0799	1.5069	0.0807	0.9874	0.0245	0.9889	0.0243
		WSELF	1.5061	0.0809	1.5158	0.0818	0.9930	0.0245	0.9946	0.0243
		PLF	1.5720	0.0932	1.5828	0.0955	1.0274	0.0258	1.0287	0.0258
30	1.5	SELF	1.5340	0.0553	1.5387	0.0548	1.0102	0.0169	1.0122	0.0163
		GELF	1.4989	0.0517	1.5031	0.0509	0.9906	0.0166	0.9927	0.0160
		WSELF	1.5053	0.0521	1.5096	0.0514	0.9945	0.0166	0.9965	0.0160
		PLF	1.5498	0.0577	1.5545	0.0574	1.0179	0.0172	1.0199	0.0166
50	1.5	SELF	1.5202	0.0306	1.5229	0.0300	1.0057	0.0100	1.0090	0.0100
		GELF	1.4984	0.0294	1.5010	0.0287	0.9938	0.0099	0.9970	0.0098
		WSELF	1.5025	0.0295	1.5051	0.0288	0.9961	0.0099	0.9994	0.0098
		PLF	1.5295	0.0315	1.5322	0.0309	1.0104	0.0101	1.0137	0.0101
100	1.5	SELF	1.5096	0.0145	1.5119	0.0150	1.0032	0.0050	1.0037	0.0049
		GELF	1.4985	0.0142	1.5008	0.0146	0.9971	0.0050	0.9976	0.0049
		WSELF	1.5006	0.0142	1.5029	0.0147	0.9983	0.0050	0.9988	0.0049
		PLF	1.5142	0.0147	1.5166	0.0152	1.0056	0.0051	1.0061	0.0049

In all cases, the biases and MSEs of the estimators decrease as the sample size n increases. It indicates that all the estimators are asymptotically unbiased and

Table 3.The simulated, means and MSEs values under different loss function for Tierney Kadaneâ€$^{\text{TM}}$s approximation parameter estimators of α and β

			Tierney-Kadane's approximation							
			$\hat{\alpha}$				$\hat{\beta}$			
			NP		GP		NP		GP	
n	α	LF	Mean	MSE	Mean	MSE	Mean	MSE	Mean	MSE
20	0.5	SELF	0.5288	0.0107	0.5348	0.0107	1.2014	0.3498	1.2423	0.3448
		GELF	0.5081	0.0091	0.5143	0.0089	0.9001	0.3049	0.9501	0.1962
		WSELF	0.5123	0.0094	0.5185	0.0092	0.9549	0.2193	1.0046	0.2093
		PLF	0.5369	0.0115	0.5428	0.0116	1.3497	0.4496	1.3823	0.4838
30	0.5	SELF	0.5175	0.0063	0.5218	0.0065	1.1420	0.2066	1.1656	0.2083
		GELF	0.5040	0.0057	0.5084	0.0058	0.9430	0.1433	0.9714	0.1393
		WSELF	0.5068	0.0058	0.5111	0.0059	0.9809	0.1485	1.0084	0.1460
		PLF	0.5229	0.0066	0.5271	0.0069	1.2317	0.2645	1.2527	0.2662
50	0.5	SELF	0.5103	0.0034	0.5132	0.0036	1.0765	0.1096	1.0982	0.1124
		GELF	0.5024	0.0032	0.5053	0.0033	0.9609	0.0888	0.9836	0.0871
		WSELF	0.5040	0.0033	0.5069	0.0034	0.9833	0.0906	1.0058	0.0898
		PLF	0.5134	0.0036	0.5163	0.0037	1.1258	0.1273	1.1470	0.1318
100	0.5	SELF	0.5045	0.0016	0.5047	0.0016	1.0534	0.0551	1.0486	0.0505
		GELF	0.5006	0.0015	0.5008	0.0016	0.9966	0.0481	0.9923	0.0443
		WSELF	0.5014	0.0016	0.5016	0.0016	1.0078	0.0490	1.0034	0.0450
		PLF	0.5060	0.0016	0.5063	0.0017	1.0769	0.0599	1.0718	0.0549
20	1.5	SELF	1.5464	0.0339	1.4050	0.0321	1.0161	0.0250	1.0184	0.0250
		GELF	1.4859	0.1329	1.3065	0.1373	0.9819	0.0246	0.9849	0.0243
		WSELF	1.4801	0.1009	1.3183	0.1119	0.9887	0.0245	0.9915	0.0243
		PLF	1.4260	0.0342	1.4348	0.0329	1.0299	0.0259	1.0320	0.0260
30	1.5	SELF	1.5436	0.0277	1.4485	0.0274	1.0101	0.0169	1.0126	0.0163
		GELF	1.4844	0.0379	1.400	0.0317	0.9882	0.0167	0.9908	0.0160
		WSELF	1.4848	0.0325	1.4114	0.0279	0.9926	0.0166	0.9952	0.0160
		PLF	1.4608	0.0285	1.4654	0.0283	1.019	0.0172	1.0213	0.0167
50	1.5	SELF	1.5229	0.0211	1.4757	0.0205	1.0057	0.0100	1.0091	0.0100
		GELF	1.4894	0.0212	1.4523	0.0206	0.9929	0.0099	0.9964	0.0098
		WSELF	1.4841	0.0211	1.4571	0.0205	0.9955	0.0099	0.9989	0.0098
		PLF	1.4822	0.0212	1.4849	0.0207	1.0108	0.0101	1.0143	0.0101
100	1.5	SELF	1.4886	0.0122	1.4908	0.0126	1.0032	0.0050	1.0037	0.0049
		GELF	1.4873	0.0124	1.4795	0.0126	0.9969	0.0050	0.9974	0.0049
		WSELF	1.4895	0.0123	1.4818	0.0126	0.9982	0.0050	0.9987	0.0049
		PLF	1.4931	0.0123	1.4953	0.0126	1.0057	0.0051	1.0062	0.0049

consistent for the parameters α and β. When the classical methods are compared with each other, for the shape parameter α, as far as bias is concerned, LSE, WLSE and TLME work the best for all sample sizes. With respect to the MSEs, for $\alpha < 1$, MMLE-II performs better than the other estimators for small sample sizes ($n < 20$) and otherwise MLE outperforms the rest. For $\alpha \geq 1$, LME works the best for small sample sizes ($n \leq 20$). For large sample sizes ($n \geq 50$), MLE and MMLE-II both works very well.
Similarly, if we compare the classical estimators for β , comparing the biases, for $\alpha < 1$, it is observed that LME and TLME work the best for particularly small sample sizes and for large sample sizes ($n > 50$), the performances of the LME and TLME are close to that of the MLE and MMLE-II. When $\alpha \geq 1$, LME and TLME work the better than the other estimators for small sample sizes ($n \leq 20$) and otherwise MLE and MMLE-II outperform the rest.
Then, if we compare the performance of Bayes estimators obtained by Lindley's method, it is clear that as far as MSE and bias are concerned, Bayes estimators under GELF and WSELF work the best in all cases. Similarly, comparing the performance of Bayes estimators obtained by Tierney Kadane's approximation, it is observed that, if $\alpha \leq 1$, Bayes estimators obtained under GELF works the best in all cases for estimating α parameter, followed by Bayes estimation under the WSELF. When $\alpha > 1$, for estimating parameter, Bayesian estimations under SELF and PLF work very well.
For estimating β parameter, Bayes estimation under WSELF performs better than the other estimators for small sample sizes ($n \leq 20$) and otherwise Bayesian estimations under WSELF and GELF give the same result.
When we compare the Bayesian and classical methods for estimating the α and β parameters, it is clear that as far as bias and MSE are concerned; Bayesian methods outperform the classical methods. Furthermore, Lindley's method works well than the Tierney-Kadane's method in the most of the cases. Also, the GP gives better estimators than the NP for all loss functions.

6. Application

In this section, an actual data set is used to illustrate the estimation procedure developed in section 3-4. The data set measured from Sivas, Turkey during 2017 was used. There were 6011 observations recorded. The data was taken from the Turkish State Meteorological Service. All measurements were made at the heights of 10m in hourly basis.
In this paper, the performance of the Weibull distribution (WD) was compared with the Gamma distribution (GD), log- normal distribution (LND) and inverse Gauss distribution (IGD). These distributions for wind speed data were analyzed seasonally and annually. To determine the distribution providing better fit to wind speed data, we computed the root mean square error (RMSE), the coefficient of

determination (R^2) and Akaike information criteria(AIC) values for each distribution, as shown in Table 4. The formulas for model selection criteria were given in Table 5. In addition to these statistical criteria, the cumulative density function of the WD, GD, IGD and LND were presented in Figure 1 for seasonal and annual wind speed data.

Table 4. RMSE, R^2 and AIC values for distributions

n		Criteria	WD	IGD	LN	GD
		RMSE	0.0215	0.0746	0.0502	0.0301
6011	Annual	R2	0.9946	0.9214	0.9664	0.989
		AIC	2.5736	2.7704	2.7488	2.6058
		RMSE	0.0215	0.0821	0.0571	0.0353
1560	Winter	R2	0.9945	0.9053	0.9564	0.9847
		AIC	6.6889	7.2235	6.9984	6.7801
		RMSE	0.025	0.0627	0.0419	0.0261
1710	Spring	R2	0.9926	0.9453	0.977	0.9918
		AIC	7.2749	7.7628	8.3423	7.3414
		RMSE	0.0236	0.0796	0.0538	0.0338
1429	Summer	R2	0.9936	0.9084	0.9606	0.986
		AIC	6.1904	6.7319	6.8202	6.298
		RMSE	0.0231	0.0721	0.0482	0.0287
1312	Autumn	R2	0.9936	0.9276	0.9693	0.99
		AIC	5.5252	5.8949	5.6003	5.5729

Table 5. The formulas of criteria for model evaluation

Criteria	Formulas
RMSE	$2k - 2\ln\alpha$
R^2	$1-\left(\sum_{i=1}^{n}\hat{F}\left(X_{(i)}\right)-\frac{i}{n+1}\right)^2/\left(\sum_{i=1}^{n}\hat{F}\left(X_i\right)-\overline{\hat{F}}\left(X_i\right)\right)^2$
AIC	$\left[\sum_{i=1}^{n}\left(\hat{F}\left(X_{(i)}\right)-\frac{i}{n+1}\right)^2/n\right]^{1/2}$

According to Table 4, Weibull distribution has the smallest RMSE, AIC values and the highest R^2 values. In Table 5, k is the number of the unknown parameters, In L is the value of log-likelihood function for each distribution, $\hat{F}$ is the estimated cumulative density function, X_i is $i-th$ order statistics, n is sample size and $\bar{\hat{F}}=\sum_{i=1}^{n}\hat{F}_i/n$.

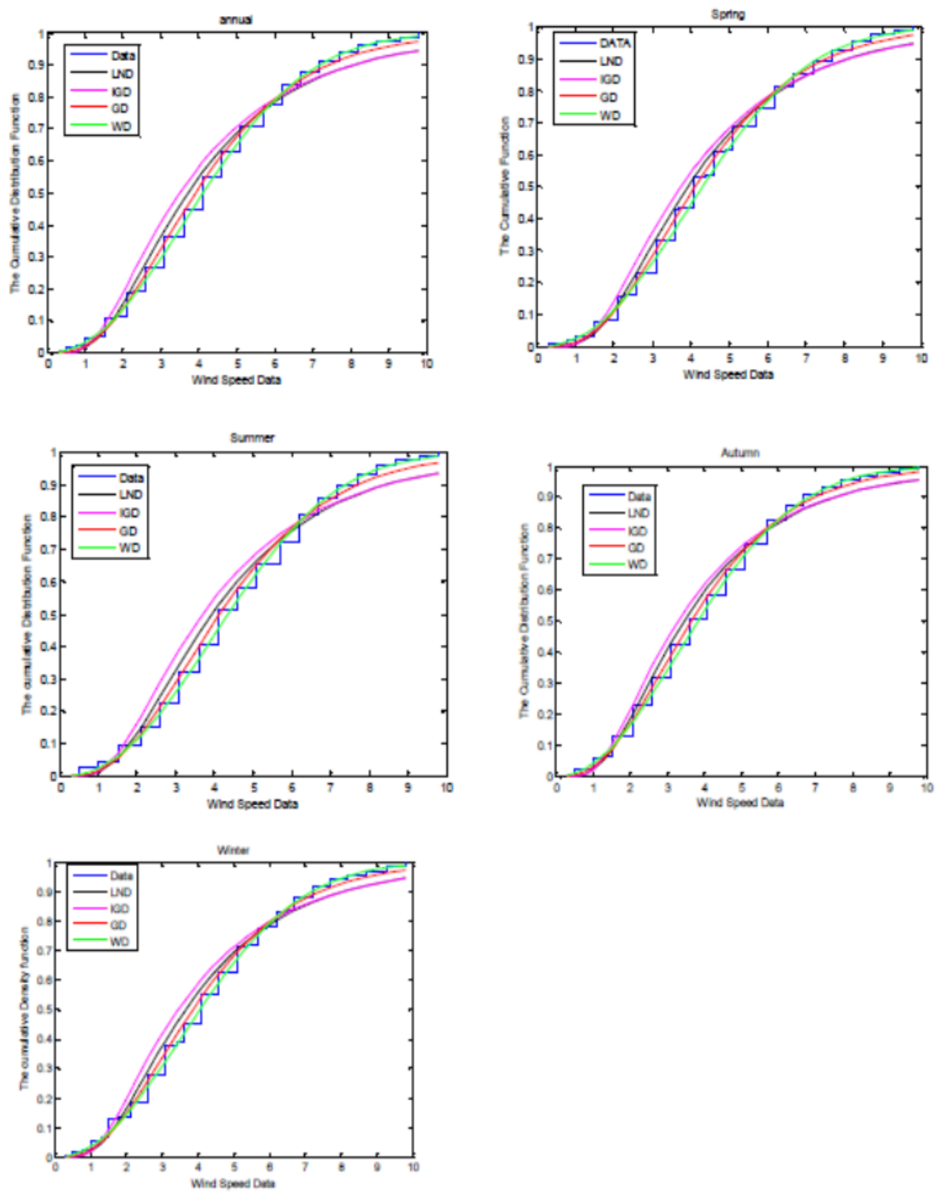

Figure 1. The cumulative density function for annual and seasonal wind speed data (m/s) for Sivas.

Table 6. Classical parameter estimations for the wind speed data.

n		Estimator	$\hat{\alpha}$	$\hat{\beta}$
6011	Annual	MLE	2.1520	4.9177
		LME	2.1422	4.9202
		TLME	2.1256	4.9326
		MMLE-I	2.1318	4.9088
		MMLE-II	3.0342	5.2794
		ME	2.1551	4.9203
		LSE	2.1083	4.9360
		WLSE	2.1571	4.9221
		PE	2.1876	4.9300
1560	Winter	MLE	2.1052	4.8413
		LME	2.1020	4.8463
		TLME	2.0534	4.8832
		MMLE-I	2.0780	4.8291
		MMLE-II	2.6032	5.0557
		ME	2.1139	4.8465
		LSE	2.1089	4.8786
		WLSE	2.1168	4.8610
		PE	2.1452	4.8600
1710	Spring	MLE	2.2734	5.0726
		LME	2.2605	5.0705
		TLME	2.3060	5.0389
		MMLE-I	2.2730	5.0724
		MMLE-II	2.6083	5.2082
		ME	2.2674	5.0704
		LSE	2.1956	5.0535
		WLSE	2.2540	5.0615
		PE	2.2746	5.0723
1429	Summer	MLE	2.2483	5.1749
		LME	2.2290	5.1816
		TLME	2.1682	5.2274
		MMLE-I	2.2058	5.1573
		MMLE-II	2.5317	5.2892
		ME	2.2541	5.1812
		LSE	2.1641	5.2318
		WLSE	2.2420	5.1998
		PE	2.3019	5.1970

Table 6. Continued.

n		Estimator	$\hat{\alpha}$	$\hat{\beta}$
1312	Autumn	MLE	2.0056	4.5226
		LME	2.0020	4.5226
		TLME	2.0117	4.5155
		MMLE-I	1.9993	4.5196
		MMLE-II	2.5919	4.7879
		ME	2.0060	4.5228
		LSE	1.9794	4.5225
		WLSE	2.0062	4.5257
		PE	2.0140	4.5285

Table 7.Lindley's and Tierney Kadane's parameter estimations under NP for the wind speed data

			Lindley's		Tierney-Kadane's	
n		LF	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
6011	Annual	SELF	2.1517	4.9178	2.1494	4.9178
		GELF	2.1515	4.9176	2.1493	4.9177
		WSELF	2.1515	4.9176	2.1492	4.9176
		PLF	2.1518	4.9179	2.1496	4.9179
1560	Winter	SELF	2.1041	4.8419	2.0963	4.8418
		GELF	2.1034	4.8412	2.0956	4.8414
		WSELF	2.1033	4.8411	2.0954	4.8413
		PLF	2.1046	4.8423	2.0967	4.8423
1710	Spring	SELF	2.2724	5.0731	2.2619	5.0731
		GELF	2.2717	5.0725	2.2611	5.0726
		WSELF	2.2716	5.0724	2.2611	5.0724
		PLF	2.2728	5.0734	2.2624	5.0734
1429	Summer	SELF	2.2471	5.1755	2.2349	5.1753
		GELF	2.2463	5.1748	2.2340	5.1749
		WSELF	2.2462	5.1747	2.2338	5.1748
		PLF	2.2476	5.1759	2.2355	5.1758
1312	Autumn	SELF	2.0044	4.5232	1.9973	4.5233
		GELF	2.0036	4.5224	1.9965	4.5223
		WSELF	2.0035	4.5223	1.9963	4.5222
		PLF	2.0049	4.5237	1.9978	4.5237

Table 8. Lindley's and Tierney Kadane's parameter estimations under GP for the wind speed

			Lindley's		Tierney-Kadane's	
n		LF	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
6011	Annual	SELF	2.1517	4.9186	2.1489	4.918
		GELF	2.1515	4.9184	2.1487	4.9131
		WSELF	2.1515	4.9184	2.1487	4.9131
		PLF	2.1518	4.9187	2.1490	4.9187
1560	Winter	SELF	2.1039	4.8365	2.0939	4.8365
		GELF	2.1034	4.8358	2.0953	4.8358
		WSELF	2.1033	4.8357	2.0951	4.8357
		PLF	2.1046	4.8369	2.0965	4.8369
1710	Spring	SELF	2.2727	5.0731	2.2623	5.0731
		GELF	2.2720	5.0725	2.2615	5.0725
		WSELF	2.2719	5.0725	2.2614	5.0725
		PLF	2.2731	5.0734	2.2627	5.0734
1429	Summer	SELF	2.2494	5.1776	2.2371	5.1777
		GELF	2.2486	5.1769	2.2361	5.1770
		WSELF	2.2484	5.1768	2.2360	5.1769
		PLF	2.2499	5.1780	2.2376	5.1781
1312	Autumn	SELF	2.0059	4.5218	1.9984	4.5218
		GELF	2.0047	4.5210	1.9976	4.5210
		WSELF	2.0046	4.5209	1.9975	4.5209
		PLF	2.0060	4.5223	1.9989	4.5223

It is clear that the results in Figure 1 are consistent with Table 4. Thus, in terms of all criteria, WD performed better than GD, IGD and LND for the seasonal and the annual wind speed data. Therefore, the two-parameter Weibull distribution was used for modelling the wind speed data. The estimators of the α and β obtained by using Bayesian and classical methods are given in Table 6-8. In light of the aforementioned information, we recommend the Bayesian estimations under WSELF and GELF for estimating the unknown parameters of Weibull distribution.

7. Conclusion

In this paper, we obtained different methods of estimation of the unknown parameters both with Bayesian and classical approximation. In classical method, the parameters α and β were estimated by using nine different method. In Bayesian method, we computed the Bayesian estimators of unknown parameters based on symmetric and asymmetric loss functions. The Bayes estimators do not have explicit form. Hence, we used the Lindley and Tierney Kadane's techniques under the assumption of Gamma priors. We also compare the performances of the estimators via simulation study. It is clear from the simulation results given in Table 1-3 that Lindley approximation under GELF and WSELF are more preferable than the other estimators according to the MSE and bias criteria in both scenarios i.e. informative prior and non-informative prior (especially for sample size $n > 50$).

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