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Infinitesimal Projective Transformations on the Tangent Bundle of a Riemannian Manifold with a Class of Lift Metrics

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ABSTRACT

Let (M, g) be a Riemannian manifold and TM be its tangent bundle. In the present paper, we study infinitesimal projective transformations on TM with respect to the Levi-Civita connection of a class of (pseudo-)Riemannian metrics $\tilde{g}$ which is a generalization of the three classical lifts of the metric g. We characterized this type of transformations and then we prove that if $(TM, \tilde{g})$ admits a non-affine infinitesimal projective transformation, then M and TM are locally flat.

Keywords: *Lift metrics; infinitesimal projective transformations; Riemannian manifold; tangent bundle; locally flat.*

AMS Subject Classification (2020): *Primary: 53B21 ; Secondary: 53C07; 53C22.*

1. Introduction

Let M be an n-dimensional $(n > 1)$ C^∞ connected manifold and TM be its tangent bundle. In this paper, we denote the set of all tensor fields of type (r, s) on M and TM by $\Im_s^r(M)$ and $\Im_s^r(TM)$, respectively. Also, we use $\sim$ for any geometric object on TM, for example, $\tilde{V}$ is a vector field on TM, but V is a vector field on M.

Let ∇ be an affine connection on a manifold M. A transformation f on M is called a projective transformation if it preserves the geodesics as set points. An affine transformation may be characterized as a projective transformation which preserves the geodesics with the affine parameter.

A vector field V on M with the local one parameter group $\{f_t\}$ is called an infinitesimal projective (affine) transformation if every f_t be a projective (affine) transformation. It is well known that a vector field V on M is an infinitesimal projective transformation if there exists an one form Ω on M such that

$$(L_V\nabla)(X, Y) = \Omega(X)Y + \Omega(Y)X,$$

for any $X, Y \in \Im_0^1(M)$, where L_V is the Lie derivation with respect to V. The one form Ω is called the associated one form of V. Also, the vector field V is an infinitesimal affine transformation, if $\Omega = 0$[16].

Let $g = (g_{ji})$ be a Riemannian metric on M. It is well-known that we can define from g several (pseudo-)Riemannian metrics on TM, where they are called the lift metrics of g, as follow: 1) complete lift metric or lift metric II is denoted by g^C, 2) diagonal lift metric or Sasaki metric or lift metric I+III is denoted by g^S, 3) lift metric I+II and 4) lift metric II+III, where I:$= g_{ji}dx^j dx^i$, II:$= 2g_{ji}dx^j\delta y^i$ and III:$= g_{ji}\delta y^j\delta y^i$ are bilinear differential forms defined globally on TM. It should be noted that in literature I:$= g_{ji}dx^j dx^i$ is called the vertical lift of g and denoted by g^V. For more details on lift metrics, one can refer to[17].

The problems of existing infinitesimal projective transformations on M and TM, have been studied by many authors, e.g. [3, 5, 6, 7, 8] and [10, 11, 12, 13, 14, 15]. These studies show that the existence of infinitesimal projective transformations on M or TM might lead to some global results. For example in [10], it is proved that if M, which is a complete Riemannian manifold with the parallel Ricci tensor, admits a non-affine infinitesimal projective transformation, then M is a space of positive constant curvature. Also it is proved in [11] that if a

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simply contact Riemannian manifold M admits a non-affine infinitesimal projective transformation, then M is isometric to a unit sphere.

In [6], [7] and [12], the following theorem is proved.

Theorem A: Let (M, g) be a complete Riemannian manifold and TM its tangent bundle. If TM, with 1) complete lift metric or 2) Sasaki metric or 3) lift metric II+III, admits a non-affine infinitesimal projective transformation, then M and TM are locally flat.

Abbassi and Sarih in [1] defined the *g-natural metrics* on TM, and in [2] studied a subclass of this metrics, that is displayed as

$$\tilde{g} = \alpha g^S + \beta g^C + \gamma g^V,$$

where α, β and γ are real constants with $\alpha > 0$ and $\alpha(\alpha + \gamma) - \beta^2 > 0$. As we said that g^S, g^C and g^V are the diagonal lift, the complete lift and the vertical lift of the Riemannian metric g, respectively. It is obvious that $\tilde{g}$ is a Riemannian metric on TM.

In [4], fiber-preserving projective vector fields with respect to the Levi-Civita connection from this subclass of g-natural metric are considered. It is proved that the Theorem A is true about of this class of metrics.

In this paper, we study the infinitesimal projective transformations on TM with respect to the Levi-Civita connection of the pseudo-Riemannian metric $\tilde{g} = \alpha g^S + \beta g^C + \gamma g^V$, where α, β and γ are real constants and $\alpha(\alpha + \gamma) - \beta^2 \neq 0$. In this case, one can see that $\tilde{g}$ is a generalization of the above metrics.

In fact, we have the following Theorems:

Theorem 1.1. *Let (M, g) be an n-dimensional Riemannian manifold and TM be its tangent bundle with (pseudo-)Riemannian metric $\tilde{g} = \alpha g^S + \beta g^C + \gamma g^V$, where α, β and γ are real constants with $\alpha \neq 0$ and $\lambda := \alpha(\alpha + \gamma) - \beta^2 \neq 0$. Then $\tilde{V}$ is an infinitesimal projective transformation with the associated one form $\tilde{\Omega}$ on TM if and only if there exist $\varphi, \psi \in C^\infty(M)$, $B = (B^h)$, $D = (D^h) \in \Im^1_0(M)$ and $A = (A^h_i)$, $C = (C^h_i) \in \Im^1_1(M)$, satisfying*

1. $(\tilde{V}^h, \tilde{V}^{\bar{h}}) = (B^h + y^a A^h_a, D^h + y^a C^h_a + y^h y^a \Phi_a)$,
2. $(\tilde{\Omega}_i, \tilde{\Omega}_{\bar{i}}) = (\partial_i \psi, \partial_i \varphi) = (\Psi_i, \Phi_i)$,
3. $\nabla_j \Psi_i = 0, \nabla_i \Phi_j = 0$,
4. $\nabla_i A^h_j = \Phi_j \delta^h_i - \frac{\alpha^2}{2\lambda} D^a R^h_{aij}$,
5. $R^h_{bja} A^a_i = 0, \;\; R^a_{jib} A^h_a = 0$,
6. $B^a \nabla_a R^h_{bji} = R^a_{bji} \nabla_a B^h - R^h_{bja} \nabla_i B^a - R^h_{aji} C^a_b - R^h_{bai} C^a_j$,
7. $\nabla_i C^h_j = \Psi_i \delta^h_j + B^a R^h_{iaj} + \frac{\alpha\beta}{2\lambda} D^a R^h_{aji}$,
8. $R^a_{kji}(\beta \nabla_a B^h - \beta C^h_a + \alpha \nabla_a D^h) = 0$,
9. $L_B \Gamma^h_{ji} = \nabla_j \nabla_i B^h + B^a R^h_{aji} = \Psi_j \delta^h_i + \Psi_i \delta^h_j - \frac{\alpha\beta}{2\lambda} D^a (R^h_{aji} + R^h_{aij})$,
10. $\nabla_j \nabla_i D^h = -\frac{\beta^2}{\lambda} D^a R^h_{jai} + \frac{\alpha(\alpha+\gamma)}{2\lambda} D^a R^h_{jia}$,
11. $\beta D^a \nabla_j R^h_{bai} = -\beta(R^h_{baj} \nabla_i D^a + R^h_{bai} \nabla_j D^a) - \beta R^a_{jib} \nabla_a D^h$
$- \beta R^h_{bai}(2\frac{\beta^2}{\alpha} \nabla_j B^a - 2\frac{\beta^2}{\alpha} C^a_j - \nabla_j D^a)$,

where $(\tilde{V}^h, \tilde{V}^{\bar{h}}) := \tilde{V}^a E_a + \tilde{V}^{\bar{a}} E_{\bar{a}} = \tilde{V}$, and $(\tilde{\Omega}_i, \tilde{\Omega}_{\bar{i}}) := \tilde{\Omega}_a dx^a + \tilde{\Omega}_{\bar{a}} \delta y^a = \tilde{\Omega}$.

Theorem 1.2. *Let (M, g) be an n-dimensional Riemannian manifold and TM be its tangent bundle with the (pseudo-)Riemannian metric $\tilde{g} = \beta g^C + \gamma g^V$, where β and γ are real constants with $\beta \neq 0$. Then $\tilde{V}$ is an infinitesimal projective transformation with the associated one form $\tilde{\Omega}$ on TM if and only if there exist $\varphi, \psi \in C^\infty(M)$, $B = (B^h)$, $D = (D^h) \in \Im^1_0(M)$ and $A = (A^h_i)$, $C = (C^h_i) \in \Im^1_1(M)$, satisfying*

1. $(\tilde{V}^h, \tilde{V}^{\bar{h}}) = (B^h + y^a A^h_a, D^h + y^a C^h_a + y^h y^a \Phi_a)$,
2. $(\tilde{\Omega}_i, \tilde{\Omega}_{\bar{i}}) = (\partial_i \psi, \partial_i \varphi) = (\Psi_i, \Phi_i)$,
3. $\nabla_j \Psi_i = 0, \nabla_i \Phi_j = 0$,

4. $\nabla_i A_j^h = \Phi_j \delta_i^h$,
5. $A_i^a R_{bja}^h = 0, \quad R_{bji}^a A_a^h = 0$,
6. $\nabla_i C_j^h = \Psi_j \delta_i^h + B^a R_{iaj}^h$,
7. $L_B \Gamma_{ji}^h = \nabla_j \nabla_i B^h + B^a R_{aji}^h = \Psi_j \delta_i^h + \Psi_i \delta_j^h$,
8. $L_D \Gamma_{ji}^h = \nabla_j \nabla_i D^h + D^a R_{aji}^h = 0$,
9. $B^a \nabla_a R_{bji}^h = -R_{aji}^h \nabla_b B^a - R_{bja}^h \nabla_i B^a - R_{bai}^h C_j^a + R_{bji}^a C_a^h$,

where $(\tilde{V}^h, \tilde{V}^{\bar{h}}) := \tilde{V}^a E_a + \tilde{V}^{\bar{a}} E_{\bar{a}} = \tilde{V}$, *and* $(\tilde{\Omega}_i, \tilde{\Omega}_{\bar{i}}) := \tilde{\Omega}_a dx^a + \tilde{\Omega}_{\bar{a}} \delta y^a = \tilde{\Omega}$.

Theorem 1.3. *Let* (M, g) *be a complete Riemannian manifold and* TM *be its tangent bundle with the (pseudo-)Riemannian metric* $\tilde{g} = \alpha g^S + \beta g^C + \gamma g^V$, *where* α, β *and* γ *are real constants with* $\alpha(\alpha+\gamma) - \beta^2 \neq 0$. *If* $(TM, \tilde{g})$ *admits a non-affine infinitesimal projective transformation, then* M *and* TM *are locally flat.*

Thus the Theorem A is true about of the (pseudo-)Riemannian metric $\tilde{g} = \alpha g^S + \beta g^C + \gamma g^V$, where $\alpha(\alpha + \gamma) - \beta^2 \neq 0$. It would be mentioned that the equation $R_{bji}^a A_a^h = 0$ is eliminated in [6], [7] and [12].

2. Preliminaries

In this section, we give the basic definitions and results on M and TM that are needed later. The details of them can be founded in [17, 18]. In here, indices $a, b, c, i, j, k, \ldots$ have range in $\{1, 2, \ldots, n\}$.

Let M be an n-dimensional C^∞ connected manifold. The coordinate systems on M are denoted by (U, x^i), where U is the coordinate neighborhood and x^i the coordinate functions. Let $T_x M$ denotes the tangent space of M at x and $TM := \bigcup_{x \in M} T_x M$ is the tangent bundle of M. The elements of TM are denoted by (x, y) where $y \in T_x M$ and the natural projection $\pi : TM \to M$ is given by $\pi(x, y) := x$.

Let (M, g) be a Riemannian manifold, ∇ be the Levi-Civita (Riemannian) connection of g and Γ_{ji}^h be the coefficients of ∇, i.e. $\nabla_{\partial_j} \partial_i = \Gamma_{ji}^h \partial_h$, with respect to the frame field $\{\partial_h := \frac{\partial}{\partial x^h}\}$.

Using the Levi-Civita Connection ∇, we define the local frame field $\{E_i, E_{\bar{i}}\}$ on each induced coordinate neighborhood $\pi^{-1}(U)$ of TM, as follow:

$$E_i := \partial_i - y^b \Gamma_{bi}^h \partial_{\bar{h}}, \quad E_{\bar{i}} := \partial_{\bar{i}},$$

where $\partial_{\bar{i}} := \frac{\partial}{\partial y^i}$. This frame field is called the adapted frame of TM. The dual frame of $\{E_i, E_{\bar{i}}\}$ is $\{dx^h, \delta y^h\}$, where $\delta y^h := dy^h + y^b \Gamma_{ab}^h dx^a$. By the straightforward calculation, we have the following lemmas.

Lemma 2.1. *The Lie brackets of the adapted frame of* TM *satisfy the following identities:*

1. $[E_j, E_i] = y^b R_{ijb}^a E_{\bar{a}}$,
2. $[E_j, E_{\bar{i}}] = \Gamma_{ji}^a E_{\bar{a}}$,
3. $[E_{\bar{j}}, E_{\bar{j}}] = 0$,
   where $R = (R_{ijb}^a)$ *is the curvature tensor of* ∇.

Lemma 2.2. *Let* $\tilde{V} = (\tilde{V}^h, \tilde{V}^{\bar{h}}) = \tilde{V}^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}}$ *be a vector field on* TM. *Then*

1. $[\tilde{V}, E_i] = -(E_i \tilde{V}^a) E_a + (\tilde{V}^c y^b R_{icb}^a - \tilde{V}^{\bar{b}} \Gamma_{bi}^a - E_i \tilde{V}^{\bar{a}}) E_{\bar{a}}$,
2. $[\tilde{V}, E_{\bar{i}}] = -(E_{\bar{i}} \tilde{V}^a) E_a + (\tilde{V}^b \Gamma_{bi}^a - E_{\bar{i}} \tilde{V}^{\bar{a}}) E_{\bar{a}}$.

From the Riemannian metric $g = (g_{ji})$ on a manifold M, one can see that

I: $g_{ji} dx^j dx^i$,
II: $2 g_{ji} dx^j \delta y^i$,
III: $g_{ji} \delta y^j \delta y^i$,

are quadratic differential forms which globally defined on TM and also

II: $2 g_{ji} dx^j \delta y^i$,

I+II: $g_{ji}dx^j dx^i + 2g_{ji}dx^j \delta y^i$,
I+III: $g_{ji}dx^j dx^i + g_{ji}\delta y^j \delta y^i$,
II+III: $2g_{ji}dx^j \delta y^i + g_{ji}\delta y^j \delta y^i$

are Riemannian or pseudo-Riemannian metrics on TM. It would be mentioned that the metric II is called the complete lift metric and denoted by g^C, the metric I+III is called the Sasakian metric and denoted by g^S, and quadratic form I is called the vertical lift and denoted by g^V. For more details, one can refer to [16].

Abbassi and Sarih in [2] studied a subclass of Riemannian g-natural metrics on TM that is denoted by $\tilde{g} = \alpha g^S + \beta g^C + \gamma g^V$, where α, β and γ are constants with $\alpha > 0$ and $\alpha(\alpha+\gamma) - \beta^2 > 0$.

Here, we consider pseudo-Riemannian metric $\tilde{g} = \alpha g^S + \beta g^C + \gamma g^V$ on TM, where α, β and γ are constants with $\alpha(\alpha+\gamma) - \beta^2 \neq 0$. In this case, one can see that $\tilde{g}$ is a generalization of the above metrics, for example, if put $\alpha = \beta = 1$ and $\gamma = -1$, then $\tilde{g} = g^S + g^C - g^V$ is the lift metric II+III.

The coefficients of Levi-Civita connection $\tilde{\nabla}$ of the metric $\tilde{g} = \alpha g^S + \beta g^C + \gamma g^V$, with respect to the adapted frame $\{E_i, E_{\bar{i}}\}$ are computed in [2]. In fact, we have the following Lemma.

Lemma 2.3. *Let $\tilde{\nabla}$ be the Levi-Civita connection of the metric $\tilde{g} = \alpha g^S + \beta g^C + \gamma g^V$ on TM, where α, β and γ are constants with $\lambda := \alpha(\alpha+\gamma) - \beta^2 \neq 0$. Then we have*

$$\tilde{\nabla}_{E_j} E_i = \{\Gamma^h_{ji} + \tfrac{\alpha\beta}{2\lambda} y^k (R^h_{kji} + R^h_{kij})\} E_h + y^k \big(\tfrac{\beta^2}{\lambda} R^h_{jki} - \tfrac{\alpha(\alpha+\gamma)}{2\lambda} R^h_{jik}\big) E_{\bar{h}},$$
$$\tilde{\nabla}_{E_j} E_{\bar{i}} = \tfrac{\alpha^2}{2\lambda} y^k R^h_{kij} E_h + (\Gamma^h_{ji} - \tfrac{\alpha\beta}{2\lambda} y^k R^h_{kij}) E_{\bar{h}},$$
$$\tilde{\nabla}_{E_{\bar{j}}} E_i = \tfrac{\alpha^2}{2\lambda} y^k R^h_{kji} E_h - \tfrac{\alpha\beta}{2\lambda} y^k R^h_{kji} E_{\bar{h}},$$
$$\tilde{\nabla}_{E_{\bar{j}}} E_{\bar{i}} = 0.$$

where Γ^h_{ji} denotes the coefficients of Riemannian connection ∇ with respect to g.

3. Proof of Theorems

In this section, we prove Theorems 1.1 and 1.3 because Theorem1.2 can be proved in a similar way.

Proof of Theorem1.1

Because the sufficient conditions are easy to proof, we only prove the necessary conditions. Let $\tilde{V} = \tilde{V}^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}}$ be an infinitesimal projective transformation and $\tilde{\Omega} = \tilde{\Omega}_h dx^h + \tilde{\Omega}_{\bar{h}} \delta y^h$ its the associated one form on TM, thus for any $\tilde{X}, \tilde{Y} \in \Im^1_0(TM)$ we have

$$(L_{\tilde{V}} \tilde{\nabla})(\tilde{X}, \tilde{Y}) = \tilde{\Omega}(\tilde{X})\tilde{Y} + \tilde{\Omega}(\tilde{Y})\tilde{X}. \tag{3.1}$$

From $(L_{\tilde{V}} \tilde{\nabla})(E_{\bar{j}}, E_{\bar{i}}) = \tilde{\Omega}(E_{\bar{j}})E_{\bar{i}} + \tilde{\Omega}(E_{\bar{i}})E_{\bar{j}}$ and Lemma 2.3 we have

$$\partial_{\bar{j}}\partial_{\bar{i}}\tilde{V}^h - \frac{\alpha^2}{2\lambda} y^k (R^h_{ika}\partial_{\bar{j}}\tilde{V}^a + R^h_{jka}\partial_{\bar{i}}\tilde{V}^a) = 0, \tag{3.2}$$

and

$$\partial_{\bar{j}}\partial_{\bar{i}}\tilde{V}^{\bar{h}} + \frac{\alpha\beta}{2\lambda} y^k (R^h_{ika}\partial_{\bar{j}}\tilde{V}^a + R^h_{jka}\partial_{\bar{i}}\tilde{V}^a) = \tilde{\Omega}_{\bar{j}}\delta^h_i + \tilde{\Omega}_{\bar{i}}\delta^h_j \tag{3.3}$$

One can see that (3.2) is rewritten as follows:

$$\partial_{\bar{j}}\partial_{\bar{i}}\tilde{V}^h = \frac{\alpha^2}{2\lambda}\big\{\partial_{\bar{j}}(y^b R^h_{iba}\tilde{V}^a) + \partial_{\bar{i}}(y^b R^h_{jba}\tilde{V}^a)\big\}. \tag{3.4}$$

By differentiaiting from (3.4) with respect to y^k, we have

$$\begin{aligned}
\partial_{\bar{k}}\partial_{\bar{j}}\partial_{\bar{i}}\tilde{V}^h &= \frac{\alpha^2}{2\lambda}\big\{\partial_{\bar{k}}\partial_{\bar{j}}(y^b R^h_{iba}\tilde{V}^a) + \partial_{\bar{k}}\partial_{\bar{i}}(y^b R^h_{jba}\tilde{V}^a)\big\} \\
&= \frac{\alpha^2}{2\lambda}\big\{\partial_{\bar{j}}\partial_{\bar{i}}(y^b R^h_{iba}\tilde{V}^a) + \partial_{\bar{j}}\partial_{\bar{k}}(y^b R^h_{jba}\tilde{V}^a)\big\} \\
&= \frac{\alpha^2}{2\lambda}\big\{\partial_{\bar{i}}\partial_{\bar{k}}(y^b R^h_{iba}\tilde{V}^a) + \partial_{\bar{i}}\partial_{\bar{j}}(y^b R^h_{jba}\tilde{V}^a)\big\}.
\end{aligned} \tag{3.5}$$

From (3.5), we obtain that

$$\partial_{\bar{k}}\partial_{\bar{j}}(\partial_{\bar{i}}\tilde{V}^h - \frac{\alpha^2}{\lambda}y^b R^h_{iba}\tilde{V}^a) = 0. \tag{3.6}$$

Thus we can put

$$P^h_{ji} := \partial_{\bar{j}}(\partial_{\bar{i}}\tilde{V}^h - \frac{a^2}{\lambda}y^b R^h_{iba}\tilde{V}^a), \tag{3.7}$$

and

$$A^h_i + y^b P^h_{bi} = \partial_{\bar{i}}\tilde{V}^h - \frac{\alpha^2}{\lambda}y^b R^h_{iba}\tilde{V}^a, \tag{3.8}$$

where P^h_{ji} and A^h_i are functions on M. By a straightforward calculation, we see that $A = (A^h_i) \in \Im^1_1(M)$ and $P = (P^h_{ji}) \in \Im^1_2(M)$.
By using (3.2), we have

$$P^h_{ji} + P^h_{ij} = 2\partial_{\bar{j}}\partial_{\bar{i}}\tilde{V}^h - \frac{\alpha^2}{\lambda}y^b(R^h_{iba}\partial_{\bar{j}}\tilde{V}^a + R^h_{jba}\partial_{\bar{i}}\tilde{V}^a) = 0. \tag{3.9}$$

This means that P^h_{ji} is antisymmetric with respect to i, j and thus we have

$$2P^h_{ji} = P^h_{ji} - P^h_{ij} = \frac{\alpha^2}{\lambda}\{\partial_{\bar{i}}(y^b R^h_{jba}\tilde{V}^a) - \partial_{\bar{j}}(y^b R^h_{iba}\tilde{V}^a)\}. \tag{3.10}$$

Therefore

$$\begin{aligned} 2y^j P^h_{ji} &= \frac{\alpha^2}{\lambda}\{y^j\partial_{\bar{i}}(y^b R^h_{jba}\tilde{V}^a) - y^j\partial_{\bar{j}}(y^b R^h_{iba}\tilde{V}^a)\} \\ &= -\frac{2\alpha^2}{\lambda}y^j R^h_{ija}\tilde{V}^a - \frac{\alpha^2}{\lambda}y^j y^b R^h_{iba}\partial_{\bar{j}}\tilde{V}^a. \end{aligned} \tag{3.11}$$

By substituting (3.11) into (3.8), we obtain

$$\partial_{\bar{i}}\tilde{V}^h = A^h_i - \frac{\alpha^2}{2\lambda}y^j y^b R^h_{iba}\partial_{\bar{j}}\tilde{V}^a, \tag{3.12}$$

so we have

$$y^i\partial_{\bar{i}}\tilde{V}^h = y^i A^h_i. \tag{3.13}$$

Substituting (3.13) into (3.12), we obtain

$$\partial_{\bar{i}}\tilde{V}^h = A^h_i - \frac{\alpha^2}{2\lambda}y^b y^c R^h_{iba}A^a_c, \tag{3.14}$$

from which

$$\partial_{\bar{j}}\partial_{\bar{i}}\tilde{V}^h = -\frac{\alpha^2}{2\lambda}y^b(R^h_{iba}A^a_j + R^h_{ija}A^a_b). \tag{3.15}$$

On the other hand, by substituting (3.14) into (3.2), we have

$$\partial_{\bar{j}}\partial_{\bar{i}}\tilde{V}^h = \frac{\alpha^2}{2\lambda}y^b(R^h_{iba}A^a_j + R^h_{jba}A^a_i) - \frac{\alpha^4}{4\lambda}y^b y^c y^d(R^h_{iba}R^a_{jce}A^e_d + R^h_{jba}R^a_{ice}A^e_d). \tag{3.16}$$

Comparing (3.15) and (3.16), we obtain

$$\alpha(2R^h_{jba}A^a_i + R^h_{jia}A^a_b + R^h_{iba}A^a_j) = 0, \tag{3.17}$$

therefore

$$\alpha(R^h_{jba}A^a_i + R^h_{iba}A^a_j) = 0. \tag{3.18}$$

By use of (3.18) and the first Bianchi identity, we have

$$\alpha(R^h_{bja}A^a_i) = 0, \tag{3.19}$$

thus

$$R^h_{bja}A^a_i = 0, \tag{3.20}$$

by virtue of $\alpha \neq 0$.
Substituting (3.20) into (3.14), we obtain

$$\tilde{V}^h = B^h + A^h_a y^a, \tag{3.21}$$

where B^h are certain functions on M. One can see that $B = (B^h) \in \Im^1_0(M)$.
Substituting (3.21) into (3.3), we have

$$\partial_{\bar{j}} \partial_{\bar{i}} \tilde{V}^{\bar{h}} = \tilde{\Omega}_{\bar{j}} \delta^h_i + \tilde{\Omega}_{\bar{i}} \delta^h_j. \tag{3.22}$$

Contracting i and h in (3.22)

$$\tilde{\Omega}_{\bar{j}} = \partial_{\bar{j}} \tilde{\varphi}, \tag{3.23}$$

where

$$\tilde{\varphi} := \frac{1}{n+1} \partial_{\bar{a}} \tilde{V}^{\bar{a}}. \tag{3.24}$$

Substituting (3.23) into (3.22), we get

$$\partial_{\bar{j}} \partial_{\bar{i}} \tilde{V}^{\bar{h}} = \partial_{\bar{j}} \tilde{\varphi} \delta^h_i + \partial_{\bar{i}} \tilde{\varphi} \delta^h_j \tag{3.25}$$

By a similar way, one can see that there exist $\Phi = (\Phi_i) \in \Im^0_1(M)$, $D = (D^h) \in \Im^1_0(M)$ and $C = (C^h_i) \in \Im^1_1(M)$, satisfying

$$\tilde{\Omega}_{\bar{i}} = \Phi_i, \tag{3.26}$$

and

$$\tilde{V}^{\bar{h}} = \Phi_a y^a y^h + C^h_a y^a + D^h. \tag{3.27}$$

From $(L_{\tilde{V}} \tilde{\nabla})(E_{\bar{j}}, E_i) = \tilde{\Omega}_{\bar{j}} E_i + \tilde{\Omega}_i E_{\bar{j}}$ and by use of (3.21), (3.26) and (3.27), we obtain

$$\begin{aligned} \Phi_j \delta^h_i &= (\nabla_i A^h_j + \frac{\alpha^2}{2\lambda} D^a R^h_{aji}) + \frac{y^b}{2\lambda} \{ \alpha^2 (B^a \nabla_a R^h_{bji} - R^a_{bji} \nabla_a B^h \\ &\quad + R^h_{bja} \nabla_i B^a + R^h_{aji} C^a_b + R^h_{bai} C^a_j) + \alpha\beta R^a_{bji} A^h_a \} \\ &\quad + \frac{y^b y^c}{2\lambda} \alpha^2 \big(A^a_c \nabla_a R^h_{bji} - R^a_{bji} \nabla_a A^h_c + R^h_{bja} \nabla_i A^a_c + 2\Phi_c R^h_{bji} \big). \end{aligned} \tag{3.28}$$

Contracting i and h in (3.28), we have

$$\Phi_i = \frac{1}{n} \nabla_a A^a_i. \tag{3.29}$$

From (3.28) we get

$$\nabla_i A^h_j = \Phi_j \delta^h_i - \frac{\alpha^2}{2\lambda} D^a R^h_{aji}, \tag{3.30}$$

$$\alpha (B^a \nabla_a R^h_{bji} - R^a_{bji} \nabla_a B^h + R^h_{bja} \nabla_i B^a + R^h_{aji} C^a_b + R^h_{bai} C^a_j) + \beta R^a_{bji} A^h_a = 0, \tag{3.31}$$

and

$$A^a_t \nabla_a R^h_{bji} + A^a_c \nabla_a R^h_{bji} = R^a_{bji} \nabla_a A^h_c + R^a_{cji} \nabla_a A^h_b - R^h_{bja} \nabla_i A^a_c - R^h_{cja} \nabla_i A^a_b - 2\Phi_c R^h_{bji} - 2\Phi_b R^h_{cji}. \tag{3.32}$$

From (3.29) and (3.30) we have

$$\Phi_i = \nabla_i A^a_a - \frac{\alpha^2}{2\lambda} R_{ai} D^a = \frac{1}{n} \nabla_a A^a_i. \tag{3.33}$$

From $(L_{\tilde{V}} \tilde{\nabla})(E_{\bar{j}}, E_i) = \tilde{\Omega}_{\bar{j}} E_i + \tilde{\Omega}_i E_{\bar{j}}$ and using (3.20), (3.21) and (3.27) we obtain

$$\begin{aligned} \tilde{\Omega}_i \delta^h_j &= (\nabla_i C^h_j - B^s R^h_{isj} - \frac{\alpha\beta}{2\lambda} R^h_{aji} D^a) - \frac{y^b}{2\lambda} \{ \alpha\beta (B^a \nabla_a R^h_{bji} + R^h_{bja} \nabla_i B^a \\ &\quad + R^h_{aji} C^a_b + R^h_{bai} C^a_j - R^a_{bji} C^h_a) + \alpha^2 R^a_{bji} \nabla_a D^h + 2\lambda (\nabla_i \Phi_j \delta^h_b + \nabla_i \Phi_b \delta^h_j) \} \\ &\quad + \frac{y^b y^c}{2\lambda} \{ \alpha^2 (R^a_{bji} B^d R^h_{adc} - R^a_{bji} \nabla_a C^h_c) - \alpha\beta (A^a_c \nabla_a R^h_{bji} + R^h_{bja} \nabla_i A^a_c \\ &\quad + R^h_{bji} \Phi_c - R^a_{bji} \Phi_a \delta^h_c) \} - y^b y^c y^d \frac{\alpha^2}{2\lambda} R^a_{bji} \nabla_a \Phi_c \delta^h_d. \end{aligned} \tag{3.34}$$

Contracting h and j in (3.34), we obtain

$$\begin{aligned} n\tilde{\Omega}_i &= (\nabla_i C^a_a + \frac{\alpha\beta}{2\lambda} R_{ai} D^a) + \frac{y^b}{2\lambda}\{\alpha\beta(B^a \nabla_a R^h_{bi} + R_{ba}\nabla_i B^a + R_{ai} C^a_b) \\ &+ \alpha^2 R^a_{bei}\nabla_a D^e + 2\lambda(n+1)\nabla_i \Phi_b\} + \frac{y^b y^c}{2\lambda}\{\alpha^2(R^a_{bei} B^d R^e_{adc} - R^a_{bei}\nabla_a C^e_c) \\ &+ \alpha\beta(A^a_c \nabla_a R_{bi} + R_{ba}\nabla_i A^a_c + R_{bi}\Phi_c)\}, \end{aligned} \tag{3.35}$$

where R_{ji} is the Ricci tensor of M which is defined by $R_{ji} := R^s_{sji}$.
Substituting (3.35) into (3.34) and comparing the both side, we get

$$\nabla_i C^h_j = \Psi_i \delta^h_j + B^a R^h_{iaj} + \frac{\alpha\beta}{2\lambda} R^h_{aji} D^a, \tag{3.36}$$

where $\Psi_i := \frac{1}{n}(\nabla_i C^a_a + \frac{\alpha\beta}{2\lambda} R_{ai} D^a)$, and

$$\begin{aligned} 2\lambda(n\nabla_i \Phi_j \delta^h_k - \nabla_i \Phi_k \delta^h_j) &= n\{-\alpha\beta(B^a\nabla_a R^h_{bji} + R^h_{bja}\nabla_i B^a + R^h_{hji} C^a_b + R^h_{bai} C^a_j - R^a_{bji} C^h_a) - \alpha^2 R^a_{bji}\nabla_a D^h\} \\ &- \delta^h_j\{\alpha\beta(B^a \nabla_a R_{bi} + R_{ba}\nabla_i B^a + R_{ai} C^a_b) + \alpha^2 R^c_{aki}\nabla_c D^a\}. \end{aligned} \tag{3.37}$$

One can see that the last part of right hand side in (3.35) vanishes.
Contracting h and k in (3.37), we obtain

$$-2\lambda(n-1)\nabla_i \Phi_j = \alpha\beta(B^a \nabla_a R_{ji} + R_{ja}\nabla_i B^a + R_{ia} C^a_j) + \alpha^2 R^c_{aji}\nabla_c D^a. \tag{3.38}$$

Using (3.38), we can rewritten (3.35) and (3.37) as follows:

$$\tilde{\Omega}_i = \Psi_i + 2y^k \nabla_i \Phi_k, \tag{3.39}$$

and

$$2\lambda(\nabla_i \Phi_b \delta^h_j - \nabla_i \Phi_j \delta^h_b) = \alpha\beta(B^a \nabla_a R^h_{bji} + R^h_{bja}\nabla_i B^a + R^h_{aji} C^a_b + R^h_{bai} C^h_j - R^a_{bji} C^h_a) + \alpha^2 R^a_{bji}\nabla_a D^h. \tag{3.40}$$

From $(L_{\tilde{V}}\tilde{\nabla})(E_j, E_i) = \tilde{\Omega}_j E_i + \tilde{\Omega}_i E_j$ and using (3.20), (3.21), (3.27) and (3.39), we obtain

$$\begin{aligned} \Psi_j \delta^h_i + \Psi_i \delta^h_j + 2y^b(\nabla_j \Phi_b \delta^h_i + \nabla_i \Phi_b \delta^h_j) &= \nabla_j \nabla_i B^h + B^a R^h_{aji} \\ &+ \frac{\alpha\beta}{2\lambda} D^a (R^h_{aji} + R^h_{aij}) + \frac{y^b}{2\lambda}\{2\lambda \nabla_j \nabla_i A^h_b + \alpha\beta\big(B^a(\nabla_a R^h_{bji} \\ &+ \nabla_a R^h_{bij}) - (R^a_{bji} + R^a_{bij})\nabla_a B^h + (R^h_{bai} + R^h_{bia})\nabla_j B^a \\ &+ (R^h_{baj} + R^h_{bja})\nabla_i B^a + (R^h_{aji} + R^h_{aij}) C^a_b\big) - 2\beta^2 R^a_{jbi} A^h_a \\ &+ \alpha(\alpha+\gamma) R^a_{jib} A^h_a + \alpha^2(R^h_{bai}\nabla_j D^a + R^h_{baj}\nabla_i D^a)\} \\ &+ \frac{y^b y^c}{2\lambda}\{\alpha\beta\big(A^a_c(\nabla_a R^h_{bji} + \nabla_a R^h_{bij}) - (R^a_{bji} + R^a_{bij})\nabla_a A^h_c \\ &+ (R^h_{bai} + R^h_{bia})\nabla_j A^a_c + (R^h_{baj} + R^h_{bja})\nabla_i A^a_c + \Phi_b(R^h_{cji} + R^h_{cij})\big) \\ &- \alpha^2(R^h_{bai} B^d R^a_{jdc} + R^h_{baj} B^d R^a_{idc} - R^h_{bai}\nabla_j C^a_c + R^h_{baj}\nabla_i C^a_c)\} \end{aligned} \tag{3.41}$$

and

$$0 = \nabla_j\nabla_i D^h + \frac{\beta^2}{\lambda}R^h_{jai}D^a - \frac{\alpha(\alpha+\gamma)}{2\lambda}R^h_{jia}D^a + \frac{y^b}{2\lambda}\{2\lambda\big(\nabla_j\nabla_i C^h_b - \nabla_j(B^c R^h_{icb})\big) + 2\beta^2(B^a\nabla_a R^h_{jbi} + R^h_{abi}\nabla_j B^a + R^h_{jba}\nabla_i B^a + R^h_{jai}C^a_b - R^a_{jbi}C^h_a) - \alpha(\alpha+\gamma)(B^a\nabla_a R^h_{jib} + R^h_{aib}\nabla_j B^a + R^h_{jab}\nabla_i B^a + R^h_{jia}C^a_b - R^a_{jib}C^h_a) - \alpha\beta\big(R^h_{bai}\nabla_i D^a + R^h_{baj}\nabla_i D^a + (R^a_{bji}+R^a_{bij})\nabla_a D^h\big)\} + \frac{y^b y^c}{2\lambda}\{(2\lambda\nabla_j\nabla_i\Phi_b + \alpha(\alpha+\gamma)R^a_{jib}\Phi_a - 2\beta^2 R^a_{jbi}\Phi_a)\delta^h_c + 2\beta^2(A^a_c\nabla_a R^h_{jbi} + R^h_{abi}\nabla_j A^a_c + R^h_{jba}\nabla_i A^a_c) - \alpha(\alpha+\gamma)(A^a_c\nabla_a R^h_{jib} + R^h_{aib}\nabla_j A^a_c + R^h_{jab}\nabla_i A^a_c) + \alpha\beta\big((R^a_{bji}+R^a_{bij})B^d R^h_{adc} + R^h_{bai}B^d R^a_{jdc} + R^h_{baj}B^d R^a_{idc} - (R^a_{bji}+R^a_{bij})\nabla_a C^h_c - R^h_{bai}\nabla_j C^a_c - R^h_{baj}\nabla_i C^a_c\big)\} - y^b y^c y^d\frac{\alpha\beta}{2\lambda}(R^a_{bji}+R^a_{bij})\nabla_a\Phi_c\delta^h_d. \tag{3.42}$$

Comparing both side of (3.41), we obtain

$$L_B\Gamma^h_{ji} = \nabla_j\nabla_i B^h + B^a R^h_{aji} = \Psi_j\delta^h_i + \Psi_i\delta^h_j - \frac{\alpha\beta}{2\lambda}D^a(R^h_{aji}+R^h_{aij}), \tag{3.43}$$

and

$$2\lambda\nabla_j\nabla_i A^h_b = -\alpha\beta\{B^a(\nabla_a R^h_{bji} + \nabla_a R^h_{bij}) + (R^a_{bji}+R^a_{bij})\nabla_a B^h - (R^h_{bai}+R^h_{bia})\nabla_j B^a - (R^h_{baj}+R^h_{bja})\nabla_i B^b - (R^h_{aji}+R^h_{aij})C^a_b\} + 2\beta^2 R^a_{jbi}A^h_a - \alpha(\alpha+\gamma)R^a_{jib}A^h_a - \alpha^2(R^h_{bai}\nabla_j D^a + R^h_{baj}\nabla_i D^a) + 4\lambda(\nabla_j\Phi_b\delta^h_i + 2\nabla_i\Phi_b\delta^h_j). \tag{3.44}$$

Substituting (3.30) into (3.44), we have

$$\lambda(4\nabla_j\Phi_b\delta^h_i + 2\nabla_i\Phi_b\delta^h_j) = \alpha\beta\{B^a(\nabla_a R^h_{bji} + \nabla_a R^h_{bij}) - (R^a_{bji} + R^a_{bij})\nabla_a B^h + (R^h_{bai}+R^h_{bia})\nabla_j B^a + (R^h_{baj}+R^h_{bja})\nabla_i B^a + (R^h_{aji}+R^h_{aij})C^a_b\} - 2\beta^2 R^a_{jbi}A^h_a + \alpha(\alpha+\gamma)R^a_{jib}A^h_a + \alpha^2(2R^h_{bai}\nabla_j D^a + R^h_{baj}\nabla_i D^a - D^a\nabla_j R^h_{abi}). \tag{3.45}$$

Contracting i and h in (3.45) and using (3.38), we get

$$-2\lambda(n+2)\nabla_j\Phi_b = \alpha\beta(B^a\nabla_a R_{bj} + R_{ba}\nabla_j B^a + R_{aj}C^a_b) - \alpha^2 R^a_{bcj}\nabla_a D^c. \tag{3.46}$$

From (3.38) and (3.46), we obtain

$$\nabla_j\Phi_k = 0. \tag{3.47}$$

From (3.39) and (3.47), we get

$$\tilde{\Omega}_i = \Psi_i. \tag{3.48}$$

Substituting (3.31) and (3.47) into (3.45)

$$\alpha^2\nabla_j(R^h_{abi}D^a) = \alpha R^h_{bai}(\alpha\nabla_j D^a - \beta C^a_j + \beta\nabla_j B^a) + \alpha R^h_{baj}(\alpha\nabla_i D^a - \beta C^a_i + \beta\nabla_i B^a) + \lambda R^a_{jib}A^h_a. \tag{3.49}$$

From (3.49), we get

$$R^a_{jib}A^h_a = 0. \tag{3.50}$$

Using from (3.31), (3.40), (3.47) and (3.50) we have

$$R^a_{bji}(\beta\nabla_a B^h - \beta C^h_a + \alpha\nabla_a D^h) = 0. \tag{3.51}$$

Contracting i and h in (3.41) and using (3.21), (3.27) and (3.47), we obtain

$$(n+1)\Psi_j = \nabla_j\nabla_a B^a - \frac{\alpha\beta}{2\lambda}R_{aj}D^a - \frac{y^b}{2\lambda}\{\alpha\beta(B^a\nabla_a R_{bj} + R_{ba}\nabla_j B^a + R_{aj}C^a_b) + \alpha^2 R^c_{baj}\nabla_c D^a\}$$
$$- \frac{y^b y^c}{2\lambda}\alpha\beta\{A^a_c\nabla_a R_{bj} + 2R_{bj}\Phi_c - \frac{\alpha^2}{2\lambda}(R_{ba}R^a_{dcj}D^d + R^d_{baj}R^a_{ecd}D^e)\} \tag{3.52}$$

Comparing (3.52) with (3.35), we get

$$\Psi_i = \frac{1}{n+1}(\nabla_i\nabla_a B^a - \frac{\alpha\beta}{2\lambda}R_{ai}D^a) = \frac{1}{n}(\nabla_i C^a_a + \frac{\alpha\beta}{2\lambda}R_{ai}D^a). \tag{3.53}$$

If we define $\psi := \frac{1}{2n+1}(\nabla_a B^a + C^a_a)$, from (3.53), one can see that

$$\partial_i\psi = \Psi_i. \tag{3.54}$$

From (3.42), we have

$$\nabla_j\nabla_i D^h = -\frac{\beta^2}{\lambda}R^h_{jai}D^a + \frac{\alpha(\alpha+\gamma)}{2\lambda}R^h_{jia}D^a, \tag{3.55}$$

and

$$2\lambda\big(\nabla_j\nabla_i C^h_b - \nabla_j(B^c R^h_{icb})\big) = -2\beta^2(B^a\nabla_a R^h_{jbi} + R^h_{abi}\nabla_j B^a + R^h_{jba}\nabla_i B^a$$
$$+ R^h_{jai}C^a_b - R^a_{jbi}C^h_a) + \alpha(\alpha+\gamma)(B^a\nabla_a R^h_{jib} + R^h_{aib}\nabla_j B^a$$
$$+ R^h_{jab}\nabla_i B^a + R^h_{jia}C^a_b - R^a_{jib}C^h_a) + \alpha\beta\big(R^h_{bai}\nabla_i D^a$$
$$+ R^h_{baj}\nabla_i D^a + (R^a_{bji} + R^a_{bij})\nabla_a D^h\big). \tag{3.56}$$

If $\beta \neq 0$, we put $\varphi = A^a_a - \frac{\alpha}{\beta}(\frac{n}{2n+1}\nabla_a B^a - \frac{n+1}{2n+1}C^a_a)$ and one can see that

$$\partial_i\varphi = \Phi_i. \tag{3.57}$$

If $\beta = 0$, from (3.55), we have

$$\nabla_j\nabla_a D^a = -\frac{\alpha(\alpha+\gamma)}{2\lambda}R_{ja}D^a. \tag{3.58}$$

Thus, we put $\varphi := A^a_a + \frac{\alpha}{\alpha+\gamma}\nabla_a D^a$ and from (3.33) and (3.58), one can see that

$$\partial_i\varphi = \Phi_i. \tag{3.59}$$

Substituting (3.31), (3.36) and (3.51) into (3.56), we get

$$2\lambda\nabla_j\Psi_i\delta^h_b = \alpha(\alpha+\gamma)(B^a\nabla_a R^h_{jib} + R^h_{aib}\nabla_j B^a + R^h_{jab}\nabla_i B^a + R^h_{jia}C^a_b$$
$$- R^a_{jib}C^h_a) - 2\beta^2(R^h_{abi}\nabla_j B^a - R^h_{abi}C^a_j) + \alpha\beta\{\nabla_j(R^h_{bai}D^a)$$
$$+ R^h_{baj}\nabla_i D^a + R^h_{bai}\nabla_j D^a - R^a_{ijb}\nabla_a D^h\}. \tag{3.60}$$

Contracting i and h, and j and h, separatly in (3.60), we have

$$2\lambda\nabla_j\Psi_b = \alpha(\alpha+\gamma)(-B^a\nabla_a R_{jb} - R_{ab}\nabla_j B^a + R^c_{jab}\nabla_c B^a$$
$$- R_{ja}C^a_b - R^c_{jab}C^a_c) - \alpha\beta(R^c_{ajb} + R^c_{abj})\nabla_c D^a, \tag{3.61}$$

and

$$2\lambda\nabla_b\Psi_i = -\alpha(\alpha+\gamma)(-B^a\nabla_a R_{ib} - R_{ab}\nabla_i B^a + R^c_{iab}\nabla_c B^a$$
$$- R_{ia}C^a_b - R^c_{iab}C^a_c) + \alpha\beta(R^c_{aib} + R^c_{abi})\nabla_c D^a. \tag{3.62}$$

From (3.61) and (3.62), we get

$$\nabla_j\Psi_i + \nabla_i\Psi_j = 0. \tag{3.63}$$

On the other hand, from (3.61) and (3.63), we have

$$4\lambda\nabla_j\Psi_i = -\alpha(\alpha+\gamma)\big(R_{ai}(\nabla_j B^a - C^a_j) - R_{ja}(\nabla_i B^a - C^a_i) - R^b_{jia}(\nabla_b B^a - C^a_b)\big). \tag{3.64}$$

Contracting h and j in (3.31),

$$R_{ba}\nabla_i B^a = -B^a\nabla_a R_{bi} - R_{ai}C^a_b - R^c_{bai}(\nabla_c B^a - C^a_c). \tag{3.65}$$

Substituting (3.65) into (3.64) and using the first Binachi identity, we get

$$\nabla_j \Psi_i = 0. \tag{3.66}$$

Substituting (3.66) into (3.60) and by use of (3.31), (3.36) and (3.51) we obtain

$$\beta D^a\nabla_j R^h_{bai} = -\beta(R^h_{baj}\nabla_i D^a + R^h_{bai}\nabla_j D^a) - \beta R^a_{jib}\nabla_a D^h - \beta R^h_{bai}(2\frac{\beta^2}{\alpha}\nabla_j B^a - 2\frac{\beta^2}{\alpha}C^a_j - \nabla_j D^a). \tag{3.67}$$

This completes the proof.

Proof of Theorem 1.3

Let $\tilde{V}$ be a non-affine infinitesimal projective transformation on TM. We put $X^h := A^h_a\Phi^a$, then from (3.30) and (3.47) we have

$$L_X g_{ji} = \nabla_j X_i + \nabla_i X_j = 2(\Phi_a\Phi^a)g_{ji},$$

where $X_i := g_{ih}X^h$.
Similarly, we define $Y^h := (\nabla_a B^h - C^h_a)\Psi^a$. Then, by using (3.36), (3.43) and (3.66), we get

$$L_Y g_{ji} = (\nabla_j\nabla_a B_i - \nabla_j C_{ai})\Psi^a + (\nabla_i\nabla_a B_j - \nabla_i C_{aj})\Psi^a = 2(\Psi_a\Psi^a)g_{ji}$$

Therefore X and Y are infinitesimal homothetic transformations.
To complete the proof of theorem, we need the following Lemma, which is proved in [9].

Lemma 3.1. *If a complete Riemannian manifold M admits a non-isometric infinitesimal homothetic transformation, then M is locally flat.*

If M is not locally flat, by use of Lemma 3.1, X and Y are infinitesimal isometric transformations and thus $\Phi_i = \Psi_i = 0$. Therefore $\tilde{V}$ is an infinitesimal affine transformation, which is a contradiction. Thus M is locally flat. It is easy to see that TM is also locally flat.

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