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Some Properties of Proper UP-Filters of UP-Algebras

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Abstract

The concept of UP-algebras was introduced and analyses by A. Iampan. In our recently published article we introduced the concept of proper UP-filter in UP-algebras in a somewhat different way than it is usual in literature. In this paper we analyse some fundamental properties of such determined proper UP-filters in UP-algebras.

1. Introduction

The concepts of UP-algebra are introduced and analyzed in [1]. The author in his article has introduced and analyzed the concepts of UP-algebra, UP-subalgebra and UP-ideal and their mutual connections. This author introduced in [2] the concept of proper UP-filter in UP-algebras on something different way then it is common in the available literature. In addition, in [2] he established the connection between UP-ideals and proper UP-filters.

In this article, the author further develops the idea of a proper UP-filter by identifying some of the fundamental features of this concept. First, we have shown two criteria (Theorem 3.1 and Theorem 3.2) that allow us to estimate whether a certain subset of UP-algebra is proper UP-filter or not. Other claims relate to a link between the proper UP-filters and UP-homomorphisms. Theorem 3.5 can be viewed as the first isomorphism theorem. For more details, see [3, 4].

The notations and notions appearing in this text are not predefined, the reader can find in the articles [1, 2, 3, 4].

2. Preliminaries

Let us recall the definition of UP-algebra.

Definition 2.1. *[[1], Definition 1.3] An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a* UP- algebra *if it satisfies the following axioms:*
(UP - 1): $(\forall x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
(UP - 2): $(\forall x \in A)(0 \cdot x = x)$,
(UP - 3): $(\forall x \in A)(x \cdot 0 = 0)$,
(UP - 4): $(\forall x, y \in A)((x \cdot y = 0 \wedge y \cdot x = 0) \Longrightarrow x = y)$.

In the following we give definition of the concept of UP-ideals of UP-algebra.

Definition 2.2. *[[1], Definition 2.1] Let A be a UP-algebra. A subset J of A is called a* UP-ideal *of A if it satisfies the following properties:*

1. $0 \in J$, *and*
2. $(\forall x, y, z \in A)(x \cdot (y \cdot z) \in J \wedge y \in J \Longrightarrow x \cdot z \in J)$.

One of fundamental properties of UP-ideals is given in statement (1) of Proposition 2.7 in the article [1]:

Proposition 2.3. *Let A be a UP-algebra and B a UP-ideal of A. Then*

$$\forall x, y \in A)((x \in B \wedge x \leq y) \Longrightarrow y \in B).$$

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Our intention in short notice [2] was to construct a substructure G in UP-algebras that will have the following property

$$(\forall x,y \in A)((y \in G \wedge x \leq y) \Longrightarrow x \in G)$$

and has a standard attitude toward the UP-ideal. This was done by introducing the concept of a proper UP-filter by the following way.

Definition 2.4 ([2], Definition 3.1)**.** *Let A be a UP-algebra. A subset G of A is called a* proper UP-filter *of A if it satisfies the following properties:*

3. $\neg(0 \in G)$, *and*
4. $(\forall x,y,z \in A)((\neg(x\cdot(y\cdot z) \in G) \wedge x\cdot z \in G) \Longrightarrow y \in G)$

In the mentioned article it was shown

Proposition 2.5. *Let A be a UP-algebra and G a proper UP-filter of A. Then*

5. $(\forall x,y \in A)((\neg(x\cdot y \in G) \wedge y \in G) \Longrightarrow x \in G)$.
6. $(\forall x,y \in A)(x\cdot y \in G \Longrightarrow y \in G)$.
7. $(\forall x,y \in A)((x \leqslant y \wedge y \in G) \Longrightarrow x \in G)$.

Proposition 2.6. *A subset G of a UP-algebra A is a proper UP-filter of A if and only if the set $A \backslash G$ is a UP-ideal of A.*

Proposition 2.7. *The family $\mathfrak{G}_A$ of all proper UP-filters in a UP-algebra A forms a completely lattice.*

Finally, the concept of UP-homomorphisms is defined by the following

Definition 2.8 ([1], Definition 4.1)**.** *Let $(A,\cdot,0_A)$ and $(B,\circ,0_B)$ be UP-algebras. A mapping f from A to B is called a UP-homomorphism if holds*

$$(\forall x,y \in A)(f(x\cdot y) = f(x)\circ f(y)).$$

In [1] it was shown that $f(A)$ is a subalgebra of algebra B (Theorem 4.5 (3)) and that $Kerf$ is an UP-ideal in A (Theorem 4.5 (6)).

3. The main results

First, for a subset G of UP-algebra A we show that from (5) and (6) follows (3) and (4) if we assume that $G \neq A$.

Theorem 3.1. *For a subset G of a UP-algebra A (5) and (6) implies (3) and (4) if we assume that $G \neq A$*

Proof. Let formulas (5) and (6) be valid for the proper subset G in A. Suppose that $\neg(x\cdot(y\cdot z) \in G)$ and $x\cdot z \in G$ is valid for arbitrary elements $x,y,z \in A$.
If we put $y = x$ in (6) we get that $0 = x\cdot x \in G$ implies $x \in G$ for any element $x \in A$. This is in a contradiction with $G \neq A$. The resulting contradiction yields $\neg(0 \in G)$. Thus, (3) is proven. From here it follows immediately that the subset G satisfies the formula (7). Indeed, if $x \leqslant y$ and $y \in G$, then we have$\neg(x\cdot y = 0 \in G)$ and $y \in G$. From here follows $x \in G$ according to (5).
First, from $x\cdot z \in G$ we have $z \in G$ by (6). Second, suppose it is $\neg(y \in G)$ holds. Thus, from $\neg(y \in G)$ and $z \in G$ follows $y\cdot z \in G$ by the contraposition of (5). Third, we have $y\cdot z \leqslant x\cdot(y\cdot z)$ by statement (6) in Theorem 1.8 in the article [1]. Now, from this and $y\cdot z \in G$ we conclude $x\cdot(y\cdot z) \in G$ by (7). This is in a contradiction with the first hypothesis. So, it has to be $y \in G$. Therefore, (4) is proven. □

Our second proposition is one more criterion for determining whether a subset G of A is a proper UP-filter or not.

Theorem 3.2. *Let A be a UP-algebra and $G \subseteq A$ such that $\neg(0 \in G)$. Then G is a proper UP-filter in A if and only if*

8. $(\forall x,y,z \in A)((\neg(y \in G) \wedge x\cdot z \in G) \Longrightarrow x\cdot(y\cdot z) \in G)$.

Proof. Let G be a proper UP-filter in a UP-algebra A and x,y,z be arbitrary elements of A. Suppose $\neg(y \in G)$ and $x\cdot z \in G$. If there were $\neg(x\cdot(y\cdot z) \in G)$ then from this and $x\cdot z \in G$ would have $y \in G$. The resulting result is in contradiction with the first hypothesis. Therefore, it must be $x\cdot(y\cdot z) \in G$.
Opposite, let for subset G of A (3) and (8) be hold for any $x,y,z \in A$. Suppose $\neg(x\cdot(y\cdot z) \in G)$ and $x\cdot z \in G$ are valid. If there were $\neg(y \in G)$ then from this and the second hypothesis would have $x\cdot(y\cdot z) \in G$ by (8). The resulting result is in contradiction with the first hypothesis. Therefore, it must be $y \in G$. □

Corollary 3.3. *Let G be a proper UP-filter in a UP-algebra A. Then*

9. $(\forall x,y \in A)((\neg(x \in G) \wedge y \in G) \Longrightarrow x\cdot y \in G)$.

Proof. If we put $x = 0$, $y = x$ and $z = y$ in (8) we will got (9). □

Theorem 3.4. *Let $(A,\cdot,0_A)$ and $(B,\circ,0_B)$ be UP-algebras and let $f : A \longrightarrow B$ be a UP-homomorphism. Then the following statements hold:*
(a) *If F is a proper UP-filter in a UP-algebra A, then $f(F)$ is a proper UP-filter in a UP-algebra $f(A)$.*
(b) *If G is a proper UP-filter of B, then $f^{-1}(G)$ is a proper UP-filter of A.*

Proof. (a) Assume that F is a proper UP-filter of A. Since $\neg(0_A \in F)$ and the statement (1) of Theorem 4.5 in article [1], we have $\neg(0_B = f(0_A) \in f(F))$.
Let $a, b, c \in f(A)$ be arbitrary elements such that $\neg(a \circ (b \circ c) \in f(F))$ and $a \circ c \in f(F)$. Then there exist elements $x, y, z \in A$ such that $f(x) = a$, $f(y) = b$ and $f(z) = c$ and $\neg(f(x) \circ (f(y) \circ f(z)) \in f(F))$ and $f(x) \circ f(z) \in f(F)$. This means $\neg(f(x \cdot (y \cdot z) \in f(F))$ and $f(x \cdot z) \in f(F)$. So, we conclude $\neg(x \cdot (y \cdot z) \in F)$ and $x \cdot z \in F$. Thus $y \in F$ by (4). Therefore, $c = f(y) \in f(F)$.
(b) Assume that G is a proper UP-filter of B. Since $\neg(0_B \in G)$, we have $\neg(f(0_A) = 0_B \in G)$. Thus $\neg(0_A \in f^{-1}(G))$.
Let $x, y, z \in A$ be arbitrary elements of A such that $\neg(x \cdot (y \cdot z) \in f^{-1}(G))$ and $x \cdot z \in f^{-1}(G)$. Then $\neg(f(x \cdot (y \cdot z)) \in G)$ and $f(x \cdot z) \in G$. Since f is a UP-homomorphism, we have $\neg(f(x) \circ (f(y) \circ f(z))) \in G)$ and $f(x) \circ f(z) \in G$. Since G is a proper UP-filter of B, we have $f(y) \in G$. Thus $y \in f^{-1}(G)$. $\square$

Without major difficulties, it can be proved that if J is a UP-ideal in a UP-algebra A and $' \sim '$ the congruence on A determined by the ideal J ([1], Proposition 3.5), then $A/J \equiv A/\sim = \{[x]_\sim : x \in A\}$ is also UP-algebra with the internal operation $' * '$ defined by

$$(\forall x, y \in A)([x]_\sim * [y]_\sim = [x \cdot y]_\sim)$$

and the fixed element J. The following claims is proven by direct verification.

Theorem 3.5. *Let $f : A \longrightarrow B$ be a UP-homomorphism between UP-algebras. Then there exists the UP-isomorphism $g : A/Ker(f) \longrightarrow f(A)$ such that $f = g \circ \pi$ where $\pi : A \longrightarrow A/Ker(f)$ is the canonical UP-epimorphism.*

Theorem 3.6. *Let $f : A \longrightarrow B$ be a UP-homomorphism between UP-algebras and J be a UP-ideal in A.*
If K is a UP-ideal in a UP-algebra A such that $J \subseteq K$, then the set $K/J = \{[x]_J \in A/J : x \in K\}$ is a UP-ideal in UP-algebra A/J.
If G is a proper UP-filter in a UP-algebra A such that $G \subseteq A \setminus J$, then the set $G/J = \{[x]_J : x \in G\}$ is a proper UP-filter in a UP-algebra A/J.

Proof. (a) It is clear that $J = [0]_J \in A/J$ is the fixed element in a UP-algebra A/J. Let $x, y, z \in A$ be arbitrary elements such that $[x]_J * ([y]_J * [z]_J) \in K/J$ and $[y]_J \in H/L$. Since $x \cdot (y \cdot z) \in K$ and $y \in K$ and since K is a UP-ideal in a UP-algebra A we conclude $x \cdot z \in K$. Thus $[x]_J * [z]_J \in K/J$. Therefore, the set K/J is a UP-ideal in a UP-algebra A/J.

(b) If there were $[0]_J \in G/J$, they would have $0 \in G$, which is a contradiction. So, we have $\neg([0]_J \in G/J)$. Let $x, y, z \in A$ be arbitrary elements such that $\neg([x]_J * ([y]_J * [z]_J) \in G/J)$ and $[x]_J * [z]_J \in G/J$. This means that $\neg(x \cdot (y \cdot z) \in G)$ and $x \cdot z \in G$. Since G is a proper UP-filter in a UP-algebra A, we have $y \in G$. Thus $[y]_J \in G/J$. Therefore, the set G/J is a proper UP-filter in a UP-algebra A/J. $\square$

Corollary 3.7. *There is a mutually unambiguous correspondence between the family $F_{A/J}$ of all proper UP-filters in a UP-algebra A/J and the family of all proper UP-filters contained in $A \setminus J$.*

4. Final observation

In the present paper, in order to continue developing the theory of proper UP-filters and UP-algebras, we given some fundamental properties of proper UP-filters in UP-algebra. The author believes that this new properties of proper UP-filters in UP-algebras enrich our knowledge about UP-algebras.

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On Semi-Invariant Submanifolds of Trans-Sasakian Finsler Manifolds

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Abstract

We define trans-Sasakian Finsler manifold $\bar{F}^{2n+1} = (\bar{\mathcal{N}}, \bar{\mathcal{N}}', \bar{F})$ and semi-invariant submanifold $F^m = (\mathcal{N}, \mathcal{N}', F)$ of a trans-Sasakian Finsler manifold $\bar{F}^{2n+1}$. Then we study mixed totally geodesic and totally umbilical semi-invariant submanifolds of trans Sasakian Finsler manifold.

1. Introduction

Oubina [1] introduced trans-Sasakian manifolds that reduced to α-Sasakian and $\beta-$Kenmotsu manifolds, in 1985. Then, trans-Sasakian manifolds are studied by many geometers like in [2]. Besides, Kobayashi studied semi-invariant submanifolds for a certain class of almost contact manifolds in [3] in 1986. Afterwards, semi invariant submanifolds of several structures are discussed like nearly trans-Sasakian and nearly Kenmotsu manifolds in [4], in 2004 and in [5], in 2009. Also, Shahid got some fundamental results on almost semi-invariant submanifolds of trans-Sasakian manifolds in [6], in 1993. Besides, Shahid et al. discussed submersion and cohomology class of semi-invariant submanifolds of trans-Sasakian manifolds in [7], in 2013.

B.B. Sinha and R. K. Yadav introduced almost Sasakian Finsler manifold and determined the set of all almost Sasakian Finsler h-connection on almost Sasakian Finsler manifold [8], In 1991. Then Yaliniz and Caliskan studied Sasakian Finsler manifolds in [9] in 2013. In this paper, we discussed mixed totally geodesic and totally umbilical semi- invariant submanifolds of trans-Sasakian Finsler manifolds.

2. Trans-Sasakian Finsler manifolds

Definition 2.1. *Suppose that $\bar{\mathcal{N}}$ be an $(2n+1)$-dimensional Finsler manifold. Then an almost contact metric structure $(\phi^V, \eta^V, \xi^V, G^V)$ on $(\bar{\mathcal{N}}')^v$ is called trans-Sasakian Finsler if the following relation is satisfied:*

$$2(\bar{\nabla}^V_X \phi)Y^V = \alpha \left\{ G^V(X^V, Y^V)\xi^V - \eta^V\left(Y^V\right)X^V \right\} + \beta \left\{ G^V(\phi X^V, Y^V)\xi^V - \eta^V\left(Y^V\right)\phi X^V \right\}$$

where α and β are functions on $(\bar{\mathcal{N}}')^v$, $\bar{\nabla}$ is the Finsler connection with respect to G^V. So, $(\bar{\mathcal{N}}')^v$ is called trans-Sasakian Finsler manifold.

2.1. Semi-invariant submanifolds of trans-Sasakian Finsler manifolds

Definition 2.2. *An m-dimensional Finsler submanifold $(\mathcal{N}')^v$ of a trans-Sasakian Finsler manifold $(\bar{\mathcal{N}}')^v$ is called a semi-invariant submanifold if $\xi^V \in V_{(u,v)}\mathcal{N}'$ and there exist on $(\bar{\mathcal{N}}')^v$ a pair of orthogonal distribution $(D, D^{\perp})$ such that*
(i) $V\mathcal{N}' = D \oplus D^{\perp} \oplus \{\xi^V\}$
(ii) $\phi D_{(u,v)} = D_{(u,v)}$, $\forall (u,v) \in (\mathcal{N}')^v, \forall u \in \mathcal{N}$

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(iii) $\phi\left(D^{\perp}_{(u,v)}\right) \subset \left(V_{(u,v)}\mathcal{N}'\right)^{\perp}$ *for all* $(u,v) \in (\mathcal{N}')^{v}$*, for tangential space* $V_{(u,v)}\mathcal{N}'$ *and normal space* $\left(V_{(u,v)}\mathcal{N}'\right)^{\perp}$ *of* $(\mathcal{N}')^{v}$ *at V with the following decomposition :*

$$V_{(x,y)}\bar{\mathcal{N}}' = (V_{(u,v)}\mathcal{N}') \oplus (V_{(u,v)}\mathcal{N}')^{\perp}$$

The distribution D (resp. $D^{\perp}$) is called the horizontal (resp. vertical) distribution. A semi-invariant Finsler submanifold $(\mathcal{N}')^{v}$ is said to be an invariant (resp. anti-invariant) submanifold if we have $D^{\perp}_{(u,v)} = \{0\}$ (resp. $D_{(u,v)} = \{0\}$) for each $(u,v) \in (\mathcal{N}')^{v}$. We also call $(\mathcal{N}')^{v}$ proper if neither D nor $D^{\perp}$ is null. It is easy to check that each hypersurface of $(\mathcal{N}')^{v}$ which is tangent to ξ^{V} inherits a structure of semi-invariant Finsler submanifold of $(\mathcal{N}')^{v}$.

We denote by G the metric tensor field of $(\bar{\mathcal{N}}')^{v}$ as well as that ınduced on $(\mathcal{N}')^{v}$. Let $\bar{\nabla}$ be a Finsler connection on $\bar{F}^{2n+1} = (\bar{\mathcal{N}}, \bar{\mathcal{N}}', \bar{F})$. Thus ∇ is a Finsler connection on $F^{m} = (\mathcal{N}, \mathcal{N}', F)$ which we call the induced Finsler connection. Also B is an $\Im(\mathcal{N}')$-bilinear mapping on $\Gamma(V\mathcal{N}') \times \Gamma(V\mathcal{N}')$ and $\Gamma(V\mathcal{N}'^{\perp})$-valued, which we call the second fundamental form of F^{m}.
Using B define the $\Im(\mathcal{N}')$-bilinear mapping:

$$h^{V} : \Gamma(V\mathcal{N}') \times \Gamma(V\mathcal{N}') \to \Gamma(V\mathcal{N}')^{\perp}$$

$$h(X^{V}, Y^{V}) = B(X^{V}, Y^{V})$$

for any $X, Y \in \Gamma(T\mathcal{N}')$. We call h^{V} the v-second fundamental form of $F^{m} = (\mathcal{N}, \mathcal{N}', F)$. From Gauss formula we get;

$$\bar{\nabla}^{V}_{X} Y^{V} = \nabla^{V}_{X} Y^{V} + h^{V}(X^{V}, Y^{V}) \tag{2.1}$$

for any $X, Y \in \Gamma(T\mathcal{N}').(X^{V}, Y^{V} \in \Gamma(V\mathcal{N}'))$.
Now, for any $X \in \Gamma(T\mathcal{N}')$ and $N \in \Gamma(V\mathcal{N}'^{\perp})$, we set

$$\bar{\nabla}_{X} N = -A_{N} X + \nabla^{\perp}_{X} N \tag{2.2}$$

where $A_{N}X \in \Gamma(V\mathcal{N}')$ and $\nabla^{\perp}_{X} N \in \Gamma(V\mathcal{N}') \perp$.
It follows that $\nabla^{\perp}$ is a linear connection on the Finsler normal bundle $(V\mathcal{N}')^{\perp}$ of F^{m}. Therefore $\nabla^{\perp}$ is a vectorial Finsler connection on $V\mathcal{N}'^{\perp}$. We call the normal Finsler connection with respect to $\bar{\nabla}$.

$$A^{V} : \Gamma(V\mathcal{N}' \perp) \times \Gamma(V\mathcal{N}') \to \Gamma(V\mathcal{N}')$$

$$A^{V}(N^{V}, X^{V}) = A_{N^{V}} X^{V}$$

is an $\Im(\mathcal{N}')-$ bilinear mapping for any $N^{V} \in \Gamma(V\mathcal{N}'^{\perp})$. We call A_{N} the shape operator (the Weingarten operator) with respect to N^{V}.
As in the case of the second fundamental form, by means of A we define for any $N^{V} \in \Gamma(V\mathcal{N}') \perp$ the $\Im(\mathcal{N}')-$ linear mappings;

$$A^{V}_{N} : \Gamma(V\mathcal{N}') \to \Gamma(V\mathcal{N}')$$

$$A^{V}_{N} X^{V} = A_{N^{V}} X^{V}$$

and call the v-shape operator. Thus from the Weingarten formula we deduce that

$$\bar{\nabla}_{X^{v}} N^{V} = -A^{V}_{N} X^{V} + \nabla^{\perp}_{X^{v}} N^{V}$$

for any $X \in \Gamma(T\mathcal{N}')$ $X^{V} \in \Gamma(V\mathcal{N}')$ and $N^{V} \in \Gamma(V\mathcal{N}'^{\perp})$.
Moreover we have

$$G(h^{V}(X^{V}, Y^{V}), N^{V}) = G(A^{V}_{N} X^{V}, Y^{V}) \tag{2.3}$$

for a vector field $X^{V} \in V\mathcal{N}'$. We put

$$X^{V} = PX^{V} + QX^{V} + \eta^{V}\left(X^{V}\right)\xi^{V} \tag{2.4}$$

where PX^{V} and QX^{V} belong to the distribution D and $D^{\perp}$ respectively.
For any vector field $N^{V} \in \Gamma(V\mathcal{N}' \perp)$, we put

$$\phi N^{V} = fN^{V} + qN^{V}$$

where fN^{V}(resp. qN^{V}) denotes the tangential (resp. normal) component of ϕN^{V}.

3. Mixed totally geodesic semi-invariant submanifolds of trans-Sasakian Finsler manifolds

Definition 3.1. *A semi-invariant Finsler submanifold is said to be mixed totally geodesic if* $h(X^V, Z^V) = 0$ *for all* $X^V \in D$ *and* $Z^V \in D^{\perp}$.

Theorem 3.2. *Let* $(\mathscr{N}')v$ *be a semi-invariant submanifold of trans-Sasakian Finsler manifold* $(\mathscr{N}')v$*. Then*

$$P\nabla_{X^V}(fN^V) - PA^V_{qN^V}X^V + \phi PA^V_{N^V}X^V = 0 \tag{3.1}$$

$$Q\nabla_{X^V}(fN^V) - QA^V_{qN^V}X^V - f\nabla^{\perp}_{X^V}N^V = 0 \tag{3.2}$$

$$h^V(X^V, fN^V) + \nabla^{\perp}_{X^V}(qN^V) + \phi QA^V_{N^V}X^V - q\nabla^{\perp}_{X^V}N^V = 0 \tag{3.3}$$

$\forall X^V \in D$ *and* $\forall N^V \in (V_{(u,v)}\mathscr{N}'^{\perp})$.

Proof.

$$\bar{\nabla}_{X^V}(\phi N^V) = \bar{\nabla}_{X^V}(fN^V + qN^V) = \bar{\nabla}_{X^V}(fN^V) + \bar{\nabla}_{X^V}(qN^V) \tag{3.4}$$

All $N^V \in (V_{(u,v)}\mathscr{N}'^{\perp})$, $\forall X^V \in D$.
For $fN \in V_{(u,v)}\mathscr{N}'$, we have from (2.1)

$$\bar{\nabla}_{X^V}(fN^V) = \nabla_{X^V}(fN^V) + h^V(X^V, fN^V) \tag{3.5}$$

for $qN \in (V_{(u,v)}\mathscr{N}'^{\perp})$; we have from (2.2)

$$\bar{\nabla}_{X^V}(qN^V) = -A^V_{qN^V}X^V + \nabla^{\perp}_{X^V}(qN^V) \tag{3.6}$$

By using (3.5) and (3.6) in (3.4), we get

$$\bar{\nabla}_{X^V}(\phi N^V) = \nabla_{X^V}(fN^V) + h^V(X^V, fN^V) - A^V_{qN^V}X^V + \nabla^{\perp}_{X^V}(qN^V) \tag{3.7}$$

where $\nabla_{X^V}(fN^V) \in \left(V_{(u,v)}\mathscr{N}'\right)$ and $A^V_{qN^V}X^V \in \left(V_{(u,v)}\mathscr{N}'\right)$, We have from (2.3)

$$\nabla_{X^V}(fN^V) = P\nabla_{X^V}(fN^V) + Q\nabla_{X^V}(fN^V) + \eta^V(\nabla_{X^V}(fN^V))\xi^V \tag{3.8}$$

and

$$A^V_{qN^V}X^V = PA^V_{qN^V}X^V + QA^V_{qN^V}X^V + \eta^V(A^V_{qN^V}X^V)\xi^V \tag{3.9}$$

by using (3.8) and (3.9) in (3.7) we obtain

$$\begin{aligned}
\bar{\nabla}_{X^V}(\phi N^V) &= (\bar{\nabla}_{X^V}\phi)N^V + \phi(\bar{\nabla}_{X^V}N^V) \\
&= P\nabla_{X^V}(fN^V) + Q\nabla_{X^V}(fN^V) + \eta^V(\nabla_{X^V}(fN^V))\xi^V \\
&\quad + h^V(X^V, fN^V) - PA^V_{qN^V}X^V - QA^V_{qN^V}X^V \\
&\quad - \eta(A^V_{qN^V}X^V)\xi^V + \nabla^{\perp}_{X^V}(qN^V)
\end{aligned}$$

where

$$\begin{aligned}
(\bar{\nabla}_{X^V}\phi)N^V &= \frac{\alpha}{2}\left\{G(X^V, N^V)\xi^V - \eta^V(N^V)X^V\right\} \\
&\quad + \frac{\beta}{2}\left\{G(\phi X^V, N^V)\xi^V - \eta^V(N^V)\phi X^V\right\}
\end{aligned}$$

Since $G(X^V, N^V) = 0 = G(N^V, \xi^V) = G(\phi X^V, N^V)$, we get $\left(\bar{\nabla}_X\phi\right)N = 0$. Thus, we using (2.3) and (2.4) from (3.10) then we obtain

$$\begin{aligned}
\bar{\nabla}_{X^V}(\phi N^V) &= \phi(\bar{\nabla}_{X^V}N^V) = \phi(-A^V_{N^V}X^V + \bar{\nabla}^{\perp N^V}_{X^V}) \\
&= -\phi A^V_{N^V}X^V + \phi\nabla^{\perp}_{X^V}N^V \\
&= -\phi PA^V_{N^V}X^V - \phi Q\phi A^V_{N^V}X^V + f\nabla^{\perp}_{X^V}N^V + q\nabla^{\perp}_{X^V}N^V
\end{aligned} \tag{3.10}$$

where $A_N \in V_{(u,v)}\mathscr{N}' and \nabla^{\perp}_{X^V}N^V \in (V_{(u,v)}\mathscr{N}'^{\perp})$. By seperating the components of $D\ D^{\perp}$ and $(V_{(u,v)}\mathscr{N}'^{\perp})$ from (3.10) and (3.10) we get (3.1),(3.2) and (3.3). □

Theorem 3.3. *Let* $(\mathscr{N}')^v$ *be a semi-invariant submanifold of trans-Sasakian Finsler manifold* $(\bar{\mathscr{N}}')^v$*. Then the following propositions are equivalent:*
(a) $(\mathscr{N}')^v$ *is a totally geodesic.*
(b) $\nabla^{\perp}_{X^V}N^V \in \phi\ D^{\perp}$ *and D is invariant with respect to* A^V_N *(all* $N^V \in \phi\ D^{\perp}$*), that is* $\nabla^{\perp}_D(\phi\ D^{\perp}) \subset \phi\ D^{\perp}$ *and* $A^V_{\phi D^{\perp}}D \subset D$.

Proof. From (2.4) we know that,

$$\phi N^V = fN^V = Y^V, \text{ all } Y^V \in D^\perp, N \in \phi D^\perp \subset (V_{(u,v)}\mathscr{N}'^\perp)$$

by using (3.2) from (3.3) we have

$$h^V(X^V,Y^V) + \nabla^\perp_{X^V}(qN^V) - \phi Q A^V_{N^V} X^V - q\nabla^\perp_{X^V} N^V = 0 \tag{3.11}$$

where, since $Y^V \in D^\perp, N \in \phi D^\perp$, we can write $qN^V = 0$. Thus from (3.11) we have

$$h^V(X^V,Y^V) = q\nabla^\perp_{X^V} N^V - \phi Q A^V_{N^V} X^V \tag{3.12}$$

Now, suppose that $(\mathscr{N}')^v$ a total geodesic. Because of $h^V(X^V,Y^V) = 0$, $\forall X^V \in D$ and $Y^V \in D^\perp$, from (3.12) we get

$$0 = q\nabla^\perp_{X^V} N^V - \phi Q A^V_{N^V} X^V$$

where $A^V_{N^V} X^V \in (V_{(u,v)}\mathscr{N}'^\perp)$, $QA^V_{N^V}X^V \in D^\perp$ and $\phi Q A^V_{N^V} X^V \in \phi D^\perp \subset (V_{(u,v)}\mathscr{N}'^\perp)$. If $q\nabla^\perp_{X^V} N^V \in \phi D^\perp$, it must be $\phi N^V = fN^V = Y^V \in D^\perp$, $\forall N^V \in \phi D^\perp$. Thus we have $\phi(q\nabla^\perp_{X^V} N^V) \in D^\perp$. Also from (2.4) we can write

$$\phi \nabla^\perp_{X^V} N^V = f\nabla^\perp_{X^V} N^V + q\nabla^\perp_{X^V} N^V, \nabla^\perp_X N \in (V_{(u,v)}\mathscr{N}'^\perp)$$

if we apply ϕ on both sides of the equation we get

$$-\nabla^\perp_{X^V} N^V = \phi(f\nabla^\perp_{X^V} N^V) + \phi(q\nabla^\perp_{X^V} N^V) \tag{3.13}$$

where if $f\nabla^\perp_{X^V} N^V \in D^\perp \subset V_{(u,v)}\mathscr{N}'$, then it means $\phi f\nabla^\perp_{X^V} N^V \in \phi D^\perp \subset (V_{(u,v)}\mathscr{N}'^\perp)$. In equation (3.13), since $\nabla^\perp_{X^V} N^V \in (V_{(u,v)}\mathscr{N}'^\perp)$ and $\phi\left(f\nabla^\perp_{X^V} N^V\right) \in \phi D^\perp$, it means that $\phi\left(q\nabla^\perp_{X^V} N^V\right) \notin D^\perp.(\phi(q\nabla^\perp_X N) \in (V_{(u,v)}\mathscr{N}'^\perp))$. If $f\nabla^\perp_{X^V} N^V \notin D^\perp$, then $\phi(f\nabla^\perp_X N) \in V\mathscr{N}'_v$, while $\nabla^\perp_X N \in (V_{(u,v)v}\mathscr{N}'^\perp)$ and $\phi(f\nabla^\perp_X N) \in V_{(u,v)}\mathscr{Y}'$ either $\phi(q\nabla^\perp_X N) \in \left(V_{(u,v)}\mathscr{N}'^\perp\right)$ or $\phi(q\nabla^\perp_X N) \in V_{(u,v)}\mathscr{N}'$. If $\phi(q\nabla^\perp_X N) \in V_{(u,v)}\mathscr{N}'$, we get the following contradiction

$$\phi(q\nabla^\perp_X N) = \phi(f\nabla^\perp_X N) \tag{3.14}$$

$$q\nabla^\perp_X N = f\nabla^\perp_X N$$

In that case $\phi(q\nabla^\perp_{X^V} N^V) \notin V_{(u,v)}\mathscr{N}'$ $(\notin D^\perp)$. Thus we get $q\nabla^\perp_X N \in \left(V_{(u,v)}\mathscr{N}'^\perp - \phi D^\perp\right)$ in (3.14). Since $q\nabla^\perp_{X^V} N^V \in \left\{(VM'^\perp_v) - \phi D^\perp\right\}$ and $\phi Q A^V_{N^V} X^V \in \phi D^\perp$, it must be $q\nabla^\perp_{X^V} N^V = 0$ and $\phi Q A^V_{N^V} X^V = 0$. Since $q\nabla^\perp_{X^V} N^V = 0$, it means that $\nabla^\perp_X N \in \phi D^\perp$ and since $QA^V_{N^V} X^V = 0$, then $A^V_{N^V} X^V \in D$. Thus we get $\nabla^\perp_D \phi D^\perp$ and $A_{\phi D^\perp} D \subset D$. □

Theorem 3.4. *Let $(\mathscr{N}')^v$ be a semi-invariant submanifolds of trans-Sasakian Finsler manifold $(\bar{\mathscr{N}}')^v$. If $\beta \neq 0$, then each $M'^\perp_v$ leaf of $D^\perp$ is not totally geodesic at $(\mathscr{N}')^v$.*

Proof. Suppose that $((\mathscr{N}')^v)^\perp$ is totally geodesic in $(\mathscr{N}')^v$. Then $\nabla_{X^V} Y^V \in D^\perp$, for each $X^V, Y^V \in D^\perp$ or equivalent to $G(\nabla_{X^V} Y^V, Z^V) = 0$, for each $Z^V \in D \oplus \left\{\xi^V\right\}$. Using the

$$\nabla^V_Y \xi^V = \frac{\beta}{2} Y^V \text{ and } h^V(Y^V,\xi^V) = -\frac{\alpha}{2}\phi Y^V$$

we get

$$G(\nabla_{X^V} Y^V, \xi^V) = -G(Y^V, \nabla_{X^V}\xi^V) = -G(Y^V, \frac{\beta}{2} X^V) = -\frac{\beta}{2} G(Y^V, X^V)$$

Thus, we find the following contradiction

$$0 = G(\nabla_{X^V} X^V, \xi^V) = -\frac{\beta}{2} G(X^V, X^V)$$

That is, $((\mathscr{N}')^v)^\perp$ is not total geodesic at $(\mathscr{N}')^v$. □

4. Totally umbilical semi-invariant submanifolds of trans- Sasakian Finsler manifolds

Definition 4.1. *$\forall X^V, Y^V \in V\mathscr{N}'$ and $N^V \in V\mathscr{N}'^{\perp}$*
(1) If $A^V_{N^V} = aI$ (for $a \in \Im(\mathscr{N}')^v)$, N^V is called umbilical section of $(\mathscr{N}')^v$. (2) If N^V is umbilical section of $(\mathscr{N}')^v$ then $(\mathscr{N}')^v$ is umbilical with respect to N^V. (3) If $(\mathscr{N}')^v$ is umbilical for each $N^V \in V_{(u,v)}\mathscr{N}'^{\perp}$ then $(\mathscr{N}')^v$ is called totally umbilical submanifold of $(\mathscr{N}')^v$. (4) Suppose that $\{E^V_1, ..., E^V_m\}$ orthonormal base of $V_{(u,v)}\mathscr{N}'$. Then

$$H = \frac{1}{m} iz(h_{(u,v)}) = \frac{1}{m}\sum_{i=1}^{m} h^V(E^V_i, E^V_i)$$

is called mean curvature vector of $(\mathscr{N}')^v$ at $u \in (\mathscr{N}')^v$.

If $\{E^V_{m+1}, ..., E^V_{2n+1}\}$ is orthonormal base of $V_{(u,v)}\mathscr{N}'^{\perp}$, then we can write

$$H = \frac{1}{m}\sum_{a=m+1}^{2n+1} iz(A^V_a)E^V_a,\ A^V_a = A^V_{E^V_a} \tag{4.1}$$

Let $(\mathscr{N}')^v$ be a semi-invariant submanifold of trans-Sasakian Finsler manifold $(\bar{\mathscr{N}}')^v$. Since

$$h^V(X^V, Y^V) = \sum_{a=m+1}^{2n+1} G(h^V(X^V, Y^V), E^V_a)E^V_a$$

and

$$G(h^V(X^V, Y^V), E^V_a) = G(A^V_{E^V_a} X^V, Y^V)$$

we have

$$h^V(X^V, Y^V) = \sum_{a=m+1}^{2n+1} G(A^V_{E^V_a} X^V, Y^V)E^V_a$$

Since $(\mathscr{N}')^v$ is totally umbilical submanifold of $(\bar{\mathscr{N}}')^v$, we have

$$A^V_{E^V_a} X^V = C_a X^V,\ C_a \in \Im(\mathscr{N}')^v \tag{4.2}$$

Thus we get

$$\begin{aligned} h^V(X^V, Y^V) &= \sum_{a=m+1}^{2n+1} G(C_a X^V, Y^V)E^V_a \\ &= \sum_{a=m+1}^{2n+1} C_a G(X^V, Y^V)E^V_a \\ &= G(X^V, Y^V)\left(\sum_{a=m+1}^{2n+1} C_a E^V_a\right) \end{aligned} \tag{4.3}$$

by using (4.1) and (4.2), we get

$$\begin{aligned} H &= \frac{1}{m}\sum_{a=m+1}^{2n+1} iz(A^V_{E^V_a})E^V_a = \frac{1}{m}\sum_{a=m+1}^{2n+1} iz(C_a I)E^V_a \\ &= \frac{1}{m}\sum_{a=m+1}^{2n+1} (mC_a)E^V_a = \sum_{a=m+1}^{2n+1} C_a E^V_a \end{aligned} \tag{4.4}$$

from (4.3) and (4.4) we obtain

$$h^V(X^V, Y^V) = G(X^V, Y^V)H \tag{4.5}$$

Theorem 4.2. *Let $(\mathscr{N}')^v$ be a semi-invariant submanifolds of trans-Sasakian Finsler manifold $(\bar{\mathscr{N}}')^v$. Then*
(a) $(\mathscr{N}')^v$ is a totally geodesic.
(b) If $\alpha \neq 0$ for every point of $(\mathscr{N}')^v$, then $(\mathscr{N}')^v$ is an invariant submanifold, that is $D^{\perp} = 0$.

Proof. For $X^V = \xi^V$, from (3.1) we get $\bar{\nabla}_{\xi^V}\xi^V = 0$. Later, we take ξ^V instead of X^V and Y^V from (2.1), we obtain

$$\bar{\nabla}_{\xi^V}\xi^V = \nabla_{\xi^V}\xi^V + h^V(\xi^V, \xi^V)$$

since $\bar{\nabla}_{\xi^V}\xi^V = 0$, we have

$$0 = \nabla_{\xi^V}\xi^V + h^V(\xi^V, \xi^V)$$

that is $\nabla_{\xi^V}\xi^V = 0$ and $h^V(\xi^V, \xi^V) = 0$. Since $(\mathcal{N}')^v$ is totally umbilical submanifold, we have from (4.5)

$$0 = h^V(\xi^V, \xi^V) = G(\xi^V, \xi^V)H$$

since $G(\xi^V, \xi^V) \neq 0$, it must be $H = 0$. Thus we have

$$h^V(X^V, Y^V) = G^V(X^V, Y^V)0 = 0$$

This means that $(\mathcal{N}')^v$ is totally geodesic. We know that $\nabla^V_Y \xi^V = \frac{\beta}{2} Y^V$ and $h^V(Y^V, \xi^V) = -\frac{\alpha}{2}\phi Y^V$ for all $Y^V \in D^\perp$. Since $(\mathcal{N}')^v$ is totally geodesic and totally umbilical, we get

$$-\frac{\alpha}{2}\phi Y^V = G^V(Y^V, \xi^V)0 = 0$$

Since $\alpha \neq 0$, this means that

$$\phi Y^V = 0 \rightarrow Y^V = 0 \rightarrow D^\perp = 0$$

□

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Coding Matrices for $GL(2,q)$

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Abstract

We use the BN-pair structure for the general linear group to write a suitable listing of the elements of the finite group $GL(2,q)$ which is then used to determine its ring of matrices. This approach of identifying finite group ring with ring of matrices has been used effectively to construct linear codes, benefiting from the ring-theoretic structure of both group rings and the ring of matrices.

1. Introduction

Group rings of finite groups became a rich source for constructing error-correcting codes and investigation, their properties since F.J. MacWilliams [1] and S.D Berman [2] considered cyclic codes as ideals in the group algebra of finite cyclic groups. R. Ferraz and Polcino Milies [3] brought the techniques and deep structure of the group algebras into play by studying idempotents which generate codes. In [4] T. Hurley proved that the group ring RG of a finite group G of order n over a ring R is isomorphic to a ring of G-matrices of size $n\times n$ over R, and this was used in many later papers to construct and analyse codes from units and zero-divisors. When R has an identity and no zero-divisors (e.g. when R is a field), Hurley used this identification to describe the unit group $U(RG)$ and zero-divisors of RG in terms of the properties of their corresponding matrices. The first (and main) step towards getting codes from a group ring RG is to choose an appropriate listing for the elements of G upon which depend other steps namely; finding the matrix of G (relative to the listing), the ring of matrices of RG and constructing unit-type and zero-divisor-type codes (all this steps are explained in [5]). The types of matrices have been determined for several classes of finite groups such as cyclic, elementary abelian and dihedral groups [4]. Matrices which appear in this identification include several types such as circulant, Toeplitz, Walsh-Toeplitz and Hankel Matrices. In this paper the linear group $G = GL(2,q)$ is considered; we shall use the BN-pair structure of G (see [6], section 69) to choose a listing for its elements suitable for determining the matrix of G and hence the ring of matrices of RG. Being the first linear group to be considered in this manner we hope this will lead to constructing new linear (unit-type and zero-divisor-type) codes.

2. The ring of matrices of a group

Let G be a finite group of order n with a given listing $G = \{g_1, g_2, \dots, g_n\}$, and let R be a ring. Consider the matrix of the group G relative to its listing, say $M(G)$, which has the following form:

$$M(G) = \begin{pmatrix} g_1^{-1}g_1 & g_1^{-1}g_2 & \cdots & g_1^{-1}g_n \\ g_2^{-1}g_1 & g_2^{-1}g_2 & \cdots & g_2^{-1}g_n \\ \vdots & \vdots & \vdots & \vdots \\ g_n^{-1}g_1 & g_n^{-1}g_2 & \cdots & g_n^{-1}g_n \end{pmatrix}$$

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Now let $u = \sum_{i=1}^{n} \alpha_{g_i} g_i$ be an element in the group ring RG. Then the RG-matrix which corresponds to u in $R_{(n\times n)}$, the ring of $(n\times n)$-matrices, is given by:

$$M(RG,u) = \begin{pmatrix} \alpha_{g_1^{-1}g_1} & \alpha_{g_1^{-1}g_2} & \cdots & \alpha_{g_1^{-1}g_n} \\ \alpha_{g_2^{-1}g_1} & \alpha_{g_2^{-1}g_2} & \cdots & \alpha_{g_2^{-1}g_n} \\ \vdots & \vdots & \vdots & \vdots \\ g_n^{-1}g_1 & g_n^{-1}g_2 & \cdots & g_n^{-1}g_n \end{pmatrix}_{n\times n}$$

Theorem 2.1. *[4] Given a listing of the elements of a group G of order n. There is a bijective ring homomorphism* $\sigma : u \longmapsto M(RG,u)$ *between the group ring RG and the ring of* $(n\times n)$ *RG-matrices over R.*

There are several types of the RG-matrix which appear as isomorphic to a certain group rings. These types include Toeplitz-type matrices, Walsh-Toeplitz matrices, circulant matrices, Toeplitz combined with Hankel-type matrices and block-type circulant matrices; see [4] for more specifics and examples.

3. BN-pair structure of $G = GL(2,q)$

Definition 3.1. *[6] A finite group* $G = (G,B,N,U,R,W)$ *is said to have a split BN-pair of rank n if the following conditions are satisfied:*

- *G has a BN-pair of rank n, such that:*
  - $G = <B,N>$,
  - $B\cap N = H \trianglelefteq N$,
  - $W = N/H$, *is the corresponding Coxeter (Weyl) group which is generated by involutions,* $W = < w_1, w_2, \ldots, w_r >$.
- *There exist a normal subgroup* $U \trianglelefteq B$ *such that* $B = U \rtimes H$ *(semidirect product),*
- $U = O_p(G)$, *and H is an abelian* p'*-group.*

We have the *Bruhat decomposition* ,see[6]

$$G = \dot{\bigcup_{w\in W}} BwB,$$

On the other hand $U = U_w^+.U_w^-$, where $U_w^+ = U\cap U^w$ and $U_w^- = U\cap U^{w_o w}$, where w_o is the unique element of the coxeter group W of maximal length.
Also we have,

$$BwB = BwU_w^-, \qquad for\ each\ w\in W.$$

Therefore,

$$G = \dot{\bigcup_{w\in W}} BwB = \dot{\bigcup_{w\in W}} BwU_w^-,$$

and each element in G can be written uniquely in the form bnu, where $b\in B$, $u\in U_w^-$, n is the coset representative of an element $w\in W$.
Now, for example, if $G = GL(n,q)$ then G has the structure of split BN-pair, where B is the subgroup of an upper triangular matrices, N is the subgroup of monomial matrices, and H is the subgroup of the diagonal matrices. In fact, the Coxeter group W of $G = GL(n,q)$ is isomorphic to the symmetric group S_n.

$$W = N/H \cong S_n.$$

We shall concentrate on the case when $n = 2$. From the split BN-pair setting, we have
$GL(2,q) = B\bigcup BwB$, where

$$U = \left\{\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \mid \lambda \in \mathbb{F}_q\right\},\ H = \left\{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x,y\in \mathbb{F}_q^*\right\}.$$

Since we have $BwB = BwU_w^-$, $\forall w\in W$, where $U_w^- = U\cap U^{w_o w}$, and since the Coxeter group W in this case is isomorphic to $S_2 = \{e\ ,\ (12)\}$. Then $w = w_o = (12)$, and $U_w^- = U\cap U^{w_o w_o} = U$. Thus,

$$G = B\cup Bw_oU = B\cup Bn_oU,$$

where $n_oH = Hn_o = w_o$. The monomial subgroup N in our case have the form;

$$N = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \bigcup \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix},$$

and for $n_o\in N$, take $n_o = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which corresponds to the permutation (12) in S_2.
Therefore,

$$G = GL(2,q) = HU \cup HUn_oU$$

$$= \left\{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \cup \begin{pmatrix} x' & 0 \\ 0 & y' \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \mid \lambda,\alpha,\beta\in\mathbb{F}_q, and\ x,y\in\mathbb{F}_q^*\right\},$$

Counting the elements we have,

$$\left|\left\{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \cup \begin{pmatrix} x' & 0 \\ 0 & y' \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} | \lambda, \alpha, \beta \in \mathbb{F}_q, and\ x, y \in \mathbb{F}_q^* \right\}\right|$$

$$= (q-1)^2.\ q \ + \ (q-1)^2.\ q^2 \ = \ (q-1)^2.\ (q+q^2) \ = \ |GL(2,q)|.$$

Notation :
We write $h(x\ ,\ y) \ = \ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in H; x,y \in \mathbb{F}_q^*$ and $u(\lambda) \ = \ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \in U;\ \lambda \in \mathbb{F}_q$.

4. Multiplications

In the light of the coset decomposition of $GL(2,q) = B \cup Bn_oU = HU \cup HUn_oU$, we shall discuss four cases of element multiplication in $G = GL(2,q)$:

$*$	HU	HUn_oU
HU	CASE 1	CASE 2
HUn_oU	CASE 3	CASE 4

Proposition 4.1. *For each $x,x',y,y' \in \mathbb{F}_q^*$ and $\lambda, \lambda', \beta' \beta' \in \mathbb{F}_q$ we define the multiplication as the following:*
Case 1*:*
$h(x'\ ,\ y')\ u(\lambda')\ \cdot h(x\ ,\ y)\ u(\lambda) = \ h(x'x\ ,\ y'y)\ u(\lambda + x^{-1}\lambda' y)$
Case 2*:*
$h(x'\ ,\ y')\ u(\lambda')\ \cdot h(x\ ,\ y)\ u(\lambda)\ n_o\ u(\beta) = \ h(x'x\ ,\ y'y)\ u(\lambda + x^{-1}\lambda' y)\ n_o\ u(\beta)$
Case 3*:*
$h(x'\ ,\ y')\ u(\lambda')\ n_o\ u(\beta')\ \cdot h(x\ ,\ y)\ u(\lambda) \ = \ h(x'y\ ,\ y'x)\ u(y^{-1}\lambda' x)\ n_o\ u(\lambda +\ x^{-1}\beta' y)$
Case 4*:*
$h(x',y')u(\lambda')\ n_o\ u(\beta')\ \cdot h(x,y)\ u(\lambda)\ n_o\ u(\beta) = \ h(-x'y\alpha^{-1}, y'x\alpha)\ u(-\alpha - \alpha^2 y^{-1}\lambda' x)\ n_o\ u(\beta + \alpha^{-1})$
where, $\alpha = \ \lambda +\ x^{-1}\beta' y$.

***Special case** :*
If $\alpha = \ \lambda +\ x^{-1}\beta' y = \ 0$ the multiplication will be as :
$h(x',y')u(\lambda')\ n_o\ u(\beta')\ \cdot h(x,y)\ u(\lambda)\ n_o\ u(\beta) = \ h(x'y\ ,\ y'x)\ u(\beta +\ y^{-1}\lambda' x)$

5. Inverses

The following proposition gives the rule for getting the inverses of the elements of $GL(2,q)$.

Proposition 5.1. *For each $x,y \in \mathbb{F}_q^*$ and $\lambda, \beta \in \mathbb{F}_q$ there are two cases for getting the inverse:*
*1- **The element of the form** $h(x\ ,\ y)\ u(\lambda)$:*
$[h(x\ ,\ y)\ u(\lambda)]^{-1} \ = h(x^{-1}, y^{-1})\ u(-x\lambda y^{-1})$
*2- **The element of the form** $h(x\ ,\ y)\ u(\lambda)\ n_o\ u(\beta)$:*
$[h(x\ ,\ y)\ u(\lambda)\ n_o\ u(\beta)]^{-1} = \ h(y^{-1}, x^{-1})\ u(-x^{-1}\beta y)\ n_o\ u(-x\lambda y^{-1})$

6. Elements listing of $G \ = \ GL(2,q)$

The listing of this group depends on the number q and we discuss two cases:
Case1: when $q = p$ is an odd-prime ($q \ = \ p > 3$):
In this case, the linear group $G = GL(2,p)$ has (p^2-1) blocks, each consists of $\left(\dfrac{p-1}{2}\right)$ matrices each of size $(2p)$, (note that $(p^2-1)(\dfrac{p-1}{2})(2p) = (p-1)^2(p^2+p) = |GL(2,p)|$) obtained by the following listing:
Type (x,x); $x \in \mathbb{Z}_p^*$ gives $(p+1)$ blocks:

(1) THE BLOCK $B(x,x)$; $x \in \mathbb{Z}_p^*$, obtained from the following listing subset:
$T(x,x): h(x,x),\ h(p-x,p-x)u(p-1),\ h(x,x)u(p-2), h(p-x,p-x)u(p-3),\ \cdots,\ \cdots,\ h(x,x)u(1),\ \cdots,\ \cdots,\ \cdots,\ h(x,x)u(2),$
$h(p-x,p-x)u(1)$.
such that, $x = 1,2,\cdots,\dfrac{p-1}{2}$.
THE BLOCKS $B(x,x)(\lambda)$**;** $\lambda = \ 0,1,2,\cdots,p-1$, obtained from the listing subsets:
$T(x,x)n_o u(\lambda);\ \lambda = \ 0,1,2,\cdots,p-1$.

Type $(x,y); x \neq y$ gives the following (p^2-p-2) blocks:

THE BLOCKS $B_i(x,y)$**;** $i = 1,2,\cdots,p-2$, obtained from the listing subsets:

$T_i(x,y)(j);\ 1 \le i \le p-2,\ 1 \le j \le \frac{p-1}{2}$; where,
$T_i(x,y)(j):\ h(j,j(p-i)),\ h(j(p-1),ji)u(i),\ h(j,j(p-i))u(2i), h(j(p-1),ji)u(3i),\ \cdots,\ \cdots,\ h(j(p-1),ji)u((p-3)i),$
$h(j,j(p-i))u((p-2)i),\ h(j(p-1),ji)u((p-1)i).$

THE BLOCKS $B_i(x,y)(\lambda)$**;** $x \ne y,\ i = 1,2,\cdots,p-2,\ 0 \le \lambda \le p-1$, obtained from the listing subsets:
$T_i(x,y)(j)n_o u(\lambda);\ 1 \le i \le p-2,\ 1 \le j \le \frac{p-1}{2}.$
The total number of blocks in this case:
$(p+1)+(p^2-p-2) = p^2-1.$

Case2, when q **is a power of p** ($q = p^n$)**, and** ($n \ge 2,\ p \ge 2$) :
In this case we take $\mathbb{F}_q^* = < a | a^{q-1} = 1 >$, then the matrix of $G = GL(2,q)$ has (q^2-1) blocks each block consists of $\left(\frac{q}{p}\right)$ matrices each of size $p(q-1)$ (note that $(q^2-1)(\frac{q}{p})p(q-1) = (q-1)^2(q^2+q) = |GL(2,q)|$) obtained by the following listing:

Type (a^i,a^i) **gives the following** $(q+1)$ **blocks:**

THE BLOCK $B(a^i,a^i)$; $i = 1,2,\cdots,q-1$ obtained by the following listing subset:
$T(a^i,a^i),\ T(a^i,a^i)(1),\ T(a^i,a^i)(2),\ \cdots,\ T(a^i,a^i)((q/p)-1).$
Where,
$T(a^i,a^i):\ h(1,1),\ h(a^{q-2}\ ,\ a^{q-2})\ u(a^{q-2})\ ,\ h(a^{q-3}\ ,\ a^{q-3})\ u(2(a^{q-2}))\ ,\ \cdots,\ \cdots,\ \cdots,\ h(a^{q-p}\ ,\ a^{q-p})\ u((p-1)(a^{q-2}))\ ,$
$h(a^{q-(p+1)}\ ,\ a^{q-(p+1)})\ ,\ h(a^{q-(p+2)}\ ,\ a^{q-(p+2)})\ u(a^{q-2})\ ,\ \cdots,\ \cdots,\ h(a\ ,\ a)\ u((p-2)(a^{q-2}))\ ,\ h(1\ ,\ 1)\ u((p-1)(a^{q-2}))\ ,$
$h(a^{q-2},a^{q-2})\ ,\cdots,\ \cdots,\ h(a,a)u((p-1)(a^{q-2})).$
Such that, $T(a^i,a^i)(s) = T(a^i,a^i)u(a^{k_s});\ s = 1,2,\cdots,(q/p)-1,\ and\ a^{k_s} \in \{a,a^2,\cdots,a^{q-3}\}.$

THE BLOCKS $B(a^i,a^i)(\lambda)$; $\lambda = 0,1,2,\cdots,a^{q-2}$, obtained by the following listing subsets:
$T(a^i,a^i)n_o u(\lambda),\ T(a^i,a^i)(1)n_o u(\lambda),\ \cdots,\ T(a^i,a^i)((q/p)-1)n_o u(\lambda).$

Type (a^i,a^j)**;** $i \ne j$ **gives the following** (q^2-q-2) **blocks:**

THE BLOCKS $B_r(a^i,a^j)$**;** $r = 1,2,\cdots,q-2$, obtained from the listing subsets:
$T_r(a^i,a^j),\ T_r(a^i,a^j)(1),\ T_r(a^i,a^j)(2),\ \cdots,\ T_r(a^i,a^j)((q/p)-1)$, such that:
$T_r(a^i,a^j)(s) = T_r(a^i,a^j)u(a^{k_s});\ s = 1,2,\cdots,(q/p)-1.$
Where:
$T_r(a^i,a^j):\ h(1,a^r)\ ,\ h(a^{q-2},a^{r-1})u(a^{r-1})\ ,\ h(a^{q-3},a^{r-2})u(2a^{r-1})\ ,\ h(a^{q-4},a^{r-3})u(3a^{r-1})\ ,\ \cdots\ ,\ \cdots\ ,\ h(a^{q-p},a^{(q-p)+r})u((p-1)a^{r-1})\ ,$
$h(a^{(q-p)-1},a^{(q-p)+(r-1)})\ ,\ h(a^{(q-p)-2},a^{(q-p)+(r-2)})u(a^{r-1})\ ,\cdots\cdots,\ \cdots\ ,\ h(a,a^{r+1})u((p-1)a^{r-1}).$

THE BLOCKS $B_r(a^i,a^j)(\lambda)$**;** $r = 1,2,\cdots,q-2,\ 0 \le \lambda \le a^{q-2}$, obtained from the listing subsets:
$T_r(a^i,a^j)n_o u(\lambda),\ T_r(a^i,a^j)(1)n_o u(\lambda),\ T_r(a^i,a^j)(2)n_o u(\lambda),\ \cdots\cdots,\ T_r(a^i,a^j)((q/p)-1)n_o u(\lambda).$
The total number of blocks in this case:
$(q+1)+(q^2-q-2) = q^2-1.$

Example 6.1. ***The special case when*** $(q = 3,\ G = GL(2,3))$ *:*
From the general theory, the matrix of this group has $3^2-1 = 8$ *blocks each block consists of* $\frac{3-1}{2} = 1$ *matrix of size* $2 \times 3 = 6$ *obtained from the following listing:*
Block $B(x,x)$***:***
$h(1,1),\ h(2,2)u(2),\ h(1,1)u(1),\ h(2,2),\ h(1,1)u(2),\ h(2,2)u(1),$
Block $B(x,x)(\lambda = 0)$***:***
$h(1,1)n_o,\ h(2,2)u(2)n_o,\ h(1,1)u(1)n_o,\ h(2,2)n_o,\ h(1,1)u(2)n_o,\ h(2,2)u(1)n_o,$
Block $B(x,x)(\lambda = 1)$***:***
$h(1,1)n_o u(1),\ h(2,2)u(2)n_o u(1),\ h(1,1)u(1)n_o u(1),\ h(2,2)n_o u(1),\ h(1,1)u(2)n_o u(1),$
$h(2,2)u(1)n_o u(1),$
Block $B(x,x)(\lambda = 2)$***:***
$h(1,1)n_o u(2),\ h(2,2)u(2)n_o u(2),\ h(1,1)u(1)n_o u(2),\ h(2,2)n_o u(2),\ h(1,1)u(2)n_o u(2),$
$h(2,2)u(1)n_o u(2),$
Block $B_1(x,y)$***:***
$h(1,2),\ h(2,1)u(1),\ h(1,2)u(2),\ h(2,1),\ h(1,2)u(1),\ h(2,1)u(2),$
Block $B_1(x,y)(\lambda = 0)$***:***
$h(1,2)n_o,\ h(2,1)u(1)n_o,\ h(1,2)u(2)n_o,\ h(2,1)n_o,\ h(1,2)u(1)n_o,\ h(2,1)u(2)n_o,$
Block $B_1(x,y)(\lambda = 1)$***:***
$h(1,2)n_o u(1),\ h(1,2)u(1)n_o u(1),\ h(2,1)u(2)n_o u(1),\ h(2,1)n_o u(1),\ h(2,1)u(1)n_o u(1),\ h(1,2)u(2)n_o u(1),$
Block $B_1(x,y)(\lambda = 2)$***:***
$h(1,2)n_o u(2),\ h(1,2)u(1)n_o u(2),\ h(2,1)u(2)n_o u(2),\ h(2,1)n_o u(2),\ h(2,1)u(1)n_o u(2),\ h(1,2)u(2)n_o u(2).$

7. The matrix of $G = GL(2,q)$

Now we shall determine the G-matrix of the group $GL(2,q)$ with respect to the listing which obtained in the previous section:
We start with the first case when $q = p$. Consider $T(1,1)$ in the first block $B(x,x)$ which has the elements $g_1, g_2, \cdots, \cdots, g_{2p-2}, g_{2p-1}, g_{2p}$ with their inverses. Then the resulting matrix will have the following form:

$$\begin{pmatrix}
g_1 & g_2 & g_3 & g_4 & g_5 & \cdots & g_{2p-3} & g_{2p-2} & g_{2p-1} & g_{2p} \\
g_{2p} & g_1 & g_2 & g_3 & g_4 & \cdots & g_{2p-4} & g_{2p-3} & g_{2p-2} & g_{2p-1} \\
g_{2p-1} & g_{2p} & g_1 & g_2 & g_3 & \cdots & \cdots & g_{2p-4} & g_{2p-3} & g_{2p-2} \\
g_{2p-2} & g_{2p-1} & g_{2p} & g_1 & g_2 & \cdots & \cdots & \cdots & g_{2p-4} & g_{2p-3} \\
g_{2p-3} & g_{2p-2} & g_{2p-1} & g_{2p} & g_1 & \cdots & \cdots & \cdots & \cdots & g_{2p-4} \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
g_5 & \cdots & \cdots & \cdots & \cdots & \cdots & g_1 & g_2 & g_3 & g_4 \\
g_4 & g_5 & \cdots & \cdots & \cdots & \cdots & g_{2p} & g_1 & g_2 & g_3 \\
g_3 & g_4 & g_5 & \cdots & \cdots & \cdots & g_{2p-1} & g_{2p} & g_1 & g_2 \\
g_2 & g_3 & g_4 & g_5 & \cdots & \cdots & g_{2p-2} & g_{2p-1} & g_{2p} & g_1
\end{pmatrix}$$

This is a circulant matrix type.
Now for any subset in the the block $B(x,x)$. Consider the elements of $T(t,t)$: $g_{t_1}, g_{t_2}, \cdots, \cdots, g_{t_{2p-1}}, g_{t_{2p}}$ with the inverses of the elements in $T(1,1)$, then we will get the following matrix:

$$\begin{pmatrix}
g_{t_1} & g_{t_2} & g_{t_3} & g_{t_4} & g_{t_5} & \cdots & g_{t_{2p-3}} & g_{t_{2p-2}} & g_{t_{2p-1}} & g_{t_{2p}} \\
g_{t_{2p}} & g_{t_1} & g_{t_2} & g_{t_3} & g_{t_4} & \cdots & g_{t_{2p-4}} & g_{t_{2p-3}} & g_{t_{2p-2}} & g_{t_{2p-1}} \\
g_{t_{2p-1}} & g_{t_{2p}} & g_{t_1} & g_{t_2} & g_{t_3} & \cdots & \cdots & g_{t_{2p-4}} & g_{t_{2p-3}} & g_{t_{2p-2}} \\
g_{t_{2p-2}} & g_{t_{2p-1}} & g_{t_{2p}} & g_{t_1} & g_{t_2} & \cdots & \cdots & \cdots & g_{t_{2p-4}} & g_{t_{2p-3}} \\
g_{t_{2p-3}} & g_{t_{2p-2}} & g_{t_{2p-1}} & g_{t_{2p}} & g_{t_1} & \cdots & \cdots & \cdots & \cdots & g_{t_{2p-4}} \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
g_{t_5} & \cdots & \cdots & \cdots & \cdots & \cdots & g_{t_1} & g_{t_2} & g_{t_3} & g_{t_4} \\
g_{t_4} & g_{t_5} & \cdots & \cdots & \cdots & \cdots & g_{t_{2p}} & g_{t_1} & g_{t_2} & g_{t_3} \\
g_{t_3} & g_{t_4} & g_{t_5} & \cdots & \cdots & \cdots & g_{t_{2p-1}} & g_{t_{2p}} & g_{t_1} & g_{t_2} \\
g_{t_2} & g_{t_3} & g_{t_4} & g_{t_5} & \cdots & \cdots & g_{t_{2p-2}} & g_{t_{2p-1}} & g_{t_{2p}} & g_{t_1}
\end{pmatrix}$$

Furthermore, if we get the elements of $T(t,t)$ as above with inverses of an arbitrary subset in $B(x,x)$, say $T(j,j)$, we will get the following matrix:

$$\begin{pmatrix}
g_{k_1} & g_{k_2} & g_{k_3} & g_{k_4} & g_{k_5} & \cdots & g_{k_{2p-3}} & g_{k_{2p-2}} & g_{k_{2p-1}} & g_{k_{2p}} \\
g_{k_{2p}} & g_{k_1} & g_{k_2} & g_{k_3} & g_{k_4} & \cdots & g_{k_{2p-4}} & g_{k_{2p-3}} & g_{k_{2p-2}} & g_{k_{2p-1}} \\
g_{k_{2p-1}} & g_{k_{2p}} & g_{k_1} & g_{k_2} & g_{k_3} & \cdots & \cdots & g_{k_{2p-4}} & g_{k_{2p-3}} & g_{k_{2p-2}} \\
g_{k_{2p-2}} & g_{k_{2p-1}} & g_{k_{2p}} & g_{k_1} & g_{k_2} & \cdots & \cdots & \cdots & g_{k_{2p-4}} & g_{k_{2p-3}} \\
g_{k_{2p-3}} & g_{k_{2p-2}} & g_{k_{2p-1}} & g_{k_{2p}} & g_{k_1} & \cdots & \cdots & \cdots & \cdots & g_{k_{2p-4}} \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
g_{k_5} & \cdots & \cdots & \cdots & \cdots & \cdots & g_{k_1} & g_{k_2} & g_{k_3} & g_{k_4} \\
g_{k_4} & g_{k_5} & \cdots & \cdots & \cdots & \cdots & g_{k_{2p}} & g_{k_1} & g_{k_2} & g_{k_3} \\
g_{k_3} & g_{k_4} & g_{k_5} & \cdots & \cdots & \cdots & g_{k_{2p-1}} & g_{k_{2p}} & g_{k_1} & g_{k_2} \\
g_{k_2} & g_{k_3} & g_{k_4} & g_{k_5} & \cdots & \cdots & g_{k_{2p-2}} & g_{k_{2p-1}} & g_{k_{2p}} & g_{k_1}
\end{pmatrix}$$

Where, $g_{t_i} * {g_{j_i}}^{-1} = g_{k_i}$, $1 \leq i \leq 2p$, such that:
$g_{k_1} = h(j^{-1}t, j^{-1}t)$, $g_{k_2} = h(j^{-1}(p-t), j^{-1}(p-t))\, u(p-1)$, $g_{k_3} = h(j^{-1}t, j^{-1}t)\, u(p-2)$,
$g_{k_4} = h(j^{-1}(p-t), j^{-1}(p-t))\, u(p-3)$, $g_{k_5} = h(j^{-1}t, j^{-1}t)\, u(p-4)$,
$\vdots$
, $g_{k_{2p-4}} = h(j^{-1}(p-t), j^{-1}(p-t))\, u(5)$, $g_{k_{2p-3}} = h(j^{-1}t, j^{-1}t)\, u(4)$,
$g_{k_{2p-2}} = h(j^{-1}(p-t), j^{-1}(p-t))\, u(3)$, $g_{k_{2p-1}} = h(j^{-1}t, j^{-1}t)\, u(2)$,
$g_{k_{2p}} = h(j^{-1}(p-t), j^{-1}(p-t))\, u(1)$,

This obtained using these tow following equations :
(1) $(p-j)^{-1}t = j^{-1}(p-t)$
(2) $(p-j)^{-1}(p-t) = j^{-1}t$.

Now, if we take the elements of $T_i(x,y)(1)(\lambda = w)$ from the block $B_i(x,y)(\lambda = w)$, $g_{i_1}, g_{i_2}, \cdots, \cdots, g_{i_{2p-1}}, g_{i_{2p}}$, with the inverses of elements of $T_j(x,y)(1)(\lambda = k)$ from block $B_j(x,y)(\lambda = k)$, $g_{j_1}^{-1}, g_{j_2}^{-1}, \cdots, \cdots, g_{j_{s-1}}^{-1}, g_{j_s}^{-1}$. The resulting matrix will have the following form:

$$\begin{pmatrix} l_1 & l_2 & l_3 & l_4 & l_5 & \dots & l_{2p-3} & l_{2p-2} & l_{2p-1} & l_{2p} \\ l_{2p} & l_1 & l_2 & l_3 & l_4 & \dots & \cdots & l_{2p-3} & l_{2p-2} & l_{2p-1} \\ l_{2p-1} & l_{2p} & l_1 & l_2 & l_3 & \dots & \dots & \cdots & l_{2p-3} & l_{2p-2} \\ l_{2p-2} & l_{2p-1} & l_{2p} & l_1 & l_2 & \dots & \dots & \dots & \cdots & l_{2p-3} \\ l_{2p-3} & l_{2p-2} & l_{2p-1} & l_{2p} & l_1 & \dots & \dots & \dots & \dots & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ l_5 & \dots & \dots & \dots & \dots & \dots & l_1 & l_2 & l_3 & l_4 \\ l_4 & l_5 & \dots & \dots & \dots & \dots & l_{2p} & l_1 & l_2 & l_3 \\ l_3 & l_4 & l_5 & \dots & \dots & \dots & l_{2p-1} & l_{2p} & l_1 & l_2 \\ l_2 & l_3 & l_4 & l_5 & \dots & \dots & l_{2p-2} & l_{2p-1} & l_{2p} & l_1 \end{pmatrix}$$

Where, $g_{i_s} * g_{j_s}^{-1} = l_s$, $1 \leq s \leq 2p$.

Also, for another matrix in the same Block. We can get the elements of $T_i(x,y)(n)(\lambda = w)$ from the Block $B_i(x,y)(\lambda = w)$ with the inverses of the elements of $T_j(x,y)(n)(\lambda = k)$ in the block $B_j(x,y)(\lambda = k)$. Then we will get the same circulant matrix which we obtained when we take the elements from the first subsets of these certain blocks. We find in this case the $G-matrix$ of the group $GL(2,q)$ with respect to the above proposed listing form $\left(q^2-1 \times q^2-1\right)$- Block Circulant matrix.

Now, for the other case when q is a power of p :
For the elements from the coset(B) consider $T_b(a^i,a^j)$ from the block $B_b(a^i,a^j)$, $b_1,b_2,\cdots,\cdots,b_{s-2},b_{s-1},b_s$, where $s = p(q-1)$ is the number of the elements in each subsets in this type, and a is the generator of the multiplicative group of the field $\mathbb{F}_q$). Together with the inverses of elements in $T(a^i,a^i)$ from the block $B(a^i,a^i)$. The resulting matrix will have the following form:

$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & \dots & b_{s-3} & b_{s-2} & b_{s-1} & b_s \\ b_s & b_1 & b_2 & b_3 & b_4 & \dots & \cdots & b_{s-3} & b_{s-2} & b_{s-1} \\ b_{s-1} & b_s & b_1 & b_2 & b_3 & \dots & \dots & \cdots & b_{s-3} & b_{s-2} \\ b_{s-2} & b_{s-1} & b_s & b_1 & b_2 & \dots & \dots & \dots & \cdots & b_{s-3} \\ b_{s-3} & b_{s-2} & b_{s-1} & b_s & b_1 & \dots & \dots & \dots & \dots & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ b_5 & \dots & \dots & \dots & \dots & \dots & b_1 & b_2 & b_3 & b_4 \\ b_4 & b_5 & \dots & \dots & \dots & \dots & b_s & b_1 & b_2 & b_3 \\ b_3 & b_4 & b_5 & \dots & \dots & \dots & b_{s-1} & b_s & b_1 & b_2 \\ b_2 & b_3 & b_4 & b_5 & \dots & \dots & b_{s-2} & b_{s-1} & b_s & b_1 \end{pmatrix}$$

The same matrix will appear when we take the elements of $T_b(a^i,a^j)(n)$ from the block $B_b(a^i,a^j)$, with the inverses of elements in $T(a^i,a^i)(n)$ from the block $B(a^i,a^i)$.
Finally for the elements from the double coset (Bn_oU) consider $T_x(a^i,a^j)(\lambda = a^z)$ from the block $B_x(a^i,a^j)(\lambda = a^z)$, $c_1,c_2,\cdots,\cdots,c_{s-1},c_s$, with the inverses of elements of $T_y(a^i,a^j)(\lambda = a^\beta)$ from the block $B_y(a^i,a^j)(\lambda = a^\beta)$ $d_1^{-1},d_2^{-1},\cdots,\cdots,d_{s-1}^{-1},d_s^{-1}$. Then the matrix will be as the following :

$$\begin{pmatrix} z_1 & z_2 & z_3 & z_4 & z_5 & \dots & z_{s-3} & z_{s-2} & z_{s-1} & z_s \\ z_s & z_1 & z_2 & z_3 & z_4 & \dots & \cdots & z_{s-3} & z_{s-2} & z_{s-1} \\ z_{s-1} & z_s & z_1 & z_2 & z_3 & \dots & \dots & \cdots & z_{s-3} & z_{s-2} \\ z_{s-2} & z_{s-1} & z_s & z_1 & z_2 & \dots & \dots & \dots & \cdots & z_{s-3} \\ z_{s-3} & z_{s-2} & z_{s-1} & z_s & z_1 & \dots & \dots & \dots & \dots & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ z_5 & \dots & \dots & \dots & \dots & \dots & z_1 & z_2 & z_3 & z_4 \\ z_4 & z_5 & \dots & \dots & \dots & \dots & z_s & z_1 & z_2 & z_3 \\ z_3 & z_4 & z_5 & \dots & \dots & \dots & z_{s-1} & z_s & z_1 & z_2 \\ z_2 & z_3 & z_4 & z_5 & \dots & \dots & z_{s-2} & z_{s-1} & z_s & z_1 \end{pmatrix}$$

Where, $c_k * d_k^{-1} = z_k$, $1 \leq k \leq s$. The same matrix will obtained when we take the elements of $T_x(a^i,a^j)(m)(\lambda = a^z)$ from the block $B_x(a^i,a^j)(\lambda = a^z)$, with the inverses of elements of $T_y(a^i,a^j)(m)(\lambda = a^\beta)$ from the block $B_y(a^i,a^j)(\lambda = a^\beta)$.

Note that :
When the matrix is appear again in a certain block (with the same elements) the order of the element maybe change (they permute) not always the same (as in the example of $GL(2,5)$) .
We find the G-matrix of the group $GL(2,q)$ with respect to the above proposed listing using the multiplication table as well as the rule for getting the inverses explained above. It turns out that the G-matrix (and hence the coding matrices) is a $\left(q^2-1 \times q^2-1\right)$-block circulant matrix.

Remark 7.1. *When $q = 2$, the general linear group $G = GL(2,q) \cong S_3$ and the matrix of this group is actually a block circulant matrix as well. Also, when $q = 3$, the matrix of $GL(2,3)$ with respect to the listing given in [example (6.1)] is a block circulant matrix.*

Summarizing we have the following,

Theorem 7.2. *: With respect to the elements listing for the group $G = GL(2,q)$, the G-matrix has the form of block circulant matrix.*

8. Examples of the G-matrices

Here we have two examples of the matrix of $G = GL(2,q)$:
1- The listing of elements of $GL(2,5)$ **:**
$|Gl(2,5)|$= 480. This will divide in to 24 blocks with 20 elements in each block .

Type $(x\ ,\ x)$:
Block $B(x,x)$**:**
T (1,1):
$l_1 = h(1,1)$, $l_2 = h(4,4)\ u(4)$, $l_3 = h(1,1)\ u(3)$, $l_4 = h(4,4)\ u(2)$, $l_5 = h(1,1)\ u(1)$,
$l_6 = h(4,4)$, $l_7 = h(1,1)\ u(4)$, $l_8 = h(4,4)\ u(3)$, $l_9 = h(1,1)\ u(2)$, $l_{10} = h(4,4)\ u(1)$.
T (2,2):
$l_{11} = h(2,2)$, $l_{12} = h(3,3)\ u(4)$, $l_{13} = h(2,2)\ u(3)$, $l_{14} = h(3,3)\ u(2)$, $l_{15} = h(2,2)\ u(1)$,
$l_{16} = h(3,3)$, $l_{17} = h(2,2)\ u(4)$, $l_{18} = h(3,3)\ u(3)$, $l_{19} = h(2,2)\ u(2)$, $l_{20} = h(3,3)\ u(1)$.
Block $B(x,x)(\lambda = 0)$**:**
T (1,1)$(\lambda = 0)$**:**
$l_{21} = h(1,1)\ n_o$, $l_{22} = h(4,4)\ u(4)\ n_o$, $l_{23} = h(1,1)\ u(3)\ n_o$, $l_{24} = h(4,4)\ u(2)\ n_o$,
$l_{25} = h(1,1)\ u(1)\ n_o$, $l_{26} = h(4,4)\ n_o$, $l_{27} = h(1,1)\ u(4)\ n_o$, $l_{28} = h(4,4)\ u(3)\ n_o$,
$l_{29} = h(1,1)\ u(2)\ n_o$, $l_{30} = h(4,4)\ u(1)\ n_o$.
T (2,2)$(\lambda = 0)$**:**
$l_{31} = h(2,2)n_o$, $l_{32} = h(3,3)u(4)n_o$, $l_{33} = h(2,2)u(3)n_o$, $l_{34} = h(3,3)u(2)n_o$, $l_{35} = h(2,2)u(1)n_o$,
$l_{36} = h(3,3)n_o$, $l_{37} = h(2,2)u(4)n_o$, $l_{38} = h(3,3)u(3)n_o$, $l_{39} = h(2,2)u(2)n_o$, $l_{40} = h(3,3)u(1)n_o$, .
Block $B(x,x)(\lambda = 1)$**:**
T (1,1)$(\lambda = 1)$**:**
$l_{41} = h(1,1)n_ou(1)$, $l_{42} = h(4,4)u(4)n_ou(1)$, $l_{43} = h(1,1)u(3)n_ou(1)$, $l_{44} = h(4,4)u(2)n_ou(1)$,
$l_{45} = h(1,1)u(1)n_ou(1)$, $l_{46} = h(4,4)n_ou(1)$, $l_{47} = h(1,1)u(4)n_ou(1)$, $l_{48} = h(4,4)u(3)n_ou(1)$,
$l_{49} = h(1,1)u(2)n_ou(1)$, $l_{50} = h(4,4)u(1)n_ou(1)$.
T (2,2)$(\lambda = 1)$**:**
$l_{51} = h(2,2)n_ou(1)$, $l_{52} = h(3,3)u(4)n_ou(1)$, $l_{53} = h(2,2)u(3)n_ou(1)$, $l_{54} = h(3,3)u(2)n_ou(1)$,
$l_{55} = h(2,2)u(1)n_ou(1)$, $l_{56} = h(3,3)n_ou(1)$, $l_{57} = h(2,2)u(4)n_ou(1)$, $l_{58} = h(3,3)u(3)n_ou(1)$,
$l_{59} = h(2,2)u(2)n_ou(1)$, $l_{60} = h(3,3)u(1)n_ou(1)$.
Block $B(x,x)(\lambda = 2)$:
T (1,1)$(\lambda = 2)$**:**
$l_{61} = h(1,1)n_ou(2)$, $l_{62} = h(4,4)u(4)n_ou(2)$, $l_{63} = h(1,1)u(3)n_ou(2)$, $l_{64} = h(4,4)u(2)n_ou(2)$,
$l_{65} = h(1,1)u(1)n_ou(2)$, $l_{66} = h(4,4)n_ou(2)$, $l_{67} = h(1,1)u(4)n_ou(2)$, $l_{68} = h(4,4)u(3)n_ou(2)$,
$l_{69} = h(1,1)u(2)n_ou(2)$, $l_{70} = h(4,4)u(1)n_ou(2)$.
T (2,2)$(\lambda = 2)$**:**
$l_{71} = h(2,2)n_ou(2)$, $l_{72} = h(3,3)u(4)n_ou(2)$, $l_{73} = h(2,2)u(3)n_ou(2)$, $l_{74} = h(3,3)u(2)n_ou(2)$,
$l_{75} = h(2,2)u(1)n_ou(2)$, $l_{76} = h(3,3)n_ou(2)$, $l_{77} = h(2,2)u(4)n_ou(2)$, $l_{78} = h(3,3)u(3)n_ou(2)$,
$l_{79} = h(2,2)u(2)n_ou(2)$, $l_{80} = h(3,3)u(1)n_ou(2)$.
Block $B(x,x)(\lambda = 3)$:
T (1,1)$(\lambda = 3)$**:**
$l_{81} = h(1,1)n_ou(3)$, $l_{82} = h(4,4)u(4)n_ou(3)$, $l_{83} = h(1,1)u(3)n_ou(3)$, $l_{84} = h(4,4)u(2)n_ou(3)$,
$l_{85} = h(1,1)u(1)n_ou(3)$, $l_{86} = h(4,4)n_ou(3)$, $l_{87} = h(1,1)u(4)n_ou(3)$, $l_{88} = h(4,4)u(3)n_ou(3)$,
$l_{89} = h(1,1)u(2)n_ou(3)$, $l_{90} = h(4,4)u(1)n_ou(3)$.
T (2,2)$(\lambda = 3)$**:**
$l_{91} = h(2,2)n_ou(3)$, $l_{92} = h(3,3)u(4)n_ou(3)$, $l_{93} = h(2,2)u(3)n_ou(3)$, $l_{94} = h(3,3)u(2)n_ou(3)$,
$l_{95} = h(2,2)u(1)n_ou(3)$, $l_{96} = h(3,3)n_ou(3)$, $l_{97} = h(2,2)u(4)n_ou(3)$, $l_{98} = h(3,3)u(3)n_ou(3)$,
$l_{99} = h(2,2)u(2)n_ou(3)$, $l_{100} = h(3,3)u(1)n_ou(3)$.
Block $B(x,x)(\lambda = 4)$:
T (1,1)$(\lambda = 4)$**:**
$l_{101} = h(1,1)n_ou(4)$, $l_{102} = h(4,4)u(4)n_ou(4)$, $l_{103} = h(1,1)u(3)n_ou(4)$, $l_{104} = h(4,4)u(2)n_ou(4)$,
$l_{105} = h(1,1)u(1)n_ou(4)$, $l_{106} = h(4,4)n_ou(4)$, $l_{107} = h(1,1)u(4)n_ou(4)$, $l_{108} = h(4,4)u(3)n_ou(4)$,
$l_{109} = h(1,1)u(2)n_ou(4)$, $l_{110} = h(4,4)u(1)n_ou(4)$.
T (2,2)$(\lambda = 4)$**:**
$l_{111} = h(2,2)n_ou(4)$, $l_{112} = h(3,3)u(4)n_ou(4)$, $l_{113} = h(2,2)u(3)n_ou(4)$, $l_{114} = h(3,3)u(2)n_ou(4)$,
$l_{115} = h(2,2)u(1)n_ou(4)$, $l_{116} = h(3,3)n_ou(4)$, $l_{117} = h(2,2)u(4)n_ou(4)$, $l_{118} = h(3,3)u(3)n_ou(4)$,
$l_{119} = h(2,2)u(2)n_ou(4)$, $l_{120} = h(3,3)u(1)n_ou(4)$, .

Type $(x\ ,\ y)$:
Block $B_1(x,y)$:
$T_1(x,y)(1)$**:**
$l_{121} = h(1,4)$, $l_{122} = h(4,1)u(1)$, $l_{123} = h(1,4)u(2)$, $l_{124} = h(4,1)u(3)$, $l_{125} = h(1,4)u(4)$,
$l_{126} = h(4,1)$, $l_{127} = h(1,4)u(1)$, $l_{128} = h(4,1)u(2)$, $l_{129} = h(1,4)u(3)$, $l_{130} = h(4,1)u(4)$.
$T_1(x,y)(2)$**:**
$l_{131} = h(2,3)$, $l_{132} = h(3,2)u(1)$, $l_{133} = h(2,3)u(2)$, $l_{134} = h(3,2)u(3)$, $l_{135} = h(2,3)u(4)$,

$l_{136}=h(3,2),\ l_{137}=h(2,3)u(1),\ l_{138}=h(3,2)u(2),\ l_{139}=h(2,3)u(3),\ l_{140}=h(3,2)u(4).$

Block $B_1(x,y)(\lambda=0)$:

$T_1(x,y)(1)(\lambda=0)$**:**

$l_{141}=h(1,4)n_o,\ l_{142}=h(4,1)u(1)n_o,\ l_{143}=h(1,4)u(2)n_o,\ l_{144}=h(4,1)u(3)n_o,\ l_{145}=h(1,4)u(4)n_o,$
$l_{146}=h(4,1)n_o,\ l_{147}=h(1,4)u(1)n_o,\ l_{148}=h(4,1)u(2)n_o,\ l_{149}=h(1,4)u(3)n_o,\ l_{150}=h(4,1)u(4)n_o.$

$T_1(x,y)(2)(\lambda=0)$**:**

$l_{151}=h(2,3)n_o,\ l_{152}=h(3,2)u(1)n_o,\ l_{153}=h(2,3)u(2)n_o,\ l_{154}=h(3,2)u(3)n_o,\ l_{155}=h(2,3)u(4)n_o,$
$l_{156}=h(3,2)n_o,\ l_{157}=h(2,3)u(1)n_o,\ l_{158}=h(3,2)u(2)n_o,\ l_{159}=h(2,3)u(3)n_o,\ l_{160}=h(3,2)u(4)n_o.$

Block $B_1(x,y)(\lambda=1)$:

$T_1(x,y)(1)(\lambda=1)$**:**

$l_{161}=h(1,4)n_ou(1),\ l_{162}=h(4,1)u(1)n_ou(1),\ l_{163}=h(1,4)u(2)n_ou(1),\ l_{164}=h(4,1)u(3)n_ou(1),$
$l_{165}=h(1,4)u(4)n_ou(1),\ l_{166}=h(4,1)n_ou(1),\ l_{167}=h(1,4)u(1)n_ou(1),\ l_{168}=h(4,1)u(2)n_ou(1),$
$l_{169}=h(1,4)u(3)n_ou(1),\ l_{170}=h(4,1)u(4)n_ou(1).$

$T_1(x,y)(2)(\lambda=1)$**:**

$l_{171}=h(2,3)n_ou(1),\ l_{172}=h(3,2)u(1)n_ou(1),\ l_{173}=h(2,3)u(2)n_ou(1),\ l_{174}=h(3,2)u(3)n_ou(1),$
$l_{175}=h(2,3)u(4)n_ou(1),\ l_{176}=h(3,2)n_ou(1),\ l_{177}=h(2,3)u(1)n_ou(1),\ l_{178}=h(3,2)u(2)n_ou(1),$
$l_{179}=h(2,3)u(3)n_ou(1),\ l_{180}=h(3,2)u(4)n_ou(1).$

Block $B_1(x,y)(\lambda=2)$:

$T_1(x,y)(1)(\lambda=2)$**:**

$l_{181}=h(1,4)n_ou(2),\ l_{182}=h(4,1)u(1)n_ou(2),\ l_{183}=h(1,4)u(2)n_ou(2),\ l_{184}=h(4,1)u(3)n_ou(2),$
$l_{185}=h(1,4)u(4)n_ou(2),\ l_{186}=h(4,1)n_ou(2),\ l_{187}=h(1,4)u(1)n_ou(2),\ l_{188}=h(4,1)u(2)n_ou(2),$
$l_{189}=h(1,4)u(3)n_ou(2),\ l_{190}=h(4,1)u(4)n_ou(2).$

$T_1(x,y)(2)(\lambda=2)$**:**

$l_{191}=h(2,3)n_ou(2),\ l_{192}=h(3,2)u(1)n_ou(2),\ l_{193}=h(2,3)u(2)n_ou(2),\ l_{194}=h(3,2)u(3)n_ou(2),$
$l_{195}=h(2,3)u(4)n_ou(2),\ l_{196}=h(3,2)n_ou(2),\ l_{197}=h(2,3)u(1)n_ou(2),\ l_{198}=h(3,2)u(2)n_ou(2),$
$l_{199}=h(2,3)u(3)n_ou(2),\ l_{200}=h(3,2)u(4)n_ou(2).$

Block $B_1(x,y)(\lambda=3)$:

$T_1(x,y)(1)(\lambda=3)$**:**

$l_{201}=h(1,4)n_ou(3),\ l_{202}=h(4,1)u(1)n_ou(3),\ l_{203}=h(1,4)u(2)n_ou(3),\ l_{204}=h(4,1)u(3)n_ou(3),$
$l_{205}=h(1,4)u(4)n_ou(3),\ l_{206}=h(4,1)n_ou(3),\ l_{207}=h(1,4)u(1)n_ou(3),\ l_{208}=h(4,1)u(2)n_ou(3),$
$l_{209}=h(1,4)u(3)n_ou(3),\ l_{210}=h(4,1)u(4)n_ou(3).$

$T_1(x,y)(2)(\lambda=3)$**:**

$l_{211}=h(2,3)n_ou(3),\ l_{212}=h(3,2)u(1)n_ou(3),\ l_{213}=h(2,3)u(2)n_ou(3),\ l_{214}=h(3,2)u(3)n_ou(3),$
$l_{215}=h(2,3)u(4)n_ou(3),\ l_{216}=h(3,2)n_ou(3),\ l_{217}=h(2,3)u(1)n_ou(3),\ l_{218}=h(3,2)u(2)n_ou(3),$
$l_{219}=h(2,3)u(3)n_ou(3),\ l_{220}=h(3,2)u(4)n_ou(3).$

Block $B_1(x,y)(\lambda=4)$:

$T_1(x,y)(1)(\lambda=4)$**:**

$l_{221}=h(1,4)n_ou(4),\ l_{222}=h(4,1)u(1)n_ou(4),\ l_{223}=h(1,4)u(2)n_ou(4),\ l_{224}=h(4,1)u(3)n_ou(4),$
$l_{225}=h(1,4)u(4)n_ou(4),\ l_{226}=h(4,1)n_ou(4),\ l_{227}=h(1,4)u(1)n_ou(4),\ l_{228}=h(4,1)u(2)n_ou(4),$
$l_{229}=h(1,4)u(3)n_ou(4),\ l_{230}=h(4,1)u(4)n_ou(4).$

$T_1(x,y)(2)(\lambda=4)$**:**

$l_{231}=h(2,3)n_ou(4),\ l_{232}=h(3,2)u(1)n_ou(4),\ l_{233}=h(2,3)u(2)n_ou(4),\ l_{234}=h(3,2)u(3)n_ou(4),$
$l_{235}=h(2,3)u(4)n_ou(4),\ l_{236}=h(3,2)n_ou(4),\ l_{237}=h(2,3)u(1)n_ou(4),\ l_{238}=h(3,2)u(2)n_ou(4),$
$l_{239}=h(2,3)u(3)n_ou(4),\ l_{240}=h(3,2)u(4)n_ou(4).$

Block $B_2(x,y)$:

$T_2(x,y)(1)$**:**

$l_{241}=h(1,3),\ l_{242}=h(4,2)u(2),\ l_{243}=h(1,3)u(4),\ l_{244}=h(4,2)u(1),\ l_{245}=h(1,3)u(3),$
$l_{246}=h(4,2)\ ,\ l_{247}=h(1,3)u(2),\ l_{248}=h(4,2)u(4),\ l_{249}=h(1,3)u(1),\ l_{250}=h(4,2)u(3).$

$T_2(x,y)(2)$**:**

$l_{251}=h(2,1),\ l_{252}=h(3,4)u(2),\ l_{253}=h(2,1)u(4),\ l_{254}=h(3,4)u(1),\ l_{255}=h(2,1)u(3),$
$l_{256}=h(3,4),\ l_{257}=h(2,1)u(2),\ l_{258}=h(3,4)u(4),\ l_{259}=h(2,1)u(1),\ l_{260}=h(3,4)u(3).$

Block $B_2(x,y)(\lambda=0)$:

$T_2(x,y)(1)(\lambda=0)$**:**

$l_{261}=h(1,3)n_o,\ l_{262}=h(4,2)u(2)n_o,\ l_{263}=h(1,3)u(4)n_o,\ l_{264}=h(4,2)u(1)n_o,\ l_{265}=h(1,3)u(3)n_o,$
$l_{266}=h(4,2)n_o,\ l_{267}=h(1,3)u(2)n_o,\ l_{268}=h(4,2)u(4)n_o,\ l_{269}=h(1,3)u(1)n_o,\ l_{270}=h(4,2)u(3)n_o.$

$T_2(x,y)(2)(\lambda=0)$**:**

$l_{271}=h(2,1)n_o,\ l_{272}=h(3,4)u(2)n_o,\ l_{273}=h(2,1)\ u(4)n_o,\ l_{274}=h(3,4)u(1)n_o,\ l_{275}=h(2,1)u(3)n_o,$
$l_{276}=h(3,4)n_o,\ l_{277}=h(2,1)u(2)n_o,\ l_{278}=h(3,4)u(4)n_o,\ l_{279}=h(2,1)u(1)n_o,\ l_{280}=h(3,4)u(3)n_o.$

Block $B_2(x,y)(\lambda=1)$:

$T_2(x,y)(1)(\lambda=1)$**:**

$l_{281}=h(1,3)n_ou(1),\ l_{282}=h(4,2)u(2)n_ou(1),\ l_{283}=h(1,3)u(4)n_ou(1),\ l_{284}=h(4,2)u(1)n_ou(1),$
$l_{285}=h(1,3)u(3)n_ou(1),\ l_{286}=h(4,2)n_ou(1),\ l_{287}=h(1,3)u(2)n_ou(1),\ l_{288}=h(4,2)u(4)n_ou(1),$
$l_{289}=h(1,3)u(1)n_ou(1),\ l_{290}=h(4,2)u(3)n_ou(1).$

$T_2(x,y)(2)(\lambda=1)$**:**

$l_{291}=h(2,1)n_ou(1),\ l_{292}=h(3,4)u(2)n_ou(1),\ l_{293}=h(2,1)u(4)n_ou(1),\ l_{294}=h(3,4)u(1)n_ou(1),$
$l_{295}=h(2,1)u(3)n_ou(1),\ l_{296}=h(3,4)n_ou(1),\ l_{297}=h(2,1)u(2)n_ou(1),\ l_{298}=h(3,4)u(4)n_ou(1),$

$l_{299} = h(2,1)u(1)n_o u(1),\ l_{300} = h(3,4)u(3)n_o u(1).$
Block $B_2(x,y)(\lambda = 2)$:
$T_2(x,y)(1)(\lambda = 2)$**:**
$l_{301} = h(1,3)n_o u(2),\ l_{302} = h(4,2)u(2)n_o u(2),\ l_{303} = h(1,3)u(4)n_o u(2),\ l_{304} = h(4,2)u(1)n_o u(2),$
$l_{305} = h(1,3)u(3)n_o u(2),\ l_{306} = h(4,2)n_o u(2),\ l_{307} = h(1,3)u(2)n_o u(2),\ l_{308} = h(4,2)u(4)n_o u(2),$
$l_{309} = h(1,3)u(1)n_o u(2),\ l_{310} = h(4,2)u(3)n_o u(2).$
$T_2(x,y)(2)(\lambda = 2)$**:**
$l_{311} = h(2,1)n_o u(2),\ l_{312} = h(3,4)u(2)n_o u(2),\ l_{313} = h(2,1)u(4)n_o u(2),\ l_{314} = h(3,4)u(1)n_o u(2),$
$l_{315} = h(2,1)u(3)n_o u(2),\ l_{316} = h(3,4)n_o u(2),\ l_{317} = h(2,1)u(2)n_o u(2),\ l_{318} = h(3,4)u(4)n_o u(2),$
$l_{319} = h(2,1)u(1)n_o u(2),\ l_{320} = h(3,4)u(3)n_o u(2).$
Block $B_2(x,y)(\lambda = 3)$:
$T_2(x,y)(1)(\lambda = 3)$**:**
$l_{321} = h(1,3)n_o u(3),\ l_{322} = h(4,2)u(2)n_o u(3),\ l_{323} = h(1,3)u(4)n_o u(3),\ l_{324} = h(4,2)u(1)n_o u(3),$
$l_{325} = h(1,3)u(3)n_o u(3),\ l_{326} = h(4,2)n_o u(3),\ l_{327} = h(1,3)u(2)n_o u(3),\ l_{328} = h(4,2)u(4)n_o u(3),$
$l_{329} = h(1,3)u(1)n_o u(3),\ l_{330} = h(4,2)u(3)n_o u(3).$
$T_2(x,y)(2)(\lambda = 3)$**:**
$l_{331} = h(2,1)n_o u(3),\ l_{332} = h(3,4)u(2)n_o u(3),\ l_{333} = h(2,1)u(4)n_o u(3),\ l_{334} = h(3,4)u(1)n_o u(3),$
$l_{335} = h(2,1)u(3)n_o u(3),\ l_{336} = h(3,4)n_o u(3),\ l_{337} = h(2,1)u(2)n_o u(3),\ l_{338} = h(3,4)u(4)n_o u(3),$
$l_{339} = h(2,1)u(1)n_o u(3),\ l_{340} = h(3,4)u(3)n_o u(3).$
Block $B_2(x,y)(\lambda = 4)$:
$T_2(x,y)(1)(\lambda = 4)$**:**
$l_{341} = h(1,3)n_o u(4),\ l_{342} = h(4,2)u(2)n_o u(4),\ l_{343} = h(1,3)u(4)n_o u(4),\ l_{344} = h(4,2)u(1)n_o u(4),$
$l_{345} = h(1,3)u(3)n_o u(4),\ l_{346} = h(4,2)n_o u(4),\ l_{347} = h(1,3)u(2)n_o u(4),\ l_{348} = h(4,2)u(4)n_o u(4),$
$l_{349} = h(1,3)u(1)n_o u(4),\ l_{350} = h(4,2)u(3)n_o u(4).$
$T_2(x,y)(2)(\lambda = 4)$**:**
$l_{351} = h(2,1)n_o u(4),\ l_{352} = h(3,4)u(2)n_o u(4),\ l_{353} = h(2,1)u(4)n_o u(4),\ l_{354} = h(3,4)u(1)n_o u(4),$
$l_{355} = h(2,1)u(3)n_o u(4),\ l_{356} = h(3,4)n_o u(4),\ l_{357} = h(2,1)u(2)n_o u(4),\ l_{358} = h(3,4)u(4)n_o u(4),$
$l_{359} = h(2,1)u(1)n_o u(4),\ l_{360} = h(3,4)u(3)n_o u(4).$
Block $B_3(x,y)$:
$T_3(x,y)(1)$**:**
$l_{361} = h(1,2),\ l_{362} = h(4,3)u(3),\ l_{363} = h(1,2)u(1),\ l_{364} = h(4,3)u(4),\ l_{365} = h(1,2)u(2),$
$l_{366} = h(4,3),\ l_{367} = h(1,2)u(3),\ l_{368} = h(4,3)u(1),\ l_{369} = h(1,2)u(4),\ l_{370} = h(4,3)u(2).$
$T_3(x,y)(2)$**:**
$l_{371} = h(2,4),\ l_{372} = h(3,1)u(3),\ l_{373} = h(2,4)u(1),\ l_{374} = h(3,1)u(4),\ l_{375} = h(2,4)u(2),$
$l_{376} = h(3,1),\ l_{377} = h(2,4)u(3),\ l_{378} = h(3,1)u(1),\ l_{379} = h(2,4)u(4),\ l_{380} = h(3,1)u(2).$
Block $B_3(x,y)(\lambda = 0)$:
$T_3(x,y)(1)(\lambda = 0)$**:**
$l_{381} = h(1,2)n_o,\ l_{382} = h(4,3)u(3)n_o,\ l_{383} = h(1,2)u(1)n_o,\ l_{384} = h(4,3)u(4)n_o,\ l_{385} = h(1,2)u(2)n_o,$
$l_{386} = h(4,3)n_o,\ l_{387} = h(1,2)u(3)n_o,\ l_{388} = h(4,3)u(1)n_o,\ l_{389} = h(1,2)u(4)n_o,\ l_{390} = h(4,3)u(2)n_o.$
$T_3(x,y)(2)(\lambda = 0)$**:**
$l_{391} = h(2,4)n_o,\ l_{392} = h(3,1)u(3)n_o,\ l_{393} = h(2,4)u(1)n_o,\ l_{394} = h(3,1)u(4)n_o,\ l_{395} = h(2,4)u(2)n_o,$
$l_{396} = h(3,1)n_o,\ l_{397} = h(2,4)u(3)n_o,\ l_{398} = h(3,1)u(1)n_o,\ l_{399} = h(2,4)u(4)n_o,\ l_{400} = h(3,1)u(2)n_o.$
Block $B_3(x,y)(\lambda = 1)$:
$T_3(x,y)(1)(\lambda = 1)$**:**
$l_{401} = h(1,2)n_o u(1),\ l_{402} = h(4,3)u(3)n_o u(1),\ l_{403} = h(1,2)u(1)n_o u(1),\ l_{404} = h(4,3)u(4)n_o u(1),$
$l_{405} = h(1,2)u(2)n_o u(1),\ l_{406} = h(4,3)n_o u(1),\ l_{407} = h(1,2)u(3)n_o u(1),\ l_{408} = h(4,3)u(1)n_o u(1),$
$l_{409} = h(1,2)u(4)n_o u(1),\ l_{410} = h(4,3)u(2)n_o u(1).$
$T_3(x,y)(2)(\lambda = 1)$**:**
$l_{411} = h(2,4)n_o u(1),\ l_{412} = h(3,1)u(3)n_o u(1),\ l_{413} = h(2,4)u(1)n_o u(1),\ l_{414} = h(3,1)u(4)n_o u(1),$
$l_{415} = h(2,4)u(2)n_o u(1),\ l_{416} = h(3,1)n_o u(1),\ l_{417} = h(2,4)u(3)n_o u(1),\ l_{418} = h(3,1)u(1)n_o u(1),$
$l_{419} = h(2,4)u(4)n_o u(1),\ l_{420} = h(3,1)u(2)n_o u(1).$
Block $B_3(x,y)(\lambda = 2)$:
$T_3(x,y)(1)(\lambda = 2)$**:**
$l_{421} = h(1,2)n_o u(2),\ l_{422} = h(4,3)u(3)n_o u(2),\ l_{423} = h(1,2)u(1)n_o u(2),\ l_{424} = h(4,3)u(4)n_o u(2),$
$l_{425} = h(1,2)u(2)n_o u(2),\ l_{426} = h(4,3)n_o u(2),\ l_{427} = h(1,2)u(3)n_o u(2),\ l_{428} = h(4,3)u(1)n_o u(2),$
$l_{429} = h(1,2)u(4)n_o u(2),\ l_{430} = h(4,3)u(2)n_o u(2).$
$T_3(x,y)(2)(\lambda = 2)$**:**
$l_{431} = h(2,4)n_o u(2),\ l_{432} = h(3,1)u(3)n_o u(2),\ l_{433} = h(2,4)u(1)n_o u(2),\ l_{434} = h(3,1)u(4)n_o u(2),$
$l_{435} = h(2,4)u(2)n_o u(2),\ l_{436} = h(3,1)n_o u(2),\ l_{437} = h(2,4)u(3)n_o u(2),\ l_{438} = h(3,1)u(1)n_o u(2),$
$l_{439} = h(2,4)u(4)n_o u(2),\ l_{440} = h(3,1)u(2)n_o u(2).$
Block $B_3(x,y)(\lambda = 3)$:
$T_3(x,y)(1)(\lambda = 3)$**:**
$l_{441} = h(1,2)n_o u(3),\ l_{442} = h(4,3)u(3)n_o u(3),\ l_{443} = h(1,2)u(1)n_o u(3),\ l_{444} = h(4,3)u(4)n_o u(3),$
$l_{445} = h(1,2)u(2)n_o u(3),\ l_{446} = h(4,3)n_o u(3),\ l_{447} = h(1,2)u(3)n_o u(3),\ l_{448} = h(4,3)u(1)n_o u(3),$
$l_{449} = h(1,2)u(4)n_o u(3),\ l_{450} = h(4,3)u(2)n_o u(3).$
$T_3(x,y)(2)(\lambda = 3)$**:**

$l_{451}=h(2,4)n_ou(3),\ l_{452}=h(3,1)u(3)n_ou(3),\ l_{453}=h(2,4)u(1)n_ou(3),\ l_{454}=h(3,1)u(4)n_ou(3),$
$l_{455}=h(2,4)u(2)n_ou(3),\ l_{456}=h(3,1)n_ou(3),\ l_{457}=h(2,4)u(3)n_ou(3),\ l_{458}=h(3,1)u(1)n_ou(3),$
$l_{459}=h(2,4)u(4)n_ou(3),\ l_{460}=h(3,1)u(2)n_ou(3).$

Block $B_3(x,y)(\lambda=4)$:

$T_3(x,y)(1)(\lambda=4)$**:**

$l_{461}=h(1,2)n_ou(4),\ l_{462}=h(4,3)u(3)n_ou(4),\ l_{463}=h(1,2)u(1)n_ou(4),\ l_{464}=h(4,3)u(4)n_ou(4),$
$l_{465}=h(1,2)u(2)n_ou(4),\ l_{466}=h(4,3)n_ou(4),\ l_{467}=h(1,2)u(3)n_ou(4),\ l_{468}=h(4,3)u(1)n_ou(4),$
$l_{469}=h(1,2)u(4)n_ou(4),\ l_{470}=h(4,3)u(2)n_ou(4).$

$T_3(x,y)(2)(\lambda=4)$**:**

$l_{471}=h(2,4)n_ou(4),\ l_{472}=h(3,1)u(3)n_ou(4),\ l_{473}=h(2,4)u(1)n_ou(4),\ l_{474}=h(3,1)u(4)n_ou(4),$
$l_{475}=h(2,4)u(2)n_ou(4),\ l_{476}=h(3,1)n_ou(4),\ l_{477}=h(2,4)u(3)n_ou(4),\ l_{478}=h(3,1)u(1)n_ou(4),$
$l_{479}=h(2,4)u(4)n_ou(4),\ l_{480}=h(3,1)u(2)n_ou(4).$

The matrix of $GL(2,5)$:

We will discuss some blocks of the $G-matrix$ for $G=GL(2,5)$.
First we take the first block $B(x,x)$. That is :

$$\left(\begin{array}{ccccccccc||ccccccccc}
l_1 & l_2 & l_3 & l_4 & l_5 & l_6 & l_7 & l_8 & \cdots & l_{11} & l_{12} & l_{13} & l_{14} & l_{15} & l_{16} & l_{17} & l_{18} & \cdots \\
l_{10} & l_1 & l_2 & l_3 & l_4 & l_5 & l_6 & l_7 & \cdots & l_{20} & l_{11} & l_{12} & l_{13} & l_{14} & l_{15} & l_{16} & l_{17} & \cdots \\
l_9 & l_{10} & l_1 & l_2 & l_3 & l_4 & l_5 & l_6 & \cdots & l_{19} & l_{20} & l_{11} & l_{12} & l_{13} & l_{14} & l_{15} & l_{16} & \cdots \\
l_8 & l_9 & l_{10} & l_1 & l_2 & l_3 & l_4 & l_5 & \cdots & l_{18} & l_{19} & l_{20} & l_{11} & l_{12} & l_{13} & l_{14} & l_{15} & \cdots \\
l_7 & l_8 & l_9 & l_{10} & l_1 & l_2 & l_3 & l_4 & \cdots & l_{17} & l_{18} & l_{19} & l_{20} & l_{11} & l_{12} & l_{13} & l_{14} & \cdots \\
l_6 & l_7 & l_8 & l_9 & l_{10} & l_1 & l_2 & l_3 & \cdots & l_{16} & l_{17} & l_{18} & l_{19} & l_{20} & l_{11} & l_{12} & l_{13} & \cdots \\
l_5 & l_6 & l_7 & l_8 & l_9 & l_{10} & l_1 & l_2 & \cdots & l_{15} & l_{16} & l_{17} & l_{18} & l_{19} & l_{20} & l_{11} & l_{12} & \cdots \\
l_4 & l_5 & l_6 & l_7 & l_8 & l_9 & l_{10} & l_1 & \cdots & l_{14} & l_{15} & l_{16} & l_{17} & l_{18} & l_{19} & l_{20} & l_{11} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline\hline
l_{16} & l_{17} & l_{18} & l_{19} & l_{20} & l_{11} & l_{12} & l_{13} & \cdots & l_1 & l_2 & l_3 & l_4 & l_5 & l_6 & l_7 & l_8 & \cdots \\
l_{15} & l_{16} & l_{17} & l_{18} & l_{19} & l_{20} & l_{11} & l_{12} & \cdots & l_{10} & l_1 & l_2 & l_3 & l_4 & l_5 & l_6 & l_7 & \cdots \\
l_{14} & l_{15} & l_{16} & l_{17} & l_{18} & l_{19} & l_{20} & l_{11} & \cdots & l_9 & l_{10} & l_1 & l_2 & l_3 & l_4 & l_5 & l_6 & \cdots \\
l_{13} & l_{14} & l_{15} & l_{16} & l_{17} & l_{18} & l_{19} & l_{20} & \cdots & l_8 & l_9 & l_{10} & l_1 & l_2 & l_3 & l_4 & l_5 & \cdots \\
l_{12} & l_{13} & l_{14} & l_{15} & l_{16} & l_{17} & l_{18} & l_{19} & \cdots & l_7 & l_8 & l_9 & l_{10} & l_1 & l_2 & l_3 & l_4 & \cdots \\
l_{11} & l_{12} & l_{13} & l_{14} & l_{15} & l_{16} & l_{17} & l_{18} & \cdots & l_6 & l_7 & l_8 & l_9 & l_{10} & l_1 & l_2 & l_3 & \cdots \\
l_{20} & l_{11} & l_{12} & l_{13} & l_{14} & l_{15} & l_{16} & l_{17} & \cdots & l_5 & l_6 & l_7 & l_8 & l_9 & l_{10} & l_1 & l_2 & \cdots \\
l_{19} & l_{20} & l_{11} & l_{12} & l_{13} & l_{14} & l_{15} & l_{16} & \cdots & l_4 & l_5 & l_6 & l_7 & l_8 & l_9 & l_{10} & l_1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right)$$

Now for another block we take the elements of block $B_2(x,y)(\lambda=2)$ with inverses of the elements of the block $B(x,x)$. So, we get :

$$\left(\begin{array}{cccccccc||cccccccc}
l_{301} & l_{302} & l_{303} & l_{304} & l_{305} & l_{306} & l_{307} & \cdots & l_{311} & l_{312} & l_{313} & l_{314} & l_{315} & l_{316} & l_{317} & \cdots \\
l_{310} & l_{301} & l_{302} & l_{303} & l_{304} & l_{305} & l_{306} & \cdots & l_{320} & l_{311} & l_{312} & l_{313} & l_{314} & l_{315} & l_{316} & \cdots \\
l_{309} & l_{310} & l_{301} & l_{302} & l_{303} & l_{304} & l_{305} & \cdots & l_{319} & l_{320} & l_{311} & l_{312} & l_{313} & l_{314} & l_{315} & \cdots \\
l_{308} & l_{309} & l_{310} & l_{301} & l_{302} & l_{303} & l_{304} & \cdots & l_{318} & l_{319} & l_{320} & l_{311} & l_{312} & l_{313} & l_{314} & \cdots \\
l_{307} & l_{308} & l_{309} & l_{310} & l_{301} & l_{302} & l_{303} & \cdots & l_{317} & l_{318} & l_{319} & l_{320} & l_{311} & l_{312} & l_{313} & \cdots \\
l_{306} & l_{307} & l_{308} & l_{309} & l_{310} & l_{301} & l_{302} & \cdots & l_{316} & l_{317} & l_{318} & l_{319} & l_{320} & l_{311} & l_{312} & \cdots \\
l_{305} & l_{306} & l_{307} & l_{308} & l_{309} & l_{310} & l_{301} & \cdots & l_{315} & l_{316} & l_{317} & l_{318} & l_{319} & l_{320} & l_{311} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline\hline
l_{316} & l_{317} & l_{318} & l_{319} & l_{320} & l_{311} & l_{312} & \cdots & l_{301} & l_{302} & l_{303} & l_{304} & l_{305} & l_{306} & l_{307} & \cdots \\
l_{315} & l_{316} & l_{317} & l_{318} & l_{319} & l_{320} & l_{311} & \cdots & l_{310} & l_{301} & l_{302} & l_{303} & l_{304} & l_{305} & l_{306} & \cdots \\
l_{314} & l_{315} & l_{316} & l_{317} & l_{318} & l_{319} & l_{320} & \cdots & l_{309} & l_{310} & l_{301} & l_{302} & l_{303} & l_{304} & l_{305} & \cdots \\
l_{313} & l_{314} & l_{315} & l_{316} & l_{317} & l_{318} & l_{319} & \cdots & l_{308} & l_{309} & l_{310} & l_{301} & l_{302} & l_{303} & l_{304} & \cdots \\
l_{312} & l_{313} & l_{314} & l_{315} & l_{316} & l_{317} & l_{318} & \cdots & l_{307} & l_{308} & l_{309} & l_{310} & l_{301} & l_{302} & l_{303} & \cdots \\
l_{311} & l_{312} & l_{313} & l_{314} & l_{315} & l_{316} & l_{317} & \cdots & l_{306} & l_{307} & l_{308} & l_{309} & l_{310} & l_{301} & l_{302} & \cdots \\
l_{320} & l_{311} & l_{312} & l_{313} & l_{314} & l_{315} & l_{316} & \cdots & l_{305} & l_{306} & l_{307} & l_{308} & l_{309} & l_{310} & l_{301} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right)$$

Now, if we take the elements of block $B_1(x,y)$ with the inverses of the elements of block $B_2(x,y)(\lambda=0)$ we get the following :

$$\left(\begin{array}{cccccccc||cccccccc}
l_{396} & l_{411} & l_{436} & l_{451} & l_{476} & l_{391} & l_{416} & \cdots & l_{381} & l_{406} & l_{421} & l_{446} & l_{461} & l_{386} & l_{401} & \cdots \\
l_{471} & l_{396} & l_{411} & l_{436} & l_{451} & l_{476} & l_{391} & \cdots & l_{466} & l_{381} & l_{406} & l_{421} & l_{446} & l_{461} & l_{386} & \cdots \\
l_{456} & l_{471} & l_{396} & l_{411} & l_{436} & l_{451} & l_{476} & \cdots & l_{441} & l_{466} & l_{381} & l_{406} & l_{421} & l_{446} & l_{461} & \cdots \\
l_{431} & l_{456} & l_{471} & l_{396} & l_{411} & l_{436} & l_{451} & \cdots & l_{426} & l_{441} & l_{466} & l_{381} & l_{406} & l_{421} & l_{446} & \cdots \\
l_{416} & l_{431} & l_{456} & l_{471} & l_{396} & l_{411} & l_{436} & \cdots & l_{401} & l_{426} & l_{441} & l_{466} & l_{381} & l_{406} & l_{421} & \cdots \\
l_{391} & l_{416} & l_{431} & l_{456} & l_{471} & l_{396} & l_{411} & \cdots & l_{386} & l_{401} & l_{426} & l_{441} & l_{466} & l_{381} & l_{406} & \cdots \\
l_{476} & l_{391} & l_{416} & l_{431} & l_{456} & l_{471} & l_{396} & \cdots & l_{461} & l_{386} & l_{401} & l_{426} & l_{441} & l_{466} & l_{381} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline\hline
l_{386} & l_{401} & l_{426} & l_{441} & l_{466} & l_{381} & l_{406} & \cdots & l_{396} & l_{411} & l_{436} & l_{451} & l_{476} & l_{391} & l_{416} & \cdots \\
l_{461} & l_{386} & l_{401} & l_{426} & l_{441} & l_{466} & l_{381} & \cdots & l_{471} & l_{396} & l_{411} & l_{436} & l_{451} & l_{476} & l_{391} & \cdots \\
l_{446} & l_{461} & l_{386} & l_{401} & l_{426} & l_{441} & l_{466} & \cdots & l_{456} & l_{471} & l_{396} & l_{411} & l_{436} & l_{451} & l_{476} & \cdots \\
l_{421} & l_{446} & l_{461} & l_{386} & l_{401} & l_{426} & l_{441} & \cdots & l_{431} & l_{456} & l_{471} & l_{396} & l_{411} & l_{436} & l_{451} & \cdots \\
l_{406} & l_{421} & l_{446} & l_{461} & l_{386} & l_{401} & l_{426} & \cdots & l_{416} & l_{431} & l_{456} & l_{471} & l_{396} & l_{411} & l_{436} & \cdots \\
l_{381} & l_{406} & l_{421} & l_{446} & l_{461} & l_{386} & l_{401} & \cdots & l_{391} & l_{416} & l_{431} & l_{456} & l_{471} & l_{396} & l_{411} & \cdots \\
l_{466} & l_{381} & l_{406} & l_{421} & l_{446} & l_{461} & l_{386} & \cdots & l_{476} & l_{391} & l_{416} & l_{431} & l_{456} & l_{471} & l_{396} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right)$$

2- The listing of elements of $GL(2,4)$:
$|Gl(2,4)|$=180, and this will divide into 15 blocks with 12 elements in each block .

Type $(a^i\ ,\ a^i)$:
Block $B(a^i,a^i)$**:**
$T(a^i,a^i)$**:**
$g_1=h(1,1),\ g_2=h(a^2,a^2)u(a^2),\ g_3=h(a,a),\ g_4=h(1,1)u(a^2),\ g_5=h(a^2,a^2),$
$g_6=h(a,a)u(a^2).$
$T(a^i,a^i)(1)$**:**
$g_7=h(1,1)u(a),\ g_8=h(a^2,a^2)u(1),\ g_9=h(a,a)u(a),\ g_{10}=h(1,1)u(1),$
$g_{11}=h(a^2,a^2)u(a),\ g_{12}=h(a,a)u(1).$
Block $B(a^i,a^i)(\lambda=0)$**:**
$T(a^i,a^i)(\lambda=0)$**:**
$g_{13}=h(1,1)n_o,\ g_{14}=h(a^2,a^2)u(a^2)n_o,\ g_{15}=h(a,a)n_o,\ g_{16}=h(1,1)u(a^2)n_o,$
$g_{17}=h(a^2,a^2)n_o,\ g_{18}=h(a,a)u(a^2)n_o.$
$T(a^i,a^i)(1)(\lambda=0)$**:**
$g_{19}=h(1,1)u(a)n_o,\ g_{20}=h(a^2,a^2)u(1)n_o,\ g_{21}=h(a,a)u(a)n_o,$
$g_{22}=h(1,1)u(1)n_o,\ g_{23}=h(a^2,a^2)u(a)n_o,\ g_{24}=h(a,a)u(1)n_o.$
Block $B(a^i,a^i)(\lambda=1)$**:**
$T(a^i,a^i)(\lambda=1)$**:**
$g_{25}=h(1,1)n_ou(1),\ g_{26}=h(a^2,a^2)u(a^2)n_ou(1),\ g_{27}=h(a,a)n_ou(1),$
$g_{28}=h(1,1)u(a^2)n_ou(1),\ g_{29}=h(a^2,a^2)n_ou(1),\ g_{30}=h(a,a)u(a^2)n_ou(1).$
$T(a^i,a^i)(1)(\lambda=1)$**:**
$g_{31}=h(1,1)u(a)n_ou(1),\ g_{32}=h(a^2,a^2)u(1)n_ou(1),\ g_{33}=h(a,a)u(a)n_ou(1),$
$g_{34}=h(1,1)u(1)n_ou(1),\ g_{35}=h(a^2,a^2)u(a)n_ou(1),\ g_{36}=h(a,a)u(1)n_ou(1).$
Block $B(a^i,a^i)(\lambda=a)$**:**
$T(a^i,a^i)(\lambda=a)$**:**
$g_{37}=h(1,1)n_ou(a),\ g_{38}=h(a^2,a^2)u(a^2)n_ou(a),\ g_{39}=h(a,a)n_ou(a),$
$g_{40}=h(1,1)u(a^2)n_ou(a),\ g_{41}=h(a^2,a^2)n_ou(a),\ g_{42}=h(a,a)u(a^2)n_ou(a).$
$T(a^i,a^i)(1)(\lambda=a)$**:**
$g_{43}=h(1,1)u(a)n_ou(a),\ g_{44}=h(a^2,a^2)u(1)n_ou(a),\ g_{45}=h(a,a)u(a)n_ou(a),$
$g_{46}=h(1,1)u(1)n_ou(a),\ g_{47}=h(a^2,a^2)u(a)n_ou(a),\ g_{48}=h(a,a)u(1)n_ou(a).$
Block $B(a^i,a^i)(\lambda=a^2)$**:**
$T(a^i,a^i)(\lambda=a^2)$**:**
$g_{49}=h(1,1)n_ou(a^2),\ g_{50}=h(a^2,a^2)u(a^2)n_ou(a^2),\ g_{51}=h(a,a)n_ou(a^2),$
$g_{52}=h(1,1)u(a^2)n_ou(a^2),\ g_{53}=h(a^2,a^2)n_ou(a^2),\ g_{54}=h(a,a)u(a^2)n_ou(a^2).$
$T(a^i,a^i)(1)(\lambda=a^2)$**:**
$g_{55}=h(1,1)u(a)n_ou(a^2),\ g_{56}=h(a^2,a^2)u(1)n_ou(a^2),\ g_{57}=h(a,a)u(a)n_ou(a^2),$
$g_{58}=h(1,1)u(1)n_ou(a^2),\ g_{59}=h(a^2,a^2)u(a)n_ou(a^2),\ g_{60}=h(a,a)u(1)n_ou(a^2).$

Type $h(a^i\ ,\ a^j)$:
Block $B_1(a^i,a^j)$**:**
$T_1(a^i,a^j)$**:**
$g_{61}=h(1,a),\ g_{62}=h(a^2,1)u(1),\ g_{63}=h(a,a^2),\ g_{64}=h(1,a)u(1),\ g_{65}=h(a^2,1),$
$g_{66}=h(a,a^2)u(1).$
$T_1(a^i,a^j)(1)$**:**
$g_{67}=h(1,a)u(a^2),\ g_{68}=h(a^2,1)u(a),\ g_{69}=h(a,a^2)u(a^2),\ g_{70}=h(1,a)u(a),$
$g_{71}=h(a^2,1)u(a^2),\ g_{72}=h(a,a^2)u(a).$

Block $B_1(a^i,a^j)(\lambda=0)$**:**
$T_1(a^i,a^j)(\lambda=0)$**:**
$g_{73}=h(1,a)n_o,\ g_{74}=h(a^2,1)u(1)n_o,\ g_{75}=h(a,a^2)n_o,\ g_{76}=h(1,a)u(1)n_o,$
$g_{77}=h(a^2,1)n_o,\ g_{78}=h(a,a^2)u(1)n_o.$
$T_1(a^i,a^j)(1)(\lambda=0)$**:**
$g_{79}=h(1,a)u(a^2)n_o,\ g_{80}=h(a^2,1)u(a)n_o,\ g_{81}=h(a,a^2)u(a^2)n_o,$
$g_{82}=h(1,a)u(a)n_o,\ g_{83}=h(a^2,1)u(a^2)n_o,\ g_{84}=h(a,a^2)u(a)n_o.$
Block $B_1(a^i,a^j)(\lambda=1)$**:**
$T_1(a^i,a^j)(\lambda=1)$**:**
$g_{85}=h(1,a)n_ou(1),\ g_{86}=h(a^2,1)u(1)n_ou(1),\ g_{87}=h(a,a^2)n_ou(1),$
$g_{88}=h(1,a)u(1)n_ou(1),\ g_{89}=h(a^2,1)n_ou(1),\ g_{90}=h(a,a^2)u(1)n_ou(1).$
$T_1(a^i,a^j)(1)(\lambda=1)$**:**
$g_{91}=h(1,a)u(a^2)n_ou(1),\ g_{92}=h(a^2,1)u(a)n_ou(1),\ g_{93}=h(a,a^2)u(a^2)n_ou(1),$
$g_{94}=h(1,a)u(a)n_ou(1),\ g_{95}=h(a^2,1)u(a^2)n_ou(1),\ g_{96}=h(a,a^2)u(a)n_ou(1).$
Block $B_1(a^i,a^j)(\lambda=a)$**:**
$T_1(a^i,a^j)(\lambda=a)$**:**
$g_{97}=h(1,a)n_ou(a),\ g_{98}=h(a^2,1)u(1)n_ou(a),\ g_{99}=h(a,a^2)n_ou(a),$
$g_{100}=h(1,a)u(1)n_ou(a),\ g_{101}=h(a^2,1)n_ou(a),\ g_{102}=h(a,a^2)u(1)n_ou(a).$
$T_1(a^i,a^j)(1)(\lambda=a)$**:**
$g_{103}=h(1,a)u(a^2)n_ou(a),\ g_{104}=h(a^2,1)u(a)n_ou(a),\ g_{105}=h(a,a^2)u(a^2)n_ou(a),$
$g_{106}=h(1,a)u(a)n_ou(a),\ g_{107}=h(a^2,1)u(a^2)n_ou(a),\ g_{108}=h(a,a^2)u(a)n_ou(a).$
Block $B_1(a^i,a^j)(\lambda=a^2)$**:**
$T_1(a^i,a^j)(\lambda=a^2)$**:**
$g_{109}=h(1,a)n_ou(a^2),\ g_{110}=h(a^2,1)u(1)n_ou(a^2),\ g_{111}=h(a,a^2)n_ou(a^2),$
$g_{112}=h(1,a)u(1)n_ou(a^2),\ g_{113}=h(a^2,1)n_ou(a^2),\ g_{114}=h(a,a^2)u(1)n_ou(a^2).$
$T_1(a^i,a^j)(1)(\lambda=a^2)$**:**
$g_{115}=h(1,a)u(a^2)n_ou(a^2),\ g_{116}=h(a^2,1)u(a)n_ou(a^2),\ g_{117}=h(a,a^2)u(a^2)n_ou(a^2),$
$g_{118}=h(1,a)u(a)n_ou(a^2),\ g_{119}=h(a^2,1)u(a^2)n_ou(a^2),\ g_{120}=h(a,a^2)u(a)n_ou(a^2).$
Block $B_2(a^i,a^j)$**:**
$T_2(a^i,a^j)$**:**
$g_{121}=h(1,a^2),\ g_{122}=h(a^2,a)u(a),\ g_{123}=h(a,1),\ g_{124}=h(1,a^2)u(a),$
$g_{125}=h(a^2,a),\ g_{126}=h(a,1)u(a).$
$T_2(a^i,a^j)(1)$**:**
$g_{127}=h(1,a^2)u(1),\ g_{128}=h(a^2,a)u(a^2),\ g_{129}=h(a,1)u(1),\ g_{130}=h(1,a^2)u(a^2),$
$g_{131}=h(a^2,a)u(1),\ g_{132}h(a,1)u(a^2).$
Block $B_2(a^i,a^j)(\lambda=0)$**:**
$T_2(a^i,a^j)(\lambda=0)$**:**
$g_{133}=h(1,a^2)n_o,\ g_{134}=h(a^2,a)u(a)n_o,\ g_{135}=h(a,1)n_o,\ g_{136}=h(1,a^2)u(a)n_o,$
$g_{137}=h(a^2,a)n_o,\ g_{138}=h(a,1)u(a)n_o.$
$T_2(a^i,a^j)(1)(\lambda=0)$**:**
$g_{139}=h(1,a^2)u(1)n_o,\ g_{140}=h(a^2,a)u(a^2)n_o,\ g_{141}=h(a,1)u(1)n_o,$
$g_{142}=h(1,a^2)u(a^2)n_o,\ g_{143}=h(a^2,a)u(1)n_o,\ g_{144}=h(a,1)u(a^2)n_o.$
Block $B_2(a^i,a^j)(\lambda=1)$**:**
$T_2(a^i,a^j)(\lambda=1)$**:**
$g_{145}=h(1,a^2)n_ou(1), g_{146}=h(a^2,a)u(a)n_ou(1),\ g_{147}=h(a,1)n_ou(1),$
$g_{148}=h(1,a^2)u(a)n_ou(1),\ g_{149}=h(a^2,a)n_ou(1),\ g_{150}=h(a,1)u(a)n_ou(1).$
$T_2(a^i,a^j)(1)(\lambda=1)$**:**
$g_{151}=h(1,a^2)u(1)n_ou(1),\ g_{152}=h(a^2,a)u(a^2)n_ou(1),\ g_{153}=h(a,1)u(1)n_ou(1),$
$g_{154}=h(1,a^2)u(a^2)n_ou(1),\ g_{155}=h(a^2,a)u(1)n_ou(1),\ g_{156}=h(a,1)u(a^2)n_ou(1).$
Block $B_2(a^i,a^j)(\lambda=a)$**:**
$T_2(a^i,a^j)(\lambda=a)$**:**
$g_{157}=h(1,a^2)n_ou(a),\ g_{158}=h(a^2,a)u(a)n_ou(a),\ g_{159}=h(a,1)n_ou(a),$
$g_{160}=h(1,a^2)u(a)n_ou(a),\ g_{161}=h(a^2,a)n_ou(a),\ g_{162}=h(a,1)u(a)n_ou(a).$
$T_2(a^i,a^j)(1)(\lambda=a)$**:**
$g_{163}=h(1,a^2)u(1)n_ou(a),\ g_{164}=h(a^2,a)u(a^2)n_ou(a),\ g_{165}=h(a,1)u(1)n_ou(a),$
$g_{166}=h(1,a^2)u(a^2)n_ou(a),\ g_{167}=h(a^2,a)u(1)n_ou(a),\ g_{168}=h(a,1)u(a^2)n_ou(a).$
Block $B_2(a^i,a^j)(\lambda=a^2)$**:**
$T_2(a^i,a^j)(\lambda=a^2)$**:**
$g_{169}=h(1,a^2)n_ou(a^2),\ g_{170}=h(a^2,a)u(a)n_ou(a^2),\ g_{171}=h(a,1)n_ou(a^2),$
$g_{172}=h(1,a^2)u(a)n_ou(a^2),\ g_{173}=h(a^2,a)n_ou(a^2),\ g_{174}=h(a,1)u(a)n_ou(a^2).$
$T_2(a^i,a^j)(1)(\lambda=a^2)$**:**
$g_{175}=h(1,a^2)u(1)n_ou(a^2),\ g_{176}=h(a^2,a)u(a^2)n_ou(a^2),\ g_{177}=h(a,1)u(1)n_ou(a^2),$
$g_{178}=h(1,a^2)u(a^2)n_ou(a^2),\ g_{179}=h(a^2,a)u(1)n_ou(a^2),\ g_{180}=h(a,1)u(a^2)n_ou(a^2).$

Where, F_4 is the unique field (up to the isomorphism) with four elements such that;

$$F_4 \cong \frac{F_2[x]}{< x^2+x+1 >},$$

where (x^2+x+1) is an irreducible polynomial in $F_2[x]$.
Now we examine some blocks of the $G-matrix$ of $GL(2,4)$. For the first block in the G-matrix we get the elements in the block $B(a^i,a^i)$ with its inverses and we get the following :

$$\left(\begin{array}{cccccc||cccccc}
g_1 & g_2 & g_3 & g_4 & g_5 & g_6 & g_7 & g_8 & g_9 & g_{10} & g_{11} & g_{12} \\
g_6 & g_1 & g_2 & g_3 & g_4 & g_5 & g_{12} & g_7 & g_8 & g_9 & g_{10} & g_{11} \\
g_5 & g_6 & g_1 & g_2 & g_3 & g_4 & g_{11} & g_{12} & g_7 & g_8 & g_9 & g_{10} \\
g_4 & g_5 & g_6 & g_1 & g_2 & g_3 & g_{10} & g_{11} & g_{12} & g_7 & g_8 & g_9 \\
g_3 & g_4 & g_5 & g_6 & g_1 & g_2 & g_9 & g_{10} & g_{11} & g_{12} & g_7 & g_8 \\
g_2 & g_3 & g_4 & g_5 & g_6 & g_1 & g_8 & g_9 & g_{10} & g_{11} & g_{12} & g_7 \\
\hline\hline
g_7 & g_8 & g_9 & g_{10} & g_{11} & g_{12} & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \\
g_{12} & g_7 & g_8 & g_9 & g_{10} & g_{11} & g_6 & g_1 & g_2 & g_3 & g_4 & g_5 \\
g_{11} & g_{12} & g_7 & g_8 & g_9 & g_{10} & g_5 & g_6 & g_1 & g_2 & g_3 & g_4 \\
g_{10} & g_{11} & g_{12} & g_7 & g_8 & g_9 & g_4 & g_5 & g_6 & g_1 & g_2 & g_3 \\
g_9 & g_{10} & g_{11} & g_{12} & g_7 & g_8 & g_3 & g_4 & g_5 & g_6 & g_1 & g_2 \\
g_8 & g_9 & g_{10} & g_{11} & g_{12} & g_7 & g_2 & g_3 & g_4 & g_5 & g_6 & g_1
\end{array}\right)$$

Now, if we consider the elements of block $B_2(a^i,a^j)(\lambda=0)$ with the inverses of the elements of block $B_1(a^i,a^j)$, we will get the following matrix:

$$\left(\begin{array}{cccccc||cccccc}
g_{73} & g_{80} & g_{75} & g_{82} & g_{77} & g_{84} & g_{76} & g_{83} & g_{78} & g_{79} & g_{74} & g_{81} \\
g_{84} & g_{73} & g_{80} & g_{75} & g_{82} & g_{77} & g_{81} & g_{76} & g_{83} & g_{78} & g_{79} & g_{74} \\
g_{77} & g_{84} & g_{73} & g_{80} & g_{75} & g_{82} & g_{74} & g_{81} & g_{76} & g_{83} & g_{78} & g_{79} \\
g_{82} & g_{77} & g_{84} & g_{73} & g_{80} & g_{75} & g_{79} & g_{74} & g_{81} & g_{76} & g_{83} & g_{78} \\
g_{75} & g_{82} & g_{77} & g_{84} & g_{73} & g_{80} & g_{78} & g_{79} & g_{74} & g_{81} & g_{76} & g_{83} \\
g_{80} & g_{75} & g_{82} & g_{77} & g_{84} & g_{73} & g_{83} & g_{78} & g_{79} & g_{74} & g_{81} & g_{76} \\
\hline\hline
g_{76} & g_{83} & g_{78} & g_{79} & g_{74} & g_{81} & g_{73} & g_{80} & g_{75} & g_{82} & g_{77} & g_{84} \\
g_{81} & g_{76} & g_{83} & g_{78} & g_{79} & g_{74} & g_{84} & g_{73} & g_{80} & g_{75} & g_{82} & g_{77} \\
g_{74} & g_{81} & g_{76} & g_{83} & g_{78} & g_{79} & g_{77} & g_{84} & g_{73} & g_{80} & g_{75} & g_{82} \\
g_{79} & g_{74} & g_{81} & g_{76} & g_{83} & g_{78} & g_{82} & g_{77} & g_{84} & g_{73} & g_{80} & g_{75} \\
g_{78} & g_{79} & g_{74} & g_{81} & g_{76} & g_{83} & g_{75} & g_{82} & g_{77} & g_{84} & g_{73} & g_{80} \\
g_{83} & g_{78} & g_{79} & g_{74} & g_{81} & g_{76} & g_{80} & g_{75} & g_{82} & g_{77} & g_{84} & g_{73}
\end{array}\right)$$

Also, if we get the elements of block $B_1(a^i,a^j)(\lambda=a)$ with the inverses of the elements of block $B_2(a^i,a^j)(\lambda=a^2)$, then the resulting matrix will have the following form:

$$\left(\begin{array}{cccccc||cccccc}
g_{71} & g_{111} & g_{67} & g_{113} & g_{69} & g_{109} & g_{139} & g_{34} & g_{141} & g_{36} & g_{143} & g_{32} \\
g_{109} & g_{71} & g_{111} & g_{67} & g_{113} & g_{69} & g_{32} & g_{139} & g_{34} & g_{141} & g_{36} & g_{143} \\
g_{69} & g_{109} & g_{71} & g_{111} & g_{67} & g_{113} & g_{143} & g_{32} & g_{139} & g_{34} & g_{141} & g_{36} \\
g_{113} & g_{69} & g_{109} & g_{71} & g_{111} & g_{67} & g_{36} & g_{143} & g_{32} & g_{139} & g_{34} & g_{141} \\
g_{67} & g_{113} & g_{69} & g_{109} & g_{71} & g_{111} & g_{141} & g_{36} & g_{143} & g_{32} & g_{139} & g_{34} \\
g_{111} & g_{67} & g_{113} & g_{69} & g_{109} & g_{71} & g_{34} & g_{141} & g_{36} & g_{143} & g_{32} & g_{139} \\
\hline\hline
g_{139} & g_{34} & g_{141} & g_{36} & g_{143} & g_{32} & g_{71} & g_{111} & g_{67} & g_{113} & g_{69} & g_{109} \\
g_{32} & g_{139} & g_{34} & g_{141} & g_{36} & g_{143} & g_{109} & g_{71} & g_{111} & g_{67} & g_{113} & g_{69} \\
g_{143} & g_{32} & g_{139} & g_{34} & g_{141} & g_{36} & g_{69} & g_{109} & g_{71} & g_{111} & g_{67} & g_{113} \\
g_{36} & g_{143} & g_{32} & g_{139} & g_{34} & g_{141} & g_{113} & g_{69} & g_{109} & g_{71} & g_{111} & g_{67} \\
g_{141} & g_{36} & g_{143} & g_{32} & g_{139} & g_{34} & g_{67} & g_{113} & g_{69} & g_{109} & g_{71} & g_{111} \\
g_{34} & g_{141} & g_{36} & g_{143} & g_{32} & g_{139} & g_{111} & g_{67} & g_{113} & g_{69} & g_{109} & g_{71}
\end{array}\right)$$

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Nonexistence of Global Solutions for the Kirchhoff-Type Equation with Fractional Damped

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Abstract

In this work, we investigate the Kirchhoff-type equation with a fractional damping term in a bounded domain. The fractional damping term plays a quenching role, which is weaker than strong damping and stronger than weak damping term. We prove a nonexistence of global solutions with negative inital energy. This result extends and improves some results in the literature.

1. Introduction

In this work, we deal with the nonexistence of solutions following Kirchhoff-type equation:

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + \partial_t^{1+\alpha} u = |u|^{p-1}u, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ \ u_t(x,0) = u_1(x), & x \in \Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, and $M(s) = \beta_1 + \beta_2 s^\gamma$. The constants $p > 1$ real number, $\gamma \geq 0$, $\beta_1, \beta_2 > 0$ and $-1 < \alpha < 1$. Without loss of generality, we choose $\beta_1 = \beta_2 = 1$ in (1.1) in this paper. The notation $\partial_t^{1+\alpha}$ stands for the Caputo's fractional derivative of order $1+\alpha$ with respect to the time variable [1, 2]. It is defined as follows

$$\partial_t^{1+\alpha} w(t) = \begin{cases} I^{-\alpha}\frac{d}{dt}w(t) & \text{for } -1 < \alpha < 0 \\ I^{1-\alpha}\frac{d^2}{dt^2}w(t) & \text{for } \quad 0 < \alpha < 1 \end{cases}$$

where $I^\beta, \beta > 0$ is fractional integral

$$I^\beta \frac{d}{dt} w(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} w_\tau(\tau)d\tau.$$

The fractional damping term plays a quenching role, which is weaker than strong damping and stronger than weak damping term [3]. The problem (1.1) is a generalization of a model introduced by Kirchhoff [4].
Ono [5] considered equation (1.1) for $\alpha = 0$. He proved that the solution blows up with negative initial energy. Later, Wu and Tsai [6] proved the blow up of the solution with positive upper bounded initial energy.
In [7] Yang et al. studied the following equation

$$u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + (-\Delta)^\alpha u + f(u) = g(x).$$

They proved the attractors for $1/2 < \alpha < 1$.
There are many literatures on the nonexistence of solutions for the Kirchhoff-type equation.
This work is organized as follows. In Section 2, we give some notations and lemmas needed for our paper. In Section 3, we prove the nonexistence of the solution for the problem (1.1) with negative initial energy. We use improved the method of [8].

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2. Preliminaries

In this part, we give some notations and material needed in our main result. Without loss of generality, we get only the case $-1<\alpha<0$. We define the energy with problem (1.1) is

$$E(t)=\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\nabla u\|^2+\frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}-\frac{1}{p+1}\|u\|_{p+1}^{p+1}.$$

Then,

$$E'(t)=-\frac{1}{\Gamma(-\alpha)}\int_\Omega u_t\int_0^t(t-\tau)^{-(\alpha+1)}u_\tau(\tau)d\tau dx.$$

Now, we define modified energy functional as

$$E_\varepsilon(t)=E(t)-\varepsilon\int_\Omega uu_t dx \tag{2.1}$$

where $0<\varepsilon<1$ is the constant which is specified later. Now a differentiation of $E_\varepsilon(t)$, with respect to time t gives

$$\begin{aligned} E_\varepsilon'(t) &= -\frac{1}{\Gamma(-\alpha)}\int_\Omega u_t\int_0^t(t-\tau)^{-(\alpha+1)}u_\tau(\tau)d\tau dx \\ &-\varepsilon\int_\Omega|u_t|^2dx-\varepsilon\int_\Omega|u|^{p+1}dx+\varepsilon\int_\Omega|\nabla u|^2dx+\varepsilon\int_\Omega|\nabla u|^{2(\gamma+1)}dx \\ &+\frac{\varepsilon}{\Gamma(-\alpha)}\int_\Omega u\int_0^t(t-\tau)^{-(\alpha+1)}u_\tau(\tau)d\tau dx. \end{aligned}\tag{2.2}$$

Also, we define the following functionals

$$H(t)=-\left(e^{-\sigma\varepsilon t}E_\varepsilon(t)+\mu F(t)+d\right),\tag{2.3}$$

$$F(t)=\int_0^t\int_\Omega G(t-\tau)e^{-\sigma\varepsilon\tau}u_\tau^2dxd\tau\tag{2.4}$$

and

$$G(t)=e^{\beta t}\int_t^\infty e^{-\beta\tau}\tau^{-(\alpha+1)}d\tau$$

where $\sigma=\frac{p+1}{2}$ and β,μ,d are positive consants.

Lemma 2.1. *Let p be sufcifiently large and $E_\varepsilon(0)<0$. Then $H'(t)>0$ and $H(t)>0$.*

Proof. By taking a derivative of (2.3) and (2.4), we get

$$H'(t)=\sigma\varepsilon e^{-\sigma\varepsilon t}E_\varepsilon(t)-e^{-\sigma\varepsilon t}E_\varepsilon'(t)-\mu F'(t),\tag{2.5}$$

$$F'(t)=\beta^\alpha\Gamma(-\alpha)e^{-\sigma\varepsilon t}\int_\Omega u_t^2dx-\int_0^t\int_\Omega(t-\tau)^{-(\alpha+1)}e^{-\sigma\varepsilon\tau}u_\tau^2dxd\tau+\beta F(t).\tag{2.6}$$

Taking into account (2.6), (2.1) and (2.2) in (2.5), we obtain

$$\begin{aligned} H'(t) &= e^{-\sigma\varepsilon t}\left(\frac{\sigma\varepsilon}{2}+\varepsilon-\mu\beta^\alpha\Gamma(-\alpha)\right)\int_\Omega|u_t|^2dx+\varepsilon e^{-\sigma\varepsilon t}\left(\frac{\sigma}{2}-1\right)\int_\Omega|\nabla u|^2dx \\ &+\varepsilon e^{-\sigma\varepsilon t}\left(\frac{\sigma}{2(\gamma+1)}-1\right)\int_\Omega|\nabla u|^{2(\gamma+1)}dx+\varepsilon e^{-\sigma\varepsilon t}\left(1-\frac{\sigma}{p+1}\right)\int_\Omega|u|^{p+1}dx \\ &-\varepsilon\sigma\varepsilon e^{-\sigma\varepsilon t}\int_\Omega uu_tdx+\frac{e^{-\sigma\varepsilon t}}{\Gamma(-\alpha)}\int_\Omega u_t\int_0^t(t-\tau)^{-(\alpha+1)}u_\tau(\tau)d\tau dx \\ &-\frac{\varepsilon e^{-\sigma\varepsilon t}}{\Gamma(-\alpha)}\int_\Omega u\int_0^t(t-\tau)^{-(\alpha+1)}u_\tau(\tau)d\tau dx \\ &+\mu\int_0^t\int_\Omega(t-\tau)^{-(\alpha+1)}e^{-\sigma\varepsilon\tau}u_\tau^2dxd\tau-\mu\beta F(t). \end{aligned}\tag{2.7}$$

Next, we estimate some terms in the right hand side of (2.7). For the sixth term on the right hand side of (2.7), using Young's inequality, we obtain

$$\begin{aligned} & e^{-\sigma\varepsilon t}\int_\Omega u_t \int_0^t (t-\tau)^{-(\alpha+1)} u_\tau(\tau)d\tau dx \\ \leq\ & \delta_1 e^{-\sigma\varepsilon t}\int_\Omega u_t^2 dx + \frac{1}{4\delta_1}e^{-\sigma\varepsilon t}\int_\Omega \left[\int_0^t (t-\tau)^{-(\alpha+1)} u_\tau(\tau)d\tau\right]^2 dx. \end{aligned}$$

Writing $-(\alpha+1) = -\frac{\alpha+1}{2} - \frac{\alpha+1}{2}$ and thanks to the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & e^{-\sigma\varepsilon t}.\int_\Omega u_t \int_0^t (t-\tau)^{-(\alpha+1)} u_\tau(\tau)d\tau dx \\ \leq\ & \delta_1 e^{-\sigma\varepsilon t}\int_\Omega u_t^2 dx + \frac{(\sigma\varepsilon)^\alpha\Gamma(-\alpha)}{4\delta_1}\int_\Omega\int_0^t (t-\tau)^{-(\alpha+1)} e^{-\sigma\varepsilon\tau}u_\tau^2 d\tau dx. \end{aligned} \tag{2.8}$$

Smilarly, we have

$$\begin{aligned} & e^{-\sigma\varepsilon t}.\int_\Omega u \int_0^t (t-\tau)^{-(\alpha+1)} u_\tau(\tau)d\tau dx \\ \leq\ & \delta_2 e^{-\sigma\varepsilon t}.\int_\Omega |u|^2 dx + \frac{1}{4\delta_2}e^{-\sigma\varepsilon t}\int_\Omega \left(\int_0^t (t-\tau)^{-(\alpha+1)} e^{\frac{-\sigma\varepsilon t}{2}} u_\tau(\tau)d\tau\right)^2 dx. \end{aligned}$$

Using Sobolev-Poincare's inequality, we arrive at

$$\begin{aligned} & e^{-\sigma\varepsilon t}\int_\Omega u \int_0^t (t-\tau)^{-(\alpha+1)} u_\tau(\tau)d\tau dx \\ \leq\ & \delta_2 e^{-\sigma\varepsilon t} C_{p_1}\int_\Omega |\nabla u|^2 dx \\ & + \frac{(\sigma\varepsilon)^\alpha\Gamma(-\alpha)}{4\delta_2}\int_\Omega\int_0^t (t-\tau)^{-(\alpha+1)} e^{-\sigma\varepsilon\tau}u_\tau^2 d\tau dx. \end{aligned} \tag{2.9}$$

Now, we estimate the fifth term in the right side of (2.7), thanks to the Young's and Sobolev-Poincare's inequalities, we have

$$\begin{aligned} \int_\Omega u u_t dx &\leq \delta_3\int_\Omega |u|^2 dx + \frac{1}{4\delta_3}\int_\Omega |u_t|^2 dx \\ &\leq \delta_3 C_{p_2}\int_\Omega |\nabla u|^2 dx + \frac{1}{4\delta_3}\int_\Omega |u_t|^2 dx. \end{aligned} \tag{2.10}$$

By (2.7), (2.8), (2.9) and (2.10), we have

$$\begin{aligned} H'(t) \geq\ & e^{-\sigma\varepsilon t}\left(\frac{\sigma\varepsilon}{2} + \varepsilon - \mu\beta^\alpha\Gamma(-\alpha) - \frac{\varepsilon^2\sigma}{4\delta_3} - \frac{\delta_1}{\Gamma(-\alpha)}\right)\int_\Omega |u_t|^2 dx \\ & + \varepsilon e^{-\sigma\varepsilon t}\left(\frac{\sigma}{2} - 1 - \delta_3 C_{p_2}\varepsilon\sigma - \frac{\delta_2 C_{p_1}}{\Gamma(-\alpha)}\right)\int_\Omega |\nabla u|^2 dx \\ & + \varepsilon e^{-\sigma\varepsilon t}\left(\frac{\sigma}{2(\gamma+1)} - 1\right)\int_\Omega |\nabla u|^{2(\gamma+1)} dx + \varepsilon e^{-\sigma\varepsilon t}\left(1 - \frac{\sigma}{p+1}\right)\int_\Omega |u|^{p+1} dx \\ & + \left(\mu - \frac{(\sigma\varepsilon)^\alpha}{4\delta_1} - \frac{(\sigma\varepsilon)^\alpha\varepsilon}{4\delta_2}\right)\int_\Omega\int_0^t (t-\tau)^{-(\alpha+1)} e^{-\sigma\varepsilon\tau}u_\tau^2 d\tau dx - \mu\beta F(t). \end{aligned}$$

Subtracting and adding $C_1H(t)$ on the right hand side of above inequality, we get

$$\begin{aligned} H'(t) &\geq C_1H(t)+e^{-\sigma\varepsilon t}\left(\frac{\sigma\varepsilon}{2}+\varepsilon-\mu\beta^{\alpha}\Gamma(-\alpha)-\frac{\varepsilon^2\sigma}{4\delta_3}-\frac{\delta_1}{\Gamma(-\alpha)}+\frac{C_1}{2}-\frac{C_1}{4\delta_3}\right)\int_\Omega u_t^2dx \\ &+\varepsilon e^{-\sigma\varepsilon t}\left(\frac{\sigma}{2}-1-\delta_3C_{p_2}\varepsilon\sigma-\frac{\delta_2C_{p_1}}{\Gamma(-\alpha)}+\frac{C_1}{2}-C_1\delta_3C_{p_2}\right)\int_\Omega|\nabla u|^2dx \\ &+e^{-\sigma\varepsilon t}\left(\frac{\varepsilon\sigma}{2(\gamma+1)}-\varepsilon+\frac{C_1}{2(\gamma+1)}\right)\int_\Omega|\nabla u|^{2(\gamma+1)}dx \\ &+e^{-\sigma\varepsilon t}\left(\varepsilon-\frac{\sigma\varepsilon}{p+1}-\frac{C_1}{p+1}\right)\int_\Omega|u|^{p+1}dx \\ &+\left(\mu-\frac{(\sigma\varepsilon)^\alpha}{4\delta_1}-\frac{(\sigma\varepsilon)^\alpha\varepsilon}{4\delta_2}\right)\int_\Omega\int_0^t(t-\tau)^{-(\alpha+1)}e^{-\sigma\varepsilon\tau}u_\tau^2d\tau dx \\ &+\mu(C_1-\beta)F(t)+C_1d. \end{aligned}$$

We choose $C_1=\frac{p+1}{2}\varepsilon,\ \delta_1=\delta_2=\frac{\Gamma(-\alpha)\varepsilon}{2},\ \delta_3=\frac{1}{2}$ and $\beta=1$, we obtain

$$\begin{aligned} H'(t) &\geq \frac{p+1}{2}\varepsilon H(t)+e^{-\sigma\varepsilon t}\left(\frac{p+1}{4}\varepsilon(1-\varepsilon)-\mu\Gamma(-\alpha)\right)\int_\Omega u_t^2dx \\ &+\varepsilon C_{p_3}e^{-\sigma\varepsilon t}\left(\frac{p-3+\varepsilon\left(p+1-C_p(2p+4)\right)}{4}\right)\int_\Omega|\nabla u|^2dx \\ &+\varepsilon e^{-\sigma\varepsilon t}\left(\frac{p+1}{2(\gamma+1)}-1\right)\int_\Omega|\nabla u|^{2(\gamma+1)}dx \\ &+\left(\mu-\frac{(p+1)^\alpha\varepsilon^{\alpha-1}}{2^{\alpha+1}\Gamma(-\alpha)}(1+\varepsilon)\right)\int_\Omega\int_0^t(t-\tau)^{-(\alpha+1)}e^{-\sigma\varepsilon\tau}u_\tau^2d\tau dx \\ &+\mu\left(\frac{p+1}{2}\varepsilon-1\right)F(t)+\frac{p+1}{2}\varepsilon d. \end{aligned}$$

We choose

$$\varepsilon<\varepsilon_1=\min\left\{1,\frac{p-3}{2\left[2(p+2)C_p-(p+1)\right]}\right\}.$$

Where $C_p>\frac{p+1}{2(p+2)}$, it appears that the third coefficient is nonnegative. Observe if that $C_p<\frac{3}{4}$ ve $p\geq\frac{1+8C_p}{1-4C_p}$, than $\frac{p-3}{2[2(p+2)C_p-(p+1)]}\geq 1$ and this condition reduces to $\varepsilon<1$. We can take μ so that the second coefficient is nonnegative and the forth coefficient is greater than $\frac{(p+1)^\alpha}{2^{\alpha+1}\varepsilon^{1-\alpha}\Gamma(-\alpha)}$. Also, if p is sufficiently large $\frac{p+1}{2}\varepsilon-1$ is positive. Consequently, we get

$$\begin{aligned} H'(t) &\geq \frac{p+1}{2}\varepsilon H(t)+\frac{p-3}{8}\varepsilon e^{-\sigma\varepsilon t}.\int_\Omega|\Delta u|^2dx \\ &+\frac{(p+1)^\alpha}{2^{\alpha+1}\varepsilon^{1-\alpha}\Gamma(-\alpha)}\int_\Omega\int_0^t(t-\tau)^{-(\alpha+1)}e^{-\sigma\varepsilon\tau}u_\tau^2d\tau dx. \end{aligned} \tag{2.11}$$

If we select $E_\varepsilon(0)<-d$, then $H(0)>0$. This completes the proof. □

3. Nonexistence of global solutions

In this part, we obtain the nonexistence of global solutions of the problem (1.1).

Theorem 3.1. *Suppose that* $-1<\alpha<0$,

$$E(0)<0 \text{ and } \int_\Omega u_1u_0dx\geq 0.$$

Then the solution of (1.1) blows up in finite time.

Proof. We define an auxiliary function

$$\Psi(t)=H^{1-\gamma}(t)+\varphi e^{-\sigma\varepsilon t}.\int_\Omega uu_tdx$$

where $\gamma=\frac{p-1}{2(p+1)}$ and φ is a positive constant to be specified later. Our aim is to show that $\Psi(t)$ satisfies the following differential inequality:

$$\Psi^{\frac{1}{1-\gamma}}(t)\leq k\Psi'(t).$$

By differentiating $\Psi(t)$ with respect to t and using (1.1), we obtain

$$\begin{aligned}\Psi'(t) &= (1-\gamma)H^{-\gamma}(t)H'(t)-\varphi\sigma\varepsilon e^{-\sigma\varepsilon t}.\int_\Omega uu_t dx+\varphi e^{-\sigma\varepsilon t}.\left(\int_\Omega u_t^2 dx+\int_\Omega uu_{tt}dx\right)\\ &= (1-\gamma)H^{-\gamma}(t)H'(t)-\varphi\sigma\varepsilon e^{-\sigma\varepsilon t}.\int_\Omega uu_t dx\\ &\quad+\varphi e^{-\sigma\varepsilon t}\left[\int_\Omega |u|^{p+1}dx+\int_\Omega |\Delta u|^2 dx-\frac{1}{\Gamma(-\alpha)}\int_\Omega u\int_0^t (t-\tau)^{-(\alpha+1)}u_\tau(\tau)d\tau dx\right]\\ &\quad+\varphi e^{-\sigma\varepsilon t}\int_\Omega u_t^2 dx.\end{aligned}$$

By using the inequalities (2.10) and (2.9) with the constant $\delta_4,\delta_5>0$, we get

$$\begin{aligned}\Psi'(t) &= (1-\gamma)H^{-\gamma}(t)H'(t)-\varphi\sigma\varepsilon\delta_4 e^{-\sigma\varepsilon t}.\int_\Omega |u|^2 dx+\varphi e^{-\sigma\varepsilon t}.\int_\Omega |u|^{p+1}dx\\ &\quad-\frac{\varphi\sigma\varepsilon e^{-\sigma\varepsilon t}}{4\delta_4}\int_\Omega u_t^2 dx+\varphi e^{-\sigma\varepsilon t}.\int_\Omega u_t^2 dx+\varphi e^{-\sigma\varepsilon t}.\int_\Omega |\Delta u|^2 dx\\ &\quad-\frac{\varphi\delta_5 e^{-\sigma\varepsilon t}}{\Gamma(-\alpha)}\int_\Omega |u|^2 dx-\frac{\varphi(\sigma\varepsilon)^\alpha}{4\delta_5}\int_\Omega\int_0^t (t-\tau)^{-(\alpha+1)}e^{-\sigma\varepsilon\tau}u_\tau^2 d\tau dx.\end{aligned}$$

By (2.11), we have

$$\begin{aligned}\Psi'(t) &\geq \left((1-\gamma)H^{-\gamma}(t)-\frac{\varphi\varepsilon\Gamma(-\alpha)}{2\delta_5}\right)H'(t)\\ &\quad+\frac{\varphi(p+1)\Gamma(-\alpha)\varepsilon^2}{4\delta_5}H(t)\\ &\quad+\varphi e^{-\sigma\varepsilon t}\left(1+\frac{(p-3)\Gamma(-\alpha)}{8\delta_5}\varepsilon^2-\left(\sigma\varepsilon\delta_4+\frac{\delta_5}{\Gamma(-\alpha)}\right)C_p\right)\int_\Omega |\Delta u|^2 dx\\ &\quad+\varphi e^{-\sigma\varepsilon t}\left(1-\frac{\sigma\varepsilon}{4\delta_4}\right)\int_\Omega u_t^2 dx+\varphi e^{-\sigma\varepsilon t}.\int_\Omega |u|^{p+1}dx.\end{aligned}$$

If we take $\delta_5=L\Gamma(-\alpha)H^\gamma(t)$, we get

$$\begin{aligned}\Psi'(t) &\geq \left((1-\gamma)-\frac{\varphi\varepsilon}{2L}\right)H^{-\gamma}(t)H'(t)+\frac{\varphi(p+1)\varepsilon^2}{4L}H^{-\gamma}(t)H(t)\\ &\quad+\varphi e^{-\sigma\varepsilon t}\left(1+\frac{(p-3)H^{-\gamma}(t)}{8L}\varepsilon^2-(\sigma\varepsilon\delta_4+LH^\gamma(t))C_p\right)\int_\Omega |\Delta u|^2 dx\\ &\quad+\varphi e^{-\sigma\varepsilon t}\left(1-\frac{\sigma\varepsilon}{4\delta_4}\right)\int_\Omega u_t^2 dx+\varphi e^{-\sigma\varepsilon t}.\int_\Omega |u|^{p+1}dx.\end{aligned}$$

If we substitute and add $H(t)$ to the right side of the equation, we arrive at

$$\begin{aligned}\Psi'(t) &\geq \left(1-\gamma-\frac{\varphi\varepsilon}{2L}\right)H^{-\gamma}(t)H'(t)\\ &\quad+\left(\frac{\varphi(p+1)\varepsilon^2}{4L}H^{-\gamma}(t)+1\right)H(t)\\ &\quad+\varphi e^{-\sigma\varepsilon t}\Bigg[\varphi+\frac{(p-3)H^{-\gamma}(t)\varphi}{8L}\varepsilon^2-\varphi(\varepsilon\sigma\delta_4+LH^\gamma(t))C_p\\ &\quad-C_p\left(\varepsilon\delta_6 C_*+\frac{1}{2}\right)\Bigg]\int_\Omega |\Delta u|^2 dx\\ &\quad+e^{-\sigma\varepsilon t}\left(\varphi-\frac{\varphi(p+1)\varepsilon}{8\delta_4}+\frac{1}{2}-\frac{\varepsilon}{4\delta_6}\right)\int_\Omega u_t^2 dx\\ &\quad+e^{-\sigma\varepsilon t}\left(\varphi-\frac{1}{p+1}\right).\int_\Omega |u|^{p+1}dx+\frac{e^{-\sigma\varepsilon t}}{2(\gamma+1)}\int_\Omega |\nabla u|^{2(\gamma+1)}dx+\mu F(t)+d.\end{aligned}\tag{3.1}$$

We take $1-\gamma-\frac{\varphi\varepsilon}{2L}\geq 0$ and $\varepsilon\leq\varepsilon_2=\frac{2L(1-\gamma)}{\varphi}$, we have

$$\frac{\varphi(p+1)\varepsilon^2}{4L}H^{-\gamma}(t)\geq 0.$$

Also, we take $\varphi=\frac{p+3}{p+1}$, $\delta_4=\delta_6=\frac{1}{2}$ and $\varepsilon<\varepsilon_3=\frac{4(p+3)}{(p+1)(p+5)}$, we have

$$\varphi-\frac{\varphi(p+1)\varepsilon}{8\delta_4}-\frac{\varepsilon}{4\delta_6}\geq 0,$$

The fifth coefficient is nonnegative as soon as ε and C_p is chosen small enough, we have

$$\varphi+\frac{(p-3)H^{-\gamma}(t)\,\varphi}{8L}\varepsilon^2-\varphi\left(\varepsilon\sigma\delta_4+LH^{\gamma}(t)\right)C_p-C_p\left(\varepsilon\delta_6C_*+\frac{1}{2}\right)\geq 0$$

and

$$\begin{aligned}&\frac{p+3}{p+1}+\frac{(p-3)(p+3)H^{-\gamma}(t)}{8L(p+1)}\varepsilon^2\\&-\frac{1}{2}C_p\left(\frac{p+3}{p+1}\left(\varepsilon\frac{p+1}{2}+2LH^{\gamma}(t)\right)+\varepsilon C_*+1\right)\\ \geq\;& 0.\end{aligned}$$

Therefore (3.1) takes the form

$$\Psi'(t)\geq H(t)+\frac{1}{2}\int_{\Omega}u_t^2dx+\frac{p+2}{p+1}\int_{\Omega}|u|^{p+1}dx. \tag{3.2}$$

By the defination $\Psi(t)$, we deduce that

$$\Psi^{\frac{1}{1-\gamma}}(t)\leq 2^{\frac{\gamma}{1-\gamma}}\left[H(t)+\varphi^{\frac{1}{1-\gamma}}\left(\int_{\Omega}uu_tdx\right)^{\frac{1}{1-\gamma}}\right].$$

By the Cauchy-Schwarz and Hölder's inequalities, we arrive at

$$\Psi^{\frac{1}{1-\gamma}}(t)\leq 2^{\frac{\gamma}{1-\gamma}}\left[H(t)+\varphi^{\frac{1}{1-\gamma}}b\left(\int_{\Omega}|u_t|^2dx+\int_{\Omega}|u|^{p+1}dx\right)\right]. \tag{3.3}$$

If we take k is large enough

$$2^{\frac{\gamma}{1-\gamma}}\leq k,$$

$$2^{\frac{\gamma}{1-\gamma}}\varphi^{\frac{1}{1-\gamma}}b\leq\frac{k}{2},$$

$$2^{\frac{\gamma}{1-\gamma}}\varphi^{\frac{1}{1-\gamma}}b\leq\frac{p+2}{p+1}k.$$

That is k has to be chosen so that

$$k\geq 2^{\frac{\gamma}{1-\gamma}}\max\left\{1,2\varphi^{\frac{1}{1-\gamma}}b,\frac{p+1}{p+2}\varphi^{\frac{1}{1-\gamma}}b\right\}.$$

Combining (3.2) and (3.3), we have

$$\Psi^{\frac{1}{1-\gamma}}(t)\leq k\Psi'(t). \tag{3.4}$$

From (3.2) it is clear that $\Psi'(t)\geq 0$. Therefore, by the definition of $\Psi(t)$ and the hypotheses on the initial data, we get

$$\Psi(t)\geq\Psi(0)>\varphi\int_{\Omega}u_1u_0dx\geq 0.$$

Thus $\Psi(t)>0$. Integrating (3.4) over $(0,t)$, we get

$$\Psi^{\frac{\gamma}{1-\gamma}}(t)\geq\frac{1}{\Psi^{\frac{-\gamma}{1-\gamma}}(0)-\frac{\gamma}{k(1-\gamma)}t}. \tag{3.5}$$

Therefore (3.5) shows that $\Psi(t)$ blows up in finite time

$$T^*\leq\frac{k(1-\gamma)\Psi^{\frac{-\gamma}{1-\gamma}}(0)}{\gamma}.$$

This completes the proof. □

Remark 3.2. *The larger $\Psi(0)$ is the quicker the blow up takes place.*

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Weak Semilocal Convergence Conditions for a Two-Step Newton Method in Banach Space

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Abstract

We present new sufficient convergence conditions for a two step Newton method (TSNM) to solve nonlinear equations in a Banach space setting. The new conditions depend on the center-Lipschitz constant instead of the Lipschitz constant. This way the applicability of (TSNM) is expanded in cases not covered before. Numerical examples are also provided in this study.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y.
Many problems from computational sciences and other disciplines can be brought in a form similar to equation (1.1) using mathematical modeling [5, 13, 18, 26, 29, 30]. The solution of these equations can rarely be found in closed form. That is why most solution methods for these equations are iterative. In applied sciences the practice of Numerical Analysis for finding solutions x^* of equation (1.1) is essentially connected to variants of Newton's method [1]-[46].
The basic idea of Newton's method is linearization. Starting from an initial guess, we can have the linear approximation of $F(x)$ in the neighborhood of $x_0 : F(x_0+h) \approx F(x_0) + F'(x_0)h$, and solve the resulting linear equation $F(x_0) + F'(x_0)h = 0$, leading to the recurrent Newton method (NM)

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \tag{1.2}$$

for each $n = 0, 1, 2, \cdots$. This is Newton's method as proposed in 1669 by I.Newton (for polynomial only) defined on the real line. It was J. Raphson, who proposed the usage of Newton's method for general functions. That is why the method is often called the Newton-Raphson method. Later in 1818, Fourier proved that the method converges quadratically in a neighborhood of the root, while Cauchy (1829, 1847) provided the multidimensional extension of Newton's method (1.3). In 1948, L.V. Kantorovich published an important paper [26], extending Newton's method for functional spaces (the Newton-Kantorovich method (NKM)). Ever, since thousands of papers have been written in a Banach space setting for the (NM) as well as Newton-type methods, and their applications. We refer the reader to the publications [5, 13] for recent results (see also, the references therein).
The study about convergence matter of iterative procedures is usually based on two types: semilocal and local convergence analysis. The semilocal convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls.

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In order to increase the order of convergence the two step Newton method (TSNM)

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= y_n - F'(y_n)^{-1}F(y_n) \end{aligned}$$

for each $n = 0,1,2,\cdots$ has been used to approximate x^* [1]-[46]. (TSNM) has convergence order four. Let $U(x,r)$ denote an open ball with center $x \in X$ and of radius $r > 0$. Let $\overline{U(x,r)}$ denote the closure of $U(x,r)$. Let also $L(X,Y)$ denote the space of bounded linear operators from X into Y.

The Newton-Kantorovich theorem is a semilocal convergence result for solving nonlinear equations using (NM) or (TSNM) asserts that if there exist $x_0 \in D$, $\eta > 0$ and $L > 0$ such that

$$F'(x_0)^{-1} \in L(Y,X), \quad \|F'(x_0)^{-1}F'(x_0)\| \le \eta \tag{1.3}$$

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \le L\|x-y\| \tag{1.4}$$

for each x and y in D

$$h = 2L\eta \le 1 \tag{1.5}$$

and

$$\overline{U(x_0,R)} \subseteq D, \tag{1.6}$$

where

$$R = \frac{1-\sqrt{1-h}}{L} \tag{1.7}$$

then, (NM) or (TSNM) converges to x^*. Error estimates on the distances can be found in [5, 6, 13], [19]-[24] and the references therein. However, there are simple examples (see numerical examples at the end of the study) where (1.4) or (1.5) are not satisfied. In these cases the Newton-Kantorovich theorem cannot guarantee that (NM) or (TSNM) converge although these methods may converge. This is happening because (1.5) is only a sufficient convergence condition for (NM) or (TSNM). In the present paper we expand the applicability of (TSNM) by weakening (1.4) or (1.5). Relevant work for (NM) or (TSNM) can be found in [1]- [24].

In particular we replace Lipschitz condition (1.4) by the center-Lipschitz condition

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \le L_0\|x-x_0\| \tag{1.8}$$

for each $x \in D$, (1.5) by

$$h_0 = (9+4\sqrt{5})L_0\eta \le 1 \tag{1.9}$$

and (1.6) by

$$\overline{U(x_0,r)} \subseteq D, \tag{1.10}$$

where

$$r = \frac{2\eta}{1+L_0\eta+\sqrt{(1+L_0\eta)^2-20L_0\eta}}. \tag{1.11}$$

Note that

$$L_0 \le L \tag{1.12}$$

holds in general and $\frac{L_0}{L}$ can be arbitrarily small (see Example 3.3).

We also have

$$h \le 1 \ \text{ and } \ (9+4\sqrt{5})L_0 \le 2L \Rightarrow h_0 \le 1 \tag{1.13}$$

and

$$\frac{h_0}{h} \to 0 \ \text{ as } \ \frac{L_0}{L} \to 0. \tag{1.14}$$

Hence, in these cases the applicability of (TSNM) is extended under weaker hypotheses since the computation of constant L_0 is less expensive than the computation of constant L. Note also that we may have

$$h_0 \le 1 \ \text{ and } \ (9+4\sqrt{5})L_0 \ge 2L \Rightarrow h \le 1 \tag{1.15}$$

Therefore, in practice we shall choose the condition that is satisfied (if any).

The paper is organized as follows. In Section 2 we present the semilocal convergence of (TSNM). The numerical examples are given in Section 3.

2. Semilocal Convergence of (TSNM)

We need the following Ostrowski-type representations for (TSNM).

Lemma 2.1. *Suppose that sequences $\{x_n\},\{y_n\}$ generated by (TSNM) are well defined for each $n=0,1,2,\cdots$. Then, the following assertions hold for each $n=1,2,\cdots$*

$$x_n-y_{n-1}=\Gamma_1+\Gamma_2, \tag{2.1}$$

where

$$\begin{aligned}\Gamma_1 &= F'(y_{n-1})^{-1}[F'(y_{n-1})(y_{n-1}-x_{n-1})-F(y_{n-1})+F(x_{n-1})]\\ &= F'(y_{n-1})^{-1}\int_0^1[F'(y_{n-1}+t(x_{n-1}-y_{n-1}))-F'(y_{n-1})]\\ &\quad\times(x_{n-1}-y_{n-1})dt \qquad (2.2)\\ \Gamma_2 &= F'(y_{n-1})^{-1}(F'(y_{n-1})-F'(x_{n-1}))(x_{n-1}-y_{n-1}) \qquad (2.3)\end{aligned}$$

and

$$y_n-x_n=\Gamma_3+\Gamma_4, \tag{2.4}$$

where

$$\begin{aligned}\Gamma_3 &= F'(x_{n-1})^{-1}[F'(x_n)(x_n-y_{n-1})-F(x_n)+F(y_{n-1})]\\ &= F'(x_{n-1})^{-1}\int_0^1[F'(x_n+t(y_{n-1}-x_n))-F'(x_n)]\\ &\quad\times(y_{n-1}-x_n)dt \qquad (2.5)\\ \Gamma_4 &= F'(x_n)^{-1}(F'(x_n)-F'(y_{n-1}))(y_{n-1}-x_n). \qquad (2.6)\end{aligned}$$

Proof. Use (TSNM) and Taylor's formula.
We can show the main semilocal convergence results for (TSNM).

Theorem 2.2. *Let $F:D\to Y$ be Fréchet differentiable. Suppose that (1.3), (1.8)- (1.11) hold. Then, sequence $\{x_n\},\{y_n\}$ generated by (TSNM) are well defined, remain in $\overline{U(x_0,r)}$ for each $n=0,1,2,\cdots$ and converge to a solution $x^*\in\overline{U(x_0,r)}$ of equation $F(x)=0$. Moreover, the following estimates hold for each $n=1,2,\cdots$*

$$\begin{aligned}\|x_n-y_{n-1}\| &\le \frac{L_0}{2(1-L_0\|y_{n-1}-x_0\|)}[5\|y_{n-1}-x_0\|\\ &\quad+3\|x_{n-1}-x_0\|]\|x_{n-1}-y_{n-1}\|, \qquad (2.7)\\ \|x_n-x_{n-1}\| &\le [1+\frac{L_0}{2(1-L_0\|y_{n-1}-x_0\|)}(5\|y_{n-1}-x_0\|\\ &\quad+3\|x_{n-1}-x_0\|)]\|x_{n-1}-y_{n-1}\|, \qquad (2.8)\\ \|x_n-y_n\| &\le \frac{L_0}{2(1-L_0\|x_{n-1}-x_0\|)}[5\|x_{n-1}-x_0\|\\ &\quad+3\|y_{n-1}-x_0\|]\|x_n-y_{n-1}\|, \qquad (2.9)\end{aligned}$$

and

$$\|x^*-x_n\| \le \frac{(1+b)b^{2n}}{1-b^2}\eta, \tag{2.10}$$

where

$$b=b(r)=\frac{4L_0r}{1-L_0r}<1. \tag{2.11}$$

Furthermore, if there exists $R\ge r$ such that

$$\overline{U(x_0,R)}\subseteq D \tag{2.12}$$

and

$$L_0(R+r)<2, \tag{2.13}$$

then the solution x^* is unique in $\overline{U(x_0,R)}$.

Proof. We have by (TSNM) for $n=0$, (1.3) and (1.10) that $\|y_0-x_0\|\leq\eta\leq r$, which shows that $y_0\in\overline{U(x_0,r)}$. Let $x\in\overline{U(x_0,r)}$. Then, using the center-Lipschitz condition we get that

$$\|F'(x_0)^{-1}(F'(x)-F'(x_0))\|\leq L_0\|x-x_0\|\leq L_0 r<1 \tag{2.14}$$

by the choice of r. It follows from (2.14) and Banach Lemma on invertible operators [5]-[26] that $F'(x)^{-1}\in L(Y,X)$ and

$$\|F'(x)^{-1}F'(x_0)\|\leq\frac{1}{1-L_0\|x-x_0\|}\leq\frac{1}{1-L_0 r}. \tag{2.15}$$

In particular (2.15) holds for $x=y_0$. Then, by (TSNM) x_1 is well defined. Let us assume that $x_k,y_k\in\overline{U(x_0,r)}$. Then, we have using Lemma 2.1, the center-Lipschitz condition and (2.15) that

$$\begin{aligned}
\|\Gamma_1\| &\leq \|F'(y_{k-1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}\int_0^1\{[F'(x_{k-1}+t(y_{k-1}-x_{k-1}))\\
&\quad -F'(x_0)]+[F'(x_0)-F'(y_{k+1})]\}(x_{k-1}-y_{k-1})dt\|\\
&\leq \frac{L_0}{1-L_0\|y_{k-1}-x_0\|}[\frac{\|y_{k-1}-x_0\|+\|x_{k-1}-x_0\|}{2}\\
&\quad +\|y_{k-1}-x_0\|]\|x_{k-1}-y_{k-1}\|\\
&\leq \frac{L_0}{2(1-L_0\|y_{k-1}-x_0\|)}(3\|y_{k-1}-x_0\|+\|x_{k-1}-x_0\|)\\
&\quad \|x_{k-1}-y_{k-1}\|,\\
\|\Gamma_2\| &\leq \|F'(y_{k-1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}(F'(y_{k-1})-F'(x_{k-1}))\|\\
&\quad \|x_{k-1}-y_{k-1}\|\\
&\leq \frac{L_0}{1-L_0\|y_{k-1}-x_0\|}[\|y_{k-1}-x_0\|+\|x_{k-1}-x_0\|]\\
&\quad \|x_{k-1}-y_{k-1}\|,
\end{aligned}$$

so, since $\|x_k-y_{k-1}\|\leq\|\Gamma_1\|+\|\Gamma_2\|$, is obtained by adding the preceding two inequalities. To show (2.8) we use (2.7) and the triangle inequality

$$\|x_k-x_{k-1}\|\leq\|x_k-y_{k-1}\|+\|y_{k-1}-x_{k-1}\|.$$

As in the computation of $\|\Gamma_1\|$ and $\|\Gamma_2\|$, we get in turn that

$$\begin{aligned}
\|\Gamma_3\| &\leq \frac{L_0}{2(1-L_0\|x_{k-1}-x_0\|)}(3\|x_{k-1}-x_0\|+\|y_{k-1}-x_0\|)\\
&\quad \|x_k-y_{k-1}\|,\\
\|\Gamma_4\| &\leq \frac{L_0}{1-L_0\|x_{k-1}-x_0\|}[\|x_k-x_0\|+\|y_{k-1}-x_0\|]\\
&\quad \|x_k-y_{k-1}\|
\end{aligned}$$

and since $\|y_k-x_k\|\leq\|\Gamma_3\|+\|\Gamma_4\|$, we obtain (2.9). It then follows from (2.9) and (2.11) that

$$\begin{aligned}
\|y_k-x_k\| &\leq \frac{L_0}{1-L_0 r}8r\|x_k-y_{k-1}\|\\
&\leq b^2\|x_{k-1}-y_{k-1}\|\leq b^{2k}\|x_0-y_0\|\\
&\leq b^{2k}\eta.
\end{aligned}$$

We also have that for $m=0,1,2,\cdots$

$$\|x_{k+m}-x_k\|\leq\|x_{k+m}-x_{m+k-1}\|+\|x_{k+m-1}-x_{m+k-2}\|+\cdots+\|x_{k+1}-x_k\|$$

but

$$\|x_{k+m}-x_{m+k-1}\|\leq(1+b)\|x_{k+m-1}-y_{m+k-1}\|\leq(1+b)b^{2(k+m-1)}\eta,$$

so,

$$\begin{aligned}
\|x_{k+m}-x_k\| &\leq (1+b)(b^{2(k+m-1)}+\cdots+b^{2k})\eta\\
&\leq (1+b)b^{2k}\frac{1-b^{2m}}{1-b^2}\eta\leq\frac{\eta}{1-b}=r.
\end{aligned}$$

It follows that sequence $\{x_k\}$ is complete in a Banach space X and as such it converges to some $x^*\in\overline{U(x_0,r)}$ (since $\overline{U(x_0,r)}$ is closed). By letting $m\to\infty$ in the preceding inequality we get (2.10). In particular the preceding inequality for $k=0$ gives that sequence $x_m\in\overline{U(x_0,r)}$ for each $m=0,1,2\cdots$. We also have that

$$\begin{aligned}
\|y_k-x_0\| &\leq \|y_k-x_k\|+\|x_k-x_0\|\\
&\leq b^{2k}\eta+(1+b)\frac{1-b^{2k}}{1-b^2}\eta\\
&\leq \frac{1-b^{2k+1}}{1-b}\eta<\frac{\eta}{1-b}=r.
\end{aligned}$$

That is $y_k \in \overline{U(x_0,r)}$ for each $k = 1,2,\cdots$.
In order for us to show that $F(x^*) = 0$, we use the approximation

$$\begin{aligned} F(x_{k+1}) &= F(x_{k+1}) - F(x_k) - F'(x_k)(y_k - x_k) \\ &= (F(x_{k+1}) - F(x_k) - F'(x_k)(x_{k+1} - x_k)) - F'(x_k)(y_k - x_{k+1}), \end{aligned}$$

so

$$\begin{aligned} F'(x_0)^{-1}F(x_{k+1}) &= F'(x_0)^{-1}\int_0^1 (F'(x_k + t(x_{k+1} - x_k)) - F'(x_k))(x_{k+1} - x_k)dt \\ &\quad + (I - F'(x_0)^{-1}(F'(x_k) - F'(x_0)))(y_k - x_{k+1}). \end{aligned}$$

Then, we get that

$$\begin{aligned} \|F'(x_0)^{-1}F(x_{k+1})\| &\leq \frac{L_0}{2}(\|x_{k+1} - x_0\| + 3\|x_k - x_0\|)\|x_{k+1} - x_k\| \\ &\quad + (1 + L_0\|x_k - x_0\|)\|x_{k+1} - y_k\| \\ &\leq 2L_0 r\|x_{k+1} - x_k\| + (1 + L_0 r)\|x_{k+1} - y_k\|. \end{aligned}$$

But $\{x_k\}$ is a complete sequence and $\|x_{k+1} - y_k\| \to 0$ as $k \to \infty$ (by (2.7)). Hence, $\|F'(x_0)^{-1}F(x_{k+1})\| \to 0$ as $k \to \infty$. That is $F(x^*) = 0$. Finally, to show the uniqueness part, let $y^* \in \overline{U(x_0,R)}$ be such that $F'(y^*) = 0$. Let $M = \int_0^1 F'(y^* + t(x^* - y^*))dt$. Then, we have that

$$\begin{aligned} \|F'(x_0)^{-1}(M - F'(x_0))\| &\leq L_0 \int_0^1 \|y^* + t(x^* - y^*) - x_0\|dt \\ &\leq \frac{L_0}{2}(\|x^* - x_0\| + \|y^* - x_0\|) \\ &\leq \frac{L_0}{2}(r + R) < 1. \end{aligned}$$

It follows that $M^{-1} \in L(Y,X)$. Then, in view of the identity

$$0 = F(x^*) - F(y^*) = M(x^* - y^*)$$

we deduce that $x^* = y^*$. The proof of the Theorem is complete.

Remark 2.3. *The Newton-Kantorovich hypothesis (1.5) has been used in the literature [1]-[46], as the sufficient convergence condition for both (NM) and the modified Newton's method (MNM)*

$$z_{n+1} = z_n - F'(z_0)^{-1}F(z_n) \quad \text{for each} \quad n = 0,1,2,\cdots.$$

In [6] we showed that (1.5) can be replaced by

$$h_1 = 2L_0\eta \leq 1 \tag{2.16}$$

and

$$\overline{U(x_0,r_1)} \subseteq D, \tag{2.17}$$

where

$$r_1 = \frac{1 - \sqrt{1 - h_1}}{L_0}. \tag{2.18}$$

Note that

$$h_0 \leq 1 \Rightarrow h_1, \quad h \leq 1 \Rightarrow h_1 \leq 1 \tag{2.19}$$

and

$$\frac{h_1}{h} \to 0 \ \text{ as } \ \frac{L_0}{L} \to 0. \tag{2.20}$$

Hence, in cases (1.5) or (1.9) are not satisfied but (2.16) is satisfied we can start with linearly convergent (MNM) until a certain iterate z_n (or (1.5)) is satisfied, then we continue with faster (TSNM) [6].

Remark 2.4. *The convergence order of (TSNM) is expected to be four. In Theorem 2.2 the error bounds are too pessimistic. That is why in practice we shall use the computational order of convergence (COC) (see eg. [14]) defined by*

$$\rho \approx \ln\left(\frac{\|x_{n+1} - x_\alpha^\delta\|}{\|x_n - x_\alpha^\delta\|}\right) / \ln\left(\frac{\|x_n - x_\alpha^\delta\|}{\|x_{n-1} - x_\alpha^\delta\|}\right).$$

The (COC) ρ will then be close to 4 which is the order of convergence of (TSNM).

3. Numerical Examples

Example 3.1. *Let $X = Y = \mathbb{R}$, $D = [0,\infty), x_0 = 1$ and define function F on D by*

$$F(x) = \frac{x^{1+\frac{1}{i}}}{1+\frac{1}{i}} + c_1 x + c_2, \tag{3.1}$$

where c_1, c_2 are real parameters and $i > 2$ an integer. Then $F'(x) = x^{1/i} + c_1$ is not Lipschitz on D. However central Lipschitz condition $(C1)'$ holds for $L_0 = 1$.
Indeed, we have

$$\begin{aligned} \|F'(x) - F'(x_0)\| &= |x^{1/i} - x_0^{1/i}| \\ &= \frac{|x - x_0|}{x_0^{\frac{i-1}{i}} + \cdots + x^{\frac{i-1}{i}}} \\ &\leq L_0|x - x_0|. \end{aligned}$$

Example 3.2. *We consider the integral equations*

$$u(s) = f(s) + \tau \int_a^b G(s,t) u(t)^{1+1/n} dt, \quad n \in \mathbb{N}. \tag{3.2}$$

Here, f is a given continuous function satisfying $f(s) > 0, s \in [a,b], \tau$ is a real number, and the kernel G is continuous and positive in $[a,b] \times [a,b]$.
For example, when $G(s,t)$ is the Green kernel, the corresponding integral equation is equivalent to the boundary value problem

$$u'' = \tau u^{1+1/n} \tag{3.3}$$

$$u(a) = f(a), u(b) = f(b). \tag{3.4}$$

These type of problems have been considered in [5], [9]-[13], [18]-[22].
Equation of the form (3.2) generalize equations of the form

$$u(s) = \int_a^b G(s,t) u(t)^n dt \tag{3.5}$$

studied in [5], [13], [20]. Instead of (3.2) we can try to solve the equation $F(u) = 0$ where

$$F : \Omega \subseteq C[a,b] \to C[a,b], \Omega = \{u \in C[a,b] : u(s) \geq 0, s \in [a,b]\},$$

and

$$F(u)(s) = u(s) - f(s) - \tau \int_a^b G(s,t) u(t)^{1+1/n} dt.$$

The norm we consider is the max-norm.
The derivative F' is given by

$$F'(u)v(s) = v(s) - \tau(1 + \frac{1}{n}) \int_a^b G(s,t) u(t)^{1/n} v(t) dt, \quad v \in \Omega.$$

First of all, we notice that F' does not satisfy a Lipschitz-type condition in Ω. Let us consider, for instance, $[a,b] = [0,1], G(s,t) = 1$ and $y(t) = 0$. Then $F'(y)v(s) = v(s)$ and

$$\|F'(x) - F'(y)\| = |\tau|(1 + \frac{1}{n}) \int_a^b x(t)^{1/n} dt.$$

If F' were a Lipschitz function, then

$$\|F'(x) - F'(y)\| \leq L_1 \|x - y\|,$$

or, equivalently, the inequality

$$\int_0^1 x(t)^{1/n} dt \leq L_2 \max_{x \in [0,1]} x(s), \tag{3.6}$$

would hold for all $x \in \Omega$ and for a constant L_2. But this is not true. Consider, for example, the functions

$$x_j(t) = \frac{t}{j}, \quad j \geq 1, \quad t \in [0,1].$$

If these are substituted into (3.6)

$$\frac{1}{j^{1/n}(1 + 1/n)} \leq \frac{L_2}{j} \Leftrightarrow j^{1-1/n} \leq L_2(1 + 1/n), \quad \forall j \geq 1.$$

This inequality is not true when $j \to \infty$.

Therefore, condition (3.6) is not satisfied in this case. However, condition (1.8) holds. To show this, let $x_0(t) = f(t)$ *and* $\gamma = \min_{s\in[a,b]} f(s), \alpha > 0$. *Then for* $v \in \Omega$,

$$\begin{aligned} \|[F'(x) - F'(x_0)]v\| &= |\tau|(1+\frac{1}{n}) \max_{s\in[a,b]} |\int_a^b G(s,t)(x(t)^{1/n} - f(t)^{1/n})v(t)dt| \\ &\leq |\tau|(1+\frac{1}{n}) \max_{s\in[a,b]} G_n(s,t) \end{aligned}$$

where $G_n(s,t) = \frac{G(s,t)|x(t)-f(t)|}{x(t)^{(n-1)/n}+x(t)^{(n-2)/n}f(t)^{1/n}+\cdots+f(t)^{(n-1)/n}}\|v\|$. *Hence,*

$$\begin{aligned} \|[F'(x) - F'(x_0)]v\| &= \frac{|\tau|(1+1/n)}{\gamma^{(n-1)/n}} \max_{s\in[a,b]} \int_a^b G(s,t)dt\|x-x_0\| \\ &\leq L_0\|x-x_0\|, \end{aligned}$$

where $L_0 = \frac{|\tau|(1+1/n)}{\gamma^{(n-1)/n}}N$ *and* $N = \max_{s\in[a,b]} \int_a^b G(s,t)dt$.

Example 3.3. *Let* $X = D(F) = \mathbb{R}, x_0 = 0$, *and define function* F *on* $D(F)$ *by*

$$F(x) = d_0x + d_1 + d_2 \sin e^{d_3x}, \tag{3.7}$$

where $d_i, i = 0,1,2,3$ *are given parameters. Then, it can easily be seen that for* d_3 *sufficiently large and* d_2 *sufficiently small,* $\frac{L_0}{L}$ *can be arbitrarily small.*

Example 3.4. *Let* $X = Y = \mathbb{R}, x_0 = 1, D = \overline{U(x_0, 1-p)}$ *for* $p \in (0, \frac{1}{2})$ *and define* F *on* D *by*

$$F(x) = x^3 - p. \tag{3.8}$$

Then, using (1.3), (1.4), (1.8) and (3.8) we obtain that

$$\eta = \frac{1-p}{3}, L_0 = 3-p < L = 2(2-p).$$

Hence, there is no guarantee that (TSNM) converges to x^* *in these cases. Then, by (1.5) and (1.9) we get that* $h > 1$ *and* $h_0 > 1$ *for each* $p \in (0, \frac{1}{2})$. *However, using (2.16) we get that*

$$h_1 \leq 1 \quad \text{for each} \quad p \in [0.418861170, 0.5).$$

Note that we have that [6]

$$\|z_{n+1} - z_n\| \leq q\|z_n - z_{n-1}\| \quad \text{for each} \quad n = 1,2,\cdots,$$

where $q = 1 - \sqrt{1-h_1}$.

Let us choose $p = 0.48$. *Then, we have that* $L_0 = 2.52, L = 3.04, \eta = 0.17333\cdots, h_1 = 0.8735999\cdots$ *and* $q = 0.644472221$. *Using the estimates*

$$\begin{aligned} \|F'(z_N)^{-1}(F'(x) - F'(y))\| &\leq \|F'(z_N)^{-1}F'(z_0)\| \\ &\quad \times\|F'(z_0)^{-1}(F'(x) - F'(y))\| \\ &\leq \frac{L}{1-L_0\|z_N - z_0\|}\|x-y\| \\ &\leq \frac{L}{1-L_0r_1}\|x-y\| \\ &= \frac{L}{\sqrt{1-h_1}}\|x-y\| \end{aligned}$$

and

$$\begin{aligned} \|F'(z_N)^{-1}(F'(x) - F'(z_0))\| &\leq \|F'(z_N)^{-1}F'(z_0)\| \\ &\quad \times\|F'(z_0)^{-1}(F'(x) - F'(z_0))\| \\ &\leq \frac{L}{1-L_0\|z_N - z_0\|}\|x-z_0\| \\ &\leq \frac{L}{1-L_0r_1}\|x-z_0\| \\ &= \frac{L}{\sqrt{1-h_1}}\|x-z_0\| \end{aligned}$$

Therefore, we can set

$$\bar{L} = \frac{L}{\sqrt{1-h_1}} \quad \text{and} \quad \bar{L_0} = \frac{L_0}{\sqrt{1-h_1}}.$$

we then have $\bar{L} = 8.550668013$ *and* $\bar{L_0} = 7.088053748$. *Hence, estimates (1.5) and (1.9) hold, respectively, if*

$$\bar{h} = 2\bar{L}\eta q^N \leq 1$$

and

$$\bar{h_0} = (9+4\sqrt{5})\bar{L_0}\eta q^N \leq 1.$$

These inequalities are satisfied, respectively for $N = 3$ *and* $N = 8$, *since they become*

$$\bar{h} = 0.793459449 < 1$$

and

$$\bar{h_0} = 0.656097755 < 1.$$

Hence, we must choose $x_0 = z_3$ *(under (1.5)) or* $x_0 = z_8$ *(under (1.9)).*

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The Number of Snakes in a Box

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Abstract

Within the class of polyominoes we work on the enumeration of two subfamilies of the family of snake polyominoes: stairs and snakes of height 2. We consider them from a graph theoretical perspective. In the process of enumeration of these graphs, we use classical ideas, as symmetries, and a new approach that connects these snakes with the partitions of integers.

1. Introduction

A *polyomino* is a planar shape made by connecting a certain number of equal-sized squares, each joined together with at least one other square along an edge. A *snake* of length $n > 1$, is a packing of n congruent geometrical objects, called *cells*, where the first and last cell have only one neighbor and all the other cells have exactly two neighbors. A *snake polyomino* is a snake where all the cells are squares. In Figure 1.1 we show all the polyominoes with six cells, the snakes have been highlighted.

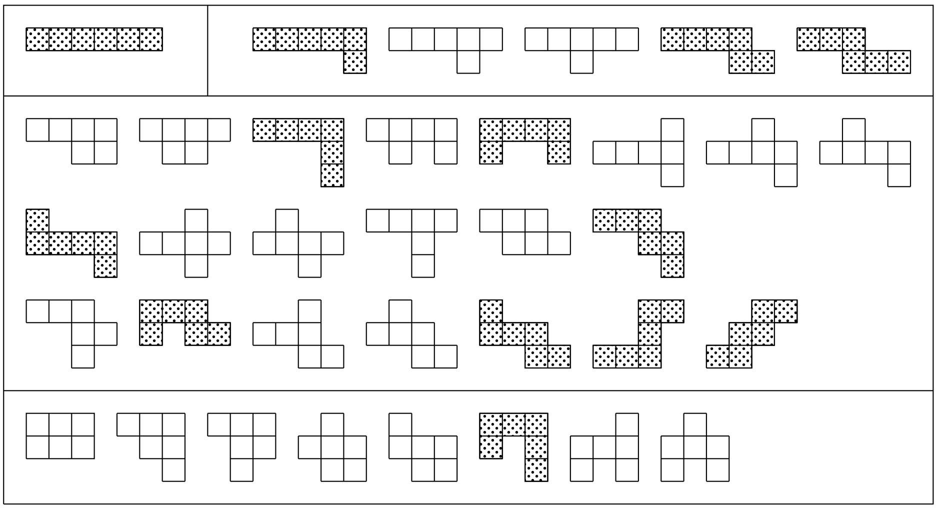

Figure 1.1: All polyominoes with six cells.

In this work, polyominoes are considered graphs, where every cell is a copy of the cycle C_4. Moreover, these graphs are embedded in the integral grid. This last restriction has implications on their number. Thus, snake polyominoes, or simply snakes, form a polyomino class, which can be described by the avoidance of the polyominoes shown in Figure 1.2. This definition is slightly different of the one given in [1]. There, Battaglino et al., only consider, as forbidden substructures, the first two shapes. We included here the third one to be consistent with the definition of snake, where the extreme cells only have one neighbor.

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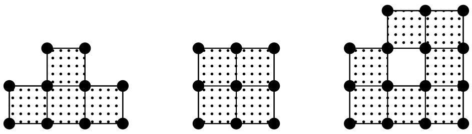

Figure 1.2: Forbidden structures in snake polyominoes.

Golomb [2] introduced the concept of polyomino in 1953, since then, there has been a number of papers centered in the enumeration of subfamilies of them. Several algorithms, that count the number of members of these subfamilies, have been created; however, the general case remains unsolved, that is, for a given value of n, it is unknown the number of polyominoes with n cells.
In Section 2 we present some general results that are used in our counting process in the coming sections. We study two subfamilies of snakes: stairs (Section 3) and snakes of height 2 (Section 4). In both cases, we consider the snakes inscribed into a box, i.e., the boundary of $P_a \times P_b$; partitions of integers are used in the enumeration process of both subfamilies.

2. General results

2.1. Quadrilateral snakes and snake polyominoes

The problem of counting snakes has been considered by several authors. Recently, Goupil et al., [3] studied the problem not only in the plane, they also considered higher dimensions. In that work as well as in [4], the authors accept the third structure in Figure 1.2 as a valid substructure of a snake. Pegg Jr. [4], called these combinatorial structures, 2-sided strip polyominoes with n cells. In Table 1 we show the first values of $p(n)$, i.e., the number of 2-sided strip polyominoes, and $\overline{p}(n)$, the number of snakes that follow our definition. As we may expect, the difference between $\overline{p}(n)$ and $p(n)$ increses with n.

n	1	2	3	4	5	6	7	8	9
$p(n)$	1	1	2	3	7	13	31	65	154
$\overline{p}(n)$	1	1	2	3	7	13	30	64	150

Table 1: Number of quadrilateral snakes and snake polyominoes.

In [5], the first author defined a *kC_n-snake* as a connected graph in which the k cells are isomorphic to the cycle C_n and the block-cutpoint graph is a path. By a *quadrilateral snake* we understand a kC_4-snake. In [6], we established a relationship between quadrilateral snakes and snake polyominoes, showing that for every snake polyomino there exists a quadrilateral snake of the same length. We also show that the converse of this statement is not valid. The reason is that when the number of cells is at least 7, there exist quadrilateral snakes which associated graph is not a snake polyomino because they have a subgraph isomorphic to the third structure in Figure 1.2. In Figure 2.1 we show three, of the 31 quadrilateral snakes of length 7, together with their associated polyomino. We can see that in the third example, the polyomino is not a snake according to our defintion, but it is according to the one used in [3] and [4]. Hence, $p(n)$ actually counts the number of quadrilateral snakes of length n. Therefore, determining a formula for $\overline{p}(n)$, as well as for $p(n)$, is still an open problem.

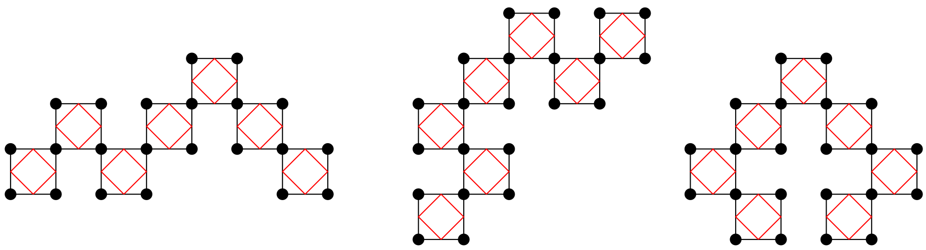

Figure 2.1: Quadrilateral snakes and associated polyominoes.

2.2. Partitioning n into k parts

It is well-known that the number $P(n,k)$ of partitions of n into k parts, where the order is taken under consideration, is given by

$$P(n,k) = C(n-1,k-1),$$

where $C(n-1,k-1)$ is the standard binomial coefficient $\binom{n-1}{k-1}$.
In order to prove this fact, the number n is represented on a line formed by n balls. There are $n-1$ spaces in between the balls where a bar (or separator) can be placed. So, to separate the balls into k groups we need to introduce $k-1$ bars. The number of ways to do this is $C(n-1,k-1)$.
In Figure 2.2 we show an example of this result, exhibiting all the 3-part partitions of 5.

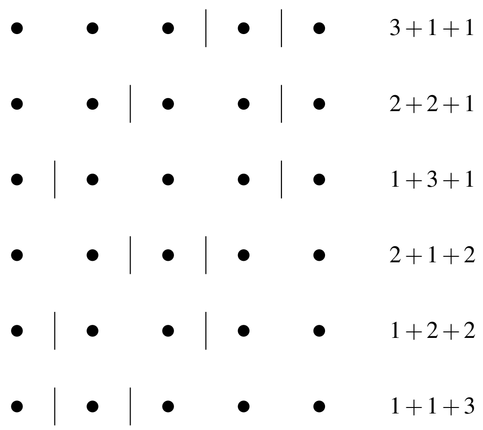

Figure 2.2: 3-part partitions of 5.

In the next sections we use this type of partitions to count the number of snakes considered in each case.

2.3. Snakes in a box

As we mentioned before, our snakes are subgraphs of $P_a \times P_b$. Suppose that a snake S of length n is a subgraph of $P_a \times P_b$ and is not a subgraph of $P_{a-1} \times P_b$ nor $P_a \times P_{b-1}$, then we say that S is inscribed in a box of *base* $b-1$ and *height* $a-1$, or that S has base $b-1$ and height $a-1$. For example, the snakes in Figure 2.1 are inside the boxes $P_4 \times P_8$, $P_6 \times P_6$, and $P_5 \times P_6$, respectively. We use the symmetries of these boxes to count the number of non-isomorphic snakes.

3. The number of stairs

By a *block of cells* of length t we understand the ladder $L_t = P_2 \times P_{t+1}$. Let $L_{p_1}, L_{p_2}, \ldots, L_{p_k}$ be a sequence of these blocks, where $p_1 + p_2 + \cdots + p_k = n$ and each $p_i \neq 0$. The *stair snake polyomino*, or simply *stair*, formed by these blocks, is the graph obtained by placing the first cell of L_{p_i} on top of the last cell of $L_{p_{i-1}}$, for each $2 \leq i \leq k$. In Figure 3.1, we show the stair with base 11 and height 5 with blocks of length 1,4,3,5,2.

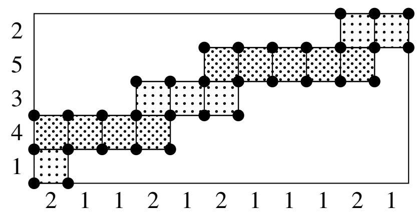

Figure 3.1: A stair of length 15.

First, we must observe that $p_1, p_2, \ldots, p_k$ is a partition of n into k parts. In addition, the construction given above establishes a bijection between the set of partitions of n into k parts, where order matters, and the set of stairs built in this way. So, in order to determine the number of stairs of length n with k steps (i.e., with k blocks of cells), we may count the distinct partitions of n into k parts.

Let S be a stair of length n with k steps built using the partition $p_1, p_2, \ldots, p_k$. Associated with this partition, there are three other partitions that form the same graph. In the case of the example given in Figure 3.1, the numbers on the left of the picture can be read from top to bottom forming a "different" partition of n. The other two partitions are obtained by reading the numbers, on the bottom, from left to right and vice versa. In general, for any given stair S, the other three partitions can be obtained using symmetries; the first one is a $180°$ rotation of S, while the other two are reflections, of S, around the two diagonals of a square centered at the center of S. In other terms, if the stair is not symmetric, there are two partitions of n into k parts and two partitions of n into $n+1-k$ parts, associated with the same stair. Therefore, we need to analyze the case where the stair is symmetric.

Consider any symmetric stair with n cells. If its first and last cell are deleted, the remaining graph is also a symmetric stair. Thus, all the symmetric stairs with $n+2$ cells can be constructed using the symmetric stairs with n cells, by attaching a new cell, to both, the first and the last cell.

It is easy to see that for $n = 1, 2$ there is only one stair with n cells. In general, every stair with n cells can be inscribed inside a rectangle, that can be a square, in such a way that the extreme cells are located in opposite corners of the rectangle.

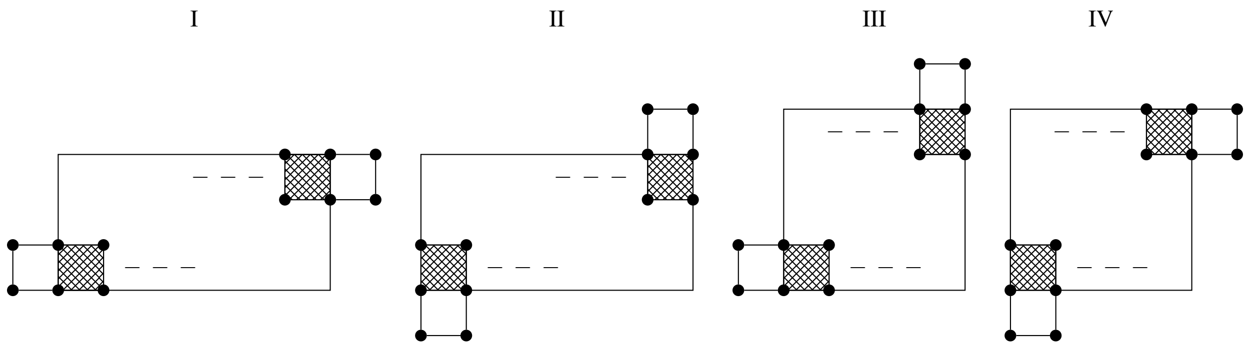

Figure 3.2: Extension schemes for symmetric stairs.

In Figure 3.2 we show the four ways, that exist, to extend a symmetric stair with n cells into a symmetric stair with $n+2$ cells. Schemes I and II show the cases where the original stair is inscribed in a rectangle (that can be a square), and the symmetry is a 180° rotation around the center of the rectangle. When the stair is inscribed in a square, schemes III and IV, the symmetry is a 180° rotation around the axis formed by the main diagonal of the square.
Independently of the case, the extreme cells have one horizontal and one vertical edge where a new cell can be attached. Thus, scheme I is the connection VV, scheme II is HH, scheme III is VH, and scheme IV is HV. Consequently, if $p_1, p_2, \ldots, p_k$ is the partition of n into k parts associated with a symmetric stair S, with n cells and k steps (or blocks of cells), then the partition of the new stair, for each case is shown in Table 2.

Connection	Partition	Number of Steps
I: VV	$1+p_1, p_2, \ldots, 1+p_k$	k
II: HH	$1, p_1, p_2, \ldots, p_k, 1$	$k+2$
III: VH	$1+p_1, p_2, \ldots, p_k, 1$	$k+1$
IV: HV	$1, p_1, p_2, \ldots, 1+p_k$	$k+1$

Table 2: Types of connections and associated partitions.

One of the consequences of this property is that if $s(n)$ is the number of symmetric stairs with n cells, then $2s(n)$ is the number of symmetric stairs with $n+2$ cells. Since, $s(1) = s(2) = 1$, we may conclude that

$$s(n) = \begin{cases} 2^{\frac{n-2}{2}}, & \text{if } n \text{ is even,} \\ 2^{\frac{n-1}{2}}, & \text{if } n \text{ is odd.} \end{cases}$$

The sequence formed by the values of $s(n)$ corresponds to the sequence A016116 in OEIS.
Summarizing, for every stair with n cells and k steps, there is a partition of n into k parts and vice versa. A non-symmetric stair is represented by four different partitions; every symmetric stair is represented by two different partitions.
Since there are 2^{n-1} partitions of n into k parts, the number $e(n)$ of non-isomorphic stairs with $n \geq 3$ cells is:
When n is even:

$$\begin{aligned} e(n) &= \frac{1}{4}\left(2^{n-1} - 2\cdot 2^{\frac{n-2}{2}}\right) + \frac{1}{2}\cdot 2\cdot 2^{\frac{n-2}{2}} \\ &= 2^{n-3} - \frac{1}{2}\cdot 2^{\frac{n-2}{2}} + 2^{\frac{n-2}{2}} \\ &= 2^{n-3} + \frac{1}{2}\cdot 2^{\frac{n-2}{2}} \\ &= 2^{n-3} + 2^{\frac{n-4}{2}}. \end{aligned}$$

When n is odd:

$$\begin{aligned} e(n) &= \frac{1}{4}\left(2^{n-1} - 2\cdot 2^{\frac{n-1}{2}}\right) + \frac{1}{2}\cdot 2\cdot 2^{\frac{n-1}{2}} \\ &= 2^{n-3} - \frac{1}{2}\cdot 2^{\frac{n-1}{2}} + 2^{\frac{n-1}{2}} \\ &= 2^{n-3} + \frac{1}{2}\cdot 2^{\frac{n-1}{2}} \\ &= 2^{n-3} + 2^{\frac{n-3}{2}}. \end{aligned}$$

The first values of $e(n)$ are shown in Table 3. For $n \geq 2$, the consecutive values of $e(n)$ form the sequence A005418 in OEIS [7].
We can go even further, using the diagrams in Figure 3.2, we can calculate the number $\sigma(n,k)$ of symmetric stair with n cells and k steps.

n	$e(n)$	n	$e(n)$	n	$e(n)$
1	1	11	272	21	262656
2	1	12	528	22	524800
3	2	13	1056	23	1049600
4	3	14	2080	24	2098176
5	6	15	4160	25	4196352
6	10	16	8256	26	8390656
7	20	17	16512	27	16781312
8	36	18	32896	28	33558528
9	72	19	65792	29	67117056
10	136	20	131328	30	134225920

Table 3: Number of non-isomorphic stairs with n cells.

Recall that $\sigma(1,1)=\sigma(2,1)=\sigma(2,2)=1$. We use the conventions that $\sigma(n,k)=0$ when $k<1$ or $k>n$, and $C(n,k)=0$ if k is not an integer.
Thus, from I and II in Figure 3.2, we know that for all values of $n\geq 3$ and $k\neq \frac{n+1}{2}$,

$$\sigma(n,k)=\sigma(n-2,k)+\sigma(n-2,k-2).$$

When $k=\frac{n+1}{2}$ we get

$$\sigma(n,k)=\sigma(n-2,k)+\sigma(n-2,k-2)+2^{\frac{n-1}{2}}.$$

The number $2^{\frac{n-1}{2}}$ comes from III and IV in Figure 3.2.

Proposition 3.1. *Let n be a positive even number and $k\in\{1,2,\ldots,n\}$. Given that $\sigma(2,1)=\sigma(2,2)=1$, the number $\sigma(n,k)$ of symmetric stairs with n cells and k steps is*

$$\sigma(n,k)=C\left(\frac{n-2}{2},\left\lfloor\frac{k-1}{2}\right\rfloor\right).$$

Proof. By induction on n. Recall that $\sigma(2,1)=\sigma(2,2)=1$; for $n=4$:

$$\begin{aligned}\sigma(4,1)=\sigma(2,1)=\sigma(2,-1)=1+0=1\\ \sigma(4,2)=\sigma(2,2)=\sigma(2,0)=1+0=1\\ \sigma(4,3)=\sigma(2,3)=\sigma(2,1)=0+1=1\\ \sigma(4,4)=\sigma(2,4)=\sigma(2,2)=0+1=1\end{aligned}$$

On the other side, $C\left(\frac{4-2}{2},\left\lfloor\frac{k-1}{2}\right\rfloor\right)=C\left(1,\left\lfloor\frac{k-1}{2}\right\rfloor\right)$. Thus,

$$\begin{aligned}C\left(1,\left\lfloor\frac{1-1}{2}\right\rfloor\right)=C(1,0)=1\\ C\left(1,\left\lfloor\frac{2-1}{2}\right\rfloor\right)=C(1,0)=1\\ C\left(1,\left\lfloor\frac{3-1}{2}\right\rfloor\right)=C(1,1)=1\\ C\left(1,\left\lfloor\frac{4-1}{2}\right\rfloor\right)=C(1,1)=1.\end{aligned}$$

Then, the proposition is correct for $n=2$ and $n=4$.
Suppose that the proposition is correct up to a certain value of n. We want to prove that is also correct for $n+2$; in other terms,

$$\sigma(n+2,k)=C\left(\frac{n}{2},\left\lfloor\frac{k-1}{2}\right\rfloor\right).$$

We know that

$$\begin{aligned}\sigma(n+2,k)&=\sigma(n,k)+\sigma(n,k-2)\\ &=C\left(\frac{n-2}{2},\left\lfloor\frac{k-1}{2}\right\rfloor\right)+C\left(\frac{n-2}{2},\left\lfloor\frac{k-3}{2}\right\rfloor\right)\\ &=C\left(\frac{n}{2},\left\lfloor\frac{k-1}{2}\right\rfloor\right).\end{aligned}$$

Therefore, the proposition is true for every even value of n. ∎

Proposition 3.2. *Let n be a positive odd number and $k \in \{1,2,\dots,n\}$. Given that $\sigma(1,1)=1$, the number $\sigma(n,k)$ of symmetric stairs with n cells and k steps is*

$$\sigma(n,k) = C\left(\frac{n-1}{2}, \frac{k-1}{2}\right) + \varepsilon(k),$$

where

$$\varepsilon(k) = \begin{cases} 2^{\frac{n-1}{2}}, & \textit{when } k = \frac{n+1}{2}, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Suppose that $n \geq 3$ is odd. Note that when k is even, $\frac{k-1}{2}$ is not an integer, then $C\left(\frac{n-1}{2}, \frac{k-1}{2}\right) = 0$. Thus, from this point we are assuming that k is odd.

The term $\varepsilon(k)$ is the number of symmetric stairs with n cells and $k = \frac{n+1}{2}$ steps that are originated by the corresponding stairs with $n-2$ cells and $\frac{n-1}{2}$ steps. Based on the diagrams III and IV in Figure 3.2 and the fact that $\sigma(1,1)=1$, we know that these stairs increase by a factor of 2 in the next generation, so $\varepsilon\left(\frac{n+1}{2}\right) = 2^{\frac{n-1}{2}}$. In addition, we must observe that this $\varepsilon(k)$ is positive only when $k = \frac{n+1}{2}$, otherwise is 0.

For any other value of k, any symmetric stair with $n-2$ cells can be inscribed into a rectangle, that is not a square, implying that this stair produces two stairs with n cells, one with k steps (diagram I) and the other one with $k+2$ steps (diagram II). Since $\sigma(1,1)=1$, we can see that the sequence of values of $\sigma(n,k)$ is exactly the sequence of binomial coefficients, adjusted conveniently. Therefore, $\sigma(n,k) = C\left(\frac{n-1}{2}, \frac{k-1}{2}\right)$ for all odd values of n and k, except when $k = \frac{n+1}{2}$ where we need to add the power $2^{\frac{n-1}{2}}$. $\square$

In Table 4 we show the first values of $\sigma(n,k)$. The triangular arrangement produced by the vales of $\sigma(n,k)$ is quite similar to the one found in the sequence A051159 in OEIS. Both triangles, only differ when n is odd and $k = \frac{n+1}{2}$, that is when we added $\varepsilon(k)$. Thus, $T(n,k) = \sigma(n,k)$ for all n and k except when n is odd and $k = \frac{n+1}{2}$, where $T(n,k)$ are the entries of the triangle in A051159.

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1																			
2	1	1																		
3	1	$0+2$	1																	
4	1	1	1	1																
5	1	0	$3+4$	0	1															
6	1	1	2	2	1	1														
7	1	0	3	$0+8$	3	0	1													
8	1	1	3	3	3	3	1	1												
9	1	0	4	0	$6+16$	0	4	0	1											
10	1	1	4	4	6	6	4	4	1	1										
11	1	0	5	0	10	$0+32$	10	0	5	0	1									
12	1	1	5	5	10	10	10	10	5	5	1	1								
13	1	0	6	0	15	0	$20+64$	0	15	0	6	0	1							
14	1	1	6	6	15	15	20	20	15	15	6	6	1	1						
15	1	0	7	0	21	0	35	$0+128$	35	0	21	0	7	0	1					
16	1	1	7	7	21	21	35	35	35	35	21	21	7	7	1	1				
17	1	0	8	0	28	0	56	0	$70+256$	0	56	0	28	0	8	0	1			
18	1	1	8	8	28	28	56	56	70	70	56	56	28	28	8	8	1	1		
19	1	0	9	0	36	0	84	0	126	$0+512$	126	0	84	0	36	0	9	0	1	
20	1	1	9	9	36	36	84	84	126	126	126	126	84	84	36	36	9	9	1	1

Table 4: $\sigma(n,k)$ number of symmetric stairs with n cells and k steps.

There is another alternative to present the problem of counting stairs. We show it for the case where $k = \frac{n+1}{2}$.

Consider the integral grid $\mathbb{N} \times \mathbb{N}$. Determine the number of non-equivalent paths between the points $(0,0)$ and (n,n). Two paths are equivalent if one can be obtained from the other by any of the symmetries of the square where it is inscribed. In Figure 3.2 we show the first instances of these paths, that is, for every $n \in \{1,2,3,4\}$.

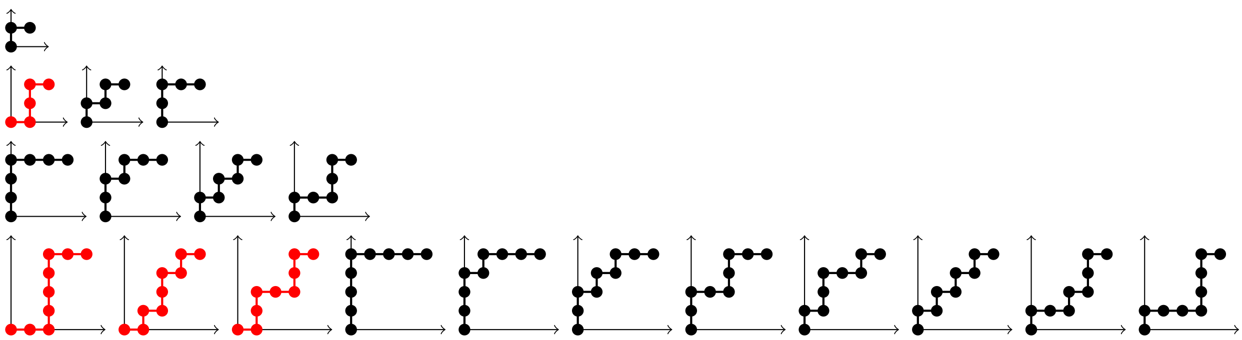

Figure 3.3: Non-equivalent paths from $(0,0)$ to (n,n)

Before closing this section, we want to note a connection between stairs and caterpillars. Suppose that the rows, of the box containing the stair, are labeled $0,1,\dots,\lambda$ from the top to the bottom, and the columns are labeled, from left to right, $\lambda+1, \lambda+2, \dots, n$. Thus, this representation of the stair corresponds to the α-labeling of a caterpillar. This labeling scheme was given by Rosa [8]; he proved that all caterpillars admit an α-labeling. Barrientos and Minion [9] used an extension of the adjacency matrix of α-labeled graphs and realized that all the adjacencies lie in a rectange. In the case of Rosa's labeling of caterpillars, the distribution of the adjacencies follows the stair pattern of these polyominoes. This fact is ratified in [7], where one of the interpretations of the sequence A005418 is that it represents the number of caterpillars of order n. For more information about labelings and, in particular, α-labelings, the interested reader is referred to [10].

4. Snakes in $P_3 \times P_{t+1}$

In this section we determine the number of snakes that can be inscribed in a box of base t and height 2. Consider the six snakes shown in Figure 4.1. Each of them is formed by blocks of cells of the form $L_p = P_2 \times P_{p+1}$, where $p \geq 1$. For example, the snake in part E is formed by the sequence of blocks L_4, L_3, L_5, L_2. In general, when $L_{p_1}, L_{p_2}, \dots, L_{p_k}$ is the sequence of blocks of cells associated to a snake polyomino of height 2, the last cell (from left to right) of L_{p_i} is adjacent to the first cell of $L_{p_{i+1}}$. We use the convention that the odd numbered blocks are placed on the top row, as shown in Figure 4.1; in this figure we show all the different posibilites for the end blocks L_{p_1} and L_{p_k}. Note that for every $2 \leq i \leq k-1$, each L_{p_i} must have at least three cells, otherwise, the associated polyomino would not be a snake because it would have a subgraph isomorphic to the second graph in Figure 1.2.

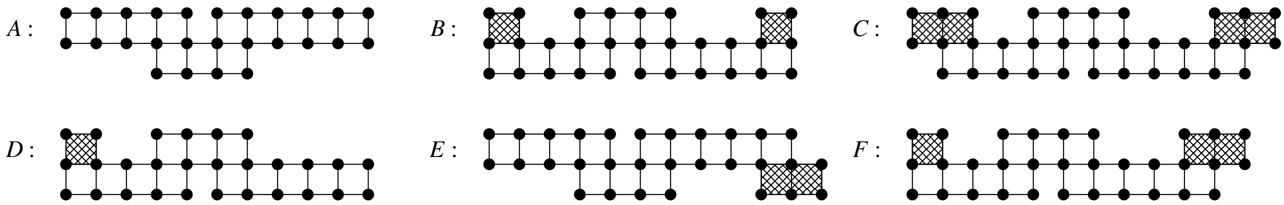

Figure 4.1: All general configurations for snakes of height 2.

Hence, if a snake of height 2 with n cells is represented by the sequence of blocks $L_{p_1}, L_{p_2}, \dots, L_{p_k}$, the associated sequence $p_1, p_2, \dots, p_k$ is a partition of n into k parts where $p_2, p_3, \dots, p_{k-1}$ are at least 3. Thus, instead of counting snakes we may count partitions that satisfy these conditions.

In [11], Deutsch showed that in the OEIS sequence A102547, the term $T(n,k)$ is the number of compositions of $n+3$ with $k+1$ parts, all at least 3. He calculated this number to be

$$T(n,k) = C(n-2k, k)$$

where $n \geq 0$ and $0 \leq k \leq \frac{n}{3}$. Adjusting this expression to our terminology we can say that for every $n \geq 3$ and $1 \leq k \leq \lfloor \frac{n}{3} \rfloor$, the number of partitions of n into k parts, where every part is at least 3 is given by

$$\pi_3(n,k) = C(n-2k-1, k-1).$$

Therefore the number of partitions of n where every part is at least three is:

$$\sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} \pi_3(n,k) = \sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} C(n-2k-1, k-1).$$

Table 5 shows the values of $\pi_3(n,k)$ from $n=3$ up to $n=29$.

Before completing the counting process, we need to calculate the number $s_3(n,k)$ of symmetric partitions of n into k parts where every part is at least three.

Let $p_1, p_2, \dots, p_k$ be a partition of n into k parts. We say that this partition is *symmetric* (or *reversible*) if for every $1 \leq i \leq k$, $p_i = p_{k_1-i}$.

Proposition 4.1. *If n is odd and k is even, then $s_3(n,k) = 0$.*

Proof. By contradiction. Suppose that $s_3(n,k) \neq 0$, that is, there exists a symmetric partition of n into k parts, where each part is at least 3. Since the partition is symmetric and k is even

$$\sum_{i=1}^{\frac{k}{2}} p_i = \sum_{i=\frac{k}{2}+1}^{k} p_i$$

and

$$\sum_{i=1}^{k} p_i = \sum_{i=1}^{\frac{k}{2}} p_i + \sum_{i=\frac{k}{2}+1}^{k} p_i = 2\sum_{i=1}^{\frac{k}{2}} p_i$$

which is even. But this is a contradiction because $n = \sum_{i=1}^{k} p_i$ is odd. Therefore $s_3(n,k) = 0$. □

Proposition 4.2. *If both n and k are odd, then $s_3(n,k) = s_3(n+1,k)$.*

$n\backslash k$	1	2	3	4	5	6	7	8	9	Total
3	1									1
4	1									1
5	1									1
6	1	1								2
7	1	2								3
8	1	3								4
9	1	4	1							6
10	1	5	3							9
11	1	6	6							13
12	1	7	10	1						19
13	1	8	15	4						28
14	1	9	21	10						41
15	1	10	28	20	1					60
16	1	11	36	35	5					88
17	1	12	45	56	15					129
18	1	13	55	84	35	1				189
19	1	14	66	120	70	6				277
20	1	15	78	165	126	21				406
21	1	16	91	220	210	56	1			595
22	1	17	105	286	330	126	7			872
23	1	18	120	364	495	252	28			1278
24	1	19	136	455	715	462	84	1		1873
25	1	20	153	560	1001	792	210	8		2745
26	1	21	171	680	1365	1287	462	36		4023
27	1	22	190	816	1820	2002	924	120	1	5896
28	1	23	210	969	2380	3003	1716	330	9	8641
29	1	24	231	1140	3060	4368	3003	792	45	12664

Table 5: Partitions of n into k parts p_i, where $p_i \geq 3$.

Proof. Suppose that both n and k are odd numbers. Let $p_1, p_2, \ldots, p_k$ be a symmetric partition of n into k parts where every part is at least 3. This partition can be transformed into a symmetric partition of $n+1$ by adding one unit to the part $p_{\frac{k+1}{2}}$. Thus, every symmetric partition of n corresponds to a symmetric partition of $n+1$, where each $p_i \geq 3$.
Let $p'_1, p'_2, \ldots, p'_k$ be a symmetric partition of $n+1$ where each $p'_i \geq 3$. Then, for every $1 \leq i \leq \frac{k-1}{2}$, $p'_i = p'_{k+1-i}$. Hence, $p'_{\frac{k+1}{2}} \geq 3$ must be an even number. So, by making $p_{\frac{k+1}{2}} = p'_{\frac{k+1}{2}} - 1$ and $p_i = p'_i$ for every $1 \leq i \leq \frac{k-1}{2}$, we obtain a symmetric partition of n. That is, every symmetric partition of $n+1$ corresponds to a symmetric partition of n.
Therefore, $s_3(n,k) = s_3(n+1,k)$ when n and k are odd. □

Proposition 4.3. *If both n and k are odd, then* $s_3(n,k) = \pi_3\left(\frac{n+3}{2}, \frac{k+1}{2}\right)$.

Proof. Let us assume that $k=3$ and p_1, p_2, p_3 is a symmetric partition of n where each part is at least 3. Because of the symmetry, we know that p_2 is odd; so $p_2 \in \{3, 5, \ldots, n-6\}$. This implies that $p_1 = p_3$ and it belongs to $\{3, 4, \ldots, \frac{n-3}{2}\}$. Then, there are $\frac{n-3}{2} - 3 + 1 = \frac{n-7}{2}$ partitions of n; that is, $s_3(n,3) = \frac{n-7}{2}$.
On the other side,

$$\pi_3\left(\frac{n+3}{2}, 2\right) = C\left(\frac{n+3}{2} - 4 - 1, 1\right) = \frac{n+3}{2} - 5 = \frac{n-7}{2}.$$

So, $s_3 = (n,3) = \pi_3\left(\frac{n+3}{2}, 2\right)$ as we claimed.
Suppose now that $k > 3$. If $p_1, p_2, \ldots, p_k$ is a symmetric partition of n into k parts where every part is at least 3. Then $p_{\frac{k+1}{2}}$ is odd and belongs to $\{3, 5, \ldots, n-6\}$. Moreover, $p_1, p_2, \ldots, p_{\frac{k-1}{2}}$ is a partition of $\frac{1}{2}\left(n - p_{\frac{k+1}{2}}\right)$ into $\frac{k-1}{2}$ parts. Thus,

$$\begin{aligned}
\sum_{p_{\frac{k+1}{2}} \in \{3,5,\ldots,n-5\}} \pi_3\left(\frac{1}{2}\left(n - p_{\frac{k+1}{2}}\right), \frac{k-1}{2}\right) &= \pi_3\left(\frac{n-3}{2}, \frac{k-1}{2}\right) + \pi_3\left(\frac{n-5}{2}, \frac{k-1}{2}\right) + \cdots + \pi_3\left(\frac{n-n+6}{2}, \frac{k-1}{2}\right) \\
&= \pi_3\left(3, \frac{k-1}{2}\right) + \pi_3\left(5, \frac{k-1}{2}\right) + \cdots + \pi_3\left(\frac{n-3}{2}, \frac{k-1}{2}\right) \\
&= \sum_{i=3}^{\frac{n-3}{2}} \pi_3\left(i, \frac{k-1}{2}\right) = \sum_{i=3}^{\frac{n+3}{2}-3} \pi_3\left(i, \frac{k+1}{2} - 1\right) \\
&= \sum_{i=3 \cdot \frac{k+1}{2} - 3}^{\frac{n+3}{2}-3} \pi_3\left(i, \frac{k+1}{2} - 1\right) = \pi_3\left(\frac{n+3}{2}, \frac{k+1}{2}\right)
\end{aligned}$$

because $\pi_3\left(i, \frac{k+1}{2}-1\right)=0$ for all the values of i such that $3 \le i < 3\left(\frac{k+1}{2}-1\right)$. □

Proposition 4.4. *For every $n < 3k$, $\pi_3(n,k)=0$.*

Proof. By contradiction. Suppose that $n < 3k$ and $\pi_3(n,k) > 0$. Then, there exists a partition of n into k parts where every part is at least 3. Let $p_1, p_2, \ldots, p_k$ be this partition. Thus, $p_1+p_2+\cdots+p_k=n$. Since $p_i \ge 3$ for every $i \in \{1,2,\ldots,k\}$, we have that $p_1+p_2+\cdots+p_k \ge 3k$. Hence $3k=n$, which is a contradiction.
Therefore, for every $n<3k$, $\pi_3(n,k)=0$. □

Proposition 4.5. *If n and k are even, then $s_3(n,k)=\pi_3\left(\frac{n}{2},\frac{k}{2}\right)$.*

Proof. Suppose that both, n and k, are even. Let $p_1, p_2, \ldots, p_k$ be a symmetric partition of n into k parts, where each part is at least 3. Then, for every $1 \le i \le \frac{k}{2}$, $p_i = p_{k+1-i}$ and $p_1, p_2, \ldots, p_{\frac{k}{2}}$ is a partition of $\frac{n}{2}$ where every part is at least 3. There are $\pi_3\left(\frac{n}{2},\frac{k}{2}\right)$ of these partitions.
Therefore, when n and k are even, $s_3(n,k)=\pi_3\left(\frac{n}{2},\frac{k}{2}\right)$. □

We summarize these results in the next theorem.

Theorem 4.6. *Let $n \ge 3$ be an integer and $1 \le k \le \left\lfloor \frac{n}{3} \right\rfloor$. The number of symmetric partitions of n into k parts where each part is at least 3 is given by*

$$s_3(n,k)=\begin{cases} 0 & \textit{if } n \textit{ is odd and } k \textit{ is even,}\\ \pi_3\left(\frac{n+3}{2},\frac{k+1}{2}\right)=C\left(\frac{n-1}{2}-k,\frac{k-1}{2}\right) & \textit{if } n \textit{ is odd and } k \textit{ is odd,}\\ \pi_3\left(\frac{n+2}{2},\frac{k+1}{2}\right)=C\left(\frac{n-2}{2}-k,\frac{k-1}{2}\right) & \textit{if } n \textit{ is even and } k \textit{ is odd,}\\ \pi_3\left(\frac{n}{2},\frac{k}{2}\right)=C\left(\frac{n-2}{2}-k,\frac{k-2}{2}\right) & \textit{if } n \textit{ is even and } k \textit{ is even.}\end{cases}$$

Table 6 contains the first values of $s_3(n,k)$. This sequence of numbers can be found in the OEIS, sequence A317489. The column of totals, obtained by adding the $s_3(n,k)$ for all possible values of k, can be also found in OEIS, sequence A226916, see [12].

$n\backslash k$	1	2	3	4	5	6	7	8	9	Total
3	1									1
4	1									1
5	1									1
6	1	1								2
7	1	0								1
8	1	1								2
9	1	0	1							2
10	1	1	1							3
11	1	0	2							3
12	1	1	2	1						5
13	1	0	3	0						4
14	1	1	3	2						7
15	1	0	4	0	1					6
16	1	1	4	3	1					10
17	1	0	5	0	3					9
18	1	1	5	4	3	1				15
19	1	0	6	0	6	0				13
20	1	1	6	5	6	3				22
21	1	0	7	0	10	0	1			19
22	1	1	7	6	10	6	1			32
23	1	0	8	0	15	0	4			28
24	1	1	8	7	15	10	4	1		47
25	1	0	9	0	21	0	10	0		41
26	1	1	9	8	21	15	10	4		69
27	1	0	10	0	28	0	20	0	1	60
28	1	1	10	9	28	21	20	10	1	101
29	1	0	11	0	36	0	35	0	5	88

Table 6: Number of symmetric partitions of n into k parts, where $p_i \ge 3$.

Similarly to what we did in the previous section, we use the values of $\pi_3(n,k)$ and $s_3(n,k)$ to find the number of non-isomorphic snake polyominoes with n cells and height 2. Note that any of these snakes must fit in exactly one of the six cases shown in Figure 4.1; so we analyze six cases:
Case I: The snake has the shape A, i.e., every block of cells has length at least 3. Thus, the number $\beta_2(n)$ of snake polyominoes of length n and height 2 is the same that the number of different partitions of n where every part is at least 3. In order to determine this number, we must

remember that the graphs produced by the partition $p_1, p_2, \ldots, p_k$ and its reverse, $p_k, p_{k-1}, \ldots, p_1$, are isomorphic; furthermore, some of these partitions are symmetric, thus for a fixed value of k

$$\frac{1}{2}\left(\pi_3(n,k) - s_3(n,k)\right)$$

is the number of different non-symmetric partitions of n into k parts where each part is at least 3. So, adding $s_3(n,k)$ to this expression we get

$$\frac{1}{2}\left(\pi_3(n,k) + s_3(n,k)\right).$$

Adding these numbers over all the possible values of k we obtain

$$\beta_2^1(n) = \frac{1}{2}\sum_{k=2}^{\lfloor \frac{n}{3} \rfloor}(\pi_3(n,k) + s_3(n,k)).$$

Note that the case $k = 1$ cannot be used here because the resulting snake has height 1.

Case II: The snake has the shape B, i.e., every block of cells has length at least 3 except the first and the last one that have length 1. Thus, the number of non-isomorphic snakes is the same that the number of different partitions of $n-2$ where every part is at least 3. Following the same steps than the previous case, we get

$$\beta_2^2(n) = \frac{1}{2}\sum_{k=1}^{\lfloor \frac{n-2}{3} \rfloor}(\pi_3(n-2,k) + s_3(n-2,k)).$$

Case III: The snake has the shape C, i.e., every block of cells has length 3 except the first and the last one that have length 2. Thus, the number of non-isomorphic snakes is the same that the number of different partitions of $n-4$ where every part is at least 3. This number is given by

$$\beta_2^3(n) = \frac{1}{2}\sum_{k=1}^{\lfloor \frac{n-4}{3} \rfloor}(\pi_3(n-4,k) + s_3(n-4,k)).$$

Case IV: The snake has the shape D, i.e., every block of cells has length 3 except the first one that has length 1. Thus, the number of non-isomorphic snakes is the same that the number of different partitions of $n-1$ where every part is at least 3. This number is given by

$$\beta_2^4(n) = \sum_{k=1}^{\lfloor \frac{n-1}{3} \rfloor}\pi_3(n-1,k).$$

Case V: The polyomino has the shape E, i.e., every block of cells has length 3 except the last one that has length 2. Thus, the number of non-isomorphic snakes is the same that the number of partitions of $n-2$ whre every part is at least 3. This number is given by

$$\beta_2^5(n) = \sum_{k=1}^{\lfloor \frac{n-2}{3} \rfloor}\pi_3(n-2,k).$$

Case VI: The polyomino has the shape F, i.e., every block of cells has length 3 except the first one that has length 1 and the last one that has length 2. Thus, the number of non-isomorphic snakes is the same that the number of different partitions of $n-3$ where every part is at least 3. This number is given by

$$\beta_2^6(n) = \sum_{k=1}^{\lfloor \frac{n-3}{3} \rfloor}\pi_3(n-3,k).$$

Adding all these quantities we obtain the total number of non-isomorphic snake polyominoes of length n and width 2. In this way we have proven the following theorem.

Theorem 4.7. *The number $\beta_2(n)$ of non-isomorphic snake polyominoes of length n and height 2 is*

$$\beta_2(n) = \sum_{i=1}^{6}\beta_2^i(n).$$

In Figure 4.2 we show a complete example for the case $n = 12$. In this case we have: $\beta_2^1(12) = 11, \beta_2^2(12) = 6, \beta_2^3(12) = 3, \beta_2^4(12) = 13, \beta_2^5(12) = 9, \beta_2^6(12) = 6$, and $\beta_2(12) = 48$.

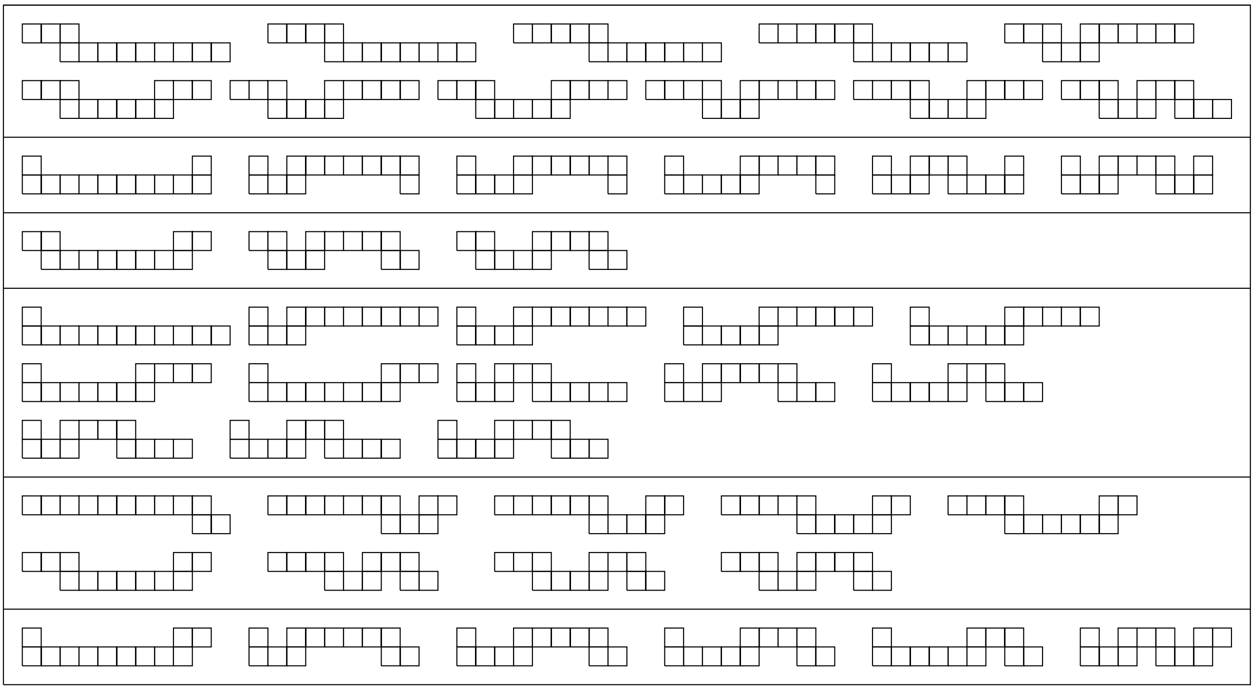

Figure 4.2: Non-isomorphic snake polyominoes of length 12 and height 2.

In Table 7 we show the inital values of these numbers. The last column, that corresponds to $\beta_2(n)$, can be obtained from A102543 in OEIS [13]. In fact, for $n \geq 5$, $\beta_2(n) = a(n+1) - 1$, where the values of $a(n)$ form the sequence A102543. We must also observe that the values of $\beta_2^4(n), \beta_2^5(n)$, and $\beta_2^6(n)$ can be found, with some shiftings, in A078012 [14].

n	$\beta_2^1(n)$	$\beta_2^2(n)$	$\beta_2^3(n)$	$\beta_2^4(n)$	$\beta_2^5(n)$	$\beta_2^6(n)$	$\beta_2(n)$
3	0	0	0	0	0	0	0
4	0	0	0	1	0	0	1
5	0	1	0	1	1	0	3
6	1	1	0	1	1	1	5
7	1	1	1	2	1	1	7
8	2	2	1	3	2	1	11
9	3	2	1	4	3	2	15
10	5	3	2	6	4	3	23
11	7	4	2	9	6	4	32
12	11	6	3	13	9	6	48
13	15	8	4	19	13	9	68
14	23	12	6	28	19	13	101
15	32	16	8	41	28	19	144
16	48	24	12	60	41	28	213
17	68	33	16	88	60	41	306
18	101	49	24	129	88	60	451
19	144	69	33	189	129	88	652
20	213	102	49	277	189	129	959
21	306	145	69	406	277	189	1392
22	451	214	102	595	406	277	2045
23	652	307	145	872	595	406	2977
24	959	452	214	1278	872	595	4370
25	1392	653	307	1873	1278	872	6375
26	2045	960	542	2745	1873	1278	9353
27	2977	1393	653	4023	2745	1873	13664
28	4370	2046	960	5896	4023	2745	20040
29	6375	2978	1393	8641	5896	4023	29306

Table 7: $\beta_2(n)$ is the number of non-isomorphic snake polyominoes of length n and height 2.

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$\mathscr{I}$-Cesàro Summability of a Sequence of Order α of Random Variables in Probability

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Abstract

In this paper, we define four types of convergence of a sequence of random variables, namely, $\mathscr{I}$-statistical convergence of order α, $\mathscr{I}$-lacunary statistical convergence of order α, strongly $\mathscr{I}$-lacunary convergence of order α and strongly $\mathscr{I}$-Cesàro summability of order α in probability where $0<\alpha<1$. We establish the connection between these notions.

1. Introduction and background

Theory of statistical convergence was firstly originated by Fast [1]. After Fridy [2] and Šalát [3] statistical convergence became a notable topic in summability theory. Lacunary statistical convergence was defined by using lacunary sequences in [4]. $\mathscr{I}$-convergence was fistly considered by Kostyrko et al. [5]. Also, Das et al. [6] gave new definitions by using ideal, such as $\mathscr{I}$-statistical convergence, $\mathscr{I}$-lacunary statistical convergence. Ulusu et al. [7] also studied asymptotically $\mathscr{I}$-Cesaro equivalence of sequences of sets.
Statistical convergence of order α $(0<\alpha<1)$ was introduced using the notion of natural density of order α where n is replaced by n^{α} in [8]. This new type convergence was different in many ways from statistical convergence. Lacunary statistical convergence of order α is studied by Sengöl and M. Et [9], $\mathscr{I}$-statistical and $\mathscr{I}$-lacunary statistical convergence of order α is studied by Das and Savas [10].
In probability theory, if for $n>0$, a random variable X_n given on space S, a probability function $P: X\to\mathbb{R}$, then we say that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of random variables and it is demonstrated by $\{X_n\}_{n\in\mathbb{N}}$.
It is important that if there exists $c\in\mathbb{R}$ for which $P(|X-c|<\varepsilon)=1$, where $\varepsilon>0$ is sufficiently small, that is, it is means that values of X lie in a very small neighbourhood of c.
New concepts have begun to be studied in probability theory by Das et al. [6], and others ([11]-[15]).

2. Main results

Definition 2.1. *$\{X_k\}_{k\in\mathbb{N}}$ is said to be $\mathscr{I}$-statistically convergent of order α in probability to a random variable X if for any ε, δ, $\gamma>0$*

$$\left\{n\in\mathbb{N}: \frac{1}{n^{\alpha}}\left|\left\{k\le n: P\left(|X_k-X|\ge\varepsilon\right)\ge\delta\right\}\right|\ge\gamma\right\}\in\mathscr{I},$$

and demonstrated by $X_k \overset{PS(\mathscr{I})^{\alpha}}{\to} X$.

Definition 2.2. *$\{X_n\}_{n\in\mathbb{N}}$ is said to be $\mathscr{I}$-lacunary statistically convergent of order α in probability to a random variable X if for any* $\varepsilon,\delta,\gamma>0$

$$\left\{r\in\mathbb{N}: \frac{1}{h_r^{\alpha}}\left|\left\{k\in I_r: P\left(|X_k-X|\ge\varepsilon\right)\ge\delta\right\}\right|\ge\gamma\right\}\in\mathscr{I},$$

and it is demonstrated by $X_k \overset{PS_{\theta}(\mathscr{I})^{\alpha}}{\to} X$.

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Definition 2.3. $\{X_k\}_{k\in\mathbb{N}}$ *is said to be strongly $\mathscr{I}$-lacunary convergent or $PV_\theta(\mathscr{I})$-convergent of order α in probability to a random variable X if for every ε, $\delta > 0$,*

$$\left\{r\in\mathbb{N} : \frac{1}{h_r^\alpha}\sum_{k\in I_r} P(|X_k - X| \geq \varepsilon) \geq \delta\right\} \in \mathscr{I},$$

and it is demonstrated by $X_k \overset{PV_\theta(\mathscr{I})^\alpha}{\to} X$.

Definition 2.4. $\{X_k\}_{k\in\mathbb{N}}$ *is said to be strongly $\mathscr{I}$-Cesàro summable of order α in probability to a random variable X if for every ε, $\delta > 0$,*

$$\left\{n\in\mathbb{N} : \frac{1}{n^\alpha}\sum_{k=1}^{n} P(|X_k - X| \geq \varepsilon) \geq \delta\right\} \in \mathscr{I},$$

and it is demonstrated by $X_k \overset{PC_1[\mathscr{I}]^\alpha}{\to} X$.

Theorem 2.5. *If $0 < \alpha \leq \beta \leq 1$ then $PS(\mathscr{I})^\alpha \subseteq PS(\mathscr{I})^\beta$.*

Proof. From the assumption, we say that

$$\frac{1}{n^\beta}\left|\{k \leq n : P(|X_k - X| \geq \varepsilon) \geq \delta\}\right| \leq \frac{1}{n^\alpha}\left|\{k \leq n : P(|X_k - X| \geq \varepsilon) \geq \delta\}\right|$$

Hence,

$$\left\{n\in\mathbb{N} : \tfrac{1}{n^\beta}\left|\{k \leq n : P(|X_k - X| \geq \varepsilon) \geq \delta\}\right| \geq \gamma\right\}$$

$$\left\{n\in\mathbb{N} : \tfrac{1}{n^\alpha}\left|\{k \leq n : P(|X_k - X| \geq \varepsilon) \geq \delta\}\right| \geq \gamma\right\}$$

for $\gamma > 0$. Therefore, we obtain $PS(\mathscr{I})^\alpha \subseteq PS(\mathscr{I})^\beta$. □

Theorem 2.6. *If* $\liminf_r q_r > 1$, *then*

$$X_k \overset{PC_1[\mathscr{I}]^\alpha}{\to} X \Rightarrow X_k \overset{PV_\theta(\mathscr{I})^\alpha}{\to} X.$$

Proof. If $\liminf_r q_r > 1$, there exists $\gamma > 0$ such that $q_r \geq 1+\gamma$ for all $r \geq 1$. Since $h_r = k_r - k_{r-1}$, we have $\frac{k_r^\alpha}{h_r^\alpha} \leq \left(\frac{1+\gamma}{\gamma}\right)^\alpha$ and $\frac{k_{r-1}^\alpha}{h_r^\alpha} \leq \left(\frac{1}{\gamma}\right)^\alpha$. Let $\varepsilon > 0$ and we define set by

$$S = \left\{k_r \in\mathbb{N} : \frac{1}{k_r^\alpha}\sum_{k=1}^{k_r} P(|X_k - X| \geq \varepsilon) < \delta\right\}.$$

Therefore, $S \in \mathscr{F}(\mathscr{I})$.

$$\begin{aligned}
\tfrac{1}{h_r^\alpha}\sum_{k\in I_r} P(|X_k - X| \geq \varepsilon) &= \tfrac{1}{h_r^\alpha}\sum_{k=1}^{k_r} P(|X_k - X| \geq \varepsilon) - \tfrac{1}{h_r^\alpha}\sum_{k=1}^{k_{r-1}} P(|X_k - X| \geq \varepsilon) \\
&= \frac{k_r^\alpha}{h_r^\alpha}\cdot\tfrac{1}{k_r^\alpha}\sum_{k=1}^{k_r} P(|X_k - X| \geq \varepsilon) - \tfrac{k_{r-1}^\alpha}{h_r^\alpha}\cdot\tfrac{1}{k_{r-1}^\alpha}\sum_{k=1}^{k_{r-1}} P(|X_k - X| \geq \varepsilon) \\
&\leq \left(\frac{1+\gamma}{\gamma}\right)^\alpha \delta - \left(\frac{1}{\delta\gamma}\right)^\alpha \delta'
\end{aligned}$$

for each $k_r \in S$. Choose $\eta = \left(\frac{1+\gamma}{\gamma}\right)^\alpha \delta - \left(\frac{1}{\delta\gamma}\right)^\alpha \delta'$. Therefore,

$$\left\{r\in\mathbb{N} : \frac{1}{h_r^\alpha}\sum_{k\in I_r} P(|X_k - X| \geq \varepsilon) < \eta\right\} \in \mathscr{F}(\mathscr{I}).$$

Hence, we get $X_k \overset{PV_\theta(\mathscr{I})^\alpha}{\to} X$. □

Theorem 2.7. *If $\{X_k\}$ is strongly $\mathscr{I}$-Cesàro summable of order α then, it is $\mathscr{I}$-statistical convergent of order α in probability to a random variable X.*

Proof. Let $X_k \overset{PC_1[\mathscr{I}]^\alpha}{\to} X$, and $\varepsilon > 0$ given. Then

$$\begin{aligned}\frac{1}{n^\alpha}\sum_{k=1}^{n} P(|X_k - X| \geq \varepsilon) &\geq \frac{1}{n^\alpha}\sum_{\substack{k=1\\ P(|X_k-X|\geq\varepsilon)}}^{n} P(|X_k - X| \geq \varepsilon)\\ &\geq \frac{\delta}{n^\alpha}.\left|\{k \leq n : P(|X_k - X| \geq \varepsilon) \geq \delta\}\right|\end{aligned}$$

and so

$$\frac{1}{\delta.n^\alpha}\sum_{k=1}^{n} P(|X_k - X| \geq \varepsilon) \geq \frac{1}{n^\alpha}\left|\{k \leq n : P(|X_k - X| \geq \varepsilon) \geq \delta\}\right|.$$

So for a given $\tau > 0$,

$$\begin{aligned}&\left\{n \in \mathbb{N} : \tfrac{1}{n^\alpha}\left|\{k \leq n : P(|X_k - X| \geq \varepsilon) \geq \delta\}\right| \geq \tau\right\}\\ &\subseteq \left\{n \in \mathbb{N} : \tfrac{1}{n^\alpha}\sum_{k=1}^{n} P(|X_k - X| \geq \varepsilon) \geq \delta.\tau\right\} \in \mathscr{I}.\end{aligned}$$

Therefore, $X_k \overset{PS(\mathscr{I})^\alpha}{\to} X$. □

Theorem 2.8. *Let a bounded $\{X_k\}$ is $\mathscr{I}$-statistical convergent of order α to X. Hence, it is strongly $\mathscr{I}$-Cesàro summable of order α to X.*

Proof. Assume that $\{X_k\}$ is bounded and $X_k \overset{PS(\mathscr{I})^\alpha}{\to} X$. Since $\{X_k\}$ is bounded, we get $P(|X_k - X| > \varepsilon) \leq M$ for all k. For $\varepsilon > 0$, we have

$$\begin{aligned}\frac{1}{n^\alpha}\sum_{k=1}^{n} P(|X_k - X| \geq \varepsilon) &= \frac{1}{n^\alpha}\sum_{\substack{k=1\\ P(|X_k-X|\geq\varepsilon)\geq\delta}}^{n} P(|X_k - X| \geq \varepsilon)\\ &+ \frac{1}{n^\alpha}\sum_{\substack{k=1\\ P(|X_k-X|\geq\varepsilon)<\delta}}^{n} P(|X_k - X| \geq \varepsilon)\\ &\leq \frac{1}{n^\alpha}M\left|\{k \leq n : P(|X_k - X| \geq \varepsilon) \geq \delta\}\right|\\ &\quad + \frac{1}{n^\alpha}n^\alpha\delta\end{aligned}$$

Then for any $\gamma > 0$,

$$\begin{aligned}&\left\{n \in \mathbb{N} : \tfrac{1}{n^\alpha}\sum_{k=1}^{n} P(|X_k - X| \geq \varepsilon) \geq \gamma\right\}\\ &\subseteq \left\{n \in \mathbb{N} : \tfrac{1}{n^\alpha}\left|\{k \leq n : P(|X_k - X| \geq \varepsilon) \geq \delta\}\right| \geq \tfrac{\gamma}{M}\right\} \in \mathscr{I}.\end{aligned}$$

Therefore $X_k \overset{PC_1[\mathscr{I}]^\alpha}{\to} X$. □

Theorem 2.9. *For $\theta = \{k_r\}$,*

(i) If $\{X_k\} \overset{PV_\theta(\mathscr{I})^\alpha}{\to} X$ then $\{X_k\} \overset{PS_\theta(\mathscr{I})^\alpha}{\to} X$, and
(ii) $PV_\theta(\mathscr{I})^\alpha$ is proper subset of $PS_\theta(\mathscr{I})^\alpha$.

Proof. (i) Let $\varepsilon, \delta > 0$ and $\{X_k\} \overset{PV_\theta(\mathscr{I})^\alpha}{\to} X$. Then, we can write

$$\begin{aligned}\frac{1}{h_r^\alpha}\sum_{k\in I_r} P(|X_k - X| \geq \varepsilon) &\geq \frac{1}{h_r^\alpha}\sum_{\substack{k\in I_r\\ P(|X_k-X|\geq\varepsilon)\geq\delta}} P(|X_k - X| \geq \varepsilon)\\ &\geq \frac{\delta}{h_r^\alpha}.\left|\{k \in I_r : P(|X_k - X| \geq \varepsilon) \geq \delta\}\right|.\end{aligned}$$

Therefore

$$\frac{1}{\delta h_r^\alpha}\sum_{k\in I_r} P(|X_k - X| \geq \varepsilon) \geq \frac{1}{h_r^\alpha}.\left|\{k \in I_r : P(|X_k - X| \geq \varepsilon) \geq \delta\}\right|.$$

which implies that for any $\gamma > 0$,

$$\begin{aligned}&\left\{r \in \mathbb{N} : \tfrac{1}{h_r^\alpha}\left|\{k \in I_r : P(|X_k - X| \geq \varepsilon) \geq \delta\}\right| \geq \gamma\right\}\\ &\subseteq \left\{r \in \mathbb{N} : \tfrac{1}{h_r^\alpha}\sum_{k\in I_r} P(|X_k - X| \geq \varepsilon) \geq \delta\gamma\right\} \in \mathscr{I}.\end{aligned}$$

Hence we get $X_k\} \overset{PS_\theta(\mathscr{I})^\alpha}{\to} X$.

(ii) Let $\{X_k\}$ be defined by

$$X_k = \begin{cases} \{-1,1\} & , \quad \text{with probability } \frac{1}{2}, \text{ if } n \text{ is the first } \left[\sqrt{h_r^\alpha}\right] \text{ integers in the interval } I_r, \\ \{0,1\} & , \quad \text{with probability } P(X_n=0) = \left(1-\frac{1}{n}\right) \text{ and } P(X_n=1)=\frac{1}{n}, \\ & \quad \text{if } n \text{ is other than the first} \\ \left[\sqrt{h_r^\alpha}\right] & \quad \text{integers in the interval } I_r. \end{cases}$$

Let $0<\varepsilon<1$ and $\delta<1$. Then, we obtain

$$P(|X_k-0| \geq \varepsilon) = \begin{cases} 1 & , \quad \text{if } n \text{ is the first } \left[\sqrt{h_r^\alpha}\right] \text{ integers in the interval } I_r, \\ \frac{1}{n} & \quad \text{if } n \text{ is other than the first } \left[\sqrt{h_r^\alpha}\right] \text{ integers in the interval } I_r. \end{cases}$$

Now

$$\frac{1}{h_r^\alpha}\left|\{k \in I_r : P(|X_k-0| \geq \varepsilon) \geq \delta\}\right| \leq \frac{\left[\sqrt{h_r^\alpha}\right]}{h_r^\alpha}$$

and for any $\gamma>0$ we get

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}\left|\{k \in I_r : P(|X_k-0| \geq \varepsilon) \geq \delta\}\right| \geq \gamma\right\} \subseteq \left\{r \in \mathbb{N} : \frac{\left[\sqrt{h_r^\alpha}\right]}{h_r^\alpha} \geq \gamma\right\}.$$

Since the set

$$\left\{r \in \mathbb{N} : \frac{\left[\sqrt{h_r^\alpha}\right]}{h_r^\alpha} \geq \gamma\right\}$$

is finite and so belongs to $\mathscr{I}$, therefore, we obtain

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}\left|\{k \in I_r : P(|X_k-0| \geq \varepsilon) \geq \delta\}\right| \geq \gamma\right\} \in \mathscr{I}$$

which means that $X_k \overset{PS_\theta(\mathscr{I})^\alpha}{\to} 0$. Also,

$$\frac{1}{h_r^\alpha}\sum_{k\in I_r} P(|X_k-0| \geq \varepsilon) = \frac{1}{h_r^\alpha}.\frac{\left[\sqrt{h_r^\alpha}\right]\left(\left[\sqrt{h_r^\alpha}\right]+1\right)}{2},$$

then

$$\begin{aligned} \left\{r \in \mathbb{N} : \tfrac{1}{h_r}\sum_{k\in I_r} P(|X_k-0| \geq \varepsilon) \geq \tfrac{1}{4}\right\} &= \left\{r \in \mathbb{N} : \tfrac{\left[\sqrt{h_r^\alpha}\right]\left(\left[\sqrt{h_r^\alpha}\right]+1\right)}{h_r} \geq \tfrac{1}{2}\right\} \\ &= \{m, m+1, m+2, \ldots\} \in \mathscr{F}(\mathscr{I}) \end{aligned}$$

for some $m \in \mathbb{N}$. Hence, $X_k \overset{PS_\theta(\mathscr{I})^\alpha}{\nrightarrow} 0$. □

Theorem 2.10. *$\mathscr{I}$-statistical convergence in probability of order α implies $\mathscr{I}$-lacunary statistical convergence in probability of order α* $\liminf_r q_r > 1$.

Proof. By assumption $\liminf_r q_r > 1$, then there exists a $\sigma>0$ such that $q_r \geq 1+\sigma$ for sufficiently large r, that is,

$$\frac{h_r}{k_r} \geq \frac{\sigma}{1+\sigma} \Rightarrow \frac{1}{h_r^\alpha} \leq \frac{1}{k_r^\alpha}\left(\frac{1+\sigma}{\sigma}\right)^\alpha$$

If $\{X_k\} \overset{PS(\mathscr{I})^\alpha}{\to} X$, then for $\varepsilon>0$ and for $r>0$, we have

$$\frac{1}{h_r^\alpha}\left|\{k \in I_r : P(|X_k-X| \geq \varepsilon) \geq \delta\}\right| \leq \frac{1}{k_r^\alpha}\left(\frac{1+\sigma}{\sigma}\right)^\alpha \left|\{k \leq k_r : P(|X_k-X| \geq \varepsilon) \geq \delta\}\right|$$

Then for any $\gamma>0$, we get

$$\begin{aligned} &\left\{r \in \mathbb{N} : \tfrac{1}{h_r^\alpha}\left|\{k \in I_r : P(|X_k-X| \geq \varepsilon)\} \geq \delta\right| \geq \gamma\right\} \\ &\quad \subseteq \left\{r \in \mathbb{N} : \frac{1}{k_r^\alpha}\left|\{k \leq k_r : P(|X_k-X| \geq \varepsilon)\} \geq \delta\right| \geq \tfrac{\gamma\sigma^\alpha}{(1+\sigma)^\alpha}\right\} \in \mathscr{I}. \end{aligned}$$

□

Theorem 2.11. *$\mathscr{I}$-lacunary statistical convergence in probability of order α implies $\mathscr{I}$-statistical convergence in probability of order α, $0<\alpha<1$, if* $\sup_r \sum_{i=0}^{r-1} \frac{h_{i+1}^{\alpha}}{(k_{r-1})^{\alpha}} = B < \infty$.

Proof. Suppose that $\{X_k\} \overset{PS_{\theta}(\mathscr{I})^{\alpha}}{\to} X$, and for ε, δ, γ_1, $\gamma_2 > 0$ define the sets

$$C = \left\{ r \in \mathbb{N} : \frac{1}{h_r^{\alpha}} \left| \{ k \in I_r : P(|X_k - X| \geq \varepsilon) \geq \delta \} \right| < \gamma_1 \right\}$$

and

$$T = \left\{ n \in \mathbb{N} : \frac{1}{n^{\alpha}} \left| \{ k \leq n : P(|X_k - X| \geq \varepsilon) \geq \delta \} \right| < \gamma_2 \right\}.$$

From our assumption we get $C \in \mathscr{F}(\mathscr{I})$. Further observe that

$$K_j = \frac{1}{h_j^{\alpha}} \left| \left\{ k \in I_j : P(|X_k - X| \geq \varepsilon) \geq \delta \right\} \right| < \gamma_1$$

for all $j \in C$. Let $n \in \mathbb{N}$ be such that $k_{r-1} < n \leq k_r$ for some $r \in C$. Hence, we obtain

$$\begin{aligned}
&\frac{1}{n^{\alpha}} \left| \{ k \leq n : P(|X_k - X| \geq \varepsilon) \geq \delta \} \right| \\
&\leq \frac{1}{k_{r-1}^{\alpha}} \left| \{ k \leq k_r : P(|X_k - X| \geq \varepsilon) \geq \delta \} \right| \\
&= \frac{1}{k_{r-1}^{\alpha}} \left| \{ k \in I_1 : P(|X_k - X| \geq \varepsilon) \geq \delta \} \right| \\
&+ \frac{1}{k_{r-1}^{\alpha}} \left| \{ k \in I_2 : P(|X_k - X| \geq \varepsilon) \geq \delta \} \right| \\
&+ \ldots + \frac{1}{k_{r-1}^{\alpha}} \left| \{ k \in I_r : P(|X_k - X| \geq \varepsilon) \geq \delta \} \right| \\
&= \frac{k_1^{\alpha}}{k_{r-1}^{\alpha}} \frac{1}{h_1^{\alpha}} \left| \{ k \in I_1 : P(|X_k - X| \geq \varepsilon) \geq \delta \} \right| \\
&+ \frac{(k_2 - k_1)^{\alpha}}{k_{r-1}^{\alpha}} \frac{1}{h_2^{\alpha}} \left| \{ k \in I_2 : P(|X_k - X| \geq \varepsilon) \geq \delta \} \right| \\
&+ \ldots + \frac{(k_r - k_{r-1})^{\alpha}}{k_{r-1}^{\alpha}} \frac{1}{h_r^{\alpha}} \left| \{ k \in I_r : P(|X_k - X| \geq \varepsilon) \geq \delta \} \right| \\
&= \frac{k_1^{\alpha}}{k_{r-1}^{\alpha}} K_1 + \frac{(k_2 - k_1)^{\alpha}}{k_{r-1}^{\alpha}} K_2 + \ldots + \frac{(k_r - k_{r-1})^{\alpha}}{k_{r-1}^{\alpha}} K_r \\
&\leq \left\{ \sup_{j \in C} K_j \right\} \sup_r \sum_{i=0}^{r-1} \frac{h_{i+1}^{\alpha}}{(k_{r-1})^{\alpha}} \\
&< \gamma_1 B.
\end{aligned}$$

Choosing $\gamma_2 = \dfrac{\gamma_1}{B}$ and by $\bigcup \{ n : k_{r-1} < n \leq k_r,\, r \in C \} \subset T$ where $C \in \mathscr{F}(\mathscr{I})$ Then the set T belongs to $\mathscr{F}(\mathscr{I})$ and this completes the proof. □

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Existence and Stability of Solutions of Katugampola-Caputo Type Implicit Fractional Differential Equations with Impulses

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Abstract

This paper investigates the existence and Ulam stability of solutions for impulsive nonlinear fractional implicit differential equations with finite delay via Katugampola fractional derivative in Caputo sense. Our results are based on some standard fixed point theorems. Some examples are presented to illustrate the main results.

1. Introduction

The interest in the study of differential equations of fractional order lies in the fact that fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. With this advantage, the fractional-order models become more realistic and practical than the classical integer-order models, in which such effects are not taken into account. As a matter of fact, fractional differential equations arise in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc., see [1]-[3]. For some recent development on the topic, see [4]-[10] and the references therein.

Impulsive differential equations, which provide a natural description of observed evolution processes, are regarded as important mathematical tools for the better understanding of several real world problems in applied sciences. The theory of impulsive differential equations of integer order has found extensive applications in realistic mathematical modeling of a wide variety of practical situations and has emerged as an important area of investigation in recent years. For the general theory and applications of impulsive differential equations, we refer the reader to the references [11]-[15]. On the other hand, the implicit differential equations with impulsive and delay have not been addressed so extensively and many aspects of these problems are yet to be explored. For some recent work on impulsive differential equations of fractional order, see [16]-[19] and the references therein. These days generalization of the derivatives of both Riemann-Liouville and Caputo types are introduced and shown the effect of utilizing it in equations of mathematical physics or related to probability. This was done using the definition of generalized fractional derivatives given by Katugampola [20]. The author initiated a new fractional integral, which generalizes the Riemann-Liouville and the Hadamard integrals into a single form. Later, Katugampola [21] introduced a new fractional derivative, which generalizes the two derivatives in question. Motivated by the papers [21]-[23], we apply Katugampola-Caputo derivative for implicit fractional differential equations.

In this paper, we investigate the existence and Ulam stability of solutions for impulsive nonlinear fractional implicit differential equations with delay via Katugampola fractional derivative given by,

$$\begin{cases} {}^{\rho}D^{\omega}_{x_m}u(x) = h(x,u_x,{}^{\rho}D^{\omega}_{x_m}u(x)), & \text{for each } x \in \mathfrak{J} := (x_m,x_{m+1}],\ m = 0,1,\dots k,, \\ \Delta u|_{x_m} = I_m(u_{x_m^-}), & m = 1,\dots,k, \\ u(x) = \psi(x), & x \in [-r,0],\ r > 0, \end{cases} \tag{1.1}$$

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where ${}^{\rho}D^{\omega}_{x_m}$ is the Katugampola fractional derivative in Caputo sense, $0<\omega\le 1$, $\rho\in\mathbb{R}^{+}$, $h:\mathfrak{J}\times\mathfrak{PC}([-r,0],\mathbb{R})\times\mathbb{R}\to\mathbb{R}$ is a given function, $I_m:\mathfrak{PC}([-r,0],\mathbb{R})\to\mathbb{R}$, and $\psi\in\mathfrak{PC}([-r,0],\mathbb{R})$, $0=x_0<x_1<\cdots<x_m<x_{m+1}=T$. $\mathfrak{PC}([-r,0],\mathbb{R})$ is a space of piecewice functions defined on $[-r,0]$ to be specified in Section 2.
For each function u defined on $[-r,T]$ and for any $x\in\mathfrak{J}$, we define by u_x the element of $\mathfrak{PC}([-r,0],\mathbb{R})$ defined by:

$$u_x(\theta)=u(x+\theta),\ \theta\in[-r,0],$$

$u_x(\cdot)$ represents the history of the state from time $x-r$ upto time x. Here $\Delta u|_{x_m}=u(x_m^{+})-u(x_m^{-})$, where $u(x_m^{+})=\lim_{l\to 0^{+}}u(x_m+l)$ and $u(x_m^{-})=\lim_{l\to 0^{-}}u(x_m+l)$ denotes the right and left limits of u_x at $x=x_m$, respectively.

2. Prerequisites

In this section, we introduce notations, definitions, lemmas and theorems that are needed for the proof of the main results.
Let $T>0$, $\mathfrak{J}=[0,T]$ and $\mathfrak{C}(\mathfrak{J},\mathbb{R})$ be the Banach space of all continuous functions from $\mathfrak{J}$ into $\mathbb{R}$ with the norm

$$\|u\|_{\infty}=\sup\{|u(x)|:x\in\mathfrak{J}\}.$$

Let $\mathfrak{J}_0=[x_0,x_1]$ and $\mathfrak{J}_m=(x_m,x_{m+1}]$, where $m=1,2,\dots k$.
Consider the set of functions

$$\mathfrak{PC}([-r,0],\mathbb{R})=\{u:[-r,0]\to\mathbb{R}:u\in\mathfrak{C}\left((t_m,t_{m+1}],\mathbb{R}\right),m=0,1,\dots,k^{'},\text{ and there exist } u(t_m^{-})\text{ and } u(t_m^{+}),\ m=1,2,\dots,k\text{ with } u(t_m^{-})=u(t_m)\}.$$

$\mathfrak{PC}([-r,0],\mathbb{R})$ is a Banach space with the norm

$$\|u\|_{\mathfrak{PC}}=\sup_{x\in[-r,0]}|u(x)|.$$

$\mathfrak{PC}([-r,T],\mathbb{R})$ is a Banach space with the norm

$$\|u\|_{\mathfrak{PC}_1}=\sup_{x\in[-r,T]}|u(x)|.$$

$\mathfrak{L}^{'}(\mathfrak{J},\mathbb{R})$ is the space of Lebesgue-integrable functions $u:\mathfrak{J}\to\mathbb{R}$ with the norm

$$\|u\|_1=\int_0^T|u(s)|\,\mathrm{d}s.$$

$\mathfrak{AC}^n(\mathfrak{J})=\{h:\mathfrak{J}\to\mathbb{R}:h,h^{'},\dots h^{(n-1)}\in\mathfrak{C}(\mathfrak{J},\mathbb{R})\text{ and } h^{(n-1)}\text{ is absolutely continuous}\}$.
In what follows $\omega>0$.

Definition 2.1. *[9, 10] The fractional(arbitrary) order integral of the function $h\in\mathfrak{L}^{'}([0,T],\mathbb{R}_+)$ of order $\omega\in\mathbb{R}_+$ is defined by*

$$I^{\omega}h(x)=\frac{1}{\Gamma(\omega)}\int_0^x(x-s)^{\omega-1}h(s)\mathrm{d}s,$$

where Γ is the Euler gamma function defined by $\Gamma(\omega)=\int_0^{\infty}x^{\omega-1}e^{-x}\mathrm{d}x$, $\omega>0$.

Definition 2.2. *[9, 10] For a function $h\in\mathfrak{AC}^n(\mathfrak{J})$, the Caputo fractional order derivative of order ω of h is defined by*

$$({}^{c}D^{\omega}_{0^+}h)(x)=\frac{1}{\Gamma(n-\omega)}\int_0^x(x-s)^{n-\omega-1}h^{n}(s)\mathrm{d}s,$$

where $n=[\omega]+1$ and $[\omega]$ denotes the integer part of the real number ω.

Definition 2.3. *[22] The generalized left-sided fractional integral ${}^{\rho}I^{\omega}_{0^+}h$ of order $\omega\in\mathbb{C}(Re(\omega)>0)$ is defined by*

$$({}^{\rho}I^{\omega}_{0^+}h)(x)=\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_0^x(x^{\rho}-s^{\rho})^{\omega-1}s^{\rho-1}h(s)\mathrm{d}s,$$

for $x>0$, if the integral exists.

Definition 2.4. *[22] The generalized fractional derivative, corresponding to the generalized fractional integral (2.1), is defined by*

$$({}^{\rho}D^{\omega}_{0^+}h(x)=\frac{\rho^{\omega-n+1}}{\Gamma(n-\omega)}\left(x^{1-\rho}\frac{d}{dx}\right)^{n}\int_0^x(x^{\rho}-s^{\rho})^{n-\omega-1}s^{\rho-1}h(s)\mathrm{d}s, \tag{2.1}$$

if the integral exists.

Lemma 2.5. *Let $\omega\ge 0$ and $n=[\omega]+1$. Then*

$${}^{\rho}I^{\omega}_{0^+}\left({}^{\rho}D^{\omega}_{0^+}h(x)\right)=h(x)-\sum_{m=0}^{n-1}\frac{h^{m}(0)}{m!}x^{m}.$$

Lemma 2.6. *Let $\omega > 0$, then the differential equation ${}^{\rho}D^{\omega}_{0^+}h(x) = 0$ has solutions*

$$h(x) = b_0 + b_1\left(\frac{x^{\rho}}{\rho}\right) + b_2\left(\frac{x^{\rho}}{\rho}\right)^2 + \cdots + b_{n-1}\left(\frac{x^{\rho}}{\rho}\right)^{(n-1)},$$

$b_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$, $n = [\omega]+1$.

Lemma 2.7. *Let $\omega > 0$, then*

$${}^{\rho}I^{\omega}_{0^+}\left({}^{\rho}D^{\omega}_{0^+}h(x)\right) = h(x) + b_0 + b_1\left(\frac{x^{\rho}}{\rho}\right) + b_2\left(\frac{x^{\rho}}{\rho}\right)^2 + \cdots + b_{n-1}\left(\frac{x^{\rho}}{\rho}\right)^{(n-1)},$$

for some $b_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$, $n = [\omega]+1$.

Lemma 2.8. *[24] Let $w : [0,T] \to [0,+\infty)$ be a real function and $\alpha(\cdot)$ is a non-negative, locally integrable function on $[0,T]$ and there are constants $a > 0$ and $0 < \omega \leq 1$ such that*

$$w(x) \leq \alpha(x) + a\int_0^x (x^{\rho} - s^{\rho})^{-\omega} s^{-\rho} w(s)\mathrm{d}s.$$

Then, there exists a constant $K = K(\omega)$ such that

$$w(x) \leq \alpha(x) + Ka\int_0^x (x^{\rho} - s^{\rho})^{-\omega} s^{-\rho} \alpha(s)\mathrm{d}s,$$

for every $x \in [0,T]$.

The following integral inequality of Gronwall type for piecewise continuous functions was introduced by Bainov and Hristova [25] which can be used in the sequel.

Lemma 2.9. *Let for $x \geq x_0 \geq 0$, the following inequality holds,*

$$u(x) \leq a(x) + \int_{x_0}^x g(x,s)u(s)\mathrm{d}s + \sum_{x_0 < x_m < x} \beta_m(x)u(x_m),$$

where $\beta_m(x)$ $(m \in \mathbb{N})$ are non-decreasing functions for $x \geq x_0$, $a \in \mathfrak{PC}([x_0,\infty),\mathbb{R}_+)$, a is non-decreasing and $g(x,s)$ is a continuous non negative function for $x,s \geq x_0$ and non decreasing with respect to x for any fixed $s \geq x_0$. Then, for $x \geq x_0$, the following inequality is valid:

$$u(x) \leq a(x)\prod_{x_0 < x_m < x}(1 + \beta_m(x))\exp\left(\int_{x_0}^x g(x,s)\mathrm{d}s\right).$$

Now, we consider the concepts of Wang et al. and refer some new concepts about Ulam-Hyers stability and Ulam-Hyers-Rassias stability for considered problem (1.1). See [24, 26, 27, 28, 29].
Let $u \in \mathfrak{PC}(\mathfrak{J},\mathbb{R})$, $\varepsilon > 0$, $\phi > 0$ and $\alpha \in \mathfrak{PC}(\mathfrak{J},\mathbb{R}_+)$ is non decreasing. We consider the set of inequalities

$$\begin{cases} |{}^{\rho}D^{\omega}u(x) - h(x,u_x,{}^{\rho}D^{\omega}u(x))| \leq \varepsilon, & x \in (x_m,x_{m+1}],\ m = 1,\ldots k, \\ \left|\Delta u|_{x_m} - I_m(u_{x_m^-})\right| \leq \varepsilon, & m = 1,\ldots,k; \end{cases} \tag{2.2}$$

the set of inequalities

$$\begin{cases} |{}^{\rho}D^{\omega}u(x) - h(x,u_x,{}^{\rho}D^{\omega}u(x))| \leq \alpha(x), & x \in (x_m,x_{m+1}],\ m = 1,\ldots k, \\ \left|\Delta u|_{x_m} - I_m(u_{x_m^-})\right| \leq \phi, & m = 1,\ldots,k; \end{cases} \tag{2.3}$$

and the set of inequalities

$$\begin{cases} |{}^{\rho}D^{\omega}u(x) - h(x,u_x,{}^{\rho}D^{\omega}u(x))| \leq \varepsilon\alpha(x), & x \in (x_m,x_{m+1}],\ m = 1,\ldots k, \\ \left|\Delta u|_{x_m} - I_m(u_{x_m^-})\right| \leq \varepsilon\phi, & m = 1,\ldots,k. \end{cases} \tag{2.4}$$

Definition 2.10. *The problem (1.1) is Ulam-Hyers stable, if there exists a real number $c_{h,k} > 0$ such that for each $\varepsilon > 0$ and for each solution $u_1 \in \mathfrak{PC}(\mathfrak{J},\mathbb{R})$ of the inequality (2.2), there exists a solution $u_2 \in \mathfrak{PC}(\mathfrak{J},\mathbb{R})$ of the problem (1.1) with*

$$|u_1(x) - u_2(x)| \leq c_{h,k}\varepsilon,\ x \in \mathfrak{J}.$$

Definition 2.11. *The problem (1.1) is generalized Ulam-Hyers stable, if there exists $\theta_{h,k} \in \mathfrak{C}(\mathbb{R}_+,\mathbb{R}_+)$, $\theta_{h,k}(0) = 0$ such that for each solution $u_1 \in \mathfrak{PC}(\mathfrak{J},\mathbb{R})$ of the inequality (2.2), there exists a solution $u_2 \in \mathfrak{PC}(\mathfrak{J},\mathbb{R})$ of the problem (1.1) with*

$$|u_1(x) - u_2(x)| \leq \theta_{h,k}(\varepsilon),\ x \in \mathfrak{J}.$$

Definition 2.12. *The problem (1.1) is Ulam-Hyers-Rassias stable with respect to (α,ϕ), if there exists $c_{h,k,\alpha} > 0$ such that for each $\varepsilon > 0$ and for each solution $u_1 \in \mathfrak{PC}(\mathfrak{J},\mathbb{R})$ of the inequality (2.4), there exists a solution $u_2 \in \mathfrak{PC}(\mathfrak{J},\mathbb{R})$ of the problem (1.1) with*

$$|u_1(x) - u_2(x)| \leq c_{h,k,\alpha}\varepsilon(\alpha(x) + \phi),\ x \in \mathfrak{J}.$$

Definition 2.13. *The problem* (1.1) *is generalized Ulam-Hyers-Rassias stable with respect to* (α,ϕ)*, if there exists* $c_{h,k,\alpha}>0$ *such that for each solution* $u_1\in\mathfrak{PC}(\mathfrak{J},\mathbb{R})$ *of the inequality* (2.3)*, there exists a solution* $u_2\in\mathfrak{PC}(\mathfrak{J},\mathbb{R})$ *of the problem* (1.1) *with*

$$|u_1(x)-u_2(x)|\le c_{h,k,\alpha}(\alpha(x)+\phi),\ x\in\mathfrak{J}.$$

Remark 2.14. *From the above definitions, we*

(i) $Definition\ 2.10\Rightarrow Definition\ 2.11$;
(ii) $Definition\ 2.12\Rightarrow Definition\ 2.13$;
(iii) $Definition\ 2.12$ *for* $\alpha(x)=\phi=1\Rightarrow$ *Definition 2.10.*

Remark 2.15. *A function* $u\in\mathfrak{PC}(\mathfrak{J},\mathbb{R})$ *is a solution of the inequality* (2.4) *if and only if there is* $\sigma\in\mathfrak{PC}(\mathfrak{J},\mathbb{R})$ *and a sequence* σ_m*,* $m=1,2,\dots k$*(which depends on u) such that*

(i) $|\sigma(x)|\le\varepsilon\alpha(x)$, $x\in(x_m,x_{m+1}]$, $m=1,2,\dots k$ *and* $|\sigma_m|\le\varepsilon\phi$, $m=1,2,\dots k$;
(ii) ${}^{\rho}D^{\omega}u(x)=h(x,u_x,{}^{\rho}D^{\omega}u(x))+\sigma(x)$, $x\in(x_m,x_{m+1}]$, $m=1,2,\dots k$;
(iii) $\Delta u|_{x_m}=I_m(u_{x_m^-})+\sigma_m$, $m=1,2,\dots k$.

Similarly, we can get remarks for inequalities (2.2) and (2.3).

Theorem 2.16. *[8] (Ascoli-Arzela's Theorem) Let* $E\subset\mathfrak{C}(\mathfrak{J},\mathbb{R})$*, E is relatively compact(i.e,*$\bar{E}$ *is compact), if:*

(1) *E is uniformly bounded, that is there exists* $N>0$ *such that* $|h(x)|<N$*, for every* $h\in E$ *and* $x\in\mathfrak{J}$.
(2) *E is equicontinuous, that is for every* $\varepsilon>0$*, there exists* $\delta>0$ *such that for each* $x_1,x_2\in\mathfrak{J}$*,* $|x_1-x_2|\le\delta$ *implies* $|h(x_1)-h(x_2)|\le\varepsilon$*, for every* $h\in E$.

Theorem 2.17. *[30] (Banach's fixed point theorem) Let* $\mathscr{C}$ *be a non empty closed subset of a Banach space* $\mathscr{X}$*, then any contraction mapping T of* $\mathscr{C}$ *into itself has a unique fixed point.*

Theorem 2.18. *[30] (Schaefer's fixed point theorem) Let* $\mathscr{X}$ *be a Banach space, and* $M:\mathscr{X}\to\mathscr{X}$ *a completely continuous operator. If the set*

$$S=\{u\in\mathscr{X}:u=\mu Mu,\ \text{for some }\mu\in(0,1)\}$$

is bounded, then M has atleast one fixed points.

3. Existence of solutions

Definition 3.1. *A function* $u\in\mathfrak{PC}([-r,T],\mathbb{R})$ *whose* ω*-derivative exists on* $\mathfrak{J}_m$ *is said to be a solution of* (1.1)*, if u satisfies the equation*

$${}^{\rho}D^{\omega}_{x_m}u(x)=h(x,u_x,{}^{\rho}D^{\omega}_{x_m}u(x)),$$

on $\mathfrak{J}_m$*, and satisfies the conditions* $\Delta u|_{x=x_m}=I_m(u_{x_m^-})$*,* $m=1,\dots,k$ *and* $u(x)=\psi(x)$*,* $x\in[-r,0]$.

The following lemma is required to prove the existence of solutions to (1.1).

Lemma 3.2. *Let* $0<\omega\le1$ *and let* $\sigma:\mathfrak{J}\to\mathbb{R}$ *be continuous. A function u is a solution of the fractional integral equation*

$$u(x)=\begin{cases}\psi(0)+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_0^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}\ \sigma(s)\mathrm{d}s, & \text{if } x\in[0,x_1],\\[2ex] \psi(0)+\sum_{i=1}^m I_i(u_{x_i^-})\\ +\frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{i=1}^m\int_{x_{i-1}}^{x_i}(x_i^\rho-s^\rho)^{\omega-1}s^{\rho-1}\ \sigma(s)\mathrm{d}s\\ +\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}\ \sigma(s)\mathrm{d}s, & \text{if } x\in(x_m,x_{m+1}],\\[2ex] \psi(x), & x\in[-r,0],\end{cases}\tag{3.1}$$

where $m=1,2,\dots k$*, if and only if u is a solution of the following fractional problem*

$$\begin{cases}{}^{\rho}D^{\omega}u(x)=\sigma(x), & x\in\mathfrak{J}_m,\\ \Delta u|_{x=x_m}=I_m(u_{x_m^-}), & m=1,2,\dots k,\\ u(x)=\psi(x), & x\in[-r,0].\end{cases}\tag{3.2}$$

Proof. Assume u satisfies (3.2). If $x\in[0,x_1]$, then ${}^{\rho}D^{\omega}u(x)=\sigma(x)$. From Lemma 2.7, we get

$$u(x)=\psi(0)+{}^{\rho}I^{\omega}\sigma(x)=\psi(0)+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_0^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}\ \sigma(s)\mathrm{d}s.$$

If $x\in(x_1,x_2]$, then from Lemma 2.7,

$$\begin{aligned}u(x)&=u(x_1^+)+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_1}^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}\ \sigma(s)\mathrm{d}s\\ &=\Delta u|_{x=x_1}+u(x_1^-)+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_1}^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}\ \sigma(s)\mathrm{d}s\\ &=\psi(0)+I_1(u_{x_1^-})+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_0^x(x_1^\rho-s^\rho)^{\omega-1}s^{\rho-1}\ \sigma(s)\mathrm{d}s+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_1}^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}\ \sigma(s)\mathrm{d}s.\end{aligned}$$

If $x \in (x_2, x_3]$, then Lemma 2.7 implies,

$$\begin{aligned}
u(x) &= u(x_2^+) + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_2}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s) \mathrm{d}s \\
&= \Delta u|_{x=x_2} + u(x_2^-) + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_2}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s) \mathrm{d}s \\
&= I_2(u_{x_2^-}) \\
&+ \left[\psi(0) + I_1(u_{x_1^-}) + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{0}^{x_1} (x_1^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s) \mathrm{d}s + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_1}^{x_2} (x_2^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s) \mathrm{d}s\right] \\
&+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_2}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s) \mathrm{d}s. \\
&= \psi(0) + \left[I_1(u_{x_1^-}) + I_2(u_{x_2^-})\right] \\
&+ \left[\frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{0}^{x_1} (x_1^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s) \mathrm{d}s + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_1}^{x_2} (x_2^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s) \mathrm{d}s\right] \\
&+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_2}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s) \mathrm{d}s.
\end{aligned}$$

Continuing this process, we get the solution $u(x)$ for $x \in (x_m, x_{m+1}]$, where $m = 1, 2, \ldots k$. Hence,

$$u(x) = \psi(0) + \sum_{i=1}^{m} I_i(u_{x_i^-}) + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{i=1}^{m} \int_{x_{i-1}}^{x_i} (x_i^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s) \mathrm{d}s + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s) \mathrm{d}s.$$

Conversely, let us assume that u satisfies the equation (3.1). If $x \in [0, x_1]$, then $u(0) = \psi(0)$ and using the concept that ${}^\rho D^\omega$ is the left inverse of ${}^\rho I^\omega$, we get ${}^\rho D^\omega u(x) = \sigma(x)$, for each $x \in [0, x_1]$. If $x \in (x_m, x_{m+1}]$, $m = 1, 2, \ldots k$ and using the fact that ${}^\rho D^\omega L = 0$, where L is a constant, we get

$$ {}^\rho D^\omega u(x) = \sigma(x), \text{ for each } x \in (x_m, x_{m+1}].$$

Also, we can show that $\Delta u|_{x=x_m} = I_m(u_{x_m^-})$, $m = 1, 2, \ldots k$. □

Now we state and prove the existence results for the problem (1.1), based on Banach's fixed point theorem.

Theorem 3.3. *Assume that*

(A1) $h : \mathfrak{J} \times \mathfrak{PC}([-r, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ *is continuous.*
(A2) There exist constants $c_1 > 0$ *and* $0 < c_2 < 1$ *such that*

$$|h(x, z_1, z_2) - h(x, \bar{z_1}, \bar{z_2})| \le c_1 \|z_1 - \bar{z_1}\|_{\mathfrak{PC}} + c_2 |z_2 - \bar{z_2}|,$$

for any $z_1, \bar{z_1} \in \mathfrak{PC}([-r, 0], \mathbb{R})$, $z_2, \bar{z_2} \in \mathbb{R}$ *and* $x \in \mathfrak{J}$.
(A3) There exists a constant $c_3 > 0$ *such that*

$$|I_m(z_1) - I_m(\bar{z_1})| \le c_3 \|z_1 - \bar{z_1}\|_{\mathfrak{PC}},$$

for each $z_1, \bar{z_1} \in \mathfrak{PC}([-r, 0], \mathbb{R})$ *and* $m = 1, 2, \ldots k$.

If

$$k c_3 + \frac{(k+1) c_1 T^{\rho\omega}}{(1 - c_2) \rho^\omega \Gamma(\omega + 1)} < 1, \tag{3.3}$$

then there exists a unique solution for the problem (1.1) *on* $\mathfrak{J}$.

Proof. Transform the problem (1.1) into a fixed point problem. Consider the operator
$M : \mathfrak{PC}([-r, T], \mathbb{R}) \to \mathfrak{PC}([-r, T], \mathbb{R})$ defined by

$$Mu(x) = \begin{cases} \psi(0) + \sum_{0 < x_m < x} I_m(u_{x_i^-}) \\ + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{0 < x_m < x} \int_{x_{m-1}}^{x_m} (x_m^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, g(s) \mathrm{d}s \\ + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, g(s) \mathrm{d}s, & x \in [0, T], \\ \\ \psi(x), & x \in [-r, 0], \end{cases} \tag{3.4}$$

where $g \in \mathfrak{C}(\mathfrak{J}, \mathbb{R})$ be such that

$$g(x) = h(x, u_x, g(x)).$$

Clearly, the fixed points of operator M are solutions of the problem (1.1). Let $y, z \in \mathfrak{PC}([-r, T], \mathbb{R})$. If $x \in [-r, 0]$, then

$$|M(y)(x) - M(z)(x)| = 0.$$

For $x \in \mathfrak{J}$, we get

$$\begin{aligned}|M(y)(x)-M(z)(x)| &\leq \frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}\left(x_m^\rho-s^\rho\right)^{\omega-1}s^{\rho-1}\,|g_1(s)-g_2(s)|\,\mathrm{d}s\\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}\left(x^\rho-s^\rho\right)^{\omega-1}s^{\rho-1}\,|g_1(s)-g_2(s)|\,\mathrm{d}s+\sum_{0<x_m<x}\left|I_m(y_{x_m^-})-I_m(z_{x_m^-})\right|,\end{aligned}$$

where $g_1, g_2 \in \mathfrak{C}(\mathfrak{J},\mathbb{R})$ be such that

$$g_1(x) = h(x, y_x, g_1(x)),$$

and

$$g_2(x) = h(x, z_x, g_2(x)).$$

By $(A2)$, we get

$$|g_1(x)-g_2(x)| = |h(x,y_x,g_1(x))-h(x,z_x,g_2(x))| \leq c_1\|y_x-z_x\|_{\mathfrak{PC}}+c_2\,|g_1(x)-g_2(x)|\,.$$

This implies,

$$|g_1(x)-g_2(x)| \leq \frac{c_1}{1-c_2}\|y_x-z_x\|_{\mathfrak{PC}}.$$

Therefore, for each $x \in \mathfrak{J}$,

$$\begin{aligned}|M(y)(x)-M(z)(x)| &\leq \frac{c_1\rho^{1-\omega}}{(1-c_2)\Gamma(\omega)}\sum_{m=1}^{k}\int_{x_{m-1}}^{x_m}\left(x_m^\rho-s^\rho\right)^{\omega-1}s^{\rho-1}\,\|y_s-z_s\|_{\mathfrak{PC}}\mathrm{d}s\\ &+ \frac{c_1\rho^{1-\omega}}{(1-c_2)\Gamma(\omega)}\int_{x_m}^{x}\left(x^\rho-s^\rho\right)^{\omega-1}s^{\rho-1}\,\|y_s-z_s\|_{\mathfrak{PC}}\mathrm{d}s+\sum_{m=1}^{k}c_3\left\|y_{x_m^-}-z_{x_m^-}\right\|_{\mathfrak{PC}}.\\ &\leq \left[kc_3+\frac{kc_1T^{\rho\omega}}{(1-c_2)\rho^\omega\Gamma(\omega+1)}+\frac{c_1T^{\rho\omega}}{(1-c_2)\rho^\omega\Gamma(\omega+1)}\right]\|y-z\|_{\mathfrak{PC}_1}.\end{aligned}$$

Thus,

$$\|M(y)-M(z)\|_{\mathfrak{PC}_1} \leq \left[kc_3+\frac{(k+1)c_1T^{\rho\omega}}{(1-c_2)\rho^\omega\Gamma(\omega+1)}\right]\|y-z\|_{\mathfrak{PC}_1}.$$

By (3.3), the operator M is a contraction. Therefore, by the Banach's contraction principle, M has a unique fixed point which is a unique solution of the problem (1.1). □

Now, Schaefer's fixed point theorem is used to prove the second result.

Theorem 3.4. *Assume that* $(A1),(A2)$ *and*

(A4) There exist $p_1, p_2, p_3 \in \mathfrak{C}(\mathfrak{J},\mathbb{R}_+)$ *with* $p_3^* = \sup_{x\in\mathfrak{J}} p_3(x) < 1$ *such that*

$$|h(x,y,z)| \leq p_1(x)+p_2(x)\,\|y\|_{PC}+p_3(x)\,|z|\,,$$

where $x \in \mathfrak{J}$, $y \in \mathfrak{PC}([-r,0],\mathbb{R})$ *and* $z \in \mathbb{R}$.

(A5) The functions $I_m : \mathfrak{PC}([-r,0],\mathbb{R}) \to \mathbb{R}$ *are continuous and there exist constants* $M_1^*, M_2^* > 0$ *with* $kM_1^* < 1$ *such that*

$$|I_m(y)| \leq M_1^*\,\|y\|_{\mathfrak{PC}}+M_2^*,$$

for each $y \in \mathfrak{PC}([-r,0],\mathbb{R})$, $m = 1,2,\ldots k$. *Then the problem* (1.1) *has at least one solution.*

Proof. Let the operator M defined in (3.4). Now we shall prove that M has atleast one fixed point by using Schaefer's fixed point theorem. The proof contains four steps.

Step 1: *N is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $\mathfrak{PC}([-r,T],\mathbb{R})$. If $x \in [-r,0]$, then

$$|M(y_n)(x)-M(y)(x)| = 0.$$

For $x \in \mathfrak{J}$, we have

$$\begin{aligned}|M(y_n)(x)-M(y)(x)| &\leq \frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}\left(x_m^\rho-s^\rho\right)^{\omega-1}s^{\rho-1}\,|g_n(s)-g(s)|\,\mathrm{d}s\\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}\left(x^\rho-s^\rho\right)^{\omega-1}s^{\rho-1}\,|g_n(s)-g(s)|\,\mathrm{d}s+\sum_{0<x_m<x}\left|I_m(y_{nx_m^-})-I_m(y_{x_m^-})\right|,\end{aligned}\tag{3.5}$$

where $g_n, g \in \mathfrak{C}(\mathfrak{J},\mathbb{R})$ such that

$$g_n(x) = h(x, y_{nx}, g_n(x)),$$

$$g(x) = h(x, y_x, g(x)).$$

By $(A2)$, we have

$$\begin{aligned} |g_n(x) - g(x)| &= |h(x, y_{nx}, g_n(x)) - h(x, y_x, g(x))| \\ &\leq c_1 \|y_{nx} - y_x\|_{\mathfrak{PC}} + c_2 |g_n(x) - g(x)|. \end{aligned}$$

Then,

$$|g_n(x) - g(x)| \leq \left(\frac{c_1}{1 - c_2}\right) \|y_{nx} - y_x\|_{\mathfrak{PC}}.$$

Since $y_n \to y$, then we get $g_n(x) \to g(x)$ as $n \to \infty$ for each $x \in \mathfrak{J}$. And let $\Omega > 0$ be such that, for each $x \in \mathfrak{J}$, we have $|g_n(x)| \leq \Omega$ and $|g(x)| \leq \Omega$. Then, we have

$$\begin{aligned} (x^\rho - s^\rho)^{\omega-1} |g_n(s) - g(s)| &\leq (x^\rho - s^\rho)^{\omega-1} [|g_n(s)| + |g(s)|] \\ &\leq 2\Omega (x^\rho - s^\rho)^{\omega-1}, \end{aligned}$$

and

$$\begin{aligned} (x_m^\rho - s^\rho)^{\omega-1} |g_n(s) - g(s)| &\leq (x_m^\rho - s^\rho)^{\omega-1} [|g_n(s)| + |g(s)|] \\ &\leq 2\Omega (x_m^\rho - s^\rho)^{\omega-1}. \end{aligned}$$

For each $x \in \mathfrak{J}$, the functions $s \to 2\Omega(x^\rho - s^\rho)^{\omega-1}$ and $s \to 2\Omega(x_m^\rho - s^\rho)^{\omega-1}$ are integrable on $[0, x]$, then by the Lebesgue Dominated Convergence Theorem and (3.5) implies that

$$|M(y_n)(x) - M(y)(x)| \to 0, \text{ as } n \to \infty.$$

and hence,

$$\|M(y_n) - M(y)\|_{\mathfrak{PC}_1} \to 0, \text{ as } n \to \infty,$$

Consequently, M is continuous.

Step 2: *M maps bounded sets into bounded sets in $\mathfrak{PC}([-r, T], \mathbb{R})$.* To prove this, it is enough to show that for any $\Omega^* > 0$, there exists a positive constant $\tilde{k}$ such that for each $y \in B_{\Omega^*} = \{y \in \mathfrak{PC}([-r, T], \mathbb{R}) : \|y\|_{\mathfrak{PC}_1} \leq \Omega^*\}$, we have $\|M(y)\|_{\mathfrak{PC}_1} \leq \tilde{k}$. We have for each $x \in \mathfrak{J}$,

$$\begin{aligned} M(y)(x) &= \psi(0) + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{0 < x_m < x} \int_{x_{m-1}}^{x_m} (x_m^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, g(s) \mathrm{d}s \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, g(s) \mathrm{d}s + \sum_{0 < x_m < x} I_m(y_{x_m^-}), \end{aligned} \tag{3.6}$$

where $g \in \mathfrak{C}(\mathfrak{J}, \mathbb{R})$ be such that

$$g(x) = h(x, y_x, g(x)).$$

By $(A4)$, for each $x \in \mathfrak{J}$, we get

$$\begin{aligned} |g(x)| &= |h(x, y_x, g(x))| \\ &\leq p_1(x) + p_2(x) \|y_x\|_{\mathfrak{PC}} + p_3(x) |g(x)| \\ &\leq p_1(x) + p_2(x) \|y\|_{\mathfrak{PC}_1} + p_3(x) |g(x)| \\ &\leq p_1(x) + p_2(x) \Omega^* + p_3(x) |g(x)| \\ &\leq p_1^* + p_2^* \Omega^* + p_3^* |g(x)|, \end{aligned}$$

where $p_1^* = \sup_{x \in \mathfrak{J}} p_1(x)$ and $p_2^* = \sup_{x \in \mathfrak{J}} p_2(x)$. Then,

$$|g(x)| \leq \frac{p_1^* + p_2^* \Omega^*}{1 - p_3^*} := N.$$

Thus (3.6) implies

$$\begin{aligned} |M(y)(x)| &\leq |\psi(0)| + \frac{kNT^{\rho\omega}}{\rho^\omega \Gamma(\omega + 1)} + \frac{NT^{\rho\omega}}{\rho^\omega \Gamma(\omega + 1)} + k\left(M_1^* \|y_{x_m^-}\|_{\mathfrak{PC}} + M_2^*\right) \\ &\leq |\psi(0)| + \frac{(k+1)NT^{\rho\omega}}{\rho^\omega \Gamma(\omega + 1)} + k\left(M_1^* \|y\|_{\mathfrak{PC}_1} + M_2^*\right) \\ &\leq |\psi(0)| + \frac{(k+1)NT^{\rho\omega}}{\rho^\omega \Gamma(\omega + 1)} + k(M_1^* \Omega^* + M_2^*) := \tilde{R}. \end{aligned}$$

And if $x \in [-r, 0]$, then

$$|M(y)(x)| \leq \|\psi\|_{\mathfrak{PC}},$$

thus

$$\|M(y)\|_{\mathfrak{PC}_1} \leq \max\{\tilde{R}, \|\psi\|_{\mathfrak{PC}}\} := \tilde{k}.$$

Step 3: *M maps bounded sets into equicontinuous sets of* $\mathfrak{PC}([-r,T],\mathbb{R})$.
Let $t_1, t_2 \in (0,T]$, $t_1 < t_2$, B_{Ω^*} be a bounded set of $\mathfrak{PC}([-r,T],\mathbb{R})$ as in Step 2, and let $y \in B_{\Omega^*}$. Then

$$\begin{aligned}
|M(y)(t_2) - M(y)(t_1)| &\leq \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_0^{t_1} \left|(t_2^\rho - s^\rho)^{\omega-1} - (t_1^\rho - s^\rho)^{\omega-1}\right| \left|s^{\rho-1}\right| |g(s)|\,\mathrm{d}s \\
&+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{t_1}^{t_2} \left|(t_2^\rho - s^\rho)^{\omega-1}\right| \left|s^{\rho-1}\right| |g(s)|\,\mathrm{d}s \\
&+ \sum_{0<x_m<t_2-t_1} \left|I_m(y_{x_m^-})\right| \\
&\leq \frac{N}{\rho^\omega \Gamma(\omega+1)} [2(t_2^\rho - t_1^\rho)^\omega + (t_2^{\rho\omega} - t_1^{\rho\omega})] \\
&+ (t_2^\rho - t_1^\rho)\left(M_1^* \|y_{x_m^-}\|_{\mathfrak{PC}} + M_2^*\right) \\
&\leq \frac{N}{\rho^\omega \Gamma(\omega+1)} [2(t_2^\rho - t_1^\rho)^\omega + (t_2^{\rho\omega} - t_1^{\rho\omega})] \\
&+ (t_2^\rho - t_1^\rho)\left(M_1^* \|y\|_{\mathfrak{PC}_1} + M_2^*\right) \\
&\leq \frac{N}{\rho^\omega \Gamma(\omega+1)} [2(t_2^\rho - t_1^\rho)^\omega + (t_2^{\rho\omega} - t_1^{\rho\omega})] \\
&+ (t_2^\rho - t_1^\rho)(M_1^* \Omega^* + M_2^*).
\end{aligned}$$

As $t_2 \to t_1$, the right hand side of the above inequality tends to zero. From Step 1 to 3 together with the Ascoli-Arzela theorem, we can conclude that $M : \mathfrak{PC}([-r,T],\mathbb{R}) \to \mathfrak{PC}([-r,T],\mathbb{R})$ is completely continuous.
Step 4: *A priori bounds*. Now, we shall show that the set

$$G = \{y \in \mathfrak{PC}([-r,T],\mathbb{R}) : y = \mu M(y), \text{ for some } 0 < \mu < 1\},$$

is bounded. Let $y \in G$, then $y = \mu M(y)$, for some $0 < \mu < 1$. Thus, for each $x \in \mathfrak{J}$, we get

$$\begin{aligned}
y(x) &= \mu\psi(0) + \frac{\mu\rho^{1-\omega}}{\Gamma(\omega)} \sum_{0<x_m<x} \int_{x_{m-1}}^{x_m} (x_m^\rho - s^\rho)^{\omega-1} s^{\rho-1} g(s)\,\mathrm{d}s \\
&+ \frac{\mu\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} g(s)\,\mathrm{d}s + \mu \sum_{0<x_m<x} I_m(y_{x_m^-}).
\end{aligned}$$

And by $(A4)$, for each $x \in \mathfrak{J}$, we get,

$$\begin{aligned}
|g(x)| &= |h(x, y_x, g(x))| \\
&\leq p_1(x) + p_2(x)\|y_x\|_{\mathfrak{PC}} + p_3(x)|g(x)| \\
&\leq p_1^* + p_2^*\|y_x\|_{\mathfrak{PC}} + p_3^*|g(x)|.
\end{aligned}$$

Thus,

$$|g(x)| \leq \frac{1}{1-p_3^*}\left(p_1^* + p_2^*\|y_x\|_{\mathfrak{PC}}\right).$$

This implies, by (3.7) and $(A5)$, that for each $x \in \mathfrak{J}$, we have

$$\begin{aligned}
|y(x)| &\leq |\psi(0)| + \frac{\rho^{1-\omega}}{(1-p_3^*)\Gamma(\omega)} \sum_{0<x_m<x} \int_{x_{m-1}}^{x_m} (x_m^\rho - s^\rho)^{\omega-1} s^{\rho-1} (p_1^* + p_2^*\|y_s\|_{\mathfrak{PC}})\mathrm{d}s \\
&+ \frac{\rho^{1-\omega}}{(1-p_3^*)\Gamma(\omega)} \int_{x_m}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} (p_1^* + p_2^*\|y_s\|_{\mathfrak{PC}})\mathrm{d}s \\
&+ k\left(M_1^*\|y_{x_m^-}\|_{\mathfrak{PC}} + M_2^*\right).
\end{aligned}$$

Now, we consider the function q defined by

$$q(x) = \sup\{|q(s)| : -r \leq s \leq x\},\ 0 \leq x \leq T,$$

then there exists $x^* \in [-r,T]$ such that $q(x) = |y(x^*)|$. If $x^* \in [0,T]$, then by the previous inequality, we have for $x \in \mathfrak{J}$,

$$\begin{aligned}
q(x) &\leq |\psi(0)| + \frac{\rho^{1-\omega}}{(1-p_3^*)\Gamma(\omega)} \sum_{0<x_m<x} \int_{x_{m-1}}^{x_m} (x_m^\rho - s^\rho)^{\omega-1} s^{\rho-1} (p_1^* + p_2^* q(s))\mathrm{d}s \\
&+ \frac{\rho^{1-\omega}}{(1-p_3^*)\Gamma(\omega)} \int_{x_m}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} (p_1^* + p_2^* q(s))\mathrm{d}s \\
&+ k(M_1^* q(x) + M_2^*).
\end{aligned}$$

Thus,

$$\begin{aligned}
q(x) &\leq \frac{|\psi(0)|+kM_2^*}{1-kM_1^*}+\frac{\rho^{1-\omega}}{(1-kM_1^*)(1-p_3^*)\Gamma(\omega)}\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}(x_m^\rho-s^\rho)^{\omega-1}s^{\rho-1}(p_1^*+p_2^*q(s))ds\\
&+\frac{\rho^{1-\omega}}{(1-kM_1^*)(1-p_3^*)\Gamma(\omega)}\int_{x_m}^{x}(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}(p_1^*+p_2^*q(s))ds\\
&\leq \frac{|\psi(0)|+kM_2^*}{1-kM_1^*}+\frac{(k+1)p_1^*T^{\rho\omega}}{(1-kM_1^*)(1-p_3^*)\rho^\omega\Gamma(\omega+1)}\\
&+\frac{(k+1)p_2^*}{(1-kM_1^*)(1-p_3^*)\Gamma(\omega)}\int_{0}^{x}(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}q(s)ds.
\end{aligned}$$

Applying Lemma 2.8, we get

$$q(x)\leq\left[\frac{|\psi(0)|+kM_2^*}{1-kM_1^*}+\frac{(k+1)p_1^*T^{\rho\omega}}{(1-kM_1^*)(1-p_3^*)\rho^\omega\Gamma(\omega+1)}\right]\times\left[1+\frac{\lambda(k+1)p_2^*T^{\rho\omega}}{(1-kM_1^*)(1-p_3^*)\rho^\omega\Gamma(\omega+1)}\right]:=\tilde{A},$$

where $\lambda=\lambda(\omega)$ a constant. If $x^*\in[-r,0]$, then $q(x)=\|\psi\|_{\mathfrak{PC}}$, thus for any $x\in[-r,T]$, $\|y\|_{\mathfrak{PC}_1}\leq q(x)$, we get

$$\|y\|_{\mathfrak{PC}_1}\leq\max\{\|\psi\|_{\mathfrak{PC}},\tilde{A}\},$$

which implies the set G is bounded. From Schaefer's fixed point theorem, we conclude that M has atleast one fixed point which is a solution of the problem (1.1). □

4. Ulam-Hyers-Rassias stability

Now, we present the following Ulam-Hyers-Rassias stable result.

Theorem 4.1. *Assume that* $(A1)$-$(A3)$, (3.3) *and*

(A6) There exists a nondecreasing function $\alpha\in\mathfrak{PC}(\mathfrak{J},\mathbb{R}_+)$ *and there exists* $\mu_\alpha>0$ *such that for any* $x\in\mathfrak{J}$:

$${}^\rho I^\omega\alpha(x)\leq\mu_\alpha\alpha(x),$$

are satisfied, then the problem (1.1) *is Ulam-Hyers-Rassias stable with respect to* (α,ϕ).

Proof. Let $v\in\mathfrak{PC}([-r,T],\mathbb{R})$ be a solution of the inequality (2.4). Denote by u the unique solution of the problem

$$\begin{cases}{}^\rho D_{x_m}^\omega u(x)=h(x,u_x,{}^\rho D_{x_m}^\omega u(x)), & \text{for each } x\in(x_m,x_{m+1}],\ m=1,\dots k;\\ \Delta u|_{x=x_m}=I_m(u_{x_m^-}), & m=1,\dots,k;\\ u(x)=v(x)=\psi(x), & x\in[-r,0],\end{cases}$$

using Lemma 3.2, we obtain for each $x\in(x_m,x_{m+1}]$,

$$\begin{aligned}
u(x) &= \psi(0)+\sum_{i=1}^{m}I_i(u_{x_i^-})+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{i=1}^{m}\int_{x_{i-1}}^{x_i}(x_i^\rho-s^\rho)^{\omega-1}s^{\rho-1}g(s)ds\\
&+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}g(s)ds,\ x\in(x_m,x_{m+1}],
\end{aligned}$$

where $g\in\mathfrak{C}(\mathfrak{J},\mathbb{R})$ be such that

$$g(x)=h(x,u_x,g(x)).$$

Since v is a solution of the inequality (2.4) and by Remark 2.15, we get

$$\begin{cases}{}^\rho D_{x_m}^\omega v(x)=h(x,v_x,{}^\rho D_{x_m}^\omega v(x))+\sigma(x), & x\in(x_m,x_{m+1}],\ m=1,\dots k;\\ \Delta v|_{x=x_m}=I_m(v_{x_m^-})+\sigma_m, & m=1,\dots,k.\end{cases}\tag{4.1}$$

Clearly the solution of (4.1) is given by,

$$\begin{aligned}
v(x) &= \psi(0)+\sum_{i=1}^{m}I_i(v_{x_i^-})+\sum_{i=1}^{m}\sigma_i\\
&+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{i=1}^{m}\int_{x_{i-1}}^{x_i}(x_i^\rho-s^\rho)^{\omega-1}s^{\rho-1}f(s)ds+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{i=1}^{m}\int_{x_{i-1}}^{x_i}(x_i^\rho-s^\rho)^{\omega-1}s^{\rho-1}\sigma(s)ds\\
&+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}f(s)ds+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}\sigma(s)ds,\ x\in(x_m,x_{m+1}],
\end{aligned}$$

where $f \in \mathfrak{C}(\mathfrak{J},\mathbb{R})$ be such that $f(x) = h(x, v_x, f(x))$. Hence for each $x \in (x_m, x_{m+1}]$, we get,

$$\begin{aligned} |v(x)-u(x)| &\leq \sum_{i=1}^{m}|\sigma_i| + \sum_{i=1}^{m}\left|I_i(v_{x_i^-}) - I_i(u_{x_i^-})\right| \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{i=1}^{m}\int_{x_{i-1}}^{x_i}(x_i^\rho - s^\rho)^{\omega-1}s^{\rho-1}|f(s)-g(s)|\,\mathrm{d}s \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{i=1}^{m}\int_{x_{i-1}}^{x_i}(x_i^\rho - s^\rho)^{\omega-1}s^{\rho-1}|\sigma(s)|\,\mathrm{d}s \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^\rho - s^\rho)^{\omega-1}s^{\rho-1}|f(s)-g(s)|\,\mathrm{d}s \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^\rho - s^\rho)^{\omega-1}s^{\rho-1}|\sigma(s)|\,\mathrm{d}s. \end{aligned}$$

Thus,

$$\begin{aligned} |v(x)-u(x)| &\leq k\varepsilon\phi + (k+1)\varepsilon\mu_\alpha\alpha(x) + \sum_{i=1}^{m}c_3\left\|v_{x_i^-} - u_{x_i^-}\right\|_{\mathfrak{PC}} \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{i=1}^{m}\int_{x_{i-1}}^{x_i}(x_i^\rho - s^\rho)^{\omega-1}s^{\rho-1}|f(s)-g(s)|\,\mathrm{d}s \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^\rho - s^\rho)^{\omega-1}s^{\rho-1}|f(s)-g(s)|\,\mathrm{d}s. \end{aligned}$$

By $(A2)$, we get

$$\begin{aligned} |f(x)-g(x)| &= |h(x,v_x,f(x)) - h(x,u_x,g(x))| \\ &\leq c_1\|v_x - u_x\|_{\mathfrak{PC}} + c_2|f(x)-g(x)|. \end{aligned}$$

Then,

$$|f(x)-g(x)| \leq \frac{c_1}{1-c_2}\|v_x - u_x\|_{\mathfrak{PC}}.$$

Therefore, for each $x \in \mathfrak{J}$,

$$\begin{aligned} |v(x)-u(x)| &\leq k\varepsilon\phi + (k+1)\varepsilon\mu_\alpha\alpha(x) + \sum_{i=1}^{m}c_3\left\|v_{x_i^-} - u_{x_i^-}\right\|_{\mathfrak{PC}} \\ &+ \frac{c_1\rho^{1-\omega}}{(1-c_2)\Gamma(\omega)}\sum_{i=1}^{m}\int_{x_{i-1}}^{x_i}(x_i^\rho - s^\rho)^{\omega-1}s^{\rho-1}\|v_s-u_s\|_{\mathfrak{PC}}\,\mathrm{d}s \\ &+ \frac{c_1\rho^{1-\omega}}{(1-c_2)\Gamma(\omega)}\int_{x_m}^{x}(x^\rho - s^\rho)^{\omega-1}s^{\rho-1}\|v_s-u_s\|_{\mathfrak{PC}}\,\mathrm{d}s. \end{aligned}$$

Thus,

$$\begin{aligned} |v(x)-u(x)| &\leq \varepsilon(\phi+\alpha(x))(k+(k+1)\mu_\alpha) + \sum_{0<x_i<x}c_3\left\|v_{x_i^-} - u_{x_i^-}\right\|_{\mathfrak{PC}} \\ &+ \frac{c_1(k+1)\rho^{1-\omega}}{(1-c_2)\Gamma(\omega)}\int_{0}^{x}(x^\rho - s^\rho)^{\omega-1}s^{\rho-1}\|v_s-u_s\|_{\mathfrak{PC}}\,\mathrm{d}s. \end{aligned} \tag{4.2}$$

Now, we consider the function q_1 defined by

$$q_1(x) = \sup\{|v(s)-u(s)| : -r \leq s \leq x\},\ 0 \leq x \leq T,$$

then, there exists $x^* \in [-r,T]$ such that $q_1(x) = |v(x^*) - u(x^*)|$. If $x^* \in [-r,0]$, then $q_1(x) = 0$. If $x^* \in [0,T]$, then by the equation (4.2), we get

$$\begin{aligned} q_1(x) &\leq \varepsilon(\phi+\alpha(x))(k+(k+1)\mu_\alpha) + \sum_{0<x_i<x}c_3q_1(x_i^-) \\ &+ \frac{c_1(k+1)\rho^{1-\omega}}{(1-c_2)\Gamma(\omega)}\int_{0}^{x}(x^\rho - s^\rho)^{\omega-1}s^{\rho-1}q_1(s)\mathrm{d}s. \end{aligned}$$

Applying Lemma 2.9, we have,

$$\begin{aligned} q_1(x) &\leq \varepsilon(\phi+\alpha(x))(k+(k+1)\mu_\alpha) \times \left[\prod_{0<x_i<x}(1+c_3)\exp\left(\int_0^x \frac{c_1(k+1)\rho^{1-\omega}}{(1-c_2)\Gamma(\omega)}(x^\rho - s^\rho)^{\omega-1}s^{\rho-1}\mathrm{d}s\right)\right] \\ &\leq l_\alpha\varepsilon(\phi+\alpha(x)), \end{aligned}$$

where

$$\begin{aligned} l_\alpha &= (k+(k+1)\mu_\alpha)\times\left[\prod_{i=1}^{k}(1+c_3)\exp\left(\frac{c_1(k+1)T^{\rho\omega}}{(1-c_2)\rho^\omega\Gamma(\omega+1)}\right)\right] \\ &= (k+(k+1)\mu_\alpha)\left[(1+c_3)\exp\left(\frac{c_1(k+1)T^{\rho\omega}}{(1-c_2)\rho^\omega\Gamma(\omega+1)}\right)\right]^k. \end{aligned}$$

Thus, the problem (1.1) is Ulam-Hyers-Rassias stable with respect to (α,ϕ). Hence the proof is complete. □

Now, we present the following Ulam-Hyers stable result.

Theorem 4.2. *Assume that* $(A1)$-$(A3)$ *and* (3.3) *are satisfied, then the problem* (1.1) *is Ulam-Hyers stable.*

Proof. Let $v\in\mathfrak{PC}([-r,T],\mathbb{R})$ be a solution of (2.2). Denote by u the unique solution of the problem.

$$\begin{cases} {}^\rho D^\omega_{x_m}u(x)=h(x,u_x,{}^\rho D^\omega_{x_m}u(x)), & x\in(x_m,x_{m+1}],\ m=1,\dots k; \\ \Delta u|_{x=x_m}=I_m(u_{x_m^-}), & m=1,\dots,k; \\ u(x)=v(x)=\psi(x), & x\in[-r,0]. \end{cases}$$

From the proof of the Theorem 4.1, we get

$$q_1(x) \le \sum_{0<x_i<x} c_3 q_1(x_i^-)+k\varepsilon+\frac{\varepsilon(k+1)T^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)}+\frac{c_1(k+1)\rho^{1-\omega}}{(1-c_2)\Gamma(\omega+1)}\int_0^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}q_1(s)\mathrm{d}s.$$

Applying Lemma 2.9, we have

$$\begin{aligned} q_1(x) &\le \varepsilon\left(\frac{k\rho^\omega\Gamma(\omega+1)+(k+1)T^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)}\right)\times\left[\prod_{0<x_i<x}(1+c_3)\exp\left(\int_0^x\frac{c_1(k+1)\rho^{1-\omega}(x^\rho-s^\rho)^{\omega-1}}{(1-c_2)\Gamma(\omega)}s^{\rho-1}\mathrm{d}s\right)\right] \\ &\le l_\alpha\varepsilon, \end{aligned}$$

where,

$$\begin{aligned} l_\alpha &= \left(\frac{k\rho^\omega\Gamma(\omega+1)+(k+1)T^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)}\right)\left[\prod_{i=1}^{k}(1+c_3)\exp\left(\frac{c_1(k+1)T^{\rho\omega}}{(1-c_2)\rho^\omega\Gamma(\omega+1)}\right)\right] \\ &= \left(\frac{k\rho^\omega\Gamma(\omega+1)+(k+1)T^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)}\right)\left[(1+c_3)\exp\left(\frac{c_1(k+1)T^{\rho\omega}}{(1-c_2)\rho^\omega\Gamma(\omega+1)}\right)\right]^k, \end{aligned}$$

which completes the proof of the theorem. □

Moreover, if we set $\gamma(\varepsilon)=l_\alpha\varepsilon$; $\gamma(0)=0$, then the problem (1.1) is generalized Ulam-Hyers stable.

5. Examples

Example 5.1. *Consider the following Katugampola-type impulsive problem,*

$$\begin{cases} {}^\rho D^{\frac{1}{2}}_{x_m}u(x)=\frac{e^{-x}}{(22+e^x)}\left[\frac{u_x}{1+u_x}-\frac{{}^\rho D^{\frac{1}{2}}_{x_m}u(x)}{1+{}^\rho D^{\frac{1}{2}}_{x_m}u(x)}\right], & \text{for each } x\in\mathfrak{J}_0\cup\mathfrak{J}_1, \\ \Delta u|_{x=\frac{1}{2}}=\frac{u(\frac{1}{2}^-)}{20+u(\frac{1}{2}^-)}, & \\ u(x)=\psi(x), & x\in[-r,0],\ r>0, \end{cases} \tag{5.1}$$

where $\psi\in\mathfrak{PC}([-r,0],\mathbb{R})$, $\mathfrak{J}_0=[0,\frac{1}{2}]$, $\mathfrak{J}_1=(\frac{1}{2},1]$, $x_0=0$, *and* $x_1=\frac{1}{2}$.

Let

$$h(x,u_1,u_2)=\frac{e^{-x}}{(22+e^x)}\left[\frac{u_1}{1+u_1}-\frac{u_2}{1+u_2}\right],$$

$x\in[0,1]$, $u_1\in\mathfrak{PC}([-r,0],\mathbb{R})$ and $u_2\in\mathbb{R}$. Clearly, the function h is jointly continuous.
Let $u_1,\bar{u_1}\in\mathfrak{PC}([-r,0],\mathbb{R})$, $u_2,\bar{u_2}\in\mathbb{R}$ and $x\in[0,1]$:

$$\begin{aligned} |h(x,u_1,u_2)-h(x,\bar{u_1},\bar{u_2})| &\le \frac{e^{-x}}{22+e^x}\left(\|u_1-\bar{u_1}\|_{\mathfrak{PC}}+|u_2-\bar{u_2}|\right) \\ &\le \frac{1}{23}\left(\|u_1-\bar{u_1}\|_{\mathfrak{PC}}+|u_2-\bar{u_2}|\right). \end{aligned}$$

Hence the condition $(A2)$ is satisfied with $c_1=c_2=\frac{1}{23}$. And let,

$$I_1(u_1)=\frac{u_1}{20+u_1},\ u_1\in\mathfrak{PC}([-r,0],\mathbb{R})$$

Let $u_1, u_2 \in \mathfrak{PC}([-r,0],\mathbb{R})$, then we have,

$$\begin{aligned} |I_1(u_1) - I_2(u_2)| &= \left| \frac{u_1}{20+u_1} - \frac{u_2}{20+u_2} \right| \\ &\leq \frac{1}{20} \|u_1 - u_2\|_{\mathfrak{PC}}. \end{aligned}$$

Let us assume $k = 1, T = 1, \rho = 1, \omega = \frac{1}{2}, c_1 = c_2 = \frac{1}{23}, c_3 = \frac{1}{20}$, Substitute these values in the inequality (3.3), we get

$$kc_3 + \frac{(k+1)c_1 T^{\rho\omega}}{(1-c_2)\rho^{\omega}\Gamma(\omega+1)} = 0.2551 < 1,$$

It follows from Theorem 3.3, we get that the problem (5.1) has a unique solution on $\mathfrak{J}$. Now, we consider for any $x \in [0,1]$, $\alpha(x) = x$, $\phi = 1, \rho = 1$. Since

$$\begin{aligned} {}^{\rho}I^{\omega}\alpha(x) &= \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_0^x (x^{\rho} - s^{\rho})^{\omega-1} s^{\rho-1} \mathrm{d}s \\ &= \frac{2}{\pi} \int_0^x (x^{\rho} - s^{\rho})^{\frac{-1}{2}} \mathrm{d}s \\ &\leq \frac{2x}{\sqrt{\pi}}, \end{aligned}$$

then the condition $(A6)$ is satisfied with $\mu_{\alpha} = \frac{2}{\sqrt{\pi}}$. Therefore, we get that the problem (5.1) is Ulam-Hyers-Rassias stable with respect to (α, ϕ).

Example 5.2. *Consider the following Katugampola-type impulsive problem,*

$$\begin{cases} {}^{\rho}D^{\frac{1}{2}}_{x_m} u(x) = \dfrac{2 + |u_x| + \left| {}^{\rho}D^{\frac{1}{2}}_{x_m} u(x) \right|}{110e^{x+3}\left(1 + |u_x| + \left| {}^{\rho}D^{\frac{1}{2}}_{x_m} u(x) \right|\right)}, & \text{for each } x \in \mathfrak{J}_0 \cup \mathfrak{J}_1, \\ \Delta u|_{x=\frac{1}{3}} = \dfrac{\left|u(\frac{1}{3}^-)\right|}{8 + \left|u(\frac{1}{3}^-)\right|}, & \\ u(x) = \psi(x), & x \in [-r,0],\ r > 0, \end{cases} \tag{5.2}$$

where $\psi \in \mathfrak{PC}([-r,0],\mathbb{R})$, $\mathfrak{J}_0 = [0, \frac{1}{3}]$, $\mathfrak{J}_1 = (\frac{1}{3}, 1]$, $x_0 = 0$, *and* $x_1 = \frac{1}{3}$.

Let

$$h(x, u_1, u_2) = \frac{2 + |u_1| + |u_2|}{110e^{x+3}(1 + |u_1| + |u_2|)},\ x \in [0,1],\ u_1 \in \mathfrak{PC}([-r,0],\mathbb{R}) \text{ and } u_2 \in \mathbb{R}.$$

Clearly, the function h is jointly continuous. For any $u_1, \bar{u_1} \in \mathfrak{PC}([-r,0],\mathbb{R})$, $u_2, \bar{u_2} \in \mathbb{R}$ and $x \in [0,1]$:

$$|h(x, u_1, u_2) - h(x, \bar{u_1}, \bar{u_2})| \leq \frac{1}{110e^3} \left(\|u_1 - \bar{u_1}\|_{\mathfrak{PC}} + |u_2 - \bar{u_2}| \right).$$

Hence the condition $(A2)$ is satisfied with $c_1 = c_2 = \frac{1}{110e^3}$. We have, for each $x \in [0,1]$,

$$|h(x, u_1, u_2)| \leq \frac{1}{110e^{x+3}} \left(2 + \|u_1\|_{\mathfrak{PC}} + |u_2| \right).$$

Thus, the condition $(A4)$ is satisfied with $p_1(x) = \frac{1}{55e^{x+3}}$ and $p_2(x) = p_3(x) = \frac{1}{110e^{x+3}}$. Let

$$I_1(u_1) = \frac{|u_1|}{8 + |u_1|},\ u_1 \in \mathfrak{PC}([-r,0],\mathbb{R}).$$

We have, for each $u_1 \in \mathfrak{PC}([-r,0],\mathbb{R})$,

$$|I_1(u_1)| \leq \frac{1}{8} \|u_1\|_{\mathfrak{PC}} + 1.$$

Thus, the condition $(A5)$ is satisfied with $M_1^* = \frac{1}{8}$ and $M_2^* = 1$. It follows from Theorem 3.4 that the problem (5.2) has at least one solution on $\mathfrak{J}$.

6. Conclusion

In this article, with the help of standard fixed point theorem of Schaefer's and Banach contraction type, we successfully developed existence of solutions of Katugampola-Caputo type implicit fractional differential equations with impulses. The obtained conditions ensure that the existence of at least one solution to the proposed problem. Further different kinds of Ulam-Hyers and Ulam-Hyers-Rassias stability have been investigated.

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On Markowitz Geometry

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Abstract

By Markowitz geometry we mean the intersection theory of ellipsoids and affine subspaces in a real finite-dimensional linear space. In the paper we give a meticulous and self-contained treatment of this arch-classical subject, which lays a solid mathematical groundwork of Markowitz mean-variance theory of efficient portfolios in economics.

1. Introduction and notation

1.1. Introduction

In this paper we solve the following extremal problem: Given a positive dimensional affine subspace $C \subset \mathbb{R}^n$, a linear form π which is not constant on C, and a positive definite quadratic form v on $\mathbb{R}^n$, find all points $x_0 \in C$ such that

$$\pi(x_0) = \max_{x \in C, v(x) \leq v(x_0)} \pi(x) \text{ and } v(x_0) = \min_{x \in C, \pi(x) \geq \pi(x_0)} v(x). \tag{1.1}$$

It turns out that the locus of solutions of (1.1) is a ray E in C whose endpoint x_0 is the foot of the perpendicular ρ_0 from the origin O of the coordinate system to the affine space C (perpendicularity is with respect to the scalar product obtained from v via polarization). Let h_r be the hyperplane with equation $\pi(x) = r$, $r \in \mathbb{R}$, and let ρ_r be the perpendicular from O to the affine subspace $C \cap h_r$. If $\pi(x_0) = r_0$, $v(x_0) = a_0$, then $E = \{\rho_r \mid r \geq r_0\}$, $\rho_0 = \rho_{r_0}$, and the levels $r = \pi(x)$ and $a = v(x)$ are quadratically related along E: $a = cr^2$.
Let $x = {}^t(x_1, \dots, x_n)$ be the generic vector in $\mathbb{R}^n$, let M be a proper subset of the set $[n] = \{1, \dots, n\}$ of indices, and let $C = \cap_{j \in M} h^{(j)}$, where $h^{(j)}$ are linearly independent hyperplanes with equations

$$\pi^{(j)}(x) = \tau_j,\ \tau_j \in \mathbb{R}, j \in M. \tag{1.2}$$

In case the hyperplane $\Pi = \{x \mid x_1 + \dots + x_n = 1\}$ is one of $h^{(j)}$'s, we may interpret $x \in C$ as an n-assets financial portfolio, subject to the linear constraints (1.2). Next, under certain conditions, see 4.2, we may interpret $\pi(x)$ as the expected return on the portfolio x and $v(x)$ as its risk. Finally, we may interpret the elements of E as efficient portfolios from Markowitz mean-variance theory in economics, considered from purely geometrical point of view. The famous pioneering work [1] is written in this fashion and the condition for nonnegativity of the variables (due to lack of short sales) distorts the picture there and forces the use of variants of simplex method in Markowitz's monograph [2]. Thus, instead of the ray E of efficient portfolios, we have to examine a more sophisticated piecewise set E_M of linear segments enclosed in the compact trace Δ of the unit simplex in Π on C. If $x_0 \in E_M \backslash E \cap \Delta$, then

$$\pi(x_0) < \max_{x \in C, v(x) \leq v(x_0)} \pi(x) \text{ or } v(x_0) > \min_{x \in C, \pi(x) \geq \pi(x_0)} v(x),$$

that is, the maximum $\pi(x_0)$ of the expected return decreases or the minimum $v(x_0)$ of the risk increases, which is our point of departure.
In section 1, Theorem 2.3, we show that the trace $Q_a \cap C$ of an ellipsoid Q_a with equation $v(x) = a$ in $\mathbb{R}^n$ on the affine space C is again an ellipsoid in case $a \geq \gamma_M(\tau)$, where $\gamma_M(\tau)$ is a positive definite quadratic form in the variables $\tau = (\tau_j)_{j \in M} \in \mathbb{R}^M$. The center of the ellipsoid $Q_a \cap C$ is the foot of the perpendicular ρ_0 from O to C, and, moreover, we find its equation in terms of appropriate coordinates on C.

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The inequality $a \geq \gamma_M(\tau)$ determines an "elliptic" cone $\hat{\gamma}_M$ in $\mathbb{R} \times \mathbb{R}^M$, which is the base of the bundle ξ described in Theorem 2.11. By dragging the ellipsoids $a = \gamma_M(\tau)$ "upward" (a is increasing) we establish a real algebraic variety Γ_M which is the frontier of $\hat{\gamma}_M$ and branch locus of ξ. The fibres of ξ over the points in the interior of $\hat{\gamma}_M$ are ellipsoids which degenerate into their centers over Γ_M. Using this bundle, we obtain that the image (the shadow) of an ellipsoid in $\mathbb{R}^n$ via projection parallel to some subspace, is again an ellipsoid — see Proposition 2.14.
In section 2 we prove some extremal properties of the tangential points of members of a family of eccentric ellipsoids and parallel hyperplanes in $\mathbb{R}^n$. These two sections stick together in section 3 where we prove that the ray E is the locus of all efficient Markowitz portfolios and give interpretation of the geometrical results in terms of Markowitz mean-variance theory.

1.2. Notation

For any positive integer n we identify the members of the real linear space $\mathbb{R}^n$ with matrices of type $n \times 1$: $x = {}^t(x_1, \ldots, x_n)$, where the sign ^{t} means the transpose of a matrix. We set $O = {}^t(0, \ldots, 0) \in \mathbb{R}^n$ and denote by $(e_i)_{i=1}^n$ the standard basis in $\mathbb{R}^n$. Say that $M = \{j_1, \ldots, j_m\}$, $j_1 < \cdots < j_m$, be a proper subset of the set of indices $[n] = \{1, \ldots, n\}$. Given a vector $x = {}^t(x_1, \ldots, x_n)$, we denote by $x^{(M)}$ the vector ${}^t(x_{j_1}, \ldots, x_{j_m}) \in \mathbb{R}^M$. Moreover, indexed Greek letters $\tau^{(M)}$, etc., mean vectors ${}^t(\tau_{j_1}, \ldots, \tau_{j_m})$, etc., from the linear space $\mathbb{R}^M$. In case K is a proper subset of the set M and we fix all τ_j, $j \in K$, and vary τ_j, $j \in L$, where $L = M \setminus K$, then, with some abuse of notation (the fixed components are supposed to be known), we write $\tau^{(M)} = \tau^{(L,K)}$.
Given a symmetric $n \times n$ matrix Q, by $Q^{(M)}$ we denote the principal $m \times m$ submatrix of Q, obtained by suppressing the rows and columns with indices which are not in M.
For a positive definite quadratic form $v(x) = {}^t xQx$ on $\mathbb{R}^n$ with matrix Q we denote $Q_a = \{x \in \mathbb{R}^n \mid v(x) = a\}$, $a \geq 0$. The set Q_a is an ellipsoid with center O in $\mathbb{R}^n$ for all $a > 0$. In case $n = 1$ the "ellipsoid" Q_a consists of two (possibly coinciding) points. We extend this terminology by defining the singleton $\{O\}$ to be an "ellipsoid" when $a = 0$ as well as in the case of zero-dimensional linear space.
For any $a \geq 0$ we denote $Q_{\leq a} = \{x \in \mathbb{R}^n \mid v(x) \leq a\}$ and $Q_{<a} = \{x \in \mathbb{R}^n \mid v(x) < a\}$. Note that $Q_{\leq a}$ and $Q_{<a}$ are strictly convex sets.
We let $\pi(x) = p_1x_1 + \cdots + p_nx_n$ be a linear form and let us denote by h_r the hyperplane in $\mathbb{R}^n$, defined by the equation $\pi(x) = r$, $r \in \mathbb{R}$. Let $h_r(\leq)$ denote the half-space $\{x \in \mathbb{R}^n \mid \pi(x) \leq r\}$. The meaning of notation $h_r(\geq)$, $h_r(<)$, and $h_r(>)$ is clear.
The standard scalar product $(x, y) = {}^t xy$ in $\mathbb{R}^n$ produces the standard norm $\|x\|$ with $\|x\|^2 = (x, x)$. We set $S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ (the unit sphere).
The scalar product $\langle x, y \rangle = {}^t xQy$ in $\mathbb{R}^n$ produces the Q-norm $\|x\|_Q$ with $\|x\|_Q^2 = \langle x, x \rangle = v(x)$ and the Q-distance $dist_Q(x, y) = \|x - y\|_Q$. Thus, the ellipsoid Q_a is a Q-sphere with Q-radius $\sqrt{a}$. Two vectors x and y are said to be Q-perpendicular, if $\langle x, y \rangle = 0$.
Throughout the rest of the paper we assume that n is a positive integer and m is a nonnegative integer with $m < n$. Moreover, we suppose that if a proper subset M of the set $[n]$ of indices is given as a list: $M = \{j_1, \ldots, j_m\}$, then $j_1 < \cdots < j_m$.

2. Ellipsoids and affine subspaces

2.1. Intersections of quadric hypersurfaces and affine subspaces

Let $M \subset [n]$ be a set of indices of size m, $M = \{j_1, \ldots, j_m\}$, and let $(h^{(j)})_{j \in M}$ be a family of linearly independent affine hyperplanes in $\mathbb{R}^n$. The system of coordinates can be chosen in such a way that the hyperplane $h^{(j)}$ has equation $x_j = \tau_j$, $\tau_j \in \mathbb{R}$. We denote by $h(\tau^{(M)})$ the intersection $\cap_{j \in M} h^{(j)}$. The family $\{h(\tau^{(M)}) \mid \tau^{(M)} \in \mathbb{R}^M\}$ consists of all $(n-m)$-dimensional affine spaces in $\mathbb{R}^n$, which are orthogonal to the m-dimensional vector subspace generated by the vectors e_j, $j \in M$.
Let $Q = (q_{ij})_{i,j=1}^n$ be a symmetric matrix. For any $j \in M$ we denote by $\rho_{-,j}^{(Q;M^c)}$ the j-th column of the $(n-m) \times n$ matrix obtained from Q by deleting the rows indexed by the elements of M. Thus, $\rho_{-,j}^{(Q;M^c)}$ is a vector in $\mathbb{R}^{n-m}$ with components $\rho_{i,j}^{(Q;M^c)} = q_{ij}$, $i \in M^c$. Given a vector $\tau^{(M)} \in \mathbb{R}^M$, $\tau^{(M)} = {}^t(\tau_{j_1}, \ldots, \tau_{j_m})$, we set $\rho_{-,\tau^{(M)}}^{(Q;M^c)} = \sum_{k=1}^m \tau_{j_k} \rho_{-,j_k}^{(Q;M^c)}$. By

$$\alpha_{(Q;M)}(x) = \sum_{j,k \in M}^{n} q_{jk} x_j x_k$$

we denote the quadratic form which corresponds to the principal submatrix $Q^{(M)}$ of Q.
Let $M^c = \{i_1, \ldots, i_{n-m}\}$. In case the submatrix $Q^{(M^c)}$ is invertible, let

$$x^{(M^c)} = {}^t(c_{i_1}^{(Q;M^c)}(\tau^{(M)}), \ldots, c_{i_{n-m}}^{(Q;M^c)}(\tau^{(M)}))$$

be the solution of the matrix equation

$$Q^{(M^c)} x^{(M^c)} = -\rho_{-,\tau^{(M)}}^{(Q;M^c)}. \tag{2.1}$$

We set

$$c^{(Q;M^c)}(\tau^{(M)}) = {}^t(c_1^{(Q;M^c)}(\tau^{(M)}), \ldots, c_n^{(Q;M^c)}(\tau^{(M)})),$$

where $c_j^{(Q;M^c)}(\tau^{(M)}) = \tau_j$ for $j \in M$. In particular, $c^{(Q;M^c)}(\tau^{(M)}) \in h(\tau^{(M)})$. In case $L \subset M$, $L = \{\ell_1, \ldots, \ell_\lambda\}$, we set

$$c_L^{(Q;M^c)}(\tau^{(M)}) = {}^t(c_{\ell_1}^{(Q;M^c)}(\tau^{(M)}), \ldots, c_{\ell_\lambda}^{(Q;M^c)}(\tau^{(M)})).$$

Note that if $M = \emptyset$, then $c^{(Q;[n])}(\tau^{(\emptyset)}) = 0$. We write $c^{(Q;M^c)}(\tau^{(M)}) = c^{(M^c)}(\tau)$, and, similarly, $\rho_{-,\tau^{(M)}}^{(Q;M^c)} = \rho_{-,\tau}^{(M^c)}$, etc., when the context allows that.

Since the vector $\rho_{-,\tau}^{(M^c)} \in \mathbb{R}^{n-m}$ depends linearly on $\tau^{(M)}$, the map

$$\psi_M : \mathbb{R}^M \to \mathbb{R}^n,\ \tau^{(M)} \mapsto c^{(M^c)}(\tau^{(M)}),$$

is an injective homomorphism of linear spaces. We set $Eff^{(Q;M^c)} = \psi_M\left(\mathbb{R}^M\right)$ and note that $Eff^{(Q;M^c)}$ is an m-dimensional subspace of $\mathbb{R}^n$. Below we use also the short notation $Eff^{(M^c)} = Eff^{(Q;M^c)}$ when the matrix Q is given by default.

Lemma 2.1. *Let K and M be proper subsets of the set of indices $[n]$ with $K \subset M$. Let $Q^{(K^c)}$ and $Q^{(M^c)}$ be invertible submatrices of Q. The following two statements are equivalent:*
(i) *One has $c^{(K^c)}(\tau) \in h(\tau^{(M)})$.*
(ii) *One has $c^{(K^c)}(\tau) = c^{(M^c)}(\tau)$.*

Proof. We have $h(\tau^{(M)}) \subset h(\tau^{(K)})$ and let us assume $K \neq M$. It is enough to prove that (i) implies (ii). Let $c^{(K^c)}(\tau) \in h(\tau^{(M)})$. We remind that the hyperplane $h^{(j)}$ has equation $h^{(j)}: x_j = \tau_j$ for any $j \in M$. In particular, for each $j \in K^c \setminus M^c = M \setminus K$ we obtain $c_j^{(K^c)}(\tau) = \tau_j$. Therefore $c^{(K^c)}(\tau)_{M^c}$ is a solution of the equation (2.1). The uniqueness of this solution implies $c^{(K^c)}(\tau) = c^{(M^c)}(\tau)$.

□

Corollary 2.2. *One has*

$$Eff^{(K^c)} \cap h\left(\tau^{(M)}\right) \subset Eff\,(M^c).$$

Now, let us fix all components of $\tau^{(M)} \in \mathbb{R}^M$, except $r = \tau_\ell$ for some $\ell \in M$, so $\tau^{(M)} = \tau^{(\{\ell\},M\setminus\{\ell\})}(r)$. When we vary $r \in \mathbb{R}$, then $\tau^{(\{\ell\},M\setminus\{\ell\})}(r)$ describes a straight line in $\mathbb{R}^M$ and hence $c^{(M^c)}(\tau^{(\{\ell\},M\setminus\{\ell\})}(r))$ describes a straight line in $\mathbb{R}^n$ which we denote by $Eff_\ell^{(Q;M^c)}$. Its ray $\{c^{(M^c)}(\tau^{(\{\ell\},M\setminus\{\ell\})}(r)) \mid r \geq b\}$, $b \in \mathbb{R}$, is denoted by $Eff_{\ell b+}^{(Q;M^c)}$.
Let us set

$$\gamma_M^{(Q)}(\tau) = \alpha_{(Q;M)}(\tau) - \alpha_{(Q;M^c)}(c_{i_1}^{(Q;M^c)}(\tau), \ldots, c_{i_{n-m}}^{(Q;M^c)}(\tau)).$$

Since $\alpha_{(Q;\emptyset)}(x) = 0$ and $c_1^{(Q;[n])}(\tau) = \cdots = c_n^{(Q;[n])}(\tau) = 0$, we obtain $\gamma_\emptyset^{(Q)}(\tau) = 0$. We write $\gamma_M^{(Q)}(\tau) = \gamma_M(\tau)$ when the matrix Q is known from the context.
It follows from Lemma A.2, (i), that $\gamma_M(\tau)$ is a quadratic form in $\tau^{(M)}$.
Let us move the origin of the coordinate system by the substitution $x = z(\tau^{(M)}) + c^{(M^c)}(\tau^{(M)})$. Then the restrictions of the components of both $x^{(M^c)}$ and $z^{(M^c)}(\tau^{(M)})$ on $h(\tau^{(M)})$ are coordinate functions in this $(n-m)$-dimensional affine space.
Let $v(x) = {}^t xQx$ be the quadratic form produced by the symmetric nonzero $n \times n$-matrix Q. Thus, $Q_a: v(x) = a$ is a quadric in $\mathbb{R}^n$ for generic $a \in \mathbb{R}$ and the real variety $q_{a,\tau^{(M)}} = Q_a \cap h(\tau^{(M)})$ is defined in $h(\tau^{(M)})$ by the equation

$$^t x^{(M^c)} Q^{(M^c)} x^{(M^c)} + 2\,{}^t\rho_{-,\tau}^{(M^c)} x^{(M^c)} + \alpha_M(\tau) - a = 0. \tag{2.2}$$

Let us set

$$v^{(M^c)}(z(\tau^{(M)})) = {}^t z^{(M^c)}(\tau^{(M)}) Q^{(M^c)} z^{(M^c)}(\tau^{(M)}). \tag{2.3}$$

In case the principal submatrix $Q^{(M^c)}$ is invertible, Lemma A.3 implies that $v(x) = v^{(M^c)}(z(\tau^{(M)})) + \gamma_M(\tau^{(M)})$ on $h^{(M)}$, and in terms of z-coordinates the equation (2.2) has the form

$$v^{(M^c)}(z(\tau^{(M)})) = a - \gamma_M(\tau^{(M)}). \tag{2.4}$$

2.2. Intersections of ellipsoids and affine subspaces

Let $v(x) = {}^t xQx$ be a positive definite quadratic form produced by the symmetric (positive definite) $n \times n$-matrix Q. This being so, $Q_a: v(x) = a$ is an ellipsoid in $\mathbb{R}^n$ for $a > 0$, $Q_0 = \{0\}$, and $Q_a = \emptyset$ for $a < 0$. In particular, $Q^{(M^c)}$ is a principal, hence positive definite, submatrice of Q. Thus, the quadratic form (2.3) is positive definite.
In accord with (2.2) and (2.4), we establish parts (ii), (iii), and (iv) of the next theorem. Part (i) is proved in Lemma A.2, (ii).

Theorem 2.3. *Let the quadratic form $v(x) = {}^t xQx$ be positive definite.*
(i) *If $M \neq \emptyset$, then the quadratic form $\gamma_M(\tau)$ is positive definite.*
(ii) *If $a > \gamma_M(\tau)$, then $q_{a,\tau^{(M)}}$ is an ellipsoid in the $(n-m)$-dimensional vector space $h(\tau^{(M)})$ with center $c^{(M^c)}(\tau)$ and $Q^{(M^c)}$-radius $\sqrt{a - \gamma_M(\tau)}$.*
(iii) *If $a = \gamma_M(\tau)$, then $q_{a,\tau^{(M)}} = \{c^{(M^c)}(\tau)\}$.*
(iv) *If $a < \gamma_M(\tau)$, then the set $q_{a,\tau^{(M)}}$ is empty.*

Remark 2.4. We remind that ellipsoid in an one-dimensional affine subspace is a set consisting of two points and its center is the midpoint.

Remark 2.5. In accord with Lemma 3.2, the affine subspace $h(\tau^{(M)})$ is tangential to the ellipsoid Q_a, $a = \gamma_M(\tau)$, at the point $x = c^{(M^c)}(\tau)$.

Remark 2.6. In view of the previous remark, Lemma 2.1 has transparent geometrical meaning: If the subspace $h(\tau^{(M)})$ of $h(\tau^{(K)})$ passes through the point $x = c^{(K^c)}(\tau)$, then $h(\tau^{(M)})$ is also tangential to Q_a at x.

We obtain immediately the following corollary:

Corollary 2.7. (i) *For any $x \in h(\tau^{(M)})$ one has $v(x) \geq \gamma_M(\tau)$ and an equality holds if and only if $x = c^{(M^c)}(\tau)$.*
(ii) *The point $c^{(M^c)}(\tau) \in h(\tau^{(M)})$ is the foot of Q-perpendicular from the origin O to the affine subspace $h(\tau^{(M)})$ and one has*

$$dist_Q\left(O, h\left(\tau^{(M)}\right)\right) = c^{(M^c)}(\tau)_Q = \sqrt{\gamma_M(\tau)}.$$

Corollary 2.8. *Let K and L be disjoint subsets of M with $K \cup L = M$. One has*
(i) *If $a = \gamma_M(\tau^{(M)})$, then the trace $q_{a,\tau^{(K)}}$ of the ellipsoid Q_a on the affine space $h(\tau^{(K)})$ is nonempty and the affine subspace $h(\tau^{(M)}) \subset h(\tau^{(K)})$ is tangential to the ellipsoid $q_{a,\tau^{(K)}}$ at the point $c^{(Q;M^c)}(\tau^{(M)})$.*

$$\text{(ii) } c^{(Q;M^c)}(\tau^{(M)}) = c^{(Q;K^c)}(\tau^{(K)}) + c^{\left(Q^{(K^c)};M^c\right)}(\tau^{(L)} - c_L^{(Q;K^c)}(\tau^{(K)}))$$

and

$$\text{(iii) } \gamma_M^{(Q)}(\tau^{(M)}) = \gamma_K^{(Q)}(\tau^{(K)}) + \gamma_L^{\left(Q^{(K^c)}\right)}(\tau^{(L)} - c_L^{(Q;K^c)}(\tau^{(K)})).$$

Proof. Both assertions hold when one of the sets M, K, or L, is empty.
(i) The equalities

$$q_{a,\tau^{(M)}} = q_{a,\tau^{(K)}} \cap h(\tau^{(L)}) = q_{a,\tau^{(K)}} \cap h(\tau^{(M)}) = Q_a \cap h(\tau^{(M)})$$

and Theorem 2.3, (ii) – (iv), yield that under the condition $a = \gamma_M(\tau^{(M)})$ we have

$$q_{a,\tau^{(K)}} \cap h(\tau^{(M)}) = \{c^{(Q;M^c)}(\tau^{(M)})\}. \tag{2.5}$$

In particular, $a \geq \gamma_K(\tau^{(K)})$ and in this case $q_{a,\tau^{(K)}}$ is an ellipsoid in the vector space $h(\tau^{(K)})$ endowed with coordinate functions $(z_s^{(K^c)}(\tau^{(K)}))_{s\in K^c}$. The point $\{c^{(Q;K^c)}(\tau^{(K)})\}$ is both the origin of the coordinates and the center of the ellipsoid $q_{a,\tau^{(K)}}$ which has equation

$${}^t z^{(K^c)}(\tau^{(K)}) Q^{(K^c)} z^{(K^c)}(\tau^{(K)}) = a - \gamma_K(\tau^{(K)}).$$

Therefore we have

$$q_{a,\tau^{(K)}} = Q^{(K^c)}_{a-\gamma_K(\tau^{(K)})}.$$

Because of (2.5), the trace $h(\tau^{(M)})$ of $h(\tau^{(L)})$ on $h(\tau^{(K)})$ is tangential to $q_{a,\tau^{(K)}}$ at the point $c^{(Q;M^c)}(\tau^{(M)})$ (Note that in case $q_{a,\tau^{(K)}} = \{c^{(Q;M^c)}(\tau^{(M)})\}$ we have $c^{(Q;M^c)}(\tau^{(M)}) = c^{(Q;K^c)}(\tau^{(K)})$ and $h(\tau^{(M)})$ is also tangential to $q_{a,\tau^{(K)}}$ at the point $c^{(Q;M^c)}(\tau^{(M)})$ — see Remark 3.1).
(ii) The affine subspace $h(\tau^{(M)})$ is defined in $h(\tau^{(K)})$ by the equations $z_s^{(K^c)}(\tau^{(K)}) = \tau_s - c_s^{(Q;K^c)}(\tau^{(K)})$, $s \in L$ (we have $L \subset K^c$). Hence the difference $c^{(Q;M^c)}(\tau^{(M)}) - c^{(Q;K^c)}(\tau^{(K)})$ of points in the affine subspace $h(\tau^{(K)}) \subset \mathbb{R}^n$ coincides with the vector $c^{\left(Q^{(K^c)};M^c\right)}(\tau^{(L)} - c_L^{(Q;K^c)}(\tau^{(K)}))$ and we have obtained part (ii). The equalities $a - \gamma_K(\tau^{(K)}) = \gamma_L^{\left(Q^{(K^c)}\right)}(\tau^{(L)} - c_L^{(Q;K^c)}(\tau^{(K)}))$ and $a = \gamma_M(\tau^{(M)})$ yield assertion (iii).

□

Remark 2.9. Since the vector $c^{\left(Q^{(K^c)}\right)}(\tau^{(K)})$ is Q-perpendicular to the affine subspace $h(\tau^{(K)})$ and since the vector $c^{\left(Q^{(K^c)};M^c\right)}(\tau^{(L)} - c_L^{(Q;K^c)}(\tau^{(K)}))$ lies in this subspace, part (ii) of the above corollary is Pythagorean theorem.

Remark 2.10. It follows from Theorem of three perpendiculars that the vector $c^{\left(Q^{(K^c)};M^c\right)}(\tau^{(L)} - c_L^{(Q;K^c)}(\tau^{(K)}))$ is Q-perpendicular to the affine subspace $h(\tau^{(M)})$.

2.3. A bundle

Let us consider the $(m+1)$-dimensional space $\mathbb{R} \times \mathbb{R}^M$ with generic vector ${}^t(a, \tau^{(M)})$, endowed with standard topology and let $\hat{\gamma}_M = \{{}^t(a, \tau^{(M)}) \in \mathbb{R} \times \mathbb{R}^M \mid a \geq \gamma_M(\tau)\}$. The set $\hat{\gamma}_M$ is the closed region in $\mathbb{R} \times \mathbb{R}^M$, which consists of all points above the graph Γ_M of the quadratic function $a = \gamma_M(\tau)$ when $M \neq \emptyset$ and $\hat{\gamma}_\emptyset = [0, \infty) \times \{0\}$. In all cases $pr_a(\hat{\gamma}_M) = [0, \infty)$. The set Γ_M is an algebraic variety (hence a closed set) in $\mathbb{R} \times \mathbb{R}^M$ and the difference $\tilde{\gamma}_M = \hat{\gamma}_M \backslash \Gamma_M$ is an open set, both being nonempty.
Let $\gamma_M(\tau) = {}^t\tau^{(M)} R \tau^{(M)}$, where R is a symmetric $M \times M$-matrix. In accord with Theorem 2.3, (i), in case $M \neq \emptyset$, the matrix R is positive definite. If $M = \emptyset$, then R is the empty matrix. Given $a \geq 0$, we set $R_a = \{\tau^{(M)} \in \mathbb{R}^M \mid \gamma_M(\tau^{(M)}) = a\}$ and note that R_a is an ellipsoid in $\mathbb{R}^M$. Any level set $\Gamma_{a,M} = \{{}^t(a, \tau^{(M)}) \in \mathbb{R} \times \mathrm{R}^M \mid a = \gamma_M(\tau)\}$, $a > 0$, is isomorphic to the ellipsoid R_a in $\mathbb{R}^M$, and $\Gamma_{0,M} = \{(0,0)\}$. Given $a \geq 0$, let us denote $Eff^{(a;M^c)} = \left\{x \in \mathbb{R}^n \mid x = c^{(M^c)}(\tau),\ {}^t\left(a, \tau^{(M)}\right) \in \Gamma_{a,M}\right\}$. We define a morphism of real algebraic varieties by the rule

$$\varphi_M : \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^M,\ x \mapsto {}^t\left(v(x), x^{(M)}\right).$$

Theorem 2.3 yields $\varphi_M(\mathbb{R}^n) = \hat{\gamma}_M$, we set $\Phi_M = \varphi_M^{-1}(\hat{\gamma}_M)$, and denote the restriction of φ_M on Φ_M by the same letter. Since $\varphi_M^{-1}({}^t(a, \tau^{(M)})) = q_{a,\tau^{(M)}}$, we establish the following:

Theorem 2.11. *Let $\xi = (\Phi_M, \varphi_M, \hat{\gamma}_M)$ be the bundle defined by the map φ_M.*
(i) *The restriction $\xi_{|\tilde{\gamma}_M}$ is a fibration with fibres $\varphi_M^{-1}({}^t(a, \tau^{(M)})) = q_{a,\tau^{(M)}}$, ${}^t(a, \tau^{(M)}) \in \tilde{\gamma}_M$, which are ellipsoids in $\mathbb{R}^{n-m}$ with centers $c^{(M^c)}(\tau)$.*
(ii) *The restriction $\xi_{|\Gamma_M}$ is an isomorphism of real algebraic m-dimensional varieties with inverse isomorphism $\Gamma_M \to Eff^{(M^c)}$, ${}^t(a, \tau^{(M)}) \mapsto c^{(M^c)}(\tau)$, which maps any level set $\Gamma_{a,M}$ onto $Eff^{(a;M^c)}$.*

Corollary 2.12. *The set $Eff^{(a;M^c)}$ is a real algebraic subvariety of Q_a, which is isomorphic via $\xi \mid \Gamma_M$ to the ellipsoid $\Gamma_{a,M}$.*

Taking into account Remark 2.5, we obtain immediately the following:

Corollary 2.13. *The family $\{h(\tau^{(M)}) \mid \tau^{(M)} \in \Gamma_{a,M}\}$ consists of all $(n-m)$-dimensional affine spaces in $\mathbb{R}^n$, which are both orthogonal to the m-dimensional vector subspace generated by the vectors e_j, $j \in M$, and tangential to the ellipsoid Q_a.*

2.4. A shadow

Let us denote by ζ_M the restriction of the second projection $pr_2\colon \mathbb{R}\times\mathbb{R}^M \to \mathbb{R}^M$ on $\hat{\gamma}_M$. The composition $\phi_M = \zeta_M \circ \varphi_M$ is the restriction on Φ_M of the projection of $\mathbb{R}^n$ parallel to the subspace W defined by $x^{(M)} = 0$: $\phi_M\colon \mathbb{R}^n \to W^\perp$, $\phi_M(x) = x^{(M)}$, and, moreover, $\phi_M^{-1}(\tau^{(M)}) = h(\tau^{(M)})$. Since the set $Eff^{(a;M^c)} \subset Q_a$ is mapped via ϕ_M onto the ellipsoid R_a in $\mathbb{R}^M$ and since the internal points of Q_a are mapped onto the internal points of R_a, we can formulate the result from Corollary 2.13 as solution of a shadow problem:

Proposition 2.14. *All $(n-m)$-dimensional affine spaces in $\mathbb{R}^n$ with common direction vector subspace W, which are also tangential to an ellipsoid Q_a in $\mathbb{R}^n$, intersect the orthogonal complement $W^\perp$ at the points of an ellipsoid R_a in $W^\perp \simeq \mathbb{R}^M$. All affine spaces in $\mathbb{R}^n$ which have nonempty intersection with the interior of Q_a and are parallel to W intersect $W^\perp$ at the internal points of R_a.*

3. Ellipsoids and hyperplanes

3.1. Ellipsoids and their tangent spaces

Let $v(x) = {}^t xQx$ be a positive definite quadratic form. The equation of the tangent space θ_{x_0} of the ellipsoid $Q_a\colon v(x) = a$, $a > 0$, at the point $x_0 \in Q_a$ is

$$\theta_{x_0}(x) = a,$$

where $\theta_{x_0}(x) = {}^t x_0 Qx$. For all $x \in Q_a$ we have $x \neq 0$ and since the matrix Q has rank n, we obtain $Qx_0 \neq 0$. In particular, θ_{x_0} is a hyperplane and Q_a is a smooth hypersurface in $\mathbb{R}^n$.

Remark 3.1. *The tangent space of the "ellipsoid" $Q_0 = \{O\}$ at its only point $x_0 = O$ is $\mathbb{R}^n$. In particular, any linear subspace of $\mathbb{R}^n$ is tangential to Q_0.*

Let $a > 0$ and let us fix a point $x_0 \in Q_a$. For any vector $u \in S^{n-1}$ we denote for short by L_u the line $\{z \in \mathbb{R}^n \mid z = x_0 + tu,\ t \in \mathbb{R}\}$.

Lemma 3.2. *One has*

$$L_u \cap Q_{\leq a} = \{x_0 + tu \mid 0 \leq t \leq -2\frac{\theta_{x_0}(u)}{v(u)}\},\ L_u \cap Q_a = \{x_0, x_0 - 2\frac{\theta_{x_0}(u)}{v(u)}u\}.$$

Proof. The inequality $v(x_0 + tu) \leq a$ is equivalent to $2\theta_{x_0}(u)t + v(u)t^2 \leq 0$ and the equality holds if and only if $t = 0$ or $t = -2\frac{\theta_{x_0}(u)}{v(u)}$. □

Lemma 3.3. *Let $x_0 \in Q_a$.*
(i) *One has $Q_{\leq a} \subset \theta_{x_0}(\leq)$.*
(ii) *One has $Q_{\leq a} \cap \theta_{x_0} = Q_a \cap \theta_{x_0} = \{x_0\}$.*
(iii) *One has $Q_{\leq a} \setminus \{x_0\} \subset \theta_{x_0}(<)$.*

Proof. (i) Let $y \in Q_{\leq a}$, $y \neq x_0$, and let $y \in L_u$. In accord with Lemma 3.2, $y = x_0 + tu$ where $0 \leq t \leq -2\frac{\theta_{x_0}(u)}{v(u)}$. We have $\theta_{x_0}(y) = \theta_{x_0}(x_0) + t\theta_{x_0}(u) = a + t\theta_{x_0}(u) \leq a - 2\frac{(\theta_{x_0}(u))^2}{v(u)} \leq a$.
(ii) Let us suppose that there exists a point y, $y \neq x_0$, with $y \in Q_{\leq a} \cap \theta_{x_0}$ and let $u = \frac{1}{\|y-x_0\|}(y - x_0)$. Then $\theta_{x_0}(u) = 0$, $y \in L_u$, and Lemma 3.2 implies $L_u \cap Q_{\leq a} = \{x_0\}$ — a contradiction with $y \in L_u \cap Q_{\leq a}$. Now, because of the inclusions $\{x_0\} \subset Q_a \cap \theta_{x_0} \subset Q_{\leq a} \cap \theta_{x_0} = \{x_0\}$, part (ii) is proved.
Parts (i) and (ii) yield part (iii).

□

We remind that h_r is a hyperplane in $\mathbb{R}^n$, defined by the equation $\pi(x) = r$, where $\pi(x)$ is a non-zero linear form, and $q_{a,r} = Q_a \cap h_r$.

Lemma 3.4. *Let $x_0 \in q_{a,r}$.*
(i) *If $Q_a \subset h_r(\leq)$, then $h_r = \theta_{x_0}$.*
(ii) *If $Q_{<a_0} \subset h_{r_0}(<)$, then $Q_a \subset h_r(\leq)$.*

Proof. (i) When y varies through $Q_a \setminus \{x_0\}$, then $u = \frac{1}{\|y-x_0\|}(y - x_0)$ varies bijectively through $S^{n-1} \cap \theta_{x_0}(<)$. On the other hand, since $Q_a \subset h_r(\leq)$, then $y \in Q_a \setminus \{x_0\}$ yields $\pi(y) \leq r$, that is, $\pi(x_0 - 2\frac{\theta_{x_0}(u)}{v(u)}u) \leq r$, and hence $\theta_{x_0}(u)\pi(u) \geq 0$ for all $u \in S^{n-1} \cap \theta_{x_0}(<)$. The last inequality also holds for all $u \in S^{n-1} \cap \theta_{x_0}(>)$ because $\theta_{x_0}(-u)\pi(-u) \geq 0$. Thus, we have $\theta_{x_0}(u)\pi(u) \geq 0$ for all $u \in S^{n-1}$, therefore for all vectors $u \in \mathbb{R}^n$. If the linear forms θ_{x_0} and π are not proportional, then after an appropriate change of the coordinates, θ_{x_0} and π can serve as coordinate functions in $\mathbb{R}^n$ — a contradiction.
(ii) Let $y \in Q_a$ and let us set $y_n = (1 - \frac{1}{n})y$ for any positive integer n. Then $y_n \in Q_{<a_0}$ and $\lim_{n\to\infty} y_n = y$. Since $Q_{<a_0} \subset h_{r_0}(<)$, we obtain $h_{r_0}(y_n) < r_0$, hence $h_{r_0}(y) \leq r_0$.

□

3.2. Some extremal properties

Let $h_r\colon\ \pi(x)=r$ be a hyperplane in $\mathbb{R}^n$, $\pi(x)=p_1x_1+\cdots+p_nx_n$, and let us set $p={}^t(p_1,\ldots,p_n)$. We denote $q_{a,r}=Q_a\cap h_r$.

Lemma 3.5. *Let $x_0\in R^n\backslash\{0\}$, $a>0$, and $r>0$. The following four statements are equivalent:*
(i) *One has $x_0\in q_{a,r}$ and $Qx_0\in\mathbb{R}p$.*
(ii) *One has $rQx_0=ap$ and $a=r^2({}^tpQ^{-1}p)^{-1}$.*
(iii) *One has $x_0\in q_{a,r}$ and $\theta_{x_0}=h_r$.*
(iv) *One has $q_{a,r}=\{x_0\}$.*

Proof. (i) $\Longrightarrow$ (ii) Let $Qx_0=bp$, $b\in\mathbb{R}$. We have

$$a=v(x_0)={}^t x_0Qx_0={}^tx_0(bp)=b{}^tpx_0=b\pi(x_0)=br,$$

therefore $rQx_0=ap$. On the other hand, we obtain

$$a={}^tx_0Qx_0=\frac{a}{r}{}^tpQ^{-1}\frac{a}{r}p=\frac{a^2}{r^2}{}^tpQ^{-1}p,$$

hence $a=r^2({}^tpQ^{-1}p)^{-1}$.
(ii) $\Longrightarrow$ (i) We have $Qx_0\in\mathbb{R}p$, and, moreover, ${}^tx_0=\frac{a}{r}{}^tpQ^{-1}$. $\pi(x_0)={}^tpx_0={}^tx_0p=\frac{a}{r}{}^tpQ^{-1}p=\frac{a}{r}\frac{r^2}{a}=r$, hence $x_0\in h_r$. Finally, $v(x_0)={}^tx_0Qx_0=\frac{a}{r}{}^tpQ^{-1}Qx_0=\frac{a}{r}{}^tpx=\frac{a}{r}\pi(x_0)=a$, therefore $x_0\in Q_a$.
The equivalence of parts (i) and (iii) is straightforward. Part (iii) and Lemma 3.3, (ii), imply part (iv).
(iv) $\Longrightarrow$ (iii) Let $L=\{x_0+tz\mid t\in\mathbb{R}\}$, $z\neq 0$, be a line in h_r, that is, $\pi(z)=0$. The roots of the quadratic equation $v(x_0+tz)=a$ correspond to the intersection points of the line L and the ellipsoid Q_a. Taking into account that $v(x_0+tz)=v(x_0)+2\theta_{x_0}(z)t+v(z)t^2$, we obtain the equivalent equation $2\theta_{x_0}(z)t+v(z)t^2=0$. Since $q_{a,r}=\{x_0\}$, this quadratic equation has a double root $t=0$, that is, $\theta_{x_0}(z)=0$. Thus, we obtain $L\subset\theta_{x_0}$ and therefore $\theta_{x_0}=h_r$.
□

Corollary 3.6. *Under conditions* (i) – (iv) *one has $\theta_{x_0}(x)=\frac{a}{r}\pi(x)$.*

Remark 3.7. If $x_0=0$, then parts (i), (ii), and (iv) of Lemma 3.5 hold for $a=r=0$.

Let us set $c_p=({}^tpQ^{-1}p)^{-1}$, $E_p^{(Q)}=\{(a,r)\mid a=c_pr^2,r\geq 0\}$, $x(a,r)=\frac{a}{r}Q^{-1}p$ for any $(a,r)\in E_p^{(Q)}$ with $r>0$, $x(0,0)=0$, and

$$Ef_p^{(Q)}=\{x\in\mathbb{R}^n\mid x=x(a,r),\ (a,r)\in E_p^{(Q)}\}.$$

Thus, the set $ef_p^{(Q)}$ consists of all vectors $x\in\mathbb{R}^n$ which satisfy the four equivalent conditions from Lemma 3.5. Note that $0\in ef_p^{(Q)}$ and if $x(a,r)\in ef_p^{(Q)}$, then $\{x(a,r)\}=q_{a,r}$. In other words, Lemma 3.5 implies

Corollary 3.8. *One has*

$$ef_p^{(Q)}=\cup_{r\geq 0,a=c_pr^2}q_{a,r}.$$

In case M is a singleton, Theorem 2.3 yields the following two corollaries:

Corollary 3.9. *Let $x,x_0\in ef_p^{(Q)}$, $x=x(a,r)$, $x_0=x(a_0,r_0)$.*
(i) *If $a=a_0$, then $q_{a,r_0}=\{x_0\}$.*
(ii) *If $a>a_0$, then q_{a,r_0} is an ellipsoid in the hyperplane h_{r_0}.*
(iii) *If $a<a_0$, then $q_{a,r_0}=\emptyset$.*

Corollary 3.10. *Let $x,x_0\in ef_p^{(Q)}$, $x=x(a,r)$, $x_0=x(a_0,r_0)$.*
(i) *If $r=r_0$, then $q_{a_0,r}=\{x_0\}$.*
(ii) *If $r<r_0$, then $q_{a_0,r}$ is an ellipsoid in the hyperplane h_{r_0}.*
(iii) *If $r>r_0$, then $q_{a_0,r}=\emptyset$.*

Corollaries 3.9 and (3.10) imply the following two equivalent propositions:

Proposition 3.11. *Let $x,x_0\in ef_p^{(Q)}$, $x=x(a,r)$, $x_0=x(a_0,r_0)$. One has*

$$r_0=\max_{q_{a_0,r}\neq\emptyset}r \text{ and } a_0=\min_{q_{a,r_0}\neq\emptyset}a.$$

Proposition 3.12. *Given $x_0\in ef_p^{(Q)}$, one has*

$$\pi(x_0)=\max_{x\in ef_p^{(Q)},v(x)\leq v(x_0)}\pi(x) \text{ and } v(x_0)=\min_{x\in ef_p^{(Q)},\pi(x)\geq\pi(x_0)}v(x).$$

It turns out that we can trow out the constraint condition $x\in ef_p^{(Q)}$ from Proposition 3.12. We have the following theorem (compare, for example, with [3, Section 2]).

Theorem 3.13. *Let $x_0 \in q_{a_0,r_0}$ and $r_0 \geq 0$. The following six statements are equivalent:*
(i) *One has $x_0 \in ef_p^{(Q)}$.*
(ii) *One has*

$$\pi(x_0) = \max_{v(x)\leq a_0} \pi(x) \text{ and } v(x_0) = \min_{\pi(x)\geq r_0} v(x).$$

(iii) *One has*

$$\pi(x_0) = \max_{v(x)\leq a_0} \pi(x). \tag{3.1}$$

(iv) *One has*

$$\pi(x_0) = \max_{v(x)=a_0} \pi(x).$$

(v) *One has*

$$v(x_0) = \min_{\pi(x)\geq r_0} v(x).$$

(vi) *One has*

$$v(x_0) = \min_{\pi(x)=r_0} v(x).$$

Proof. Below we prove only these implications which are not straightforward.
If $r_0 = 0$ and $x_0 = x(a_0, 0) \in ef_p^{(Q)}$, then $a_0 = 0$, $x_0 = 0$, and the equivalences hold. Now, let $r_0 > 0$. In particular, we have $x_0 \neq 0$.
(i) $\Longrightarrow$ (ii) According to Lemma 3.5, (iii), and Corollary 3.6 we have $x_0 \in q_{a_0,r_0}$ and $\theta_{x_0}(x) = \frac{a_0}{r_0}\pi(x)$. Let us suppose $v(x) \leq a_0$ for $x \in \mathbb{R}^n$. Then Lemma 3.3, (i), imply $\pi(x) \leq r_0$. Now, let $\pi(x) \geq r_0$, that is, $\theta_{x_0}(x) \geq a_0$ for some $x \in \mathbb{R}^n$. In this case Lemma 3.3, (iii), yields $v(x) \geq a_0$.
(iii) $\Longrightarrow$ (i) Let x_0 satisfies condition (3.1). Lemma 3.4, (i), imply $\theta_{x_0} = h_{r_0}$. Now Lemma 3.5, (iii), finishes the proof.
(v) $\Longrightarrow$ (i). Since $Q_{<a_0} \subset h_{r_0}(<)$, Lemma 3.4 yields $\theta_{x_0} = h_{r_0}$. In accord with Lemma 3.5, (iii), part (i) holds.

□

4. Markowitz geometry

In this section we unite the results from the previous two sections and give complete characterization of the tangent points of a family of concentric ellipsoids and a family of parallel hyperplanes in an affine subspace of $\mathbb{R}^n$.

4.1. The equality

Let $M \neq \emptyset$, $\ell \in M$, and let us set $L = \{\ell\}$, $K = M \setminus L$. Let us fix all components of $\tau^{(K)} \in \mathbb{R}^K$: $\tau^{(K)} = \mu^{(K)}$, and set $h^{(K)} = h(\mu^{(K)})$, $\rho^{(K)} = c^{(Q;K^c)}(\mu^{(K)})$, $\gamma^{(K)} = \gamma_K(\mu^{(K)})$. We denote $r = \tau_\ell$, $\rho = \rho_\ell^{(K)}$, $r' = r - \rho$, so $\tau^{(M)} = \tau^{(L,K)}(r)$. Finally, we set $a = \gamma_M(\tau^{(L,K)}(r))$.
We remind that after the translation $z = x - \rho^{(K)}$ of the coordinate system, $(z_s)_{s\in K^c}$, where $z_s = z_s^{(K^c)}$, is a system of coordinate functions on the affine subspace $h^{(K)}$ with origin $\rho^{(K)}$. In this case $h(\tau^{(M)}) = h(\tau^{(L,K)}(r))$ is a hyperplane in $h^{(K)}$ with equation $z_\ell = r'$. In particular, the corresponding ℓ-th coordinate vector $p \in \mathbb{R}^{K^c}$ (the ℓ-th component of p is 1 and all other components are zeroes) is a normal vector of $h(\tau^{(L,K)}(r))$ in $h^{(K)}$. We set $\pi(x) = x_\ell$, $\pi^{(K^c)}(z) = z_\ell$, and note that the linear form $\pi^{(K^c)}(z)$ is the restriction on $h^{(K)}$ of the linear form $\pi(x)$, written in terms of z. It follows from Corollary 2.8, (i), that the trace $q_{a,\mu^{(K)}}$ of the ellipsoid Q_a on affine space $h^{(K)}$ is nonempty and the hyperplane $h(\tau^{(L,K)}(r))$ is tangential to the ellipsoid $q_{a,\mu^{(K)}}$ at the point $c^{(Q;M^c)}(\tau^{(L,K)}(r))$.
In order to stick together notation from sections 2 and 3 in this case, we set $a' = a - \gamma^{(K)}$, $h(\tau^{(L,K)}(r)) = h_{r'}$, $q_{a',r'} = q_{a,\mu^{(K)}} \cap h_{r'} = Q_{a'}^{(K^c)} \cap h_{r'}$.

Theorem 4.1. (i) *If $r' \geq 0$, then*

$$x(a', r') = c^{(Q;M^c)}(\tau^{(L,K)}(r)) \tag{4.1}$$

and $x(0,0) = \rho^{(K)}$.
(ii) *One has*

$$eff_{\ell\rho+}^{(Q;M^c)} = ef_p^{(Q^{(K^c)})}.$$

Proof. (i) The affine space $h(\tau^{(L,K)}(r))$ is a hyperplane in $h^{(K)}$, which is tangential to the ellipsoid $q_{a,\mu^{(K)}}$ at the point $c^{(Q;M^c)}(\tau^{(L,K)}(r))$. In particular, $Q_{a'}^{(K^c)} \cap h_{r'} = \{c^{(Q;M^c)}(\tau^{(L,K)}(r))\}$ and Lemma 3.5, (ii), yields $a' = c_p r'^2$ for $c_p = ({}^t p Q^{-1} p)^{-1}$. Therefore, when $r' \geq 0$, we have $(a', r') \in E_p^{(Q)}$ and the equality (4.1) holds. In addition, if $r' = 0$, then $a' = 0$, $\gamma_M(\tau^{(L,K)}(r)) = \gamma^{(K)}$, and Corollary 2.8, (ii), (iii), implies $\gamma_L^{(Q^{(K^c)})}(\tau^{(L)} - c_L^{(Q;K^c)}(\mu^{(K)})) = 0$, hence

$$c^{(Q^{(K^c)};M^c)}((\tau^{(L)} - c_L^{(Q;K^c)}(\mu^{(K)}))) = 0.$$

In other words,

$$c^{(Q;M^c)}(\tau^{(L,K)}(\rho)) = c^{(Q;K^c)}(\mu^{(K)}).$$

This shows that $x(0,0) = c^{(Q;K^c)}(\mu^{(K)}) = \rho^{(K)}$ and the equality (4.1) proves part (i) which, in turn, yields part (ii).

□

Theorem 4.2. *Let $x_0 = c^{(Q;M^c)}(\tau^{(L,K)}(r_0)) \in eff^{(Q;M^c)}_{\ell\rho+}$. One has $r_0 = \pi(x_0)$ and if $a_0 = v(x_0)$, then*

$$\pi(x_0) = \max_{x\in h^{(K)}, v(x)\le a_0} \pi(x) \text{ and } v(x_0) = \min_{x\in h^{(K)}, \pi(x)\ge r_0} v(x). \tag{4.2}$$

Proof. According to Theorem 4.1, we have $r'_0 = r_0 - \rho \ge 0$, hence $x_0 = x(a_0, r_0) \in ef^{(Q^{(K^c)})}_p$. Let $x_0 = z_0 + \rho^{(K)}$. Theorem 3.13, (i), (ii), implies

$$\pi^{(K^c)}(z_0) = \max_{z\in h^{(K)}, v^{(K^c)}(z)\le a'_0} \pi^{(K^c)}(z)$$

and

$$v^{(K^c)}(z_0) = \min_{z\in h^{(K)}, \pi^{(K^c)}(z)\ge r'_0} v^{(K^c)}(z).$$

Since $\pi^{(K^c)}(x) = \pi(z) + \rho$, $v(x) = v^{(K^c)}(z) + \gamma^{(K)}$ on $h^{(K)}$, and since $r'_0 = r_0 - \rho$, $a'_0 = a_0 - \gamma^{(K)}$, we establish the extremal property (4.2). □

4.2. The interpretation

Let k, m, and n be integers with $n \ge 2$, $0 \le k < n-1$, $m = k+1$, and let $M = \{n-k, n-k+1, \dots, n\}$, $K = \{n-k+1, \dots, n\}$, $L = \{n-k\}$. Let $h^{(j)}: \pi^{(j)}(y) = \tau_j,\ j \in M$, be linearly independent affine hyperplanes in $\mathbb{R}^n$. We fix $h^{(n)}: y_1 + \cdots + y_n = 1$, so $\tau_n = 1$, and denote this hyperplane by Π. Since $\pi^{(j)}(y)$ are linearly independent linear forms, we can change the coordinates in $\mathbb{R}^n$: $y = Ax$, in such a way that the hyperplane $h^{(j)}$ has equation $x_j = \tau_j$, $j \in M$, and, moreover, $x_i = y_i$, $i \in [n] \setminus M$.
We fix $\tau^{(K)}$: $\tau^{(K)} = \mu^{(K)}$ ($\mu_n = 1$), and interpret $h^{(n)} = \Pi$ as the hyperplane consisting of all *financial portfolios* with n assets (here y_s is the relative amount of money invested in the s-th asset, $s = 1, \dots, n$). The affine subspace $h^{(n-k+1)} \cap \ldots \cap h^{(n-1)}$ (which is equal to $\mathbb{R}^n$ if $m = 2$) represents several additional *linear constrain conditions* and its trace on Π is the affine space $C = h^{(K)} = h^{(n-k+1)} \cap h^{(n-1)} \cap \ldots \cap \Pi$ of *linear constrain conditions on* Π.
We denote $\ell = n-k$, $\pi^{(\ell)}(y) = \pi(y)$ and let $r = \tau_\ell$ be variable. When the coefficient in front of y_s in the linear form $\pi(y)$ is the expected return on s-th asset, $s = 1, \dots, n$, the trace of the hyperplane $h = h^{(\ell)}$, $h: \pi(y) = r$, on Π may be interpreted as the set of all financial portfolios with *expected return* r. Moreover, the trace of the hyperplane h on C may be interpreted as the set of all financial portfolios with *expected return* r, that obey the above linear constrain conditions on Π.
On the other hand, if $v(x) = {}^txQx$, where ${}^tA^{-1}QA^{-1}$ is the $n \times n$ covariance matrix produced by the expected returns of the individual assets, we may interpret $v(x)$ as the risk of the portfolio x. Theorem 4.2 yields that the ray $E = eff^{(Q;M^c)}_{ell\rho+}$ with endpoint $\rho^{(K)}$ is the locus of all Markowitz efficient portfolios which satisfy the linear constraint conditions C. It turns out that the value $v(\rho^{(K)})$ is the absolute minimum of the risk and in terms of x-coordinates the ℓ-th component of $\rho^{(K)}$ is the absolute minimum of the corresponding expected return r under the given constrains.
In order to relate this approach to the classical one, we have to study the intersection $E \cap \Delta$, where Δ is the trace of the unit simplex in Π on C, because the members of $E \cap \Delta$ are the efficient portfolios that have no short sales. Moreover, the properties of this intersection characterize the financial market.

A. Appendix

In this appendix we use freely notation introduced in the main body of the paper.

A.1. Three lemmas

The partition $M^c \cup M = [n]$ of the set of indices $[n]$ produces the following partitioned matrices: Any vector $x = {}^t(x_1, \dots, x_n) \in \mathbb{R}^n$ can be visualized as $x = {}^t(x^{(M^c)}, x^{(M)})$ and any $n \times n$-matrix Q can be visualized as

$$\begin{pmatrix} Q^{(M^c)} & Q^{(M^c\times M)} \\ Q^{(M\times M^c)} & Q^{(M)} \end{pmatrix}.$$

Lemma A.1. *Let Q be a symmetric $n \times n$-matrix and let $v(x) = {}^txQx$ be the corresponding quadratic form. One has*

$$v(x) = {}^tx^{(M^c)}Q^{(M^c)}x^{(M^c)} + 2{}^tx^{(M^c)}Q^{(M^c\times M)}x^{(M)} + {}^tx^{(M)}Q^{(M)}x^{(M)}.$$

Proof. We have

$$v(x) = {}^txQx = ({}^tx^{(M^c)}, {}^tx^{(M)}) \begin{pmatrix} Q^{(M^c)} & Q^{(M^c\times M)} \\ {}^tQ^{(M^c\times M)} & Q^{(M)} \end{pmatrix} {}^t(x^{(M^c)}, x^{(M)}) =$$

$${}^tx^{(M^c)}Q^{(M^c)}x^{(M^c)} + 2{}^tx^{(M^c)}Q^{(M^c\times M)}x^{(M)} + {}^tx^{(M)}Q^{(M)}x^{(M)}.$$

□

Below we assume that $Q^{(M^c)}$ is an invertible matrix.

Lemma A.2. *Let*

$$c_{M^c}^{(M^c)}(x^{(M)}) = -(Q^{(M^c)})^{-1}Q^{(M^c\times M)}x^{(M)},\ c^{(M^c)}(x^{(M)}) = {}^t(c_{M^c}^{(M^c)}(x^{(M)}),x^{(M)}),$$

and let $\gamma_M(x^{(M)}) = -{}^t c_{M^c}^{(M^c)}(x^{(M)})Q^{(M^c)}c_{M^c}^{(M^c)}(x^{(M)}) + {}^t x^{(M)}Q^{(M)}x^{(M)}$.

(i) $\gamma_M(x^{(M)})$ *is a quadratic form in* $x^{(M)}$*,*

$$\gamma_M(x^{(M)}) = {}^t x^{(M)}[Q^{(M)} - {}^t Q^{(M^c\times M)}(Q^{(M^c)})^{-1}Q^{(M^c\times M)}]x^{(M)},$$

and one has $\gamma_M(x^{(M)}) = v(c^{(M^c)}(x^{(M)}))$.
(ii) *If* $v(x)$ *is a positive definite quadratic form in* x*, then* $\gamma_M(x^{(M)})$ *is a positive definite quadratic form in* $x^{(M)}$.

Proof. (i) We begin by noting that since

$${}^t c_{M^c}^{(M^c)}(x^{(M)})Q^{(M^c)}c_{M^c}^{(M^c)}(x^{(M)}) = {}^t x^{(M)t}Q^{(M^c\times M)}(Q^{(M^c)})^{-1}Q^{M^c\times M}x^{(M)},$$

we obtain the above expression for $\gamma_M(x^{(M)})$. On the other hand, Lemma A.1 implies

$$v(c^{(M^c)}(x^{(M)})) =$$
$${}^t c_{M^c}^{(M^c)}(x^{(M)})Q^{(M^c)}c_{M^c}^{(M^c)}(x^{(M)}) + 2^t c_{M^c}^{(M^c)}(x^{(M)})Q^{(M^c\times M)}x^{(M)} + {}^t x^{(M)}Q^{(M)}x^{(M)}.$$

Taking into account that $Q^{(M^c\times M)}x^{(M)} = -Q^{(M^c)}c_{M^c}^{(M^c)}(x^{(M)})$, we establish the identity.
(ii) In is enough to note that $c^{(M^c)}(x^{(M)}) = 0$ if and only if $x^{(M)} = 0$.

□

Now, let us translate the system of coordinates by the rule

$$z(\tau^{(M)}) = x - c^{(M^c)}(\tau^{(M)}).$$

Lemma A.3. *If* $x^{(M)} = \tau^{(M)}$*, then*

$$v(x) = {}^t z^{(M^c)}Q^{(M^c)}z^{(M^c)} + \gamma_M(\tau^{(M)}).$$

Proof. In accord with Lemma A.1, we have

$$v(x) = {}^t x^{(M^c)}Q^{(M^c)}x^{(M^c)} + 2^t x^{(M^c)}Q^{(M^c\times M)}\tau^{(M)} + {}^t\tau^{(M)}Q^{(M)}\tau^{(M)} =$$
$${}^t x^{(M^c)}Q^{(M^c)}x^{(M^c)} - 2^t x^{(M^c)}Q^{(M^c)}c_{M^c}^{(M^c)}(\tau^{(M)}) + {}^t\tau^{(M)}Q^{(M)}\tau^{(M)} =$$
$${}^t(z^{(M^c)} + c_{M^c}^{(M^c)}(\tau^{(M)}))Q^{(M^c)}(z^{(M^c)} + c_{M^c}^{(M^c)}(\tau^{(M)}))$$
$$-2^t(z^{(M^c)} + c_{M^c}^{(M^c)}(\tau^{(M)}))Q^{(M^c)}c_{M^c}^{(M^c)}(\tau^{(M)}) + {}^t\tau^{(M)}Q^{(M)}\tau^{(M)} =$$
$${}^t z^{(M^c)}Q^{(M^c)}z^{(M^c)} + {}^t c_{M^c}^{(M^c)}Q^{(M^c)}c_{M^c}^{(M^c)} + 2^t z^{(M^c)}Q^{(M^c)}c_{M^c}^{(M^c)} -$$
$$2^t z^{(M^c)}Q^{(M^c)}c_{M^c}^{(M^c)} - 2^t c_{M^c}^{(M^c)}Q^{(M^c)}c_{M^c}^{(M^c)} + {}^t\tau^{(M)}Q^{(M)}\tau^{(M)} =$$
$${}^t z^{(M^c)}Q^{(M^c)}z^{(M^c)} + \gamma_M(\tau^{(M)}).$$

□

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Explicit Solutions of a Class of (3+1)-Dimensional Nonlinear Model

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Abstract

In this article, we employ Lie group analysis to obtain symmetry reduction of a class of (3+1)-dimensional nonlinear model. This nonlinear model plays a critical role in the study of nonlinear sciences. By the $\exp(-\varphi(z))$-expansion method, we construct explicit solutions for the proposed equation. Four types of explicit solutions are obtained, which are hyperbolic, exponential, trigonometric and rational function solutions.

1. Introduction

Consider the following (3+1)-dimensional nonlinear differential equation (NLDE):

$$u_t + b_1 u^2 u_x + b_2 u_{xxx} + b_3 u_{xyy} + b_4 u_{xss} + b_5 u u_x = 0. \tag{1.1}$$

where b_i $(i = 1, 2, \cdots, 5)$ are arbitrary constants.
It is know that many famous NLDEs are the special cases of Eq.(1.1). For example, if $b_1 = b_3 = b_4 = 0$, then Eq.(1.1) is the Korteweg-de Vries (KdV) equation [1, 2]. If $b_1 = b_4 = 0$, then Eq.(1.1) is the Zakharov-Kuznetsov (ZK) equation [3]. If $b_3 = b_4 = b_5 = 0$, then Eq.(1.1) is the modified KdV equation [4]. If $b_3 = b_4 = 0$, then Eq.(1.1) is the Gardner equation [5]. If $b_4 = b_5 = 0$, then Eq.(1.1) is the modified ZK equation [6].
Eq.(1.1) is a significant nonlinear model which can be used to depict important phenomena and dynamic processes in physics and engineering. It is an interesting and meaningful subject to find exact solutions of NLDEs. During the past few years, there has been extraordinary progress in constructing explicit solutions of NLDEs, for instance, the sine-cosine method [7], the modified simple equation method [8], the bifurcation method of dynamic systems [9], the enhanced $(\frac{G'}{G})$-expansion method [10], the complex method [11]-[15], the $\exp(-\varphi(z))$-expansion method [16]-[18], and the Lie group method [19]-[21] and so on. More related works are in Ref. [22]-[25].
The paper is organized as follows: The algorithm of the $\exp(-\varphi(z))$-expansion method have been introduced in Section 2. Symmetry reduction of the mentioned (3+1)-dimensional NLDE are obtained in Section 3. By the proposed method, we gain explicit solutions of this kind of (3+1)-dimensional NLDE in Section 4. In Section 5, some computer simulations will be given to illustrate our results, and conclusions are presented in the last Section.

2. Algorithm of the $\exp(-\varphi(z))$-expansion method

We consider a nonlinear PDE as follows:

$$F(u, u_x, u_y, u_t, u_{xx}, u_{yy}, u_{tt}, \cdots) = 0, \tag{2.1}$$

where F is a polynomial of an unknown function $u(x, y, t)$ and its derivatives, and it contains highest order derivatives and nonlinear terms are involved.

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Step 1. Substitute traveling wave transformation

$$u(x,y,t) = w(z), \quad z = kx + ly + rt, \tag{2.2}$$

into Eq.(2.1) to convert it to the ODE,

$$P(w, w', w'', w''', \cdots) = 0, \tag{2.3}$$

where P is a polynomial of w and its derivatives, while $' := \frac{d}{dz}$.
Step 2. Suppose that Eq.(2.3) has the exact solutions as follows:

$$w(z) = \sum_{j=0}^{n} B_j (\exp(-\varphi(z)))^j, \tag{2.4}$$

where B_j, $(0 \le j \le n)$ are constants to be determined latter, such that $B_n \neq 0$ and $\varphi = \varphi(z)$ satisfies the ODE as below:

$$\varphi'(z) = \gamma + \exp(-\varphi(z)) + \mu \exp(\varphi(z)). \tag{2.5}$$

Eq.(2.5) has the solutions as follows:
When $\gamma^2 - 4\mu > 0$, $\mu \neq 0$,

$$\varphi(z) = \ln\left(\frac{-\sqrt{(\gamma^2-4\mu)}\tanh(\frac{\sqrt{\gamma^2-4\mu}}{2}(z+a)) - \gamma}{2\mu}\right), \tag{2.6}$$

$$\varphi(z) = \ln\left(\frac{-\sqrt{(\gamma^2-4\mu)}\coth(\frac{\sqrt{\gamma^2-4\mu}}{2}(z+a)) - \gamma}{2\mu}\right). \tag{2.7}$$

When $\gamma^2 - 4\mu < 0$, $\mu \neq 0$,

$$\varphi(z) = \ln\left(\frac{\sqrt{(4\mu-\gamma^2)}\tan(\frac{\sqrt{(4\mu-\gamma^2)}}{2}(z+a)) - \gamma}{2\mu}\right), \tag{2.8}$$

$$\varphi(z) = \ln\left(\frac{\sqrt{(4\mu-\gamma^2)}\cot(\frac{\sqrt{(4\mu-\gamma^2)}}{2}(z+a)) - \gamma}{2\mu}\right). \tag{2.9}$$

When $\gamma^2 - 4\mu > 0$, $\gamma \neq 0$, $\mu = 0$,

$$\varphi(z) = -\ln\left(\frac{\gamma}{\exp(\gamma(z+a)) - 1}\right). \tag{2.10}$$

When $\gamma^2 - 4\mu = 0$, $\gamma \neq 0$, $\mu \neq 0$,

$$\varphi(z) = \ln\left(-\frac{2(\gamma(z+a)+2)}{\gamma^2(z+a)}\right). \tag{2.11}$$

When $\gamma^2 - 4\mu = 0$, $\gamma = 0$, $\mu = 0$,

$$\varphi(z) = \ln(z+a). \tag{2.12}$$

Where a is an arbitrary constant and $B_n \neq 0, \gamma, \mu$ are constants in Eq.(2.6)-Eq.(2.12). We determine the positive integer n through considering the homogeneous balance between highest order derivatives and nonlinear terms of Eq.(2.3).

Step 3. Inserting Eq.(2.4) into Eq.(2.3) and then considering the function $\exp(-\varphi(z))$ yields a polynomial of $\exp(-\varphi(z))$. Let the coefficients of same power about $\exp(-\varphi(z))$ equal to zero, then we get a set of algebraic equations. We solve the algebraic equations to obtain the values of $B_n \neq 0, \gamma, \mu$ and then we put these values into Eq.(2.4) along with Eq.(2.6)-Eq.(2.12) to finish the determination of the solutions for the given PDE.

3. Symmetry reduction

With the aim of obtaining the symmetry $\sigma = \sigma(x,y,s,t,u)$ of Eq.(1.1), we let

$$\sigma = au_x + bu_y + cu_s + du_t + eu + f, \tag{3.1}$$

where u is the solution of Eq.(1.1), a,b,c,d,e,f are unknown functions of real variables x,y,s,t. By the Lie group method [19, 20], σ satisfies

$$\sigma_t + b_1\sigma^2 u_x + b_1 u^2\sigma_x + b_2\sigma_{xxx} + b_3\sigma_{xyy} + b_4\sigma_{xss} + b_5\sigma u_x + b_5 u\sigma_x = 0. \tag{3.2}$$

Putting Eq.(3.1) into Eq.(3.2), we obtain a new differential equation, where

$$b_2u_{xxx} = -b_1u^2u_x - b_3u_{xyy} - b_4u_{xss} - b_5uu_x - u_t. \tag{3.3}$$

By Eq.(3.1), Eq.(3.2) and Eq.(3.3), we get

$$a = a_5, b = (a_2s + a_3), c = (a_4 - \frac{b_4}{b_3}a_2y), d = a_1, e = 0, f = 0, \tag{3.4}$$

where $a_i(i = 1,2,\cdots,5)$ are real constants. Inserting Eqs.(3.4) into Eq.(3.1), we obtain the symmetry of Eq.(1.1)

$$\sigma = a_5u_x + (a_2s + a_3)u_y + (a_4 - \frac{b_4}{b_3}a_2y)u_s + a_1u_t.$$

To solve the above characteristic equation of σ

$$\frac{dx}{a_5} = \frac{dy}{a_2s + a_3} = \frac{ds}{a_4 - \frac{b_4}{b_3}a_2y} = \frac{dt}{a_1} = \frac{du}{0},$$

we get symmetry reduced equations.
Setting $a_1 = a_3 = a_4 = a_5 = 0$, $a_2 = 1$, we obtain one similarity solution of Eq.(1.1)

$$u = \phi(\xi,\eta), \tag{3.5}$$

in which $\eta = \frac{y^2}{2b_3} + \frac{s^2}{2b_4}$, $\xi = x + t$. Substituting Eq.(3.5) into Eq.(1.1), we get one symmetry reduced equation of Eq.(1.1), which is

$$\phi_\xi + b_1\phi^2\phi_\xi + (b_2 + b_3)\phi_{\xi\xi\xi} + 2\phi_{\xi\eta\eta} + b_5\phi\phi_\xi = 0.$$

Setting $a_1 = a_2 = 0$, $a_3 = a_4 = a_5 = 1$, solving $\sigma = 0$, we obtain the other similarity solution of Eq.(1.1)

$$u = \phi(\xi,\eta), \tag{3.6}$$

in which $\eta = s$, $\xi = x + y$. Substituting Eq.(3.6) into Eq.(1.1), we get the other symmetry reduced equation of Eq.(1.1), which is

$$b_1\phi^2\phi_\xi + (b_2 + b_3)\phi_{\xi\xi\xi} + b_4\phi_{\xi\eta\eta} + b_5\phi\phi_\xi = 0. \tag{3.7}$$

4. Application of the exp$(-\varphi(z))$-expansion method to the nonlinear model

Substitute traveling wave transform

$$\phi(\xi,\eta) = w(z), \quad z = k\xi + l\eta,$$

into Eq.(3.7), and integrate it with respect to z, then

$$(b_2k^2 + b_3k^2 + b_4l^2)w'' + \frac{b_5}{2}w^2 + \frac{b_1}{3}w^3 - \lambda = 0, \tag{4.1}$$

where λ is the integration constant.
Taking the homogeneous balance between w^3 and w'' in Eq.(4.1), we have

$$w(z) = B_0 + B_1\exp(-\varphi(z)), \tag{4.2}$$

where $B_1 \neq 0$, B_0 are constants.
Substituting w^2, w^3, w'' into Eq.(4.1) and equating the coefficients of $\exp(-\varphi(z))$ to zero, we get

$$B_1b_4l^2\mu\gamma + B_1k^2b_3\mu\gamma + B_1k^2b_2\mu\gamma + \frac{1}{3}b_1{B_0}^3 + \frac{1}{2}b_5{B_0}^2 - \lambda = 0,$$

$$B_1l^2b_4\gamma^2 + B_1b_2k^2\gamma^2 + B_1b_3k^2\gamma^2 + 2B_1b_2k^2\mu + 2B_1b_3k^2\mu + 2B_1l^2b_4\mu$$
$$+{B_0}^2B_1b_1 + B_0B_1b_5 = 0,$$

$$3B_1b_4l^2\gamma + b_1B_0{B_1}^2 + \frac{1}{2}b_5{B_1}^2 + 3B_1k^2b_2\gamma + 3B_1k^2b_3\gamma = 0,$$

$$2B_1b_4l^2 + 2B_1k^2b_2 + 2B_1k^2b_3 + \frac{1}{3}b_1{B_1}^3 = 0.$$

Solving the above algebraic equations yields

$$\lambda=-\sqrt{\frac{(4\mu-\gamma^2)^3(b_2k^2+b_3k^2+b_4l^2)^3}{18b_1}},$$

$$B_0=\frac{\sqrt{-6b_1\left(b_2k^2+b_3k^2+b_4l^2\right)}\gamma-\sqrt{2b_1(4\mu-\gamma^2)\left(b_4l^2+b_3k^2+b_2k^2\right)}}{2b_1},$$

$$B_1=\sqrt{\frac{-6\left(b_4l^2+b_3k^2+b_2k^2\right)}{b_1}}, \tag{4.3}$$

where γ and μ are arbitrary constants.
We substitute Eqs.(4.3) into Eq.(4.2), then

$$w(z)=\frac{\sqrt{-6b_1\left(b_2k^2+b_3k^2+b_4l^2\right)}\gamma-\sqrt{2b_1(4\mu-\gamma^2)\left(b_2k^2+b_3k^2+b_4l^2\right)}}{2b_1}$$

$$+\sqrt{\frac{-6\left(b_2k^2+b_3k^2+b_4l^2\right)}{b_1}}\exp(-\varphi(z)). \tag{4.4}$$

Using Eq.(2.6) to Eq.(2.12) into Eq.(4.4) respectively, we gain traveling wave solutions to the nonlinear model in the following.
When $\gamma^2-4\mu>0$, $\mu\neq 0$,

$$w_1(z)=\frac{\sqrt{-6b_1\left(b_2k^2+b_3k^2+b_4l^2\right)}\gamma-\sqrt{2b_1(4\mu-\gamma^2)\left(b_2k^2+b_3k^2+b_4l^2\right)}}{2b_1}$$
$$-\sqrt{\frac{-6\left(b_2k^2+b_3k^2+b_4l^2\right)}{b_1}}\frac{2\mu}{\sqrt{(\gamma^2-4\mu)}\tanh(\frac{\sqrt{\gamma^2-4\mu}}{2}(z+a))+\gamma},$$

$$w_2(z)=\frac{\sqrt{-6b_1\left(b_2k^2+b_3k^2+b_4l^2\right)}\gamma-\sqrt{2b_1(4\mu-\gamma^2)\left(b_2k^2+b_3k^2+b_4l^2\right)}}{2b_1}$$
$$-\sqrt{\frac{-6\left(b_2k^2+b_3k^2+b_4l^2\right)}{b_1}}\frac{2\mu}{\sqrt{(\gamma^2-4\mu)}\coth(\frac{\sqrt{\gamma^2-4\mu}}{2}(z+a))+\gamma}.$$

When $\gamma^2-4\mu<0$, $\mu\neq 0$,

$$w_3(z)=\frac{\sqrt{-6b_1\left(b_2k^2+b_3k^2+b_4l^2\right)}\gamma-\sqrt{2b_1(4\mu-\gamma^2)\left(b_2k^2+b_3k^2+b_4l^2\right)}}{2b_1}$$
$$+\sqrt{\frac{-6\left(b_2k^2+b_3k^2+b_4l^2\right)}{b_1}}\frac{2\mu}{\sqrt{(4\mu-\gamma^2)}\tan(\frac{\sqrt{4\mu-\gamma^2}}{2}(z+a))-\gamma},$$

$$w_4(z)=\frac{\sqrt{-6b_1\left(b_2k^2+b_3k^2+b_4l^2\right)}\gamma-\sqrt{2b_1(4\mu-\gamma^2)\left(b_2k^2+b_3k^2+b_4l^2\right)}}{2b_1}$$
$$+\sqrt{\frac{-6\left(b_2k^2+b_3k^2+b_4l^2\right)}{b_1}}\frac{2\mu}{\sqrt{(4\mu-\gamma^2)}\cot(\frac{\sqrt{4\mu-\gamma^2}}{2}(z+a))-\gamma}.$$

When $\gamma^2-4\mu>0$, $\gamma\neq 0$, $\mu=0$,

$$w_5(z)=\frac{\sqrt{-6b_1\left(b_2k^2+b_3k^2+b_4l^2\right)}\gamma-\sqrt{-2b_1\gamma^2\left(b_2k^2+b_3k^2+b_4l^2\right)}}{2b_1}$$
$$+\sqrt{\frac{-6\left(b_2k^2+b_3k^2+b_4l^2\right)}{b_1}}\frac{\gamma}{\exp(\gamma(z+a))-1}.$$

When $\gamma^2-4\mu=0$, $\gamma\neq 0$, $\mu\neq 0$,

$$w_6(z)=\sqrt{\frac{-3\left(b_2k^2+b_3k^2+b_4l^2\right)}{2b_1}}\gamma-\sqrt{\frac{-6\left(b_2k^2+b_3k^2+b_4l^2\right)}{b_1}}\frac{\gamma^2(z+a)}{2(\gamma(z+a)+2)}.$$

When $\gamma^2-4\mu=0$, $\gamma=0$, $\mu=0$,

$$w_7(z)=\sqrt{\frac{-6\left(b_2k^2+b_3k^2+b_4l^2\right)}{b_1}}\frac{1}{z+a}.$$

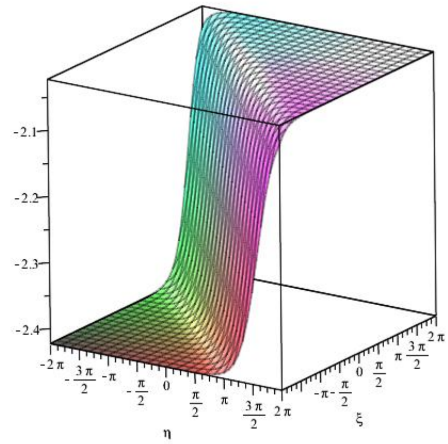

Figure 5.1: 3D profile of $w_1(z)$ for $b_4 = 1$, $b_3 = 1$, $b_2 = -1$, $b_1 = -6$, $k = 1$, $l = 1$, $\gamma = 4$, and $\mu = 3$.

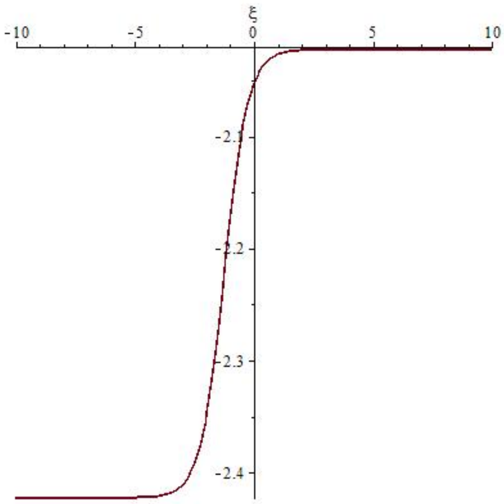

Figure 5.2: 2D profile of $w_1(z)$ for $b_4 = 1$, $b_3 = 1$, $b_2 = -1$, $b_1 = -6$, $k = 1$, $l = 1$, $\gamma = 4$, $\mu = 3$ and $\eta = 0$.

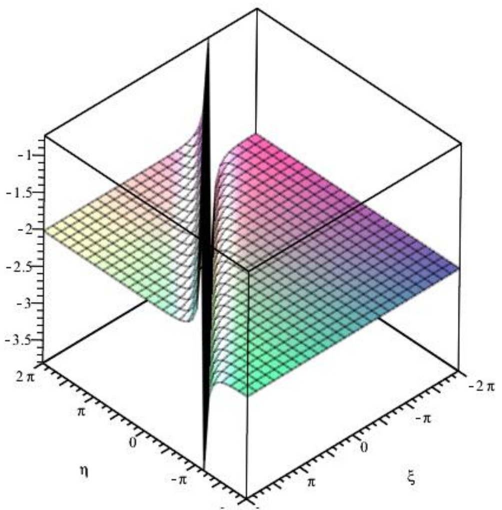

Figure 5.3: 3D profile of $w_2(z)$ for $b_4 = 1$, $b_3 = 1$, $b_2 = -1$, $b_1 = -6$, $k = 1$, $l = 1$, $\gamma = 2$, and $\mu = 2$.

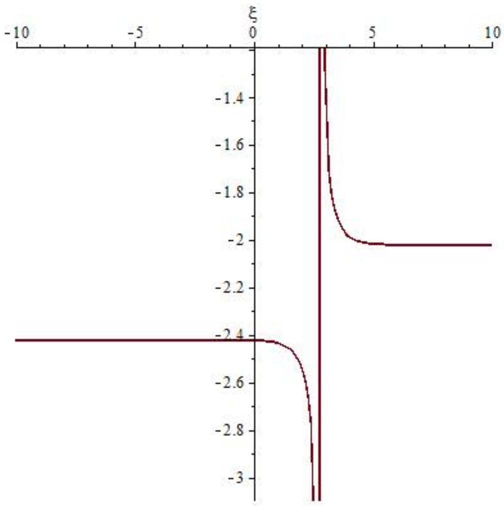

Figure 5.4: 2D profile of $w_2(z)$ for $b_4 = 1$, $b_3 = 1$, $b_2 = -1$, $b_1 = -6$, $k = 1$, $l = 1$, $\gamma = 2$, $\mu = 2$ and $\eta = 0$.

5. Computer simulations

In this section, the computer simulations are given to illustrate our results by the figures.

6. Conclusion

The $\exp(-\varphi(z))$-expansion method allows us to express the explicit solutions of NLDEs as a polynomial of $\exp(-\varphi(z))$, in which $\varphi(z)$ satisfies the ODE (2.5). We can determine the degree of the polynomial via the homogeneous balance and get the coefficients of the polynomial via the simple calculation from the process of this method, and then we obtain the exact solutions.
In this article, symmetry reduction of a class of (3+1)-dimensional nonlinear model are obtained via Lie group analysis. Then, we achieve to reduce the dimension of the NLDEs that is meaningful in engineering and mathematical physics. By the $\exp(-\varphi(z))$-expansion method, we obtain four kinds of explicit solutions. The results demonstrate that the applied method is direct and efficient method, which allow us to do tedious and complicated algebraic calculation.

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Minimum Degree and Size Conditions for Hamiltonian and Traceable Graphs

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Abstract

A graph is called Hamiltonian (resp. traceable) if the graph has a Hamiltonian cycle (resp. path), a cycle (resp. path) containing all the vertices of the graph. In this note, we present sufficient conditions involving minimum degree and size for Hamiltonian and traceable graphs. One of the sufficient conditions strengthens the result obtained by Nikoghosyan in [1].

1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [2]. For a graph $G = (V, E)$, we use n and e to denote its order $|V|$ and size $|E|$, respectively. The complement of a graph G is denoted by G^c. we use G_r to denote any graph of order r. A graph G is empty if the graph G does not have any edge. We use $G_1 \vee G_2$ to denote the the join of two disjoint graphs G_1 and G_2. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path. We define

$$\mathscr{A}(n) := \{G : G \text{ is } G_{\frac{n-2}{2}} \vee (K^c_{\frac{n-2}{2}} \cup K_2)\},$$

$$\mathscr{B}(n) := \{G : G \text{ is } G_{\frac{n-2}{2}} \vee K^c_{\frac{n+2}{2}}\},$$

$$\mathscr{C}(n) := \{G : G \text{ is } G_{\frac{n-1}{2}} \vee K^c_{\frac{n+1}{2}}\}$$

and

$$\mathscr{D}(n) := \mathscr{S}(n) \cup \mathscr{T}(n),$$

where $\mathscr{S}(n) := \{G : G \text{ is } w \vee (P \cup Q)$, where w is a vertex cut such that $G - \{w\}$ has exactly two components of P and Q which are complete graphs of order $\frac{n-1}{2}\}$,
$\mathscr{T}(n) := \{G : G$ has a vertex cut w such that $G - \{w\}$ has exactly two components of P and Q, where P is a complete graph of order $\frac{n-2}{2}$ and w is adjacent to each vertex in P, Q is a graph of order $\frac{n}{2}$ with $\delta(Q) \geq \frac{n-4}{2}$, and $\delta(G) \geq \frac{n-2}{2}\}$.

$$\mathscr{X}(n) := \{G : G \text{ is } K_{\frac{n}{2}} \cup K_{\frac{n}{2}}\}.$$

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$$\mathscr{Y}(n) := \{G : G \text{ is } K_{\frac{n-1}{2}} \cup H, \text{ where } H \text{ is a } \left(\frac{n-3}{2}\right) - \text{ regular graph of order } \frac{n+1}{2}\}.$$

Nikoghosyan obtained the following sufficient condition for Hamiltonian graphs in [1] (also see [3]).

Theorem 1.1. *Let G be a graph of order $n \geq 3$, size e, and minimum degree δ. If $\delta^2 + \delta \geq e + 1$, then G is Hamiltonian.*

Motivated by Nikoghosyan's result above, we in this note strengthen Theorem 1.1 to the following Theorem 1.2 and present an analogous sufficient condition for the traceable graphs.

Theorem 1.2. *Let G be a graph of order $n \geq 3$, size e, and minimum degree δ. If $\delta^2 + \delta \geq e$, then G is empty or G is Hamiltonian or $G \in \mathscr{A}(n) \cup \mathscr{B}(n) \cup \mathscr{C}(n) \cup \mathscr{D}(n) \cup \mathscr{X}(n)$.*

Theorem 1.3. *Let G be a graph of order $n \geq 2$, size e, and minimum degree δ. If $\delta^2 + \frac{3\delta}{2} \geq e$, then G is empty or G is traceable or $G \in \mathscr{X}(n) \cup \mathscr{Y}(n)$.*

2. Lemmas

In order to prove Theorem 1.1 and Theorem 1.2, we need the following results as our lemmas. The first one follows from Theorem 2 proved by Zhao in [4].

Lemma 2.1. *If G is a connected graph of order $n \geq 3$ and $\delta \geq \frac{n-2}{2}$, then G is Hamiltonian or $G \in \mathscr{A}(n) \cup \mathscr{B}(n) \cup \mathscr{C}(n) \cup \mathscr{D}(n)$.*

Notice that the statements in Lemma 2.1 are slightly different from the statements in Theorem 2 in [4]. The reason for this is the convenience when we use Lemma 2.1 in our proofs.
The second one is Theorem 2.5 proved by Cranston and O in [5].

Lemma 2.2. *Every connected k-regular graph with at most $3k+3$ vertices has a Hamiltonian path.*

3. Proofs

Proof of Theorem 1.2 Let G be a graph satisfying the conditions in Theorem 1.2. If $\delta = 0$, then G is empty. From now on, we assume that $\delta \geq 1$. Suppose that G is not Hamiltonian. Then, from the conditions in Theorem 1.2, we have that

$$\delta^2 + \delta \geq e \geq \frac{\sum_{v \in V(G)} d(v)}{2} \geq \frac{n\delta}{2}.$$

Therefore $\delta \geq \frac{n-2}{2}$.

Case 1 G is disconnected.

Suppose G consists of k ($k \geq 2$) components G_1 of order n_1, G_2 of order n_2, $\cdots$, G_k of order n_k. Without loss of generality, we assume that $n_1 \leq n_2 \leq \cdots \leq n_k$. Then we have $2n_1 \leq \sum_{i=1}^{k} n_i = n$. Thus $n_1 \leq \frac{n}{2}$. Therefore $\frac{n-2}{2} \leq \delta \leq d(x) \leq n_1 - 1 \leq \frac{n-2}{2}$, where x is any vertex in G_1. Hence $\frac{n-2}{2} = \delta = n_1 - 1 = \frac{n-2}{2}$. So $\delta^2 = \frac{n-2}{2}\delta$ and $\delta^2 + \delta = \frac{n\delta}{2}$. Now we have

$$\frac{n\delta}{2} \leq \frac{\sum_{v \in V(G)}}{2} \leq e \leq \delta^2 + \delta = \frac{n\delta}{2}.$$

Thus G is δ-regular graph with $\delta = \frac{n-2}{2}$ and $e = \delta^2 + \delta$. Notice that $\frac{n}{2} = n_1 \leq n_2 \leq \cdots \leq n_k$. We must have $k = 2$, $n_2 = \frac{n}{2}$, and G_1 and G_2 are complete graphs of order $\frac{n}{2}$. Therefore $G \in \mathscr{X}(n)$.

Case 2 G is connected.

From Lemma 2.1, we have $G \in \mathscr{A}(n) \cup \mathscr{B}(n) \cup \mathscr{C}(n) \cup \mathscr{D}(n)$.

Hence, the proof of Theorem 1.2 is complete.

Proof of Theorem 1.3 Let G be a graph satisfying the conditions in Theorem 1.3. Notice that G is empty when $\delta = 0$ and G is empty or traceable when $n = 2$ or 3. From now on, we assume that $\delta \geq 1$ and $n \geq 4$. Suppose that G is not traceable. Then, from the conditions in Theorem 1.2, we have that

$$\delta^2 + \frac{3\delta}{2} \geq e \geq \frac{\sum_{v \in V(G)} d(v)}{2} \geq \frac{n\delta}{2}.$$

Therefore $\delta \geq \frac{n-3}{2}$.

Case 1 G is disconnected.

Suppose G consists of k ($k \geq 2$) components G_1 of order n_1, G_2 of order n_2, $\cdots$, G_k of order n_k. Without loss of generality, we assume that $n_1 \leq n_2 \leq \cdots \leq n_k$. Then we have $2n_1 \leq \sum_{i=1}^{k} n_i = n$. Thus $n_1 \leq \frac{n}{2}$. Therefore $\delta \leq d(x) \leq n_1 - 1 \leq \frac{n-2}{2}$, where x is any vertex in G_1.

Case 1.1 $\delta=\frac{n-2}{2}$.

Thus $\frac{n-2}{2}\leq\delta\leq d(x)\leq n_1-1\leq\frac{n-2}{2}$, where x is any vertex in G_1. Therefore $\frac{n-2}{2}=\delta=d(x)=n_1-1=\frac{n-2}{2}$, where x is any vertex in G_1. Hence G_1 is a complete graph of order $\frac{n}{2}$. Notice that $\frac{n}{2}=n_1\leq n_2\leq\cdots\leq n_k$. We must have $k=2$ and $n_2=\frac{n}{2}$. Since $n_2=\frac{n}{2}$ and $\frac{n-2}{2}=n_2-1\geq d(y)\geq\delta=\frac{n-2}{2}$ for any vertex y in G_2, G_2 is a complete graph of order $\frac{n}{2}$. So $G\in\mathscr{X}(n)$.

Case 1.2 $\delta=\frac{n-3}{2}$.

Thus $\delta^2=\frac{n-3}{2}\delta$ and $\delta^2+\frac{3\delta}{2}=\frac{n\delta}{2}$. Now we have

$$\frac{n\delta}{2}\leq\frac{\sum_{v\in V(G)}}{2}\leq e\leq\delta^2+\frac{3\delta}{2}=\frac{n\delta}{2}.$$

Thus G is δ-regular graph with $\delta=\frac{n-3}{2}$ and $e=\delta^2+\frac{3\delta}{2}$. Notice now that n is odd. Then $n_1\leq\frac{n}{2}$ implies that $n_1\leq\frac{n-1}{2}$. Thus for any vertex x in G_1 we have $\frac{n-3}{2}=d(x)\leq n_1-1\leq\frac{n-3}{2}$. Therefore G_1 is a complete graph of order $\frac{n-1}{2}$. Notice that $\frac{n-1}{2}=n_1\leq n_2\leq\cdots\leq n_k$. We must have $k=2$ and $n_2=\frac{n+1}{2}$. Hence G_2 is a $\left(\frac{n-3}{2}\right)$-regular graph of order $\frac{n+1}{2}$. So $G\in\mathscr{Y}(n)$.

Case 2 G is connected.

Case 2.1 n is even.

Then $\delta\geq\frac{n-3}{2}$ implies that $\delta\geq\frac{n-2}{2}$. From Lemma 2.1, we have G is Hamiltonian or $G\in\mathscr{A}(n)\cup\mathscr{B}(n)\cup\mathscr{C}(n)\cup\mathscr{D}(n)$.

First, we prove that it is impossible that $G\in\mathscr{B}(n)$. Suppose, to the contrary, that $G\in\mathscr{B}(n)$. Then $\delta=\frac{n-2}{2}$. Clearly, $e\geq\frac{n^2-4}{4}$. Then we can get a contradiction from

$$\delta^2+\frac{3\delta}{2}\geq e\geq\frac{n^2-4}{4}.$$

Obviously, G is traceable when G is Hamiltonian. It is easy to verify that G is traceable when $G\in\mathscr{A}(n)\cup\mathscr{C}(n)\cup\mathscr{S}(n)$. When $G\in\mathscr{T}(n)$, notice that $\delta(Q)\geq\frac{|V(Q)|}{2}$ when $n\geq 8$. Thus Q is Hamiltonian when $n\geq 8$. It is easy to verify that G is traceable when $n\geq 8$. When $n=4$ or 6, we can also verify that G is traceable. Hence we arrive at a contradiction.

Case 2.2 n is odd.

Then $\delta\geq\frac{n-3}{2}+1=\frac{n-1}{2}$ or $\delta=\frac{n-3}{2}$.

When $\delta\geq\frac{n-3}{2}+1=\frac{n-1}{2}$, then $G\notin\mathscr{A}(n)\cup\mathscr{B}(n)\cup\mathscr{T}(n)$. From Lemma 2.1, we have G is Hamiltonian or $G\in\mathscr{C}(n)\cup\mathscr{S}(n)$. Obviously, G is traceable when G is Hamiltonian or $G\in\mathscr{C}(n)\cup\mathscr{S}(n)$. Hence we arrive at a contradiction.

When $\delta=\frac{n-3}{2}$, then $\delta^2=\frac{n-3}{2}\delta$ and $\delta^2+\frac{3\delta}{2}=\frac{n\delta}{2}$. Now we have

$$\frac{n\delta}{2}\leq\frac{\sum_{v\in V(G)}}{2}\leq e\leq\delta^2+\frac{3\delta}{2}=\frac{n\delta}{2}.$$

Thus G is δ-regular graph with $\delta=\frac{n-3}{2}$ and $e=\delta^2+\frac{3\delta}{2}$. From Lemma 2.2, we have that G is traceable, a contradiction.

Hence, the proof of Theorem 1.3 is complete.

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Qualitative Behavior of Two Rational Difference Equations

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Abstract

Obtaining the exact solutions of most rational recursive equations is sophisticated sometimes. Therefore, a considerable number of nonlinear difference equations is often investigated by studying the qualitative behavior of the governing forms of these equations. The prime purpose of this work is to analyse the equilibria, local stability, global stability character, boundedness character and the solution behavior of the following fourth order fractional difference equations:

$$x_{n+1}=\frac{\alpha x_n x_{n-3}}{\beta x_{n-3}-\gamma x_{n-2}}, \quad x_{n+1}=\frac{\alpha x_n x_{n-3}}{-\beta x_{n-3}+\gamma x_{n-2}}, \quad n=0,1,\ldots,$$

where the constants $\alpha,\ \beta,\ \gamma\in\mathbb{R}^+$ and the initial values $x_{-3},\ x_{-2},\ x_{-1}$ and x_0 are required to be arbitrary non zero real numbers. Furthermore, some numerical figures will be obviously shown in this paper.

1. Introduction

The present paper aims to offer a significant analysis about local asymptotic stability, global attractivity and periodicity of the following rational recursive equations:

$$x_{n+1}=\frac{\alpha x_n x_{n-3}}{\beta x_{n-3}-\gamma x_{n-2}}, \quad x_{n+1}=\frac{\alpha x_n x_{n-3}}{-\beta x_{n-3}+\gamma x_{n-2}}, \quad n=0,1,\ldots,$$

where the initial data $x_{-3},\ x_{-2},\ x_{-1}$ and x_0 are required to be arbitrary non zero real numbers. Moreover, the parameters $\alpha,\ \beta$ and γ are required to be positive arbitrary values.

The theory of nonlinear difference equations has been extraordinarily developed in recent decades. Obviously, this development can be evidently seen in the studies which have been published on difference equations. Take, for instance, the following ones. Avotina [1] investigated the periodicity of three special cases from the fractional difference equation given by

$$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}}.$$

Bajo et al. [2] analyzed the global character of the following second order recursive equation:

$$x_{n+1}=\frac{x_{n-1}}{a+bx_n x_{n-1}}.$$

Çınar [3] provided the solution of the next fractional recursive relation

$$x_{n+1}=\frac{ax_{n-1}}{1+bx_n x_{n-1}}.$$

Din [4] explored some qualitative behaviors such as the stability and the periodicity of the following system:

$$x_{n+1}=\frac{ay_n}{b+cy_n}, \quad y_{n+1}=\frac{dy_n}{e+fx_n}.$$

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El-Moneam et al. [5] explored the qualitative behavior of the difference equation

$$x_{n+1}=Ax_n+Bx_{n-k}+Cx_{n-l}+Dx_{n-\sigma}+\frac{bx_{n-k}+hx_{n-l}}{dx_{n-k}+ex_{n-l}}.$$

Elsayed [6] obtained the forms of the solutions of the recursive relations given on the form:

$$x_{n+1}=\frac{x_n}{x_{n-1}(x_n\pm 1)}.$$

Ibrahim [7] examined the global and local stability of the second order recursive relation on the form:

$$x_{n+1}=\frac{ax_{n-1}}{-1+bx_nx_{n-1}}.$$

More details on this aspect can be simply found in refs. [8], [9]-[14], [15].

2. On the recursive relation $x_{n+1}=\frac{\alpha x_n x_{n-3}}{\beta x_{n-3}-\gamma x_{n-2}}$

This section underlines widely some aspects and properties of the recursive equation

$$x_{n+1}=\frac{\alpha x_n x_{n-3}}{\beta x_{n-3}-\gamma x_{n-2}},\quad n=0,1,2,\ldots, \tag{2.1}$$

where the initial values are required to be arbitrary constants. The parameters $\alpha,\ \beta$ and γ are as mentioned above.

2.1. Local stability analysis

The local behaviour of the fixed point of our equation will be proved under an intrinsic hypothesis in this subsection. The equilibrium point of Eq.(2.1) can be evaluated from the following equation:

$$\overline{x}=\frac{\alpha\overline{x}\overline{x}}{\beta\overline{x}-\gamma\overline{x}}=\frac{\alpha\overline{x}}{\beta-\gamma}.$$

This implies that

$$\overline{x}=0.$$

Assume that a function $h:(0,\infty)^3\longrightarrow(0,\infty)$ is described by the following form:

$$h(t,s,z)=\frac{\alpha tz}{\beta z-\gamma s}, \tag{2.2}$$

from which we can obtain that

$$\begin{aligned}\frac{\partial h(t,s,z)}{\partial t}&=\frac{\alpha z}{\beta z-\gamma s},\\ \frac{\partial h(t,s,z)}{\partial s}&=\frac{\alpha\gamma tz}{(\beta z-\gamma s)^2},\\ \frac{\partial h(t,s,z)}{\partial z}&=-\frac{\alpha\gamma ts}{(\beta z-\gamma s)^2}.\end{aligned} \tag{2.3}$$

These partial derivatives can be obviously calculated at $\overline{x}=0$, as follows:

$$\begin{aligned}\frac{\partial h(\overline{x},\overline{x},\overline{x})}{\partial t}&=\frac{\alpha\overline{x}}{\beta\overline{x}-\gamma\overline{x}}=\frac{\alpha}{\beta-\gamma}=-p_2,\\ \frac{\partial h(\overline{x},\overline{x},\overline{x})}{\partial s}&=\frac{\alpha\gamma\overline{x}\overline{x}}{(\beta\overline{x}-\gamma\overline{x})^2}=\frac{\alpha\gamma}{(\beta-\gamma)^2}=-p_1,\\ \frac{\partial h(\overline{x},\overline{x},\overline{x})}{\partial z}&=-\frac{\alpha\gamma\overline{x}\overline{x}}{(\beta\overline{x}-\gamma\overline{x})^2}=-\frac{\alpha\gamma}{(\beta-\gamma)^2}=-p_0.\end{aligned}$$

Now, the corresponding linearized form of Eq.(2.1) about $\overline{x}=0$, is given by

$$y_{n+1}+p_2y_n+p_1y_{n-2}+p_0y_{n-3}=0.$$

Theorem 2.1. *Let*

$$(\beta-\gamma)^2>\max\{\alpha(\beta+\gamma),\ \alpha(3\gamma-\beta)\}.$$

Then, the fixed point of Eq.(2.1) is locally asymptotically stable.

Proof. According to Theorem A in **[16]**, Eq.(2.1) is said to be asymptotically stable if

$$|p_0|+|p_1|+|p_2|<1.$$

This expression leads to

$$\left|-\frac{\alpha\gamma}{(\beta-\gamma)^2}\right|+\left|\frac{\alpha\gamma}{(\beta-\gamma)^2}\right|+\left|\frac{\alpha}{\beta-\gamma}\right|<1.$$

- If $\beta>\gamma$, then

$$\frac{2\alpha\gamma}{(\beta-\gamma)^2}+\frac{\alpha}{\beta-\gamma}<1,$$

which can be easily simplified as

$$\alpha(\beta+\gamma)<(\beta-\gamma)^2. \tag{2.4}$$

- If $\beta<\gamma$, then

$$\frac{2\alpha\gamma}{(\beta-\gamma)^2}+\frac{\alpha}{\gamma-\beta}<1.$$

Therefore,

$$\alpha(3\gamma-\beta)<(\beta-\gamma)^2. \tag{2.5}$$

Combining condition (2.4) with condition (2.5) gives us

$$(\beta-\gamma)^2>\max\{\alpha(\beta+\gamma),\ \alpha(3\gamma-\beta)\}.$$

This achieves the proof completely. □

2.2. Global stability analysis

Here, we will present an approach to determine the global behavior of Eq.(2.1). In this equation, two different cases will emerge as illustrated in the following fundamental theorem.

Theorem 2.2. *The fixed point of Eq.(2.1) is said to be a global attractor if* $\alpha\neq\gamma$.

Proof. Suppose that $r_1,\ r_2\in\mathbb{R}$ and let $h:[r_1,r_2]^3\longrightarrow[r_1,r_2]$ be a function defined by Eq.(2.2). Then, we take into consideration the following situations.
Case 1: Let $\beta z<\gamma s$ be true. Then, equations (2.3) tell us that Eq.(2.2) is nondecreasing in s and nonincreasing in t and z. Next, let (φ,χ) be a solution of the following system:

$$\begin{aligned}\varphi &= h(\chi,\varphi,\chi)=\frac{\alpha\chi^2}{\beta\chi-\gamma\varphi},\\ \chi &= h(\varphi,\chi,\varphi)=\frac{\alpha\varphi^2}{\beta\varphi-\gamma\chi}.\end{aligned}$$

Or,

$$\beta\varphi\chi-\gamma\varphi^2=\alpha\chi^2, \tag{2.6}$$

$$\beta\varphi\chi-\gamma\chi^2=\alpha\varphi^2. \tag{2.7}$$

Subtracting Eq.(2.6) from Eq.(2.7) gives

$$\gamma(\chi^2-\varphi^2)=\alpha\left(\chi^2-\varphi^2\right).$$

Now, if $\gamma\neq\alpha$, we have

$$\varphi=\chi.$$

As claimed by Theorem B in **[17]**, the fixed point of Eq.(2.1) is a global attractor.
Case 2: This case shows the global behaviour when $\beta z>\gamma s$. The proof of this case is similar to the previous one. □

Remark 2.3. *Eq.(2.1) is not prime period two.*

2.3. Special case of eq.(2.1)

In the following paragraph, we will specify an effective theorem to verify the periodicity of the solution of the following fourth order recursive relation:

$$x_{n+1}=\frac{x_nx_{n-3}}{x_{n-3}-x_{n-2}}, \tag{2.8}$$

where the initial values are as illustrated above.

Theorem 2.4. *Each solution of Eq.(2.8) is periodic with period eighteen.*

Proof. We assume that $\{x_n\}_{n=-3}^{\infty}$ is a solution of Eq.(2.8), then

$$\begin{aligned}
x_{n+1} &= \frac{x_nx_{n-3}}{x_{n-3}-x_{n-2}},\\
x_{n+2} &= \frac{x_{n+1}x_{n-2}}{x_{n-2}-x_{n-1}}=\frac{\left(\frac{x_nx_{n-3}}{x_{n-3}-x_{n-2}}\right)x_{n-2}}{x_{n-2}-x_{n-1}}=\frac{x_{n-3}x_{n-2}x_n}{\left(x_{n-3}-x_{n-2}\right)\left(x_{n-2}-x_{n-1}\right)},\\
x_{n+3} &= \frac{x_{n+2}x_{n-1}}{x_{n-1}-x_n}=\frac{\left(\frac{x_{n-3}x_{n-2}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})}\right)x_{n-1}}{x_{n-1}-x_n}\\
&= \frac{x_{n-3}x_{n-2}x_{n-1}x_n}{\left(x_{n-3}-x_{n-2}\right)\left(x_{n-2}-x_{n-1}\right)\left(x_{n-1}-x_n\right)},\\
x_{n+4} &= \frac{x_{n+3}x_n}{x_n-x_{n+1}}=\frac{\left(\frac{x_{n-3}x_{n-2}x_{n-1}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}\right)x_n}{x_n-\frac{x_nx_{n-3}}{x_{n-3}-x_{n-2}}}\\
&= -\frac{x_{n-3}x_{n-1}x_n}{\left(x_{n-2}-x_{n-1}\right)\left(x_{n-1}-x_n\right)},\\
x_{n+5} &= \frac{x_{n+4}x_{n+1}}{x_{n+1}-x_{n+2}}=\frac{\left(-\frac{x_{n-3}x_{n-1}x_n}{(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}\right)\left(\frac{x_nx_{n-3}}{x_{n-3}-x_{n-2}}\right)}{\frac{x_nx_{n-3}}{x_{n-3}-x_{n-2}}-\frac{x_{n-3}x_{n-2}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})}}=\frac{x_{n-3}x_n}{\left(x_{n-1}-x_n\right)},\\
x_{n+6} &= \frac{x_{n+5}x_{n+2}}{x_{n+2}-x_{n+3}}\\
&= \frac{\left(\frac{x_{n-3}x_n}{x_{n-1}-x_n}\right)\left(\frac{x_{n-3}x_{n-2}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})}\right)}{\frac{x_{n-3}x_{n-2}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})}-\frac{x_{n-3}x_{n-2}x_{n-1}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}}=-x_{n-3},\\
x_{n+7} &= \frac{x_{n+6}x_{n+3}}{x_{n+3}-x_{n+4}}\\
&= \frac{-x_{n-3}\left(\frac{x_{n-3}x_{n-2}x_{n-1}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}\right)}{\frac{x_{n-3}x_{n-2}x_{n-1}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}+\frac{x_{n-3}x_{n-1}x_n}{(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}}=-x_{n-2},\\
x_{n+8} &= \frac{x_{n+7}x_{n+4}}{x_{n+4}-x_{n+5}}=\frac{-x_{n-2}\left(-\frac{x_{n-3}x_{n-1}x_n}{(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}\right)}{-\frac{x_{n-3}x_{n-1}x_n}{(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}-\frac{x_{n-3}x_n}{(x_{n-1}-x_n)}}=-x_{n-1},\\
x_{n+9} &= \frac{x_{n+8}x_{n+5}}{x_{n+5}-x_{n+6}}=\frac{-x_{n-1}\left(\frac{x_{n-3}x_n}{x_{n-1}-x_n}\right)}{\frac{x_{n-3}x_n}{x_{n-1}-x_n}+x_{n-3}}=-x_n,\\
x_{n+10} &= \frac{x_{n+9}x_{n+6}}{x_{n+6}-x_{n+7}}=\frac{-x_n\left(-x_{n-3}\right)}{-x_{n-3}+x_{n-2}}=-\frac{x_nx_{n-3}}{x_{n-3}-x_{n-2}},\\
x_{n+11} &= \frac{x_{n+10}x_{n+7}}{x_{n+7}-x_{n+8}}=\frac{\left(-\frac{x_nx_{n-3}}{x_{n-3}-x_{n-2}}\right)\left(-x_{n-2}\right)}{-x_{n-2}+x_{n-1}}\\
&= -\frac{x_{n-3}x_{n-2}x_n}{\left(x_{n-3}-x_{n-2}\right)\left(x_{n-2}-x_{n-1}\right)},
\end{aligned}$$

$$
\begin{aligned}
x_{n+12} &= \frac{x_{n+11}x_{n+8}}{x_{n+8}-x_{n+9}} = \frac{\left(-\frac{x_{n-3}x_{n-2}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})}\right)(-x_{n-1})}{-x_{n-1}+x_n} \\
&= -\frac{x_{n-3}x_{n-2}x_{n-1}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}, \\
x_{n+13} &= \frac{x_{n+12}x_{n+9}}{x_{n+9}-x_{n+10}} = \frac{\left(-\frac{x_{n-3}x_{n-2}x_{n-1}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}\right)(-x_n)}{-x_n+\frac{x_nx_{n-3}}{x_{n-3}-x_{n-2}}} \\
&= \frac{x_{n-3}x_{n-1}x_n}{(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}, \\
x_{n+14} &= \frac{x_{n+13}x_{n+10}}{x_{n+10}-x_{n+11}} = \frac{\left(\frac{x_{n-3}x_{n-1}x_n}{(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}\right)\left(-\frac{x_nx_{n-3}}{x_{n-3}-x_{n-2}}\right)}{-\frac{x_nx_{n-3}}{x_{n-3}-x_{n-2}}+\frac{x_{n-3}x_{n-2}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})}} \\
&= -\frac{x_{n-3}x_n}{(x_{n-1}-x_n)}, \\
x_{n+15} &= \frac{x_{n+14}x_{n+11}}{x_{n+11}-x_{n+12}} \\
&= \frac{\left(-\frac{x_{n-3}x_n}{x_{n-1}-x_n}\right)\left(-\frac{x_{n-3}x_{n-2}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})}\right)}{-\frac{x_{n-3}x_{n-2}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})}+\frac{x_{n-3}x_{n-2}x_{n-1}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}} = x_{n-3},
\end{aligned}
$$

$$
\begin{aligned}
x_{n+16} &= \frac{x_{n+15}x_{n+12}}{x_{n+12}-x_{n+13}} \\
&= \frac{x_{n-3}\left(-\frac{x_{n-3}x_{n-2}x_{n-1}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}\right)}{-\frac{x_{n-3}x_{n-2}x_{n-1}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}-\frac{x_{n-3}x_{n-1}x_n}{(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}} = x_{n-2}, \\
x_{n+17} &= \frac{x_{n+16}x_{n+13}}{x_{n+13}-x_{n+14}} = \frac{x_{n-2}\left(\frac{x_{n-3}x_{n-1}x_n}{(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}\right)}{\frac{x_{n-3}x_{n-1}x_n}{(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}+\frac{x_{n-3}x_n}{(x_{n-1}-x_n)}} = x_{n-1}, \\
x_{n+18} &= \frac{x_{n+17}x_{n+14}}{x_{n+14}-x_{n+15}} = \frac{x_{n-1}\left(-\frac{x_{n-3}x_n}{(x_{n-1}-x_n)}\right)}{-\frac{x_{n-3}x_n}{(x_{n-1}-x_n)}-x_{n-3}} = x_n.
\end{aligned}
$$

The proof has been completely done. □

2.4. Numerical confirmation

To confirm our theoretical outcomes in the previous subsections, we will provide some concrete numerical examples in this subsection.

Example 2.5. *Figure 2.1 is sketched according to the following values:* $\alpha=\gamma=1,\ \beta=6,\ x_{-3}=x_0=0.2$, *and* $x_{-1}=-x_{-2}=0.1$.

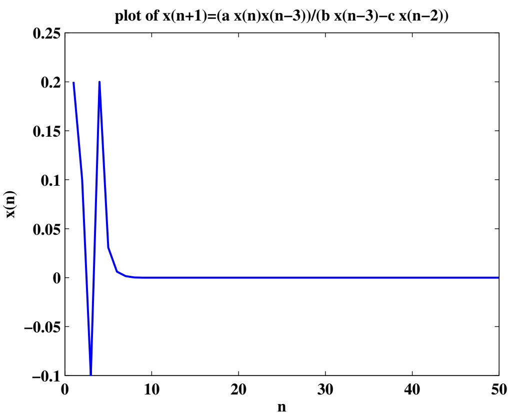

Figure 2.1

Example 2.6. *We consider* $\alpha=10,\ \ \beta=2,\ \gamma=1,\ x_{-3}=0.5,\ x_{-2}=x_0=1$ *and* $x_{-1}=-1$, *to depict the Figure 2.2.*

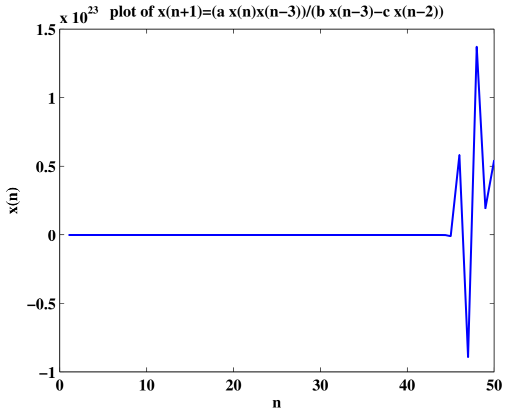

Figure 2.2

Example 2.7. *This example illustrates the periodicity of the special case equation when we take* $x_{-3} = x_{-1} = -0.1$ *and* $x_{-2} = x_0 = 0.1$. *See Figure 2.3.*

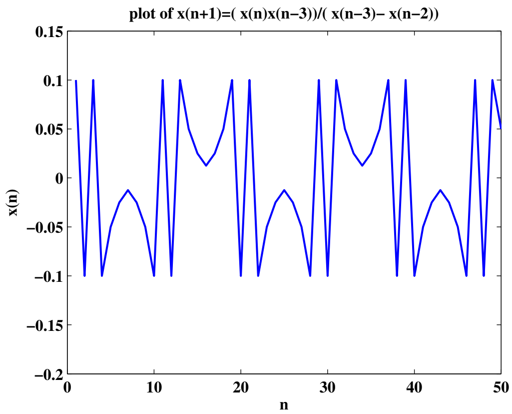

Figure 2.3

3. On the recursive relation $x_{n+1} = \frac{\alpha x_n x_{n-3}}{-\beta x_{n-3}+\gamma x_{n-2}}$

This section will offer various mathematical aspects of the following recursive form:

$$x_{n+1} = \frac{\alpha x_n x_{n-3}}{-\beta x_{n-3} + \gamma x_{n-2}}, \quad n = 0, 1, \dots . \tag{3.1}$$

The initial data and the arbitrary constants are as mentioned above.

3.1. Local stability analysis

In this part, the behaviour of the solutions in the neighbourhood of the fixed point will be established via a key theorem. The fixed point of Eq.(3.1) can be simply found from the equation given by

$$\bar{x} = \frac{\alpha \bar{x}\bar{x}}{-\beta \bar{x} + \gamma \bar{x}} = \frac{\alpha \bar{x}}{-\beta + \gamma}.$$

This gives us

$$\bar{x} = 0.$$

Assume that a function $h : (0,\infty)^3 \longrightarrow (0,\infty)$ is described as follows:

$$h(t,s,z) = \frac{\alpha t z}{-\beta z + \gamma s}. \tag{3.2}$$

Then,

$$\begin{aligned} \frac{\partial h(t,s,z)}{\partial t} &= \frac{\alpha z}{-\beta z+\gamma s}, \\ \frac{\partial h(t,s,z)}{\partial s} &= -\frac{\alpha \gamma t z}{(-\beta z+\gamma s)^2}, \\ \frac{\partial h(t,s,z)}{\partial z} &= \frac{\alpha \gamma t s}{(-\beta z+\gamma s)^2}. \end{aligned} \tag{3.3}$$

Finding these partial derivatives at $\bar{x}=0$, yields

$$\begin{aligned} \frac{\partial h(\bar{x},\bar{x},\bar{x})}{\partial t} &= \frac{\alpha \bar{x}}{-\beta \bar{x}+\gamma \bar{x}} = \frac{\alpha}{\gamma-\beta} = -p_2, \\ \frac{\partial h(\bar{x},\bar{x},\bar{x})}{\partial s} &= -\frac{\alpha \gamma \bar{x}\bar{x}}{(-\beta \bar{x}+\gamma \bar{x})^2} = -\frac{\alpha\gamma}{(\gamma-\beta)^2} = -p_1, \\ \frac{\partial h(\bar{x},\bar{x},\bar{x})}{\partial z} &= \frac{\alpha \gamma \bar{x}\bar{x}}{(-\beta \bar{x}+\gamma \bar{x})^2} = \frac{\alpha\gamma}{(\gamma-\beta)^2} = -p_0. \end{aligned}$$

Following, the corresponding linearized scheme of Eq.(3.1) about $\bar{x}=0$, is

$$y_{n+1}+p_2 y_n+p_1 y_{n-2}+p_0 y_{n-3}=0.$$

Theorem 3.1. *Assume that*

$$(\gamma-\beta)^2 > \max\{\alpha(\beta+\gamma),\ \alpha(3\gamma-\beta)\}.$$

Then, the point $\bar{x}=0$, is locally asymptotically stable.

Proof. As stated by Theorem A in **[16]**, Eq.(3.1) is said to be asymptotically stable if

$$|p_0|+|p_1|+|p_2|<1,$$

which implies that

$$\left|\frac{\alpha\gamma}{(\gamma-\beta)^2}\right| + \left|-\frac{\alpha\gamma}{(\gamma-\beta)^2}\right| + \left|\frac{\alpha}{\gamma-\beta}\right| < 1.$$

- If $\beta<\gamma$, then

$$\frac{2\alpha\gamma}{(\gamma-\beta)^2}+\frac{\alpha}{\gamma-\beta}<1.$$

Therefore,

$$\alpha(3\gamma-\beta)<(\gamma-\beta)^2. \tag{3.4}$$

- If $\beta>\gamma$, then

$$\frac{2\alpha\gamma}{(\gamma-\beta)^2}-\frac{\alpha}{\gamma-\beta}<1,$$

which can be easily reduced to

$$\alpha(\gamma+\beta)<(\gamma-\beta)^2. \tag{3.5}$$

Finally, combining condition (3.4) with condition (3.5) leads to

$$(\gamma-\beta)^2 > \max\{\alpha(\beta+\gamma),\ \alpha(3\gamma-\beta)\},$$

which is what we require to prove. □

3.2. Global stability analysis

We now turn to analyze the global attractivity of Eq.(3.1), in which two various cases are arisen.

Theorem 3.2. *The fixed point of Eq.(3.1) is a global attractor.*

Proof. Assume that $r_1,\ r_2 \in \mathbb{R}$ and let $h : [r_1, r_2]^3 \longrightarrow [r_1, r_2]$ be a function defined by Eq.(3.2). Then, we examine the next two cases.
Case 1: Let $\beta z < \gamma s$ be true. Then, from equations (3.3) we observe that Eq.(3.2) is nondecreasing in t and z and nonincreasing in s. Now, suppose that (φ, χ) is a solution of the following rational system:

$$\begin{aligned}\varphi &= h(\varphi,\chi,\varphi) = \frac{\alpha\varphi^2}{-\beta\varphi+\gamma\chi},\\ \chi &= h(\chi,\varphi,\chi) = \frac{\alpha\chi^2}{-\beta\chi+\gamma\varphi}.\end{aligned}$$

Obviously, this system can be written as

$$-\beta\varphi^2+\gamma\varphi\chi = \alpha\varphi^2, \tag{3.6}$$

$$-\beta\chi^2+\gamma\varphi\chi = \alpha\chi^2. \tag{3.7}$$

Subtracting Eq.(3.6) from Eq.(3.7) leads to

$$\beta(\chi^2-\varphi^2) = \alpha(\varphi^2-\chi^2).$$

Hence,

$$(\beta+\gamma)(\chi-\varphi)(\chi+\varphi) = 0.$$

This implies that

$$\varphi = \chi.$$

As claimed by Theorem B in **[17]**, the point $\bar{x} = 0$, is a global attractor.
Case 2: In this case we consider $\beta z > \gamma s$. The proof can be achieved in a similar way to the previous one. □

Remark 3.3. *Eq.(3.1) is not prime period two.*

3.3. Special case of eq.(3.1)

Now, we will formulate the solution of the recursive equation which is given as follows:

$$x_{n+1} = \frac{x_n x_{n-3}}{x_{n-2}-x_{n-3}}, \quad n = 0,1,\dots. \tag{3.8}$$

The initial values are required to be nonzero real numbers.

Theorem 3.4. *Suppose that $\{x_n\}_{n=-3}^{\infty}$ is a solution of Eq.(3.8) and satisfying $x_{-3} = a,\ x_{-2} = b,\ x_{-1} = c$ and $x_0 = d$. Then, for $n = 0,1,\dots$*

$$\begin{aligned}x_{3n-3} &= \frac{(-1)^{n-1}abcd}{(f_{n-1}a-f_{n-2}b)(f_{n-1}b-f_{n-2}c)(f_{n-1}c-f_{n-2}d)},\\ x_{3n-2} &= \frac{(-1)^{n}abcd}{(f_{n}a-f_{n-1}b)(f_{n-1}b-f_{n-2}c)(f_{n-1}c-f_{n-2}d)},\\ x_{3n-1} &= \frac{(-1)^{n+1}abcd}{(f_{n}a-f_{n-1}b)(f_{n}b-f_{n-1}c)(f_{n-1}c-f_{n-2}d)},\end{aligned}$$

where $\{f_n\}_{n=-2}^{\infty}$, is called Fibonacci sequence.

Proof. It can be clearly seen that the solution is confirmed for $n = 0$. Next, we assume that $n > 0$ and the above-mentioned results hold for $n-1$. This leads to that

$$\begin{aligned}x_{3n-7} &= \frac{(-1)^{n-1}abcd}{(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-3}c-f_{n-4}d)},\\ x_{3n-6} &= \frac{(-1)^{n-2}abcd}{(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)},\\ x_{3n-5} &= \frac{(-1)^{n-1}abcd}{(f_{n-1}a-f_{n-2}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)},\\ x_{3n-4} &= \frac{(-1)^{n}abcd}{(f_{n-1}a-f_{n-2}b)(f_{n-1}b-f_{n-2}c)(f_{n-2}c-f_{n-3}d)}.\end{aligned}$$

Next, from Eq. (3.8) we have

$$
\begin{aligned}
x_{3n-3} &= \frac{x_{3n-4}x_{3n-7}}{x_{3n-6}-x_{3n-7}} \\
&= \frac{\left(\frac{(-1)^{n}abcd}{(f_{n-1}a-f_{n-2}b)(f_{n-1}b-f_{n-2}c)(f_{n-2}c-f_{n-3}d)}\right)\left(\frac{(-1)^{n-1}abcd}{(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-3}c-f_{n-4}d)}\right)}{\left[\begin{array}{c}\frac{(-1)^{n-2}abcd}{(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)}- \\ \frac{(-1)^{n-1}abcd}{(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-3}c-f_{n-4}d)}\end{array}\right]} \\
&= \frac{\frac{(-1)^{2n-1}(abcd)^2(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)}{(f_{n-2}c-f_{n-3}d)(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-3}c-f_{n-4}d)}}{\begin{array}{c}(f_{n-1}a-f_{n-2}b)(f_{n-1}b-f_{n-2}c)(f_{n-2}c-f_{n-3}d)(f_{n-2}a-f_{n-3}b) \\ (f_{n-2}b-f_{n-3}c)(f_{n-3}c-f_{n-4}d)(abcd) \\ \left[\begin{array}{c}(-1)^{n-2}(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-3}c-f_{n-4}d) \\ -(-1)^{n-1}(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)\end{array}\right]\end{array}} \\
&= \frac{(-1)^{2n-1}(abcd)(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)}{\begin{array}{c}(f_{n-1}a-f_{n-2}b)(f_{n-1}b-f_{n-2}c)(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c) \\ \left[(-1)^{n-2}(f_{n-3}c-f_{n-4}d)-(-1)^{n-1}(f_{n-2}c-f_{n-3}d)\right]\end{array}} \\
& \frac{(-1)^{2n-1}(abcd)}{(f_{n-1}a-f_{n-2}b)(f_{n-1}b-f_{n-2}c)(-1)^{n}\left[(f_{n-3}c-f_{n-4}d)+(f_{n-2}c-f_{n-3}d)\right]} \\
&= \frac{(-1)^{n-1}(abcd)}{(f_{n-1}a-f_{n-2}b)(f_{n-1}b-f_{n-2}c)\left[(f_{n-1}c-f_{n-2}d)\right]}.
\end{aligned}
$$

We now turn to prove the second solution of our equation. Again, from Eq. (3.8) we have

$$
\begin{aligned}
x_{3n-2} &= \frac{x_{3n-3}x_{3n-6}}{-x_{3n-6}+x_{3n-5}} \\
&= \frac{\left(\frac{(-1)^{n-1}abcd}{(f_{n-1}a-f_{n-2}b)(f_{n-1}b-f_{n-2}c)(f_{n-1}c-f_{n-2}d)}\right)\left(\frac{(-1)^{n-2}abcd}{(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)}\right)}{\left[\begin{array}{c}\left(-\frac{(-1)^{n-2}abcd}{(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)}\right)+ \\ \left(\frac{(-1)^{n-1}abcd}{(f_{n-1}a-f_{n-2}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)}\right)\end{array}\right]} \\
&= \frac{\frac{(-1)^{2n-3}(abcd)^2(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)}{(f_{n-2}c-f_{n-3}d)(f_{n-1}a-f_{n-2}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)}}{\begin{array}{c}(f_{n-1}a-f_{n-2}b)(f_{n-1}b-f_{n-2}c)(f_{n-1}c-f_{n-2}d)(f_{n-2}a-f_{n-3}b) \\ (f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)(abcd) \\ \left[\begin{array}{c}-(-1)^{n-2}(f_{n-1}a-f_{n-2}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d) \\ +(-1)^{n-1}(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)\end{array}\right]\end{array}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(-1)^{2n-3}(abcd)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)}{\begin{array}{c}(f_{n-1}b-f_{n-2}c)(f_{n-1}c-f_{n-2}d)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d) \\ \left[-(-1)^{n-2}(f_{n-1}a-f_{n-2}b)+(-1)^{n-1}(f_{n-2}a-f_{n-3}b)\right]\end{array}} \\
&= \frac{(-1)^{2n-3}abcd}{(f_{n-1}b-f_{n-2}c)(f_{n-1}c-f_{n-2}d)(-1)^{n-1}\left[(f_{n}a-f_{n-1}b)\right]} \\
&= \frac{(-1)^{n}abcd}{(f_{n-1}b-f_{n-2}c)(f_{n-1}c-f_{n-2}d)(f_{n}a-f_{n-1}b)}.
\end{aligned}
$$

Finally, we will show the last part of the solution. Eq.(3.8) leads to

$$
\begin{aligned}
x_{3n-1} &= \frac{x_{3n-2}x_{3n-5}}{-x_{3n-5}+x_{3n-4}} \\
&= \frac{\left(\frac{(-1)^{n}abcd}{(f_{n-1}b-f_{n-2}c)(f_{n-1}c-f_{n-2}d)(f_{n}a-f_{n-1}b)}\right)\left(\frac{(-1)^{n-1}abcd}{(f_{n-1}a-f_{n-2}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)}\right)}{-\left[\begin{array}{c}\frac{(-1)^{n-1}abcd}{(f_{n-1}a-f_{n-2}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)}+\\ \frac{(-1)^{n}abcd}{(f_{n-1}a-f_{n-2}b)(f_{n-1}b-f_{n-2}c)(f_{n-2}c-f_{n-3}d)}\end{array}\right]} \\
&= \frac{\begin{array}{c}(-1)^{2n-1}(abcd)^{2}(f_{n-1}a-f_{n-2}b)(f_{n-2}b-f_{n-3}c)\\ (f_{n-2}c-f_{n-3}d)(f_{n-1}a-f_{n-2}b)(f_{n-1}b-f_{n-2}c)(f_{n-2}c-f_{n-3}d)\end{array}}{\begin{array}{c}(f_{n-1}b-f_{n-2}c)(f_{n-1}c-f_{n-2}d)(f_{n}a-f_{n-1}b)(f_{n-1}a-f_{n-2}b)\\ (f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)(abcd)\\ \left[\begin{array}{c}-(-1)^{n-1}(f_{n-1}a-f_{n-2}b)(f_{n-1}b-f_{n-2}c)(f_{n-2}c-f_{n-3}d)\\ +(-1)^{n}(f_{n-1}a-f_{n-2}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)\end{array}\right]\end{array}} \\
&= \frac{(-1)^{2n-1}(abcd)(f_{n-1}a-f_{n-2}b)(f_{n-2}c-f_{n-3}d)}{\begin{array}{c}(f_{n}a-f_{n-1}b)(f_{n-1}c-f_{n-2}d)(f_{n-1}a-f_{n-2}b)\\ (f_{n-2}c-f_{n-3}d)(-1)^{n}\left[(f_{n-2}b-f_{n-3}c)+(f_{n-1}b-f_{n-2}c)\right]\end{array}} \\
&= \frac{(-1)^{n-1}abcd}{(f_{n}a-f_{n-1}b)(f_{n-1}c-f_{n-2}d)\left[(f_{n}b-f_{n-1}c)\right]} \\
&= \frac{(-1)^{n+1-2}abcd}{(f_{n}a-f_{n-1}b)(f_{n-1}c-f_{n-2}d)\left[(f_{n}b-f_{n-1}c)\right]} \\
&= \frac{(-1)^{n+1}abcd}{(f_{n}a-f_{n-1}b)(f_{n-1}c-f_{n-2}d)(f_{n}b-f_{n-1}c)}.
\end{aligned}
$$

□

3.4. Numerical confirmation

This subsection is included to verify and confirm the results we obtained in this work.

Example 3.5. *This example pictured the stability of the fixed point when we take* $\alpha=\beta=1,\ \gamma=7,\ x_{-3}=-3,\ x_{-2}=3,\ x_{-1}=-5$ *and* $x_0=5$. *See Figure 3.1.*

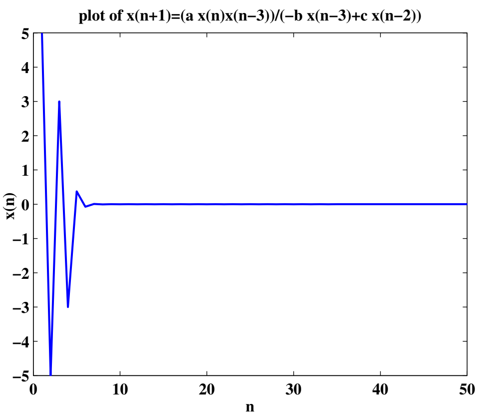

Figure 3.1

Example 3.6. *In Figure 3.2, we consider* $\alpha=15,\ \beta=1,\ \gamma=14,\ x_{-3}=0.1,\ x_{-2}=-0.5,\ x_{-1}=1$ *and* $x_0=-1$.

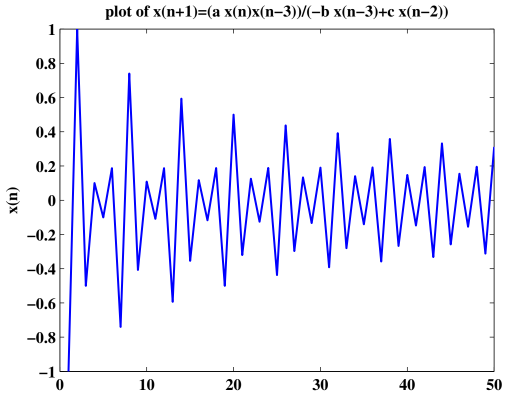

Figure 3.2

Example 3.7. *The stability of Eq.(3.8) is shown in Figure 3.3, when we let* $x_{-3} = 5,\ x_{-2} = -8,\ x_{-1} = 10$ *and* $x_0 = -10$.

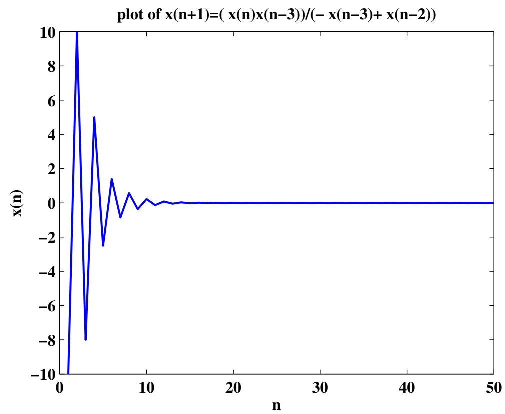

Figure 3.3

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Existence and Iteration of Monotone Positive Solution for a Fourth-Order Nonlinear Boundary Value Problem

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Abstract

This paper is concerned with the following fourth-order three-point boundary value problem BVP

$$u^{(4)}(t) = f(t, u(t)), \quad t \in [0,1],$$
$$u'(0) = u''(0) = u(1) = 0, \; u'''(\eta) + \alpha u(0) = 0,$$

where $f \in C([0,1] \times [0,+\infty), [0,+\infty))$, $\alpha \in [0,6)$ and $\eta \in \left[\frac{2}{3}, 1\right)$. Although corresponding Green's function is sign-changing, we still obtain the existence of monotone positive solution under some suitable conditions on f by applying iterative method. An example is also given to illustrate the main results.

1. Introduction

Fourth-order ordinary differential equations have attracted a lot of attention due to their applications in engineering, physics, material mechanics, fluid mechanics and so on. Many approaches, such as the Leray– Schauder nonlinear alternative, fixed point index theory in cones, the method of upper and lower solutions, degree theory, Guo-Krasnoselskii's fixed point theorem, Leggett-Williams fixed-point theorem, are used to study the existence of single or multiple positive solutions to some fourth-order boundary value problem, see [1]-[13]. However, all the above-mentioned papers are achieved when corresponding Green's functions are nonnegative, which is a very important condition.

Recently, the existence of positive solutions of the boundary value problems with sign-changing Green's function has received increasing interest.

In 2008, Palamides and Smyrlis [14] studied the existence of at least one positive solution to the singular third-order three-point BVP with an indefinitely signed Green's function

$$u'''(t) = a(t) f(t, u(t)) = 0, \quad t \in (0,1),$$
$$u(0) = u(1) = u''(\eta) = 0,$$

where $\eta \in \left(\frac{17}{24}, 1\right)$. Their technique was a combination of the Guo-Krasnoselskii's fixed point theorem [15, 16] and properties of the corresponding vector field.

In 2018, Zhang et al [17] studied the existence of at least $n-1$ decreasing positive solutions of the problem

$$u^{(4)}(t) = f(t, u(t)) = 0, \quad t \in [0,1],$$
$$u(0) = u(1) = u''(\eta) = 0,$$

their main tool is the fixed point index theory.

It is worth mentioning that there are other types of works on sign-changing Green's functions which prove the existence of sign-changing solutions, positive in some cases; see [18]-[22].

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Motivated and inspired by the above-mentioned works, in this paper we will study the following nonlinear fourth-order three-point BVP with sign-changing Green's function

$$\begin{gathered} u^{(4)}(t)=f(t,u(t)) \quad t\in[0,1], \\ u'(0)=u''(0)=u(1)=0,\ u'''(\eta)+\alpha u(0)=0, \end{gathered} \tag{1.1}$$

by applying iterative method. Throughout this paper, we always assume that $f\in C([0,1]\times[0,+\infty),[0,+\infty))$, $\alpha\in[0,6)$ and $\eta\in\left[\frac{2}{3},1\right)$. By imposing some suitable conditions on f and η, we obtain the existence of monotone positive solution for the BVP (1.1). Moreover, our iterative scheme starts off with zero function, which implies that the iterative scheme is feasible.

2. Main results

Let Banach space $E=C[0,1]$ be equipped with the norm $\|u\|=\max_{t\in[0,1]}|u(t)|$.

Lemma 2.1. *The BVP*

$$\begin{gathered} u^{(4)}(t)=0 \quad t\in[0,1], \\ u'(0)=u''(0)=u(1)=0,\ u'''(\eta)+\alpha u(0)=0 \end{gathered}$$

has only trivial solution.

Proof. It is simple to check. □

Now, for any $y\in E$, we consider the BVP

$$\begin{gathered} u^{(4)}(t)=y(t) \quad t\in[0,1], \\ u'(0)=u''(0)=u(1)=0,\ u'''(\eta)+\alpha u(0)=0. \end{gathered}$$

After a direct computation, one may obtain the expression of Green's function $G(t,s)$ of the BVP as follows: for $s\geq\eta$,

$$G(t,s)=\begin{cases} -\frac{(6-\alpha t^3)(1-s)^3}{6(6-\alpha)}, & 0\leq t\leq s\leq 1 \\ \frac{(t-s)^3}{6}-\frac{(6-\alpha t^3)(1-s)^3}{6(6-\alpha)}, & 0\leq s\leq t\leq 1 \end{cases}$$

and for $s<\eta$,

$$G(t,s)=\begin{cases} -\frac{(6-\alpha t^3)(1-s)^3}{6(6-\alpha)}+\frac{1-t^3}{6-\alpha}, & 0\leq t\leq s\leq 1 \\ \frac{(t-s)^3}{6}-\frac{(6-\alpha t^3)(1-s)^3}{6(6-\alpha)}+\frac{1-t^3}{6-\alpha}, & 0\leq s\leq t\leq 1. \end{cases}$$

Remark 2.2. *$G(t,s)$ has the following properties:*

$$G(t,s)\geq 0 \quad for \quad 0\leq s<\eta \qquad and \quad G(t,s)\leq 0 \quad for \quad \eta\leq s\leq 1.$$

Moreover, for $s\geq\eta$,

$$\max\{G(t,s): t\in[0,1]\}=G(1,s)=0,$$

$$\min\{G(t,s): t\in[0,1]\}=G(0,s)=-\frac{(1-s)^3}{6-\alpha}\geq-\frac{(1-\eta)^3}{6-\alpha}$$

and for $s<\eta$,

$$\max\{G(t,s): t\in[0,1]\}=G(0,s)=\frac{s^3+3s-3s^2}{6-\alpha}\leq\frac{\eta^3+3\eta-3\eta^2}{6-\alpha},$$

$$\min\{G(t,s): t\in[0,1]\}=G(1,s)=0.$$

So, if we let $M=\max\{|G(t,s)|: t,s\in[0,1]\}$, then

$$M=\max\left\{\frac{(1-\eta)^3}{6-\alpha},\frac{\eta^3+3\eta-3\eta^2}{6-\alpha}\right\}<\frac{1}{6-\alpha}.$$

Let

$$K=\{y\in E: y(t) \text{ is nonnegative and decreasing on } [0,1]\}.$$

Then K is a cone in E. Note that this induces an order relation "$\precsim$" in E by defining $u\precsim v$ if and only if $v-u\in K$. In the remainder of this paper, we always assume that f satisfies the following two conditions:

(H_1) for each $u\in[0,+\infty)$, the mapping $t\longmapsto f(t,u)$ is decreasing;

(H_2) for each $t\in[0,1]$, the mapping $u\longmapsto f(t,u)$ is increasing.

Now, we define an operator T as follows:

$$(Tu)(t)=\int_0^1 G(t,s)f(s,u(s))\,ds, \quad u\in K,\ t\in[0,1].$$

Obviously, if u is a fixed point of T in K, then u is a nonnegative and decreasing solution of the BVP (1.1).

Lemma 2.3. *$T : K \to K$ is completely continuous.*

Proof. Let $u \in K$. Then, for $t \in [0,\eta]$, we have

$$(Tu)(t) = \int_0^t \left[\frac{(t-s)^3}{6} + \frac{1-t^3}{6-\alpha} - \frac{\left(6-\alpha t^3\right)(1-s)^3}{6(6-\alpha)}\right] f(s,u(s))\,ds$$
$$+\int_t^\eta \left[\frac{1-t^3}{6-\alpha} - \frac{\left(6-\alpha t^3\right)(1-s)^3}{6(6-\alpha)}\right] f(s,u(s))\,ds + \int_\eta^1 \left[-\frac{\left(6-\alpha t^3\right)(1-s)^3}{6(6-\alpha)}\right] f(s,u(s))\,ds,$$

which together with (H_1) and (H_2) implies that

$$\begin{aligned}
(Tu)'(t) &= \int_0^t \left[\frac{(t-s)^2}{2} - \frac{3t^2}{6-\alpha} + \frac{\alpha t^2(1-s)^3}{2(6-\alpha)}\right] f(s,u(s))\,ds + \int_t^\eta \left[-\frac{3t^2}{6-\alpha} + \frac{\alpha t^2(1-s)^3}{2(6-\alpha)}\right] f(s,u(s))\,ds \\
&\quad + \int_\eta^1 \left[\frac{\alpha t^2(1-s)^3}{2(6-\alpha)}\right] f(s,u(s))\,ds \\
&= \int_0^t \left[\frac{t^2}{2} + \frac{s^2-2ts}{2} - \frac{3t^2}{6-\alpha} + \frac{\alpha t^2(1-s)^3}{2(6-\alpha)}\right] f(s,u(s))\,ds + \int_t^\eta \left[-\frac{3t^2}{6-\alpha} + \frac{\alpha t^2(1-s)^3}{2(6-\alpha)}\right] f(s,u(s))\,ds \\
&\quad + \int_\eta^1 \left[\frac{\alpha t^2(1-s)^3}{2(6-\alpha)}\right] f(s,u(s))\,ds \\
&= \int_0^\eta \frac{\alpha t^2\left(-3s+3s^2-s^3\right)}{2(6-\alpha)} f(s,u(s))\,ds - \frac{t^2}{2}\int_t^\eta f(s,u(s))\,ds + \int_0^t \frac{s^2-2ts}{2} f(s,u(s))\,ds \\
&\quad - \frac{t^2}{2}\int_t^\eta f(s,u(s))\,ds + \int_0^t \frac{s^2-2ts}{2} f(s,u(s))\,ds \\
&\le f(\eta,u(\eta))\,[\ \frac{\alpha t^2}{2(6-\alpha)}\int_0^\eta \left(-3s+3s^2-s^3\right)ds - \frac{t^2}{2}\int_t^\eta ds + \int_0^t \frac{s^2-2ts}{2}\,ds + \frac{\alpha t^2}{2(6-\alpha)}\int_\eta^1 (1-s)^3\,ds\ ] \\
&= \frac{t^2}{2} f(\eta,u(\eta))\,[\ \frac{\alpha t^2}{(6-\alpha)}\left(\frac{1}{4}-\eta\right) - \eta + \frac{t}{3}\ ] \\
&\le \frac{t^2}{2} f(\eta,u(\eta)) \left[\frac{\alpha t^2(1-4\eta)}{(6-\alpha)} - \frac{2\eta}{3}\right] \\
&\le 0.
\end{aligned}$$

For $t \in [\eta,1]$, we have

$$(Tu)(t) = \int_0^\eta \left[\frac{(t-s)^3}{6} + \frac{1-t^3}{6-\alpha} - \frac{\left(6-\alpha t^3\right)(1-s)^3}{6(6-\alpha)}\right] f(s,u(s))$$
$$+\int_\eta^t \left[\frac{(t-s)^3}{6} - \frac{\left(6-\alpha t^3\right)(1-s)^3}{6(6-\alpha)}\right] f(s,u(s)) + \int_t^1 \left[-\frac{\left(6-\alpha t^3\right)(1-s)^3}{6(6-\alpha)}\right] f(s,u(s)),$$

which together with (H_1) and (H_2) implies that

$$\begin{aligned}
(Tu)'(t) &= \int_0^\eta \left[\frac{(t-s)^2}{2} - \frac{3t^2}{6-\alpha} + \frac{\alpha t^2(1-s)^3}{2(6-\alpha)}\right] f(s,u(s))\,ds + \int_\eta^t \left[\frac{(t-s)^2}{2} + \frac{\alpha t^2(1-s)^3}{2(6-\alpha)}\right] f(s,u(s))\,ds \\
&\quad + \int_t^1 \frac{\alpha t^2(1-s)^3}{2(6-\alpha)} f(s,u(s))\,ds \\
&= \frac{\alpha t^2}{2(6-\alpha)}\int_0^\eta \left(-3s+3s^2-s^3\right) f(s,u(s))\,ds + \int_0^\eta \left(\frac{s^2-ts}{2}\right) f(s,u(s))\,ds \\
&\quad + \int_\eta^t \frac{(t-s)^2}{2} f(s,u(s))\,ds + \int_\eta^1 \frac{\alpha t^2(1-s)^3}{2(6-\alpha)} f(s,u(s))\,ds \\
&\le \frac{\alpha t^2}{2(6-\alpha)} f(\eta,u(\eta))\,[\ \int_0^\eta \left(-3s+3s^2-s^3\right)ds + \int_0^\eta \left(\frac{s^2-ts}{2}\right)ds + \int_\eta^t \frac{(t-s)^2}{2}\,ds + \int_\eta^1 \frac{\alpha t^2(1-s)^3}{2(6-\alpha)}\,ds\ ] \\
&= \frac{t^2}{2} f(\eta,u(\eta))\,[\ \frac{\alpha t^2(1-4\eta)}{(6-\alpha)} + \frac{t-3\eta}{3}\ ] \\
&= \frac{t^2}{2} f(\eta,u(\eta))\,[\ \frac{\alpha t^2(1-4\eta)}{(6-\alpha)} + \frac{1-3\eta}{3}\ ] \\
&\le 0.
\end{aligned}$$

So, $(Tu)(t)$ is decreasing on $[0,1]$. At the same time, since $(Tu)(1) = 0$, we know that $(Tu)(t)$ is nonnegative on $[0,1]$. This indicates that $Tu \in K$.

Now, we assume that $D \subset K$ is a bounded set. Then there exists a constant $C_1 > 0$ such that $\|u\| \leq C_1$ for any $u \in D$. In what follows, we will prove that $T(D)$ is relatively compact.
Let

$$C_2 = \sup\{f(t,u) : (t,u) \in [0,1] \times [0,C_1]\}.$$

Then for any $y \in T(D)$, there exists $u \in D$ such that $y = Tu$, and so,

$$\begin{aligned}
|y(t)| = |(Tu)(t)| &= \left|\int_0^1 G(t,s) f(s,u(s))\,ds\right| \\
&\leq \int_0^1 |G(t,s)| f(s(,u(s)))\,ds \\
&\leq M \int_0^1 f(s,u(s))\,ds \leq MC_2, \quad t \in [0,1],
\end{aligned}$$

which implies that $T(D)$ is uniformly bounded. On the other hand, when $\varepsilon > 0$, if we choose $0 < \tau < \min\left\{1-\eta, \frac{\varepsilon}{12C_2(M+1)}\right\}$, then, for any $u \in D$,

$$\int_{\eta-\tau}^{\eta+\tau} f(s,u(s))\,ds \leq 2C_2\tau < \frac{\varepsilon}{6(M+1)}. \tag{2.1}$$

Since $G(t,s)$ is uniformly continuous on $[0,1] \times [0,\eta-\tau]$ and $[0,1] \times [\eta+\tau,1]$, there exists $\delta > 0$ such that for any $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta$,

$$|G(t_1,s) - G(t_2,s)| < \frac{\varepsilon}{3(C_2+1)(\eta-\tau)}, \; s \in [0,\eta-\tau] \tag{2.2}$$

and

$$|G(t_1,s) - G(t_2,s)| < \frac{\varepsilon}{3(C_2+1)(1-\eta-\tau)}, \quad s \in [\eta+\tau,1]. \tag{2.3}$$

In view of (2.1), (2.2) and (2.3), for any $y \in T(D)$ and $t_1, t_2 \in [0,1]$ with
$|t_1 - t_2| < \delta$,

$$\begin{aligned}
|y(t_1) - y(t_2)| &= |T(t_1) - T(t_2)| \\
&= \left|\int_0^1 (G(t_1,s) - G(t_2,s)) f(s,u(s))\,ds\right| \\
&\leq \int_0^1 |(G(t_1,s) - G(t_2,s))| f(s,u(s))\,ds \\
&= \int_0^{\eta-\tau} |(G(t_1,s) - G(t_2,s))| f(s,u(s))\,ds + \int_{\eta-\tau}^{\eta+\tau} |(G(t_1,s) - G(t_2,s))| f(s,u(s))\,ds \\
&\quad + \int_{\eta+\tau}^1 |(G(t_1,s) - G(t_2,s))| f(s,u(s))\,ds \\
&\leq C_2 \frac{\varepsilon}{3(C_2+1)(\eta-\tau)}(\eta-\tau) + \frac{\varepsilon}{3(M+1)}M + C_2 \frac{\varepsilon}{3(C_2+1)(1-\eta-\tau)}(1-\eta-\tau) \\
&= \frac{C_2\varepsilon}{3(C_2+1)} + \frac{M\varepsilon}{3(M+1)} + \frac{C_2\varepsilon}{3(C_2+1)} = \varepsilon,
\end{aligned}$$

which implies that $T(D)$ is equicontinuous. By Arzela-Ascoli theorem, we know that $T(D)$ is relatively compact. Thus, we have shown that T is a compact operator.
Finally, we prove that T is continuous. Suppose that $u_n\,(n = 1,2,\ldots)$, $u_0 \in K$ and $\|u_n - u_0\| \to 0\;(n \to 0)$. Then there exists $C_3 > 0$ such that for any n, $\|u_n\| \leq C_3$.
Let

$$C_4 = \sup\{f(t,u) : (t,u) \in [0,1] \times [0,C_3]\}.$$

Then for any n and $t \in [0,1]$, we have

$$G(t,s) f(s,u_n(s)) \leq MC_4, \quad s \in [0,1].$$

By applying Lebesgue Dominated Convergence theorem, we obtain

$$\begin{aligned}
\lim_{n\to\infty} (Tu_n)(t) &= \lim_{n\to\infty} \int_0^1 G(t,s) f(s,u_n(s))\,ds \\
&= \int_0^1 G(t,s) \lim_{n\to\infty} f(s,u_n(s))\,ds \\
&= \int_0^1 G(t,s) f(s,u_0(s))\,ds = T(u_0)(t), \; t \in [0,1],
\end{aligned}$$

which indicates that T is continuous.Therefore, $T : K \to K$ is completely continuous. □

Theorem 2.4. *Assume that $f(t,0)\not\equiv 0$ for $t\in[0,1]$ and there exist two positive constants a and b such that the following conditions are satisfied:*
(H_3) $f(0,a)\le(6-\alpha)a$;
(H_4) $b(u_2-u_1)\le f(t,u_2)-f(t,u_1)\le 2b(u_2-u_1)$, $0\le t\le 1$,
$0\le u_1\le u_2\le a$. *If we construct an iterative sequence $v_{n+1}=Tv_n$, $n=0,1,2,\dots$, where $v_0(t)\equiv 0$ fot $t\in[0,1]$, then $\{v_n\}_{n=1}^{\infty}$ converges to v^* in E and v^* is a decreasing and positive solution of the BVP (1.1)*

Proof. Let $K_a=\{u\in K:\ \|u\|\le a\}$. Then it follows from Lemma 2.3 that $Tu\in K$. In view of (H_3) and $0\le u(s)\le 1$ for $s\in[0,1]$, we have

$$\begin{aligned}0\le (Tu)(t)&=\int_0^1 G(t,s)f(s,u(s))\,ds\\&\le\int_0^1|G(t,s)|f(0,a)\,ds\\&\le(6-\alpha)aM\le a,\quad t\in[0,1],\end{aligned}$$

which shows that $\|Tu\|\le a$. So, $T:K_a\to K_a$. Now, we prove that $\{v_n\}_{n=1}^{\infty}$ converges to v^*in E and v^* is a decreasing and positive solution of the BVP (1.1). Indeed, in view of $v_0\in K_a$ and $T:K_a\to K_a$, we have $v_n\in K_a$, $n=0,1,2,\dots$. Since the set $\{v_n\}_{n=0}^{\infty}$ is bounded and T is completely continuous, we know that the set $\{v_n\}_{n=0}^{\infty}$ is relatively compact. In what follows, we prove that $\{v_n\}_{n=0}^{\infty}$ is monotone by induction. First, it is obvious that $v_1-v_0=v_1\in K$, which shows that $v_0\precsim v_1$. Next, we assume that $v_{k-1}\precsim v_k$. Then it follows from (H_4) that for $0\le t\le\eta$, we obtain

$$\begin{aligned}&v'_{k+1}(t)-v'_k(t)\\&=(Tv_k)'(t)-(Tv_{k-1})'(t)\\&=\int_0^1\tfrac{\partial G(t,s)}{\partial t}\left[f(s,v_k(s))-f(s,v_{k-1}(s))\right]ds\\&=\tfrac{\alpha t^2}{2(6-\alpha)}\int_0^\eta\left(3s^2-3s-s^3\right)\left[f(s,v_k(s))-f(s,v_{k-1}(s))\right]ds\\&\quad+\int_0^t\left(\tfrac{s^2-2ts}{2}\right)\left[f(s,v_k(s))-f(s,v_{k-1}(s))\right]ds-\tfrac{t^2}{2}\int_t^\eta\left[f(s,v_k(s))-f(s,v_{k-1}(s))\right]ds\\&\quad+\tfrac{\alpha t^2}{2(6-\alpha)}\int_\eta^1(1-s)^3\left[f(s,v_k(s))-f(s,v_{k-1}(s))\right]ds\\&\le\tfrac{b\alpha t^2}{2(6-\alpha)}\int_0^\eta\left(3s^2-3s-s^3\right)\left[v_k(s)-v_{k-1}(s)\right]ds+b\int_0^t\left(\tfrac{s^2-2ts}{2}\right)\left[v_k(s)-v_{k-1}(s)\right]ds\\&\quad-\tfrac{bt^2}{2}\int_t^\eta\left[f(s,v_k(s))-f(s,v_{k-1}(s))\right]ds+\tfrac{2b\alpha t^2}{2(6-\alpha)}\int_\eta^1(1-s)^3\left[v_k(s)-v_{k-1}(s)\right]ds\\&\le b\left[v_k(\eta)-v_{k-1}(\eta)\right]\left[\tfrac{\alpha t^2}{2(6-\alpha)}\int_0^\eta\left(3s^2-3s-s^3\right)ds+\int_0^t\left(\tfrac{s^2-2ts}{2}\right)ds-\tfrac{t^2}{2}\int_t^\eta ds+\tfrac{\alpha t^2}{(6-\alpha)}\int_\eta^1(1-s)^3ds\right]\\&=\tfrac{t^2}{2}b\left[v_k(\eta)-v_{k-1}(\eta)\right]\left[\tfrac{\alpha\left(\eta^4-4\eta^3+6\eta^2-8\eta+2\right)}{4(6-\alpha)}-\eta+\tfrac{t}{3}\right]\\&\le\tfrac{t^2}{2}b\left[v_k(\eta)-v_{k-1}(\eta)\right]\left[\tfrac{\alpha(-3\eta+2)}{4(6-\alpha)}-\tfrac{2\eta}{3}\right]\\&\le\tfrac{t^2}{2}b\left[v_k(\eta)-v_{k-1}(\eta)\right]\left[\tfrac{\alpha(-3\eta+2)}{4(6-\alpha)}-\tfrac{2\eta}{3}\right]\le 0.\end{aligned}$$

For $\eta\le t\le 1$, we get

$$\begin{aligned}&v'_{k+1}(t)-v'_k(t)\\&=(Tv_k)'(t)-(Tv_{k-1})'(t)\\&=\int_0^1\tfrac{\partial G(t,s)}{\partial t}\left[f(s,v_k(s))-f(s,v_{k-1}(s))\right]ds\\&=\tfrac{\alpha t^2}{2(6-\alpha)}\int_0^\eta\left(3s^2-3s-s^3\right)\left[f(s,v_k(s))-f(s,v_{k-1}(s))\right]ds+\int_0^\eta\left(\tfrac{s^2-2ts}{2}\right)\left[f(s,v_k(s))-f(s,v_{k-1}(s))\right]ds\\&\quad+\int_\eta^t\tfrac{(t-s)^2}{2}\left[f(s,v_k(s))-f(s,v_{k-1}(s))\right]ds+\int_\eta^1\tfrac{\alpha t^2(1-s)^3}{2(6-\alpha)}\left[f(s,v_k(s))-f(s,v_{k-1}(s))\right]ds\\&\le\tfrac{b\alpha t^2}{2(6-\alpha)}\int_0^\eta\left(3s^2-3s-s^3\right)\left[v_k(s)-v_{k-1}(s)\right]ds+b\int_0^\eta\left(\tfrac{s^2-2ts}{2}\right)\left[v_k(s)-v_{k-1}(s)\right]ds\\&\quad+2b\int_\eta^t\tfrac{(t-s)^2}{2}\left[v_k(s)-v_{k-1}(s)\right]ds+2b\int_\eta^1\tfrac{\alpha t^2(1-s)^3}{2(6-\alpha)}\left[v_k(s)-v_{k-1}(s)\right]ds]\\&\le b\times\left[v_k(\eta)-v_{k-1}(\eta)\right]\times\left[\tfrac{\alpha t^2}{2(6-\alpha)}\int_0^\eta\left(3s^2-3s-s^3\right)ds+\int_0^\eta\left(\tfrac{s^2-2ts}{2}\right)ds+2\int_\eta^t\tfrac{(t-s)^2}{2}ds+2b\int_\eta^1\tfrac{\alpha t^2(1-s)^3}{2(6-\alpha)}ds\right]\\&=b\times\left[v_k(\eta)-v_{k-1}(\eta)\right]\times\left[\tfrac{\alpha t^2\left(\eta^4-4\eta^3+6\eta^2-8\eta+2\right)}{8(6-\alpha)}-\tfrac{\eta^3}{6}+\tfrac{t\eta^2}{2}+\tfrac{t^3}{3}-t^2\eta\right]\\&\le\tfrac{t^2}{2}b\times\left[v_k(\eta)-v_{k-1}(\eta)\right]\times\left[\tfrac{\alpha t^2\left(\eta^4-4\eta^3+6\eta^2-8\eta+2\right)}{4(6-\alpha)}+\tfrac{2t}{3}-\eta\right]\\&\le\tfrac{t^2}{2}b\times\left[v_k(\eta)-v_{k-1}(\eta)\right]\times\left[\tfrac{\alpha t^2(-3\eta+2)}{4(6-\alpha)}+\tfrac{2-3\eta}{3}\right]\le 0,\end{aligned}$$

hence

$$v'_{k+1}(t) - v'_k(t) \leq 0,\ t \in [0,1], \tag{2.4}$$

that is $v_{k+1}(t) - v_k(t)$ is decreasing on $[0,1]$. At the same time, it is easy to see that

$$v_{k+1}(1) - v_k(1) = \int_0^1 G(1,s)\left[f(s, v_k(s) - v_{k-1}(s))\right] ds = 0,$$

the last equation implies that

$$v_{k+1}(t) - v_k(t) \geq 0,\ t \in [0,1]. \tag{2.5}$$

It follows from (2.4) and (2.5) that $v_{k+1} - v_k \in K$, which indicates that $v_{k+1} \precsim v_k$. Thus, we have shown that $v_{k+1} \precsim v_k$, $n = 0,1,2,...$ Since $\{v_n\}_{n=1}^{\infty}$ is relatively compact and monotone, there exists a $v^* \in K_a$ such that $\lim_{n\to\infty} v_n = v^*$, which together with the continuity of T and the fact that $v_{n+1} = Tv_n$ implies that $v^* = Tv^*$. This indicates that v^* is a decreasing nonnegative solution of (1.1). Moreover, in view of $f(t,0) \neq 0$ for $t \in [0,1]$, we know that zero function is not a solution of (1.1), which shows that is v^* a positive solution of (1.1). □

3. An example

Consider the boundary value problem

$$\begin{gathered} u^{(4)}(t) = f(t,u(t)) \quad t \in [0,1], \\ u'(0) = u''(0) = u(1) = 0,\ u'''(\eta) + \alpha u(0) = 0, \end{gathered} \tag{3.1}$$

If we let $\eta = \frac{3}{4}$, $\alpha = 4$ and $f(t,u) = \frac{1}{2}u^2(t) + t$, $(t,u) \in [0,1] \times [0,+\infty)$, then all the hypotheses of Theorem 2.4 are fulfilled with $a = 3$ and $b = \frac{3}{4}$. Therefore, it follows from Theorem 2.4 that the BVP (3.1) has a decreasing and positive solution. Moreover, the iterative scheme is $v_0(t) \equiv 0$ for $t \in [0,1]$ and

$$v_{n+1}(t) = \begin{cases} \int_0^t \left[\frac{(t-s)^3}{6} + \frac{1-t^3}{2} - \frac{(3-2t^3)(1-s)^3}{6}\right] \times \left[\frac{1}{2}(v_n(s))^2 + s\right] ds \\ \quad + \int_t^{\frac{3}{4}} \left[\frac{1-t^3}{2} - \frac{(3-2t^3)(1-s)^3}{6}\right] \times \left[\frac{1}{2}(v_n(s))^2 + s\right] ds \\ \quad + \int_{\frac{3}{4}}^1 \left[-\frac{(3-2t^3)(1-s)^3}{6}\right] \times \left[\frac{1}{2}(v_n(s))^2 + s\right] ds \\ \qquad if \quad t \in \left[0, \frac{3}{4}\right],\ n = 0,1,2... \\ \int_0^{\frac{3}{4}} \left[\frac{(t-s)^3}{6} + \frac{1-t^3}{2} - \frac{(3-2t^3)(1-s)^3}{6}\right] \times \left[\frac{1}{2}(v_n(s))^2 + s\right] ds \\ \quad + \int_{\frac{3}{4}}^t \left[\frac{(t-s)^3}{6} - \frac{(3-2t^3)(1-s)^3}{6}\right] \times \left[\frac{1}{2}(v_n(s))^2 + s\right] ds \\ \quad \int_t^1 \left[-\frac{(3-2t^3)(1-s)^3}{6}\right] \times \left[\frac{1}{2}(v_n(s))^2 + s\right] ds \\ \qquad if \quad t \in \left[\frac{3}{4}, 1\right],\ n = 0,1,2... \end{cases}$$

The first, second, third, and fourth terms of this scheme are as follows:

$$v_0(t) \equiv 0,$$

$$v_1(t) = \frac{7t^5}{120} - \frac{119t^3}{480} + \frac{37}{160}$$

$$v_2(t) = \frac{7t^{14}}{49420800} - \frac{833t^{12}}{342144000} - \frac{7427t^{11}}{20275200} - \frac{184253t^{10}}{165888000} + \frac{37t^9}{4147200}$$

$$-\frac{49069t^7}{102400} + \frac{t^5}{60} + \frac{1369t^4}{614400} - \frac{147553086840691879t^3}{298491637137408000} + \frac{143787255710603}{1554643943424000}$$

$$v_3(t) = \frac{49t^{32}}{2107902249507225600000} - \frac{833t^{30}}{794386238570496000000} - \frac{7427t^{29}}{40798108054978560000}$$

$$-\frac{268461101t^{28}}{4273250110930944000000000} + \frac{26846981t^{27}}{633074090508288000000} + \frac{26815806199t^{26}}{6892640985415680000000}$$

$$+\frac{400171550569t^{25}}{1792086656208076800000000} + \frac{371462295299t^{24}}{77197579036655616000000} + \frac{114032891993t^{23}}{1045383882788044800000}$$

$$+\frac{3453761875703t^{22}}{1727155980258508800000} + \frac{78497989670046547290 71t^{21}}{105946677085599448222924800000} - \frac{272903089t^{20}}{1527724965888000000}$$

$$-\frac{1851000739420343895193t^{19}}{4750136730870832403841024000000}+\frac{3618768882947953403120899t^{18}}{1155588818738167415203430400000}$$

$$+\frac{2718808325190382897978 7t^{17}}{1414182120833421661962240000000}-\frac{34723371605213907361t^{15}}{516309342522414465024000000}$$

$$+\frac{9775873386667785160449 41t^{14}}{495656968821517886423040000000}+\frac{840630767232295533851240096179654 3t^{13}}{267291772322910140198018875392000000}$$

$$-\frac{29501725604687291t^{12}}{2127648389515444224 0000}-\frac{166598650952378994752 3t^{11}}{1452474633909209923584 00000}$$

$$+\frac{21771913436216758949940023416550641t^{10}}{449050177502489035532671710658560000000}$$

$$+\frac{143787255710603t^{9}}{14103729854742528 0000}+\frac{19684475306781550 7t^{8}}{8023455206253527040 00000}$$

$$-\frac{2121625342845137375045831629 3037t^{7}}{19490025065212197722772209664 0000000}+\frac{t^{5}}{60}$$

$$+\frac{20674774904786335034486623609t^{4}}{5800602697979820750825062 4000000}$$

$$-\frac{22999424791465727671649714973089070426023581506911t^{3}}{92131073987503901166490340551548382425907200000000}$$

$$+\frac{310661312414757109061653185761538923825439093587}{133523295634063624878971508045722293370880000000000}$$

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$\alpha_\kappa-$Implicit Contraction in non-AMMS with Some Applications

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Abstract

In this article, we establish $\alpha_\kappa-$implicit contraction and provide some fixed point results in non-AMMS. Our results progress and generalize some famous consequences in a suitable resource. As an implementation, we study stability in the sense of Ulam-Hyers and a fixed point problem's well-posedness. In addition, some examples are given for new concepts. Also, an application to integral equations is discussed.

1. Some basic concepts and definitions

In this work, we will write MMS to modular metric space and non-AMMS to non-Archimedean modular metric space. In 2010, Chistyakov [1], [2] defined a new generalized space which is a modular metric space and introduced basic concepts and topological properties.
Let M be a nonempty set, a function $\kappa:(0,\infty)\times M\times M\to[0,\infty]$ be defined

$$\kappa_\lambda(\xi,\eta)=\kappa(\lambda,\xi,\eta)$$

for all $\lambda>0$ and $\xi,\eta\in M$.

Definition 1.1. *A function $\kappa:(0,\infty)\times M\times M\to[0,\infty]$ is named a modular metric if the following conditions are supplied:*

(i) $\xi=\eta\Leftrightarrow\kappa_\lambda(\xi,\eta)=0,$ *for all* $\lambda>0$*;*
(ii) $\kappa_\lambda(\xi,\eta)=\kappa_\lambda(\eta,\xi)$*, for all* $\lambda>0$ *and* $\xi,\eta\in M$*;*
(iii) $\kappa_{\lambda+\mu}(\xi,\eta)\le\kappa_\lambda(\xi,\nu)+\kappa_\mu(\nu,\eta)$*, for all* $\lambda,\ \mu>0$ *and* $\xi,\eta,\nu\in M$*.*

Then, M_κ is named an MMS.

In the above definition, if we make use of the condition:

(i_1) $\kappa_\lambda(\xi,\xi)=0$ for all $\lambda>0$ and $\xi\in M$,

instead of (i), then M_κ is a pseudomodular metric space. M_κ is called regular if the condition (i) is supplied as:

$$\xi=\eta\quad\text{if and only if}\quad\kappa_\lambda(\xi,\eta)=0\quad\text{for some}\quad\lambda>0.$$

The space M_κ is named convex if for $\lambda,\ \mu>0$ and $\xi,\eta,\nu\in M$, the condition supplies:

$$\kappa_{\lambda+\mu}(\xi,\eta)\le\frac{\lambda}{\lambda+\mu}\kappa_\lambda(\xi,\nu)+\frac{\mu}{\lambda+\mu}\kappa_\mu(\nu,\eta).$$

Definition 1.2. *[1], [2] recognised that κ be a pseudomodular on M and $\xi_0\in M$ and fixed. The sets:*

$$M_\kappa=M_\kappa(\xi_0)=\{\xi\in M:\kappa_\lambda(\xi,\xi_0)\quad as\quad\lambda\to\infty\}$$

and

$$\mathrm{M}^*_\kappa=\mathrm{M}^*_\kappa(\xi_0)=\{\xi\in M:\exists\lambda=\lambda(\xi)>0\text{ such that }\kappa_\lambda(\xi,\xi_0)<\infty\}$$

are identified modular spaces (around ξ_0).

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It is trivial that $M_\kappa \subset M_\kappa^*$. Suppose that κ is a modular on M; from [1], [2], it can be obtained that the modular space M_κ can be settled with a (nontrivial) metric, induced by κ and given by:

$$d_\kappa(\xi,\eta) = \inf\{\lambda > 0 : \kappa_\lambda(\xi,\eta) < \lambda\},$$

for all $\xi,\eta \in M_\kappa$.
Consider that if κ is a convex modular on M, then specify [1], [2], the two modular space coincide, i.e., $M_\kappa = M_\kappa^*$, and this common set can be defined with the metric d_κ^* given by:

$$d_\kappa^*(\xi,\eta) = \inf\{\lambda > 0 : \kappa_\lambda(\xi,\eta) < 1\},$$

for all $\xi,\eta \in M_\kappa$. These distances are named Luxemburg distances.

Definition 1.3. *[3] Let M_κ be a MMS, A be a subset and $(s_n)_{n\in N}$ be a sequence in M_κ. Therefore:*

(1) $(s_n)_{n\in N}$ is named $\kappa-$convergent to $\xi \in M_\kappa$ if and only if $\kappa_\lambda(s_n,\xi) \to 0$ as $n \to \infty$ for all $\lambda > 0$. ξ will be called the $\kappa-$limit of (s_n).
(2) If for all $\lambda > 0$, $\kappa_\lambda(s_n,s_m) \to 0$, as $m,n \to \infty$, $(s_n)_{n\in N}$ is called $\kappa-$Cauchy.
(3) A is called $\kappa-$closed if the $\kappa-$limit of $\kappa-$convergent of A always belong to A.
(4) If any $\kappa-$Cauchy sequence in A is $\kappa-$convergent, then A is named $\kappa-$complete.
(5) A is called $\kappa-$bounded if for all $\lambda > 0$, we have

$$\delta_\omega(A) = \sup\{\kappa_\lambda(\xi,\eta); \xi,\eta \in A\} < \infty.$$

Paknazar et al. [4] modified the third condition of MMS.

Definition 1.4. *If in Definition 1.1, we exchange (iii) by:*

(iv) $\kappa_{\max\{\lambda,\mu\}}(\xi,\eta) \le \kappa_\lambda(\xi,\nu) + \kappa_\mu(\nu,\eta)$,

for all λ, $\mu > 0$ and $\xi,\eta,\nu \in M_\kappa$, then, M_κ is called non-AMMS.

Now, denote N the set of positive integers, the set of real numbers R and Ψ the set of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying:

(ψ_1) ψ is nondecreasing,
(ψ_2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each R^+, where ψ^n is the nth iterate of ψ.

Remark 1.5. *It is trivial that if $\psi \in \Psi$, then $\psi(t) < t$ for any $t > 0$.*

Definition 1.6. *[5] Let Γ be the set of all functions $\wp(t_1,...,t_6) : R_+^6 \to R$ satisfying:*

($\wp_1$) $\wp$ is nondecreasing in variable t_1 and nonincreasing in variable t_5,
($\wp_2$) there exists $\psi \in \Psi$ such that for all $u,v \ge 0$, $\wp(u,v,v,u,u+v,0) \le 0$ implies $u \le \psi(v)$, and $\wp(u,v,u,v,0,u+v) \le 0$ implies $u \le \psi(v)$.

Samet et al. [6] characterize a new notion by defining $\alpha-$admissible mapping.

Definition 1.7. *[6] Let $\alpha : M \times M \to [0,\infty)$ be a function. A mapping $\hbar : M \to M$ satisfying*

$$\alpha(\xi,\eta) \ge 1 \quad \Rightarrow \quad \alpha(\hbar\xi,\hbar\eta) \ge 1, \tag{1.1}$$

if for all $\xi,\eta \in M$, is called as $\alpha-$admissible mapping.

Example 1.8. *[6] Let $M = (0,\infty)$ and define $\hbar : M \to M$ and $\alpha : M \times M \to [0,\infty)$ by*

$$\hbar\xi = \ln\xi, \quad \text{for all } \xi \in M$$

and

$$\alpha(\xi,\eta) = \begin{cases} 2 & \text{if } \xi \ge \eta, \\ 0 & \text{if } \xi < \eta. \end{cases}$$

Then, $\hbar$ is an $\alpha-$admissible mapping.

Such papers related to above concept imagined to obtain some fixed and common fixed point results (see [7] [8], [9], [10]).

2. $\alpha_\kappa-$implicit contraction and fixed point results

In the sequel the function κ is convex and regular.

Definition 2.1. *Let M_κ be a non-AMMS. A mapping given as $\hbar : M_\kappa \to M_\kappa$ is called $\alpha_\kappa-$implicit contraction if there are two functions $\alpha : M_\kappa \times M_\kappa \to [0,\infty)$ and $\Gamma \in \wp$ in such a way that*

$$\begin{aligned} \wp(\alpha(\xi,\eta)\kappa_\lambda(\hbar\xi,\hbar\eta), \kappa_\lambda(\xi,\eta), \kappa_\lambda(\xi,\hbar\xi), \\ \kappa_\lambda(\eta,\hbar\eta), \kappa_\lambda(\xi,\hbar\eta), \kappa_\lambda(\eta,\hbar\xi)) \le 0, \end{aligned} \tag{2.1}$$

for all $\xi,\eta \in M_\kappa$.

Theorem 2.2. *Let M_κ be a complete non-AMMS and $\hbar : M_\kappa \to M_\kappa$ be a $\alpha_\kappa-$implicit contraction. Assume that:*

(i) $\hbar$ satisfies (1.1),

(ii) there is $\xi_0 \in M_\kappa$ in such a manner that $\alpha(\xi_0, \hbar\xi_0) \geq 1$,
(iii) $\hbar$ is continuous.

Then, $\hbar$ has a fixed point.

Proof. Let $\xi_0 \in M_\kappa$ be in such a way that $\alpha(\xi_0, \hbar\xi_0) \geq 1$ and let $\{\xi_n\}$ be a Picard sequence starting at ξ_0, that is $\xi_n = \hbar^n \xi_0 = \hbar\xi_{n-1}$ for all $n \in N$. First, imagine that $\kappa_\lambda(\xi_{n_0}, \xi_{n_0+1}) = 0$ for some $n_0 \in N$, since κ is regular, we get $\xi_{n_0} = \xi_{n_0+1} = \hbar\xi_{n_0}$. So, ξ_{n_0} is a fixed point of $\hbar$. Hence, we approve that $\xi_n \neq \xi_{n+1}$ such that $\kappa_\lambda(\xi_n, \xi_{n+1}) > 0$. Now, since the mapping $\hbar$ is $\alpha-$admissible and $\alpha(\xi_0, \xi_1) = \alpha(\xi_0, \hbar\xi_0) \geq 1$, we deduce that $\alpha(\hbar\xi_0, \hbar\xi_1) = \alpha(\xi_1, \xi_2) \geq 1$. Using the iterative method, we achieve

$$\alpha(\xi_n, \xi_{n+1}) \geq 1, \qquad \text{for all } \ n \in N. \tag{2.2}$$

From (2.1) with $\xi = \xi_n$ and $\eta = \xi_{n+1}$, we have

$$\begin{gathered}\wp(\alpha(\xi_n, \xi_{n+1})\,\kappa_\lambda(\hbar\xi_n, \hbar\xi_{n+1}),\ \kappa_\lambda(\xi_n, \xi_{n+1}), \kappa_\lambda(\xi_n, \hbar\xi_n)\\ \kappa_\lambda(\xi_{n+1}, \hbar\xi_{n+1}), \hbar_\lambda(\xi_n, \hbar\xi_{n+1}), \kappa_\lambda(\xi_{n+1}, \hbar\xi_n)) \leq 0,\end{gathered}$$

that is,

$$\begin{gathered}\wp(\alpha(\xi_n, \xi_{n+1})\,\kappa_\lambda(\xi_{n+1}, \xi_{n+2}),\ \kappa_\lambda(\xi_n, \xi_{n+1}), \kappa_\lambda(\xi_n, \xi_{n+1})\\ \kappa_\lambda(\xi_{n+1}, \xi_{n+2}), \kappa_\lambda(\xi_n, \xi_{n+2}), \kappa_\lambda(\xi_{n+1}, \xi_{n+1})) \leq 0.\end{gathered}$$

By using the conditions, (iv), (2.2) and $(\wp_1)$ we get

$$\begin{aligned}&\wp\big(\kappa_\lambda(\xi_{n+1}, \xi_{n+2}),\ \kappa_\lambda(\xi_n, \xi_{n+1}), \kappa_\lambda(\xi_n, \xi_{n+1})\\ &\qquad \kappa_\lambda(\xi_{n+1}, \xi_{n+2}), \kappa_{\max\{\lambda,\lambda\}}(\xi_n, \xi_{n+2}), 0\big) \leq 0\\ &= \wp(\kappa_\lambda(\xi_{n+1}, \xi_{n+2}),\ \kappa_\lambda(\xi_n, \xi_{n+1}), \kappa_\lambda(\xi_n, \xi_{n+1})\\ &\qquad \kappa_\lambda(\xi_{n+1}, \xi_{n+2}), \kappa_\lambda(\xi_n, \xi_{n+1}) + \kappa_\lambda(\xi_{n+1}, \xi_{n+2}), 0) \leq 0.\end{aligned}$$

Due to $(\wp_2)$, we obtain

$$\kappa_\lambda(\xi_{n+1}, \xi_{n+2}) \leq \psi(\kappa_\lambda(\xi_n, \xi_{n+1})), \quad \text{for all } \ n \in N. \tag{2.3}$$

From (2.3), it is easy to derive that

$$\kappa_\lambda(\xi_{n+1}, \xi_{n+2}) \leq \psi^{n+1}(\kappa_\lambda(\xi_0, \xi_1)), \quad \text{for all } \ n \in N. \tag{2.4}$$

Next, we illustrate that $\{\xi_n\}$ is a Cauchy sequence in M_κ. Take $m > n$; by the condition (iv) and (2.4), we write

$$\begin{aligned}\kappa_\lambda(\xi_n, \xi_m) = &\ \kappa_{\max\{\lambda,\lambda\}}(\xi_n, \xi_m)\\ &\leq \kappa_\lambda(\xi_n, \xi_{n+1}) + \kappa_\lambda(\xi_{n+1}, \xi_m)\\ &= \kappa_\lambda(\xi_n, \xi_{n+1}) + \kappa_{\max\{\lambda,\lambda\}}(\xi_{n+1}, \xi_m)\\ &\leq \kappa_\lambda(\xi_n, \xi_{n+1}) + \kappa_\lambda(\xi_{n+1}, \xi_{n+2}) + \kappa_\lambda(\xi_{n+2}, \xi_m)\\ &\ \vdots\\ &\leq \kappa_\lambda(\xi_n, \xi_{n+1}) + \kappa_\lambda(\xi_{n+1}, \xi_{n+2}) + \ldots + \kappa_\lambda(\xi_{m-1}, \xi_m)\\ &\leq \left(\psi^n + \psi^{n-1} + \ldots + \psi^{m-1}\right)\kappa_\lambda(\xi_0, \xi_1)\\ &\leq \sum_{k=n}^{\infty} \psi^k(\kappa_\lambda(\xi_0, \xi_1)).\end{aligned} \tag{2.5}$$

From (2.5) and (ψ_2) the series $\sum\limits_{k=n}^{\infty} \psi^k(\kappa_\lambda(\xi_0, \xi_1))$ is convergent and so $\{\xi_n\}$ is a Cauchy sequence in M_κ. Because M_κ is a complete non-AMMS, then there exists a point $\nu \in M_\kappa$ such that $\kappa_\lambda(\xi_n, \nu) \to 0$ as $n \to \infty$. Thus, $\kappa_\lambda(\hbar\xi_n, \hbar\nu) \to$ as $n \to \infty$, because $\hbar$ is a $\kappa-$continuous. Then, by (iv) we obtain

$$\begin{aligned}\kappa_\lambda(\nu, \hbar\nu) &= \kappa_{\max\{\lambda,\lambda\}}(\nu, \hbar\nu)\\ &\leq \kappa_\lambda(\nu, \hbar\xi_n) + \kappa_\lambda(\hbar\xi_n, \hbar\nu)\\ &= \kappa_\lambda(\nu, \xi_{n+1}) + \kappa_\lambda(\hbar\xi_n, \hbar\nu).\end{aligned}$$

As $n \to \infty$, we get $\kappa_\lambda(\nu, \hbar\nu) = 0$. Since κ is regular, we deduce that $\hbar\nu = \nu$ and hence ν is a fixed point of $\hbar$.

□

If we turn into the continuity of $\hbar$ with the condition (H), we attain the other result.

(H) If $\{\xi_n\}$ is a sequence in M_κ such that $\alpha(\xi_n, \xi_{n+1}) \geq 1$ for all $n \in N$ and $\xi_n \to \xi$ as $n \to \infty$, there exists a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ such that $\alpha(\xi_{n_k}, \xi) \geq 1$ for all $k \in N$.

Theorem 2.3. *Let M_κ be a complete non-AMMS and $\hbar : M_\kappa \to M_\kappa$ be an $\alpha_\kappa-$implicit contraction. Granted that:*

(i) $\hbar$ satisfies (1.1),

(ii) there exists $\xi_0 \in M_\kappa$ in such a way that $\alpha(\xi_0, \hbar\xi_0) \geq 1$,
(iii) (H) is supplied.

Then, $\hbar$ has a fixed point.

Proof. Due to Theorem 2.2, we acquire that the sequence $\{\xi_n\}$, defined by $\xi_n = \hbar\xi_{n-1}$ for all $n \in N$, is a Cauchy sequence with $\alpha(\xi_n, \xi_{n+1}) \geq 1$ for all $n \in N$, which converges to some $v \in M_\kappa$. Next, from the condition (iii), there is a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ in such a manner that $\alpha(\xi_{n_k}, \xi) \geq 1$ for all $k \in N$. We need to show that $\hbar v = v$. Since $\hbar$ is $\alpha_\kappa-$type implicit contraction with $\xi = \xi_{n_k}$ and $\eta = v$ and (iv), we obtain

$$\begin{aligned}
&\wp(\alpha(\xi_{n_k}, v)\,\kappa_\lambda(\hbar\xi_{n_k}, \hbar v),\, \kappa_\lambda(\xi_{n_k}, v), \kappa_\lambda(\xi_{n_k}, \hbar\xi_{n_k}) \\
&\quad \kappa_\lambda(v, \hbar v), \kappa_\lambda(\xi_{n_k}, \hbar v), \kappa_\lambda(v, \hbar\xi_{n_k})) \leq 0 \\
&= \wp(\alpha(\xi_{n_k}, v)\,\kappa_\lambda(\xi_{n_k+1}, \hbar v),\, \kappa_\lambda(\xi_{n_k}, v), \kappa_\lambda(\xi_{n_k}, \xi_{n_k+1}) \\
&\quad \kappa_\lambda(v, \hbar v), \kappa_{\max\{\lambda,\lambda\}}(\xi_{n_k}, \hbar v), \omega_\lambda(v, \xi_{n_k+1})\Big) \leq 0 \\
&\leq \wp(\alpha(\xi_{n_k}, v)\,\kappa_\lambda(\xi_{n_k+1}, \hbar v),\, \kappa_\lambda(\xi_{n_k}, v), \kappa_\lambda(\xi_{n_k}, \xi_{n_k+1}) \\
&\quad \kappa_\lambda(v, \hbar v), \kappa_\lambda(\xi_{n_k}, v) + \kappa_\lambda(v, \hbar v), \kappa_\lambda(v, \xi_{n_k+1})) \leq 0.
\end{aligned}$$

Letting k tends to infinity and using the continuity of $\wp$ and $\alpha(\xi_{n_k}, \xi) \geq 1$, we get

$$\wp(\kappa_\lambda(v, \hbar v), 0, 0, \kappa_\lambda(v, \hbar v), \kappa_\lambda(v, \hbar v), 0) \leq 0.$$

Finally, by condition $(\wp_2)$, it follows that $\kappa_\lambda(v, \hbar v) \leq 0$ which implies $\hbar v = v$. □

We need extra conditions to obtain uniqueness of fixed point.

(U) For all $u, v \in \mathrm{Fix}(\hbar)$, we attain $\alpha(u, v) \geq 1$, where $\mathrm{Fix}(\hbar)$ gives the set of all fixed points of $\hbar$.
$(\wp_3)$ There exists $\psi \in \Psi$ in such a way that for all $u, v > 0$,

$$\wp(\mathrm{u}, \mathrm{u}, 0, 0, \mathrm{u}, \mathrm{v}) \leq 0 \quad \text{implies} \quad \mathrm{u} \leq \psi(\mathrm{v}).$$

Theorem 2.4. *Adding conditions (U) and $(\wp_3)$ to the hypotheses of Theorem 2.2 (resp Theorem 2.3), we deduce that $\hbar$ has a unique fixed point.*

Proof. We discuss by contradiction, that is, there exist $u, v \in M_\kappa$ in such a way that $u = \hbar u$ and $v = \hbar v$ with $u \neq v$. From (1.1), we obtain

$$\begin{aligned}
&\wp(\alpha(u, v)\;\kappa_\lambda(\hbar u, \hbar v), \kappa_\lambda(u, v), \kappa_\lambda(u, \hbar u), \\
&\quad \kappa_\lambda(v, \hbar v), \kappa_\lambda(u, \hbar v), \kappa_\lambda(v, \hbar u)) \leq 0.
\end{aligned}$$

Then, by condition (U), we have

$$\wp(\kappa_\lambda(u, v),\, \kappa_\lambda(u, v), 0, 0, \kappa_\lambda(u, v), \kappa_\lambda(v, u)) \leq 0.$$

Since $\wp$ satisfies the property $(\wp_3)$, then

$$\kappa_\lambda(u, v) \leq \psi(\kappa_\lambda(u, v)) < \kappa_\lambda(u, v),$$

which is a contradiction and hence $u = v$. □

Now, we give some corollaries from above results.

Corollary 2.5. *Let M_κ be a complete non-AMMS and $\hbar : M_\kappa \to M_\kappa$ be a function. If there is a function $\alpha : M_\kappa \times M_\kappa \to [0, \infty)$ in such a manner that*

$$\begin{aligned}
\alpha(\xi, \eta)\,\kappa_\lambda(\hbar\xi, \hbar\eta) \leq{}& p\kappa_\lambda(\xi, \eta) + q\kappa_\lambda(\xi, \hbar\xi) + r\kappa_\lambda(\eta, \hbar\eta) \\
&+ s\kappa_\lambda(\xi, \hbar\eta) + t\kappa_\lambda(\eta, \hbar\xi),
\end{aligned}$$

for all $\xi, \eta \in M_\kappa$, where $p, q, r, s, t > 0$, $p+q+r+s+t < 1$. Assume also that:

(i) $\hbar$ satisfies (1.1),
(ii) there is $\xi_0 \in M_\kappa$ in such a way that $\alpha(\xi_0, \hbar\xi_0) \geq 1$,
(iii) $\hbar$ is continuous or the condition (H) holds true.

Then, $\hbar$ has a fixed point. Additionally, if $p+r+s < 1$ and the conditions (U) and $(\wp_3)$ hold true, then $\hbar$ has a unique fixed point.

Corollary 2.6. *Let M_κ be a complete non-AMMS and $\hbar : M_\kappa \to M_\kappa$ be a function. If there is a function $\alpha : M_\kappa \times M_\kappa \to [0, \infty)$ in such a manner that*

$$\begin{aligned}
\alpha(\xi, \eta)\,\kappa_\lambda(\hbar\xi, \hbar\eta) \leq k\max\{&\kappa_\lambda(\xi, \eta),\, \kappa_\lambda(\xi, \hbar\xi), \kappa_\lambda(\eta, \hbar\eta), \\
&\kappa_\lambda(\xi, \hbar\eta), \kappa_\lambda(\eta, \hbar\xi)\},
\end{aligned}$$

for all $\xi, \eta \in M_\kappa$, where $k \in \left[0, \frac{1}{2}\right)$. Furthermore:

(i) $\hbar$ satisfies (1.1),
(ii) there is $\xi_0 \in M_\kappa$ such that $\alpha(\xi_0, \hbar\xi_0) \geq 1$,
(iii) $\hbar$ is continuous or the property (H) is satisfied.

Then, $\hbar$ has a fixed point. Moreover, the conditions (U) and $(\wp_3)$ hold true, then $\hbar$ has a unique fixed point.

Example 2.7. *$M_\kappa = R$ endowed with the non-Archimedean modular metric $\kappa_\lambda(\xi,\eta) = \frac{1}{\lambda}|\xi-\eta|$, for all $\xi,\eta \in M_\kappa$ and $\lambda > 0$. Obviously, M_κ is an $\kappa-$complete non-AMMS.*
Consider the self-map $\hbar : M_\kappa \to M_\kappa$ defined by $\hbar\xi = \frac{\xi}{6}$.
Also define

$$\alpha(\xi,\eta) = \begin{cases} 1, & \text{if } \xi,\eta \in [0,1] \\ 0, & \text{otherwise,} \end{cases}$$

and $\wp : R_+^6 \to R_+$ defined by

$$\wp(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \frac{3}{4}\max\left\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\right\}.$$

Let $\alpha(\xi,\eta) \geq 1$, then $\xi,\eta \in [0,1]$. Also, $\hbar\xi \in [0,1]$, for all $\xi \in [0,1]$ and so $\alpha(\hbar\xi,\hbar\eta) \geq 1$. Therefore $\hbar$ is an $\alpha-$admissible mapping. Let $\xi,\eta \in [0,1]$, we have

$$\wp\left(\alpha(\xi,\eta)\kappa_\lambda(\hbar\xi,\hbar\eta),\kappa_\lambda(\xi,\eta),\kappa_\lambda(\xi,\hbar\xi),\kappa_\lambda(\eta,\hbar\eta),\tfrac{\kappa_\lambda(\xi,\hbar\eta)+\kappa_\lambda(\eta,\hbar\xi)}{2}\right)$$

$$= \alpha(\xi,\eta)\kappa_\lambda(\hbar\xi,\hbar\eta) - \tfrac{3}{4}\max\{\kappa_\lambda(\xi,\eta),\kappa_\lambda(\xi,\hbar\xi),\kappa_\lambda(\eta,\hbar\eta),$$

$$\tfrac{\kappa_\lambda(\xi,\hbar\eta)+\kappa_\lambda(\eta,\hbar\xi)}{2}\Big\}$$

$$\leq \tfrac{1}{6\lambda}|\xi-\eta| - \tfrac{3}{4}\max\left\{\tfrac{1}{\lambda}|\xi-\eta|,\tfrac{6}{5\lambda}|\xi|,\tfrac{6}{5\lambda}|\eta|,\ \tfrac{1}{12\lambda}(|6\xi-\eta|+|6\eta-\xi|)\right\}$$

$$\leq 0.$$

Similarly, it is obvious that contractive condition (2.1) holds in the case ($\xi,\eta \notin [0,1]$ and ξ or η is not in $[0,1]$.) Thus, $\hbar$ is $\alpha_\kappa-$type implicit contraction. Next, it is easy to illustrate that conditions $\hbar$ is $\kappa-$continuous, (H) and (U) are satisfied.
Thus, the axioms of the Theorem 2.2, Theorem 2.3, and Theorem 2.4 are supplied and 0 is a unique fixed point.

3. Stability problem in the sense of Ulam-Hyers

Now, we obtain the stability problem in the sense of Ulam-Hyers of fixed point. That this problem correspondences to Corollary 2.5.
Let M_κ be a non-AMMS and $\hbar : M_\kappa \to M_\kappa$ be a function. Imagine the fixed point problem

$$\xi = \hbar\xi \tag{3.1}$$

and the inequality (for $\varepsilon > 0$)

$$\kappa_\lambda(\hbar\eta,\eta) < \varepsilon. \tag{3.2}$$

We are said to be a $\hbar$ is stable in the sense of Ulam-Hyers in non-AMMS if there are $L > 0$ such that for each $\varepsilon > 0$ and a $\varepsilon-$solution $v^* \in M_\kappa$, that is, v^* supplies the condition (3.2), there is a solution $u^* \in M_\omega$ of the fixed point equation (3.1) such that

$$\kappa_\lambda(u^*,v^*) < L\varepsilon. \tag{3.3}$$

Theorem 3.1. *Let M_κ be a non-AMMS. Suppose that all the hypotheses of Corollary 2.5 hold and $\alpha(u,v) \geq 1$ for all $\varepsilon-$solution u and v, then the equation (3.1) is stable in the sense of Ulam-Hyers.*

Proof. By Corollary 2.5, we have a unique $u \in M_\kappa$ such that $u = \hbar u$, that is, $u \in M_\kappa$ is a solution of the fixed point equation (3.1). Let $\varepsilon > 0$ and $v \in M_\kappa$ be an $\varepsilon-$solution, that is,

$$\kappa_\lambda(\hbar v,v) \leq \varepsilon.$$

Since $\kappa_\lambda(u,\hbar u) = \kappa_\lambda(u,u) = 0 \leq \varepsilon$, u and v are $\varepsilon-$solutions. By hypotheses, we get $\alpha(u,v) \geq 1$ and from (3.3), so

$$\begin{aligned}
\kappa_\lambda(u,v) &= \kappa_\lambda(\hbar u,v) \\
&= \kappa_{\max\{\lambda,\lambda\}}(\hbar u,v) \\
&\leq \kappa_\lambda(\hbar u,\hbar v) + \kappa_\lambda(\hbar v,v) \\
&= \alpha(u,v)\kappa_\lambda(\hbar u,\hbar v) + \varepsilon \\
&\leq a\kappa_\lambda(u,v) + b\kappa_\lambda(u,\hbar u) + c\kappa_\lambda(v,\hbar v) \\
&\quad + d\kappa_\lambda(u,\hbar v) + e\kappa_\lambda(v,\hbar u) + \varepsilon \\
&= a\kappa_\lambda(u,v) + b\kappa_\lambda(u,\hbar u) + c\kappa_\lambda(v,\hbar v) \\
&\quad + d\kappa_{\max\{\lambda,\lambda\}}(u,\hbar v) + e\kappa_{\max\{\lambda,\lambda\}}(v,\hbar u) + \varepsilon \\
&\leq a\kappa_\lambda(u,v) + b\kappa_\lambda(u,\hbar u) + c\kappa_\lambda(v,\hbar v) \\
&\quad + d(\kappa_\lambda(u,v) + \kappa_\lambda(v,\hbar v)) + e(\kappa_\lambda(v,u) + \kappa_\lambda(u,\hbar u)) + \varepsilon.
\end{aligned}$$

We deduce

$$\kappa_\lambda(u,v) \leq \left(\frac{1+c+d}{1-a-d-e}\right)\varepsilon = L\varepsilon,$$

where $L = \left(\frac{1+c+d}{1-a-d-e}\right) > 0$. Thus, $\hbar$ is Ulam-Hyers stable. □

4. Well posedness of the fixed point problem

Now, we show well-posedness of a function $\hbar$ on non-AMMS.

Definition 4.1. *Let M_κ be a non-AMMS and let $\hbar : M_\kappa \to M_\kappa$, $\alpha : M_\kappa \times M_\kappa \to [0,\infty)$ be two functions. $\hbar$ is well-posedness if:*

(i) $u \in M_\kappa$ is the unique fixed point when $\alpha(u,\hbar u) \geq 1$,
(ii) there exists a sequence $\{\xi_n\}$ in such a manner that $\kappa_\lambda(\xi_n,\hbar\xi_n) \to 0$ as $n \to \infty$, then $\kappa_\lambda(\xi_n,u) \to 0$ as $n \to \infty$.

We define a new condition which needs to be the following result.

(R) If $\{\xi_n\}$ is a sequence in M_κ in such a way that $\kappa_\lambda(\xi_n,\hbar\xi_n) \to 0$ as $n \to \infty$, then $\alpha(\xi_n,\hbar\xi_n) \geq 1$ for all $n \in N$.

Theorem 4.2. *Let M_κ be a non-AMMS. If all the conditions of Corollary 2.5 and the condition (R) hold, hence (3.1) is well posed.*

Proof. By Corollary 2.5, we have a unique $u \in M_\kappa$ in such a manner that $u = \hbar u$ and $\alpha(u,\hbar u) \geq 1$. Let $\{\xi_n\}$ is a sequence in M_κ in such a way that $\kappa_\lambda(\xi_n,\hbar\xi_n) \to 0$ as $n \to \infty$. By condition (R), we get $\alpha(\xi_n,\hbar\xi_n) \geq 1$. Now, we have

$$\begin{aligned}
\kappa_\lambda(\xi_n,u) &= \kappa_\lambda(\xi_n,\hbar u)\\
&= \kappa_{\max\{\lambda,\lambda\}}(\xi_n,\hbar u)\\
&\leq \kappa_\lambda(\xi_n,\hbar\xi_n) + \kappa_\lambda(\hbar\xi_n,\hbar u)\\
&\leq \alpha(\xi_n,u)\,\kappa_\lambda(\hbar\xi_n,\hbar u) + \kappa_\lambda(\xi_n,\hbar\xi_n)\\
&\leq a\kappa_\lambda(\xi_n,u) + b\kappa_\lambda(\xi_n,\hbar\xi_n) + c\kappa_\lambda(u,\hbar u) + d\kappa_\lambda(\xi_n,\hbar u)\\
&\quad + e\kappa_\lambda(u,\hbar\xi_n) + \kappa_\lambda(\xi_n,\hbar\xi_n)\\
&\leq a\kappa_\lambda(\xi_n,u) + b\kappa_\lambda(\xi_n,\hbar\xi_n) + c\kappa_\lambda(u,\hbar u) + d\kappa_{\max\{\lambda,\lambda\}}(\xi_n,\hbar u)\\
&\quad + e\kappa_{\max\{\lambda,\lambda\}}(u,\hbar\xi_n) + \kappa_\lambda(\xi_n,\hbar\xi_n)\\
&\leq a\kappa_\lambda(\xi_n,u) + b\kappa_\lambda(\xi_n,\hbar\xi_n) + c\kappa_\lambda(u,\hbar u) + d(\kappa_\lambda(\xi_n,u) + \kappa_\lambda(u,\hbar u))\\
&\quad + e(\kappa_\lambda(u,\xi_n) + \kappa_\lambda(\xi_n,\hbar\xi_n)) + \kappa_\lambda(\xi_n,\hbar\xi_n).
\end{aligned}$$

Hence

$$\kappa_\lambda(\xi_n,u) \leq \left(\frac{1+b+e}{1-a-d-e}\right)\kappa_\lambda(\xi_n,\hbar\xi_n).$$

Since $\kappa_\lambda(\xi_n,\hbar\xi_n) \to 0$ as $n \to \infty$, it implies that $\kappa_\lambda(\xi_n,u) \to 0$ as $n \to \infty$. Thus, $\hbar$ is well posed. □

5. Consequences

Next, we will obtain non-AMMS version of some fixed point results.
In the Definition of 1.6, if we take $\psi(t) = ht$, $h \in [0,1)$, we get Berinde's results in [11].
Let Γ be the set of all continuous real functions $\wp : R_+^6 \to R_+$, for which we consider the following conditions:

$(\wp_{1a})$ F is non-increasing in the fifth variable and

$$\wp(\xi,\eta,\eta,\xi,\xi+\eta,0) \leq 0, \text{ for } \xi,\eta \geq 0 \;\Rightarrow\; \exists h \in [0,1) \text{ such that } \xi \leq h\eta;$$

$(\wp_{1b})$ $\wp$ is non-increasing in the fourth variable and

$$\wp(\xi,\eta,0,\xi+\eta,\xi,\eta) \leq 0, \text{ for } \xi,\eta \geq 0 \;\Rightarrow\; \exists h \in [0,1) \text{ such that } \xi \leq h\eta;$$

$(\wp_{1c})$ $\wp$ is non-increasing in the third variable and

$$\wp(\xi,\eta,\xi+\eta,0,\eta,\xi) \leq 0, \text{ for } \xi,\eta \geq 0 \;\Rightarrow\; \exists h \in [0,1) \text{ such that } \xi \leq h\eta;$$

$(\wp_2)$ $\wp(\xi,\xi,0,0,\xi,\xi) > 0$, for all $\xi > 0$.

Example 5.1. *The function $\wp \in \Gamma$, given by*

$$\wp(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2,$$

where $a \in [0,1)$, satisfies $(\wp_{1a})$-$(\wp_{1c})$ and $(\wp_2)$,with $h = a$.

Example 5.2. *The function $\wp \in \Gamma$, given by*

$$\wp(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - b(t_3+t_4),$$

where $b \in \left[0,\frac{1}{2}\right)$, satisfies $(\wp_{1a})$-$(\wp_{1c})$ and $(\wp_2)$,with $h = \frac{b}{1-b} < 1$.

Example 5.3. *The function $\wp \in \Gamma$, given by*

$$\wp(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - c(t_5+t_6),$$

where $c \in \left[0,\frac{1}{2}\right)$, satisfies $(\wp_{1a})$-$(\wp_{1c})$ and $(\wp_2)$,with $h = \frac{c}{1-c} < 1$.

Example 5.4. *The function $\wp \in \Gamma$, given by*

$$\wp(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a\max\left\{t_2,\frac{t_3+t_4}{2},\frac{t_5+t_6}{2}\right\},$$

where $a \in [0,1)$, satisfies $(\wp_{1a})$-$(\wp_{1c})$ and $(\wp_2)$,with $h = a$.

Example 5.5. *The function $\wp \in \Gamma$, given by*

$$\wp(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2 - b(t_3+t_4) - c(t_5+t_6),$$

where $a,b,c \geq 0$ and $a+2b+2c<1$ satisfies $(\wp_{1a})$-$(\wp_{1c})$ and $(\wp_2)$, with $h = \frac{a+b+c}{1-b-c} < 1$.

Corollary 5.6. *Let M_κ be a non-Archimedean modular metric space, $\hbar : M_\kappa \to M_\kappa$ be a self map for which $\wp \in \Gamma$ such that for all $\xi,\eta \in M_\kappa$,*

$$\wp(\kappa_\lambda(\hbar\xi,\hbar\eta), \kappa_\lambda(\xi,\eta), \kappa_\lambda(\xi,\hbar\xi), \kappa_\lambda(\eta,\hbar\eta), \kappa_\lambda(\xi,\hbar\eta), \kappa_\lambda(\eta,\hbar\xi)) \leq 0.$$

If $\wp$ satisfies $(\wp_{1a})$ and $(\wp_2)$, then $\hbar$ has a unique fixed point.

Proof. It suffices to take $\alpha(\xi,\eta) = 1$ and $\psi(t) = kt,\ k \in [0,1)$ in Theorem 2.2. □

6. Application to integral equation

Next, we give implementation to show the nonlinear integral equation.

$$\xi(z) = \int_a^t K(z,p,\xi(p))\,dp, \tag{6.1}$$

where $\xi \in I = [a,b]$ and $K : I \times I \times R \to R$ is continuous. Let $M = C(I,R)$ with the usual supremum norm, that is,

$$\|\xi\| = \max_{z \in I} |\xi(z)|,$$

and the metric

$$\kappa_\lambda(\xi,\eta) = \frac{1}{\lambda}\|\xi - \eta\| = \frac{1}{\lambda} d(\xi,\eta),$$

for all $\xi,\eta \in M$. For $r > 0$ and $\xi \in M$ we denote by

$$B_\lambda(\xi,r) = \{v \in M : \kappa_\lambda(\xi,\eta) \leq r\},$$

the closed ball concerned at ξ and of radius r. Note that M_κ is a $\kappa-$complete non-AMMS.
Now, imagine the mapping $\hbar : M_\kappa \to M_\kappa$

$$\hbar\xi(z) = \int_a^z K(z,p,\xi(p))\,dp. \tag{6.2}$$

Notice that (6.1) has a solution if and only if $\hbar$ has a fixed point in (6.2).

Theorem 6.1. *Let $r > 0$ and we granted that the following conditions are supplied:*

(i) if $y \in B_\lambda(\xi,r)$, $\lambda > 0$, then

$$|K(z,p,\xi(p)) - K(z,p,\eta(p))| \leq \frac{q(z,p)}{b-a}|\xi(p) - \eta(p)|,$$

for all $z,p \in I$, $\xi,\eta \in R$ and for some continuous function $q : I \times I \to R_+$;

(ii) $\sup_{z \in I} q(z,p) = k < 1$.

Hence, (6.1) has a solution.

Proof. Since $\eta \in B_\lambda(\xi,r)$ and from (ii), we have

$$\begin{aligned}
|\hbar\xi(z) - \hbar\eta(z)| &\leq \left|\int_a^z [K(z,p,\xi(p)) - K(z,p,\eta(p))]dp\right| \\
&\leq \int_a^t |K(t,p,\xi(p)) - K(z,p,\eta(p))|dp \\
&\leq \int_a^b |K(z,p,\xi(p)) - K(z,p,\eta(p))|dp \\
&\leq \int_a^b \frac{q(z,p)}{b-a}|\xi(p) - \eta(p)|dp \\
&\leq \|\xi(p) - \eta(p)\| \int_a^b \frac{k}{b-a}dp \\
&= k\|\xi(p) - \eta(p)\|.
\end{aligned} \tag{6.3}$$

This implies that

$$\begin{aligned}
\kappa_\lambda(\hbar\xi,\hbar\eta) &= \tfrac{1}{\lambda}\|\hbar\xi - \hbar\eta\| \\
&\leq \tfrac{1}{\lambda}|\hbar\xi(z) - \hbar\eta(z)| \\
&\leq \tfrac{1}{\lambda}k\|\xi(p) - \eta(p)\| \\
&\leq k\kappa_\lambda(\xi,\eta).
\end{aligned}$$

Now, $\wp : R_+^6 \to R_+$ defined by

$$\wp(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - kt_2,$$

where $k \in [0,1)$, and so the integral operator $\hbar$ satisfies all conditions of Corollary 5.6. Thus, $\hbar$ has a fixed point, i.e., (6.1) has a solution in M_κ. □

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Fluid Flow Characteristics for a Diverging-Converging Magnetohydrodynamic Electric Current Configuration

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Abstract

The effects of variations in flow field due to the presence of electromagnetic rotational forces on a transient incompressible and electrically conducting fluid flow are sought. These variations result from interactions between the electric currents with a nonuniform magnetic field. The governing equations are coupled and nonlinear and are discretized using the finite difference technique. Numerical results illustrating the development of secondary flows by the rotational electromagnetic force field are displayed, as well as the effects on the streamline axial velocity profiles by the magnetic pressure number and the flow Reynolds numbers.

1. Introduction

Magnetohyroynamics (MHD) deals with the interaction between electrically-conducting fluids and applied electromagnetic fields. The coupling between the two fields results in some exciting physics among which are; the development of secondary flows due to the presence of rotational force fields, the development of electric current due to the interaction of the magnetic field and a conducting fluid, and the generation of the Lorentz force arising from the presence of a current and a magnetic field. The effect of this force is dynamical, because it acts on the conducting fluid and modifies its motion. The motion in turn modifies the field, which also modifies the motion.
The description of MHD flows involves the solution of the equations of fluid dynamics, the so called Navier-Stokes equations, and the equations of electrodynamics. These two equations are mutually coupled via the Lorentz force, and the Ohm's Law; hence it is useful to understand the influence of an externally generated body force on a conducting fluid in various dimensions for time dependent applications. The conversion of electrical energy directly into a body force, defines the magnetohydrodynamic concept. Fundamental to this, is the interaction of an electromagnetic field and conducting fluid which may be gas or liquid. From a unified viewpoint, plasma physics can be considered a special case of magnetohydrodynamics because of its strong dependence on the kinetic theory fluid model, involving gases in the plasma regime. However for the purposes of this study, our emphasis will be concentrated more on the electromagnetic-fluid interaction model.
MHD has always attracted a keen research interest. The stimulus for much of this interest lies in the desire to further understand the influence of electric and magnetic fields on heat and momentum transfer as they affect fluid flow. There are a couple of applications of MHD that hold great potential for future use for example energy conversion devices, flight and energy control of space re-entry vehicles, nuclear fusion control, electricity generation, etc. In a coal-fired MHD power generator, gas produced by combusting coal expands through a nozzle and interacts with a magnetic field to produce electricity. The MHD power generator is beginning to take on an added importance because of the current global energy crisis and environmental pollution. A conducting fluid moves across a magnetic field and in the process generates electrical energy which is tapped by making suitable connections to an external load. Some obvious advantages arising from this type of power generation include less pollution, and cheaper operational costs. Also, the reliable prediction of MHD flows coupled with strong magnetic fields is a key factor for the design of liquid metal blankets for use in fusion reactors.
It has been shown that the passage of electric current through a flowing conducting fluid radically alters the flow profile(Chow and Uberoi[1]. Uberoi[2] adopted a linear analysis to study the effect of an axial current on the motion of an incompressible inviscid fluid through an insulated axisymmetric tube of varying cross-sectional area. The slowing down of the central flow when approaching the tube contraction was attributed to the electromagnetic pinch effect. For example, when draining a current-carrying fluid from the apex of a conical tube, only the

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fluid in the narrow region near the wall can go through the apex. The oncoming flow along the tube axis experiences rapid deceleration, that forces the fluid to become stagnant before reaching the throat (Narain and Uberoi[3]). This type of flow was found to result in recirculation downstream of the stagnation point.
Given the vast range of MHD applications and the variability of the dimensionless numbers involved; it is not possible to arrive at complete numerical or analytical solutions of the governing differential equations. Hence the challenge lies in developing both numerical and analytical techniques to deal with each problem depending on the physics and accompanying rigor.
The flow of conducting fluid in pipes in the presence of electromagnetic forces has been studied analytically by various authors. A good account of this can be found in Sherclif[4], Di Pizza and Buhler[5]. Uberoi and Chow [6] reported solutions for large scale motions in electrical discharges. Subsequent work on channel flow can be found in the thesis by Ritter[7],and the papers by Chamkha [8]-[10], Onyejekwe[11]. Pantokratoras[12], studied the fully developed flow between parallel plates in an electrically conducting fluid under the action of induced magnetic field for the case where both the magnetic and electric fields are situated on the lower plate. Making some reasonable assumptions, he obtained one-dimensional exact analytical solutions for the velocity, flow rate, and wall shear stress at the plates for weakly electrically conducting fluids. His results though restricted in scope, could be used as adequate starting point from more complex considerations of MHD flows. The application of MHD to microfluidic devices is another blossoming area of research. The magnetohydrodynamic (MHD) pumping provides an efficient, cheap, reliable and easily controllable method for pumping various liquids for the purposes of testing samples of blood, DNA and drugs in nano or microscales. Kabbani et al.[13] proposed approximate solutions for the velocity profile of steady incompressible MHD flows in a rectangular microchannel driven by Lorentz force. Their solutions were found to compare favorably with existing computational and analytical results.

2. Problem formulation

We consider a circular tube with screen electrodes positioned at the entrance and middle of the tube. A conducting fluid can move freely through the end electrodes. Both of them are separated by the same distance z_0 from the center of the tube. The tube axis is at the center of the duct and its axis is parallel to the side wall and orthogonal to the streamwise direction [Fig. 4.1]. A converging and diverging current flow is obtained by applying potential differences across the electrodes. A coordinate system is used with the origin positioned at the center of the circular tube, and the coordinates r, and z are aligned respectively along the channel height and width. The channel surface can be taken as nonconducting. It is our goal to determine the flow and electromagnetic fields as well as secondary flows that can exist for this configuration, given the appropriate governing differential equations, initial and boundary conditions.
On the assumptions that (a) fluid **B** flow is incompressible (b) the induced magnetic field is negligibly smaller than the applied magnetic field and hence much smaller than the total magnetic field . This is because the magnetic Reynolds number is considered small. (c) Both the displacement current and the free charge density are also considered negligible. (d) The Lorenz force is the only body force on the fluid (e) the velocity of flow is regarded as too small compared to the velocity of light as a consequence of this relativistic effects are ignored. The governing differential equations of motion for an incompressible electrically conducting fluid in a tube can be expressed together with the Maxwell equations in the following form
When an electric current is passed through the conducting fluid, a magnetic field is generated and a current density **J** transmits through the fluid. Relativistic effects are ignored for cases where the velocity of flow is small compared to the velocity of light. For steady state, even if the fluid were moving, the electromagnetic equations can be written in the form:

$$\frac{\partial \bar{u}}{\partial t} + \nabla \times (\bar{u} \bullet \nabla)\bar{u} = -\nabla(\frac{p}{\rho} + gr) + \nu(\nabla^2 \bar{u})\frac{\mu_m}{\rho}(\bar{j} \times \bar{B}) \tag{2.1}$$

The curl of equation (2.1)

$$\nabla \times \frac{\partial \bar{u}}{\partial t} + \nabla \times (\bar{u} \bullet \nabla)\bar{u} = -\nabla \times \nabla(\frac{p}{\rho} + gr) + \nabla \times (\nu \nabla^2 \bar{u}) + \nabla \times [(\nabla \times \bar{B}) \times \bar{B}]$$

where

$$\begin{aligned}
&\nabla \times \tfrac{\partial \bar{u}}{\partial t} = \tfrac{\partial(\nabla \times \bar{u})}{\partial t} = \tfrac{\partial \bar{\omega}}{\partial t}\\
&\nabla \times (\bar{u} \bullet \nabla)\bar{u} = \nabla \times \nabla(\bar{u} \times u) - \nabla \times [u \times (\nabla \bar{u})]\\
&\nabla(\bar{u} \bullet \bar{u}) = 2(\bar{u} \bullet \nabla)\bar{u} + 2\bar{u} \times (\nabla \bar{u})\\
&\bar{u}(u \bar{\bullet} \nabla) = \tfrac{1}{2}\nabla(u^2) - (\bar{u} \times \bar{\omega})\\
&\nabla \times (\bar{u} \bullet \nabla)\bar{u} = \nabla \times \nabla(\tfrac{u^2}{2}) - \nabla \times (\bar{u} \times \bar{\omega})\\
&\nabla \times (\bar{u} \bullet \nabla)\bar{u} = -\nabla \times (\bar{u} \times \bar{\omega}) = \nabla \times (\bar{\omega} \times \bar{u})\\
&\nabla \times [\nabla(\tfrac{p}{\rho} + gr)] = 0\\
&\nabla \times (\nu \nabla^2 \bar{u}) = \nu \nabla^2(\nabla \times \bar{u}) = \nu \nabla^2 \omega\\
&\nabla \times [(\nabla \times \bar{B}) \times \bar{B}] = \nabla \times (\bar{h} \times \bar{B}) = (\bar{B} \bullet \nabla)\bar{h} - (\bar{h} \bullet \nabla)\bar{B} + \bar{h}(\nabla \bullet \bar{B}) + \bar{B}(\nabla \bullet \bar{h})
\end{aligned}$$

The system of Maxwell equations can be written in the form:

$$\mu \bar{j} = \nabla \times \bar{B}, \;\; \nabla \bullet \bar{j} = 0, \;\; \nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}, \;\; \nabla \bullet \bar{B} = 0$$

Ohm's law can be written as:

$$\bar{j} = \sigma(\bar{E} + \mu \bar{u} \times \bar{B})$$

The body force term or the Lorentz force is given by: $\bar{f} = \bar{j} \times \mu_m \bar{B}$. It not only represents the force per unit volume that accounts for the coupling between the fluid motion and the magnetic field, but also contributes in no small measure to the interesting physics of MHD flows. Passage of electric current through a fluid will set the fluid in motion since in general, the potential; pressure forces can not be balanced by the rotational electromagnetic forces. Because of the axisymmetric geometry of tubular flow, the streamfunction can be utilized to satisfy the continuity equation, and is expressed as

$$u_z = \frac{1}{r}\frac{\partial \psi}{\partial r}, \quad u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}$$

After carrying out the chores for nondimensionalization, the governing equations are given as:

$$U = \frac{1}{R}\frac{\partial \psi}{\partial R}, \quad V = -\frac{1}{R}\frac{\psi}{\partial R} \tag{2.2}$$

$$(\frac{\partial^2 \psi}{\partial^2 R} - \frac{1}{R}\frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial R^2})\frac{1}{R} = \omega \tag{2.3}$$

$$(\frac{\partial^2 \psi}{\partial^2 R} - \frac{1}{R}\frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial R^2})RB = 0 \tag{2.4}$$

$$\frac{\partial \omega}{\partial t} + \frac{(\partial U \omega)}{\partial Z} + \frac{\partial (V\psi)}{\partial R} = \zeta \frac{B}{R}\frac{\partial B}{Z} + \frac{1}{Re}(\frac{\partial^2 \omega}{\partial R^2} + \frac{1}{R}\frac{\partial \omega}{R} - \frac{\omega}{R^2} + \frac{\partial^2 \omega}{\partial Z^2}) \tag{2.5}$$

An upwind scheme is used to handle the nonlinear convection terms in equation (2.5). It would seem as if the computational overhead associated with the governing equations will be quite intense, however the computational rigor is simplified if we take into consideration the symmetrical nature of the electromagnetic flow and find the solution in the region $(-z_0 \leq z \leq 0)$ to the left of the central electrode and use the mirror image for the right of the electrode. This however does not apply to a net flow through the tube where the whole geometry is considered because the flow pattern should not be symmetric about a center point. Our first consideration, involves the fluid dynamics of an unsteady flow of an electrically conducting viscous fluid through orifices positioned on the left and right hand sides $(z = \pm 1)$ of an insulated tube (Fig. 4.1). The tube openings at the left and the tight sides are positioned at distances $0.2r_0$ from the centerline. No-slip boundary conditions are set up at the tube walls, and by assuming that the flow is purely axial, we impose a zero perpendicular velocity (V=0) at entrance and exit. Boundary conditions for the stream functions comprise, $\psi = 0$ along the tube axis, and along the wall $\psi = \frac{1}{2}$, at the entrance and exit $\frac{\partial \psi}{\partial Z} = 0$ respectively. We assume uniform flow through the tube axis. As a consequence, the axial velocity in this region is specified as: $U_{i,1} = 2\psi_{i,2}/h$.
Equations (2.2) and (2.3) as defined by the conservation of mass, show a similarity between the electromagnetic flow term RB and the stream function . This allows for constant values of RB to behave like stream function ψ. At the tube centerline and top wall, RB=0, and RB=0.5 respectively. For the centrally placed electrode, $0 < R < r_1/r_0$, $\partial(BR)/\partial Z = 0$, and for $r_1/r_0 \leq R \leq -r_1/r_0$ RB=0.5 and at the left orifice $Z = -z_0/r_0$, $\partial(RB)/\partial Z = 0$ Vorticity is specified at zero at the walls as well its derivative with respect to the horizontal direction at the entrance and exit are both given zero values. To set up the initial flow conditions, uniform flow for a unit axial velocity is assumed. The fluid can not remain stationary in the presence of a rotational electromagnetic body force and a predominantly axial motion is motivated by an electromagnetic force per unit volume $f = \mu_m B$ set up in the tube. It should be noted that the this body force is rotational and can not be balanced by viscous forces unless at steady state
The numerical strategy for solving this problem are enumerated as follows:
Non dimensional electrode and radial sizes of $r_1/r_0 = 0.1 \quad and \quad z_0/r_0$ together with a spatial increment of h= 0.1 are chosen. The governing differential equations are replaced by appropriate difference equations. Time and spatial coordinates are approximated by forward differencing in time and central differencing in space. Iterative procedures are adopted to approximate equations (2.3), (2.4) and (2.5). Two additional grid lines at the entrance and exit are deployed to handle derivative boundary conditions. The unsteady nonlinear equations together with the given boundary conditions are solved using an implicit, iterative finite difference scheme similar to the one described in Soundalgekar et al.[14]. Square meshes are chosen to cover the problem domain. Since the absolute dimensionless distance between the central electrode and the inlet or exit is unity, the each grid has a value of $1/(n-1)$. After a series of trial, a 21x11 grid was chosen for computation for a time step $\Delta\tau = 0.01$. **RB** is computed by solving equation (2.4) iteratively to satisfy the boundary conditions. The **RB** scalar field gives us an idea of the 'streamlines of the current flow' in the problem domain. Having obtained R**B**, the influence of the Lorentz force on the conducting fluid is determined by computing the source term $(\zeta \frac{\mathbf{B}}{R}\frac{\partial \mathbf{B}}{\partial Z})$ in equation (2.5) for all interior grid points, where the magnetic pressure number is given as:

$$\zeta = \frac{\mu_m r^2 J_0^2}{\frac{1}{2}\rho u_0^2}$$

The flow profile is determined by assuming a uniform flow of unit dimensionless speed and integrating equation (2.2) for all the grid points. Since the flow field unlike the electromagnetic filed is not symmetrical, we have to pay due cognizance to the boundary conditions at the two ends of the tube containing the orifices. At any instant in time, the velocity and vorticity distributions are obtained from the conditions at the previous time step, eventhough at the beginning, they are obtained from a prescribed initial conditions. Streamfunction are computed based on the vorticity distribution by solving equation(2.3). Velocity components are computed based on equation (2.2) once the values of the streamfunction are known. These are again used to determine the vorticity field in the interior region for the next time step; and their boundary values updated appropriately. The same process is repeated at each of the time steps until the time counter reaches an apriorily specified value of maximum time or the computed scalar field satisfies the criterion for steady state. Our numerical results will then represent steady state profiles induced by an unsteady converging-diverging current field.

A second numerical experimentation involves replacing the orifices with two large screen electrodes covering the ends of the nonconducting tube. The governing equations for the electromagnetic field and the flow are the same, but appropriate changes are made to the boundary conditions to reflect the new configuration. Just like in the first example, the electromagnetic field is symmetrical about the central electrode so it is computed half the horizontal direction of the nonconducting tube and then reflected on the other side. Dirichlet boundary conditions of 0 and 0.5 are specified for the tube centerline which constitutes the lower boundary as well as the top of the nonconducting tube. This amounts to setting up an electromagnetic barrier to the flow. We assume that the electromagnetic flow through the central electrode is purely horizontal, this is defined by setting up a Neumann or zero flux boundary condition at this point i.e. $\partial RB/\partial Z = 0$. To guarantee that continuity requirements are satisfied, the total current through the central electrode (i.e. from its top the top of the tube) should be 0.5. Lastly in order to ensure that the electromagnetic field through the central electrode should be purely horizontal, a Neuman boundary condition is imposed on the left screen electrode ($\partial(RB)/\partial Z = 0$) Since the stream function ψ is the analog of BR by virtue of equations (2.3) and (2.4), their boundary conditions are essentially the same except that for the flow boundary conditions care must be taken to reflect the fact the flow pattern is no longer symmetrical and hence must have to be specified for the entire problem domain. Since the stream function essentially represents the volume of fluid per unit time between a given point and a reference plane, it is given values of 0 and 0.5 at the tubes lower boundary (the centerline) and the top boundary. The vorticity is set to zero at all boundaries. Axial flow through the end electrodes is guaranteed by setting the derivatives of the stream function and vorticity equal to zero ($\partial\psi/\partial Z = \partial\omega/\partial Z = 0$). This last equation, in combination with equations (2.2) and (2.3) determine the entire flow field (the radial and axial velocity components) in the problem geometry. For example at the centerline flow is purely axial and $\psi = 0 \Rightarrow V = 0$. At the top wall $\psi = 1/2 \Rightarrow \partial\psi/\partial R$, in terms of the velocity components $V = 0,\ \ \partial U/\partial R = 0$ at the top wall. For the two electrodes at the exit and inlet $\partial\psi/\partial Z = \partial\omega/\partial Z = 0$ [15]. We need to interpret this in terms of the velocity components.

$$\frac{\partial\psi}{\partial Z} = 0 \Rightarrow V = 0 \ \ at \ \ Z \pm z_0/r_0 \Rightarrow V_{2,j} = V_{m,j} \ \ j = 2,3,...n-1 \tag{2.6}$$

For the U velocity component we differentiate the above condition with respect to R

$$\begin{array}{l} \frac{\partial}{\partial R}(\frac{\partial\psi}{\partial Z}) = \frac{\partial}{\partial Z}(\frac{\partial\psi}{\partial R}) = \frac{\partial}{\partial Z}(RU) = R\frac{\partial U}{\partial Z} \\ R\frac{\partial U}{\partial Z} = 0 \Rightarrow \frac{\partial U}{\partial Z} = 0; \Rightarrow (\frac{\partial U}{\partial Z})_{2,j} = (\frac{\partial U}{\partial Z})_{m,j} \end{array}$$

The second condition at the wall, $\frac{\partial\Omega}{\partial Z} = 0$ can also be written in terms of the velocity components by noting that equation (2.3) can be expressed as:

$$\omega = \frac{1}{R}(R\frac{\partial U}{\partial R} - U - R\frac{\partial V}{\partial Z})$$

Hence

$$\begin{array}{l} \frac{\partial\omega}{\partial Z} = \frac{1}{R}(\frac{\partial}{\partial Z}\{R\frac{\partial U}{\partial R}\} - \frac{\partial U}{\partial Z}) - R\{\frac{\partial^2 V}{\partial Z^2}\}) = 0 \\ \frac{\partial\omega}{\partial Z} = R(\frac{\partial^2 V}{\partial Z^2}) = 0 \Rightarrow (\frac{\partial^2 V}{\partial Z^2})_{2,j} = (\frac{\partial^2 V}{\partial Z^2})_{m,j} = 0 \end{array} \tag{2.7}$$

Equations (2.6) to (2.7) are approximated by finite differences to read

$$U_{1,j} = U_{3,j} \ \ for \ \ LHS \ \ BC \ \ and \ \ U_{m+1,j} = U_{m-1,j} \ \ for \ \ RHS \ \ BC$$

$$V_{1,j} = -V_{3,j} \ \ for \ \ LHS \ \ BC \ \ and \ \ V_{m+1,j} = -V_{m-1,j} \ \ for \ \ RHS \ \ BC$$

Vorticity distribution is computed from equation (2.5). Along the tube axis and the walls vorticity is set to zero except at the exit, except at the inlet and the exit where the zero gradient specification leads to a finite difference approximation expressed as

$$\omega_{1,j} = \omega_{3,j}, \ \ and \ \ \omega_{m+1,j} = \omega_{m-1,j}$$

3. Results and discutions

Fig.4.2 shows the profile of an electric current through an orifice in the presence of a conducting fluid. The electric current converges at the central electrode where higher RB values are registered. This is in agreement with the specified boundary conditions as well as the fact that the tube walls are non-conducting. The central electrode discharges towards the inlet and the exit, and produces an axisymmetrical configuration of the current streamlines. Electromagnetic forces result in this process, and are vital in the modification of the flow of fluid through the tube.
Fig.4.3 is the profile of the electromagnetic force field. The rotational electromagnetic force associated with the current is displayed as two oppositely revolving force fields. These forces are rotational, and are not balanced by body forces arising from pressure gradient unless at steady state. This imbalance gives rise to acceleration that impact on the dynamical features of the conducting fluid.
Rotational forces will most likely develop secondary flows when the vorticity production becomes appreciable. It can be observed from Fig.4.4 that secondary flow is initiated in the region starting from the tip of the orifice and towards the upper wall at t= 10 for C = 0.3 and for a Reynolds number value of 100. Fig.4.4 also depicts the axial velocity profiles of descending magnitude as the upper boundary wall is approached. A zero velocity value corresponds to the position of the upper lip of the aperture. As the flow enters and exits through the orifices, it impacts the solid boundaries at the inlet and exit. This results in a flow reversal which is indicated in the magnitude of the velocity values in this vicinity.
Fig.4.5 displays the vorticity field generated with ζ given as 0.3 and Reynolds number of 50 at a time of 2.5. As the fluid enters the orifice, and makes the first impact with the lip of the orifice, vorticity is generated along the solid wall leading to the upper boundary. We observe an increase of vorticity away from the wall in the axial direction. At this point in time, there is more vorticity generation at the upstream of the central electrode than in the downstream as the hydrodynamic effects of the fluid contact with the orifice has not been sufficiently felt

downstream. Fig.4.6 displays the vorticity field at a later time (t=5). The impact of the vorticity produced by the central electrode is observed as the fluid moves towards the exit. In addition contributions from both the solid walls and electromagnetic field are merged, with those of the central electrode. The overall picture, indicates that more vorticity is produced closer to the walls and in the vicinity of the central electrode. In the second experiment, it is found that the strength of the magnetic field as indicated by the magnetic pressure number and the size area of the electrode in the middle of the nonconducting tube introduce some vital electromagnetic effects that produce different flow patterns. This is demonstrated by the flow streamline patterns along the tube axis (Figs. 4.6 and 4.7). The increase in the gradient of the streamlines over the central electrode with respect to the radial coordinate indicates an increase of the axial velocity. As a fact, flows of this kind mimic those of ordinary flows when they encounter solid bodies as can be seen for flows past a plate aligned normal to the freestream direction. However one thing that needs to be pointed out in this case is that there is a continuous decrease in flow as it moves towards the current constriction created by the central electrode(Figs 4.8 and 4.9). Minimum speed is observed just before the central electrode as shown by the axial velocity profile at the tube's centerline. However in order to satisfy continuity requirements, the flow accelerates to speeds higher than the entry speed and having made up for the continuity of mass, it decelerates once again to the speed of entry. This occurrence is noticeable as the value of the magnetic pressure number ζ is increased. It is worthwhile to note that nonuniformities in the flow electromagnetic field enhance their interaction with the dynamics of a conducting fluid and the eventual generation of Lorenz forces which are basically damping in nature. Figure 4.9 shows that as the electromagnetic pressure number is increased, the flow becomes dispossessed of enough kinetic energy to flow over the pressure hump created by the central electrode. This is shown by how far below zero the tube's axial velocity goes just before it gets to the central electrode.

The presence of negative values of axial velocity before the approach to the central electrode is indicative of the formation of secondary flows. These observations show in a simple manner how the Lorenz force acts as an impediment to flow and as a consequence generates vorticity and secondary flows. The value of ζ plays a significant role in flow configuration. For relatively small values of ζ , the flow is only slightly deviated from its oncoming direction towards the central electrode, while an increase in ζ creates stagnation zones in front of the central electrode. Figs.4.10 and 4.11 show that for $\zeta = 0.8$, counter-rotating vortices appears in the region of the central electrode. Both diagrams also show that the diffusive-convective transport of vorticity involves a considerable portion of the flow domain both upstream and downstream of the central electrode.

The so called 'pinch effect' in the vicinity of the obstacle (central electrode) is indicative of a non-smooth transition. It offers a region of intense hydrodynamic activity as illustrated by a deficit in the fluid axial velocity(Figs.4.12 and 4.13) and a consequential build up of velocity in the radial direction (Figs. 4.14 and 4.15). Figure 4.16 shows that near the central electrode, the current streamlines are distributed uniformly and directed towards the negative radial direction before it arrives at the obstacle. This shows that the Lorentz force opposes fluid motion in this region and as a result causes its deviation from the streamwise direction and reverses this direction immediately after the obstacle.

4. Conclusion

Most of the flows found in scientific literature dealing with nonuniform magnetic fields are related to the magnetohydrodynamics of duct flows in the streamwise direction arising from the interaction of electric and magnetic fields[16],[17],[18]. The current study in addition to this, examines the variations arising from orifice flow exposed to an electromagnetic field. In all cases this study has further illustrated how the Lorentz force produced by the interaction of electric and magnetic fields slows down the flow and generates vorticity. This may be of practical importance where mixing or heat enhancement is needed. The governing continuum equations that comprise the balance laws of mass and momentum are modified to include the electromagnetic effects. These have been solved numerically using the finite difference methods. The correctness of the numerical algorithm developed herein was confirmed by noting the closeness of some of the results with those from literature(Chow[15]). It is hoped that this work will help in further understanding of flows produced by localized forces and nonuniform magnetic fields and their concomitant effects in providing mixing, vorticity and heat enhancement.

NOMENCLATURE

$\bar{B}$	magnetic field
$\bar{E}$	electric field intensity
f	electromagnetic body force per unit volume
g	gravitational acceleration
h	magnetic field intensity
h_θ	component of the magnetic field intensity in the azimuthal direction
$\bar{j}$	current density
$\bar{j}_0$	reference current density
p	pressure
r	radial coordinate
r_0	reference coordinate
r_1	radius of aperture
r_2	radial distance from top of aperture to upper wall
R	dimensionless radial coordinate
R_0	Reynolds number
RB	analog of streamlines for current flow
t	time coordinate
u_z	axial velocity
u_0	reference velocity
U	dimensionless axial velocity
u_r	radial velocity

V dimensionless radial velocity
z axial coordinate
Z dimensionless axial coordinate
Greek Symbols
∇ gradient
μ_m magnetic permeability of fluid medium
ν kinematic viscosity
ψ stream function
Ψ dimensionless stream function
ρ density
τ dimensionless time
ω dimensionless vorticity
μ magnetic permeability
σ electrical conductivity
ζ magnetic pressure number
Subscripts
i,j node counters in axial and radial directions

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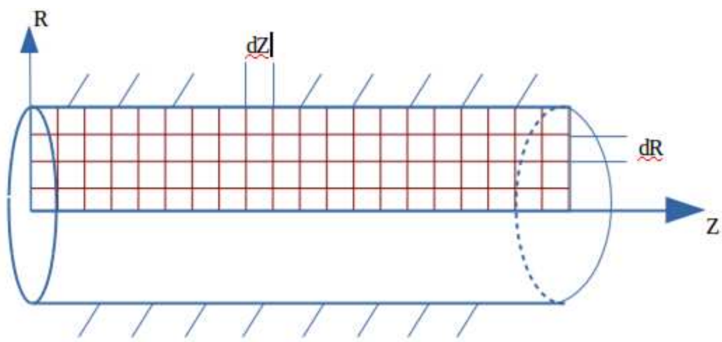

Figure 4.1: Problem configuration

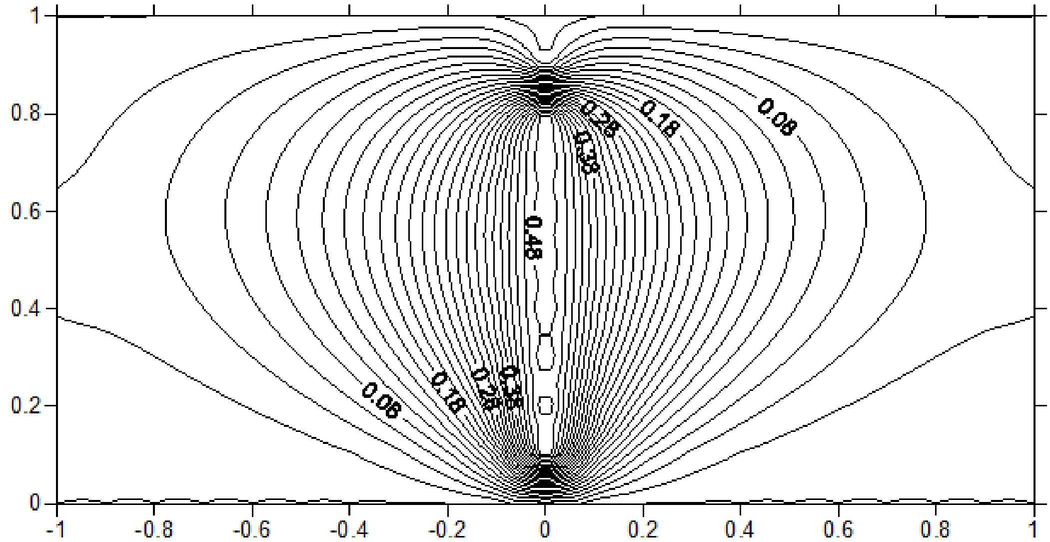

Figure 4.2: Current streamlines profile for orifice flow

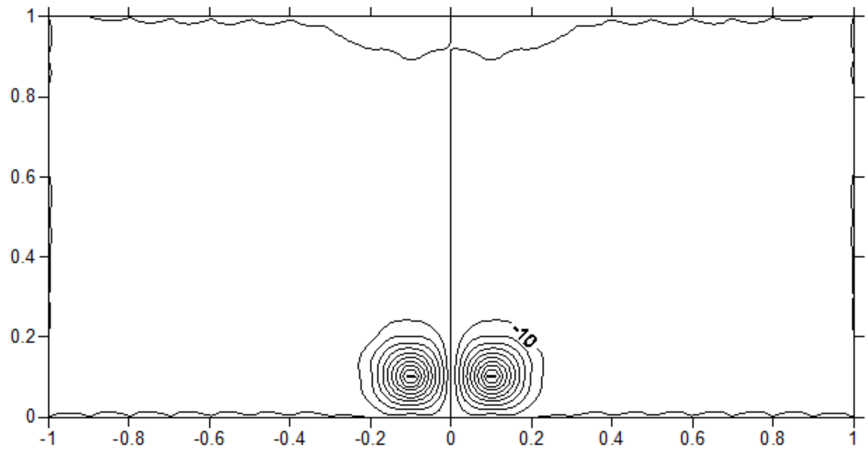

Figure 4.3: Profile of electromagnetic force for orifice flow

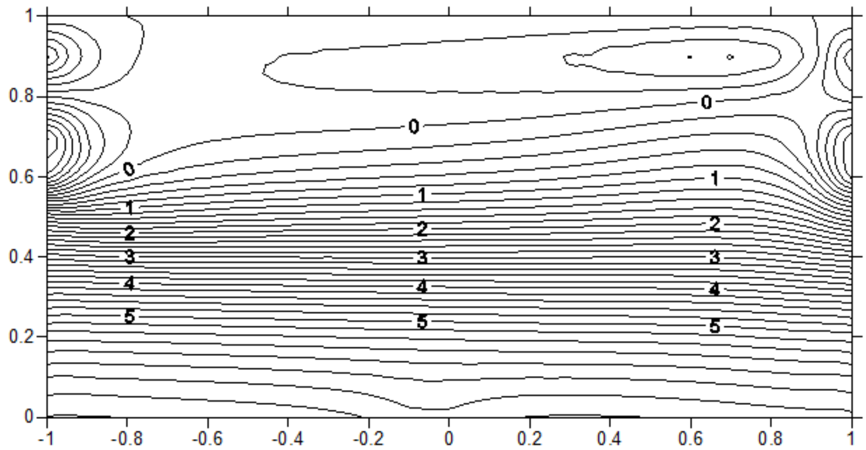

Figure 4.4: Secondary flow profile for orifice flow (ζ=0.3, Re=100, time=10)

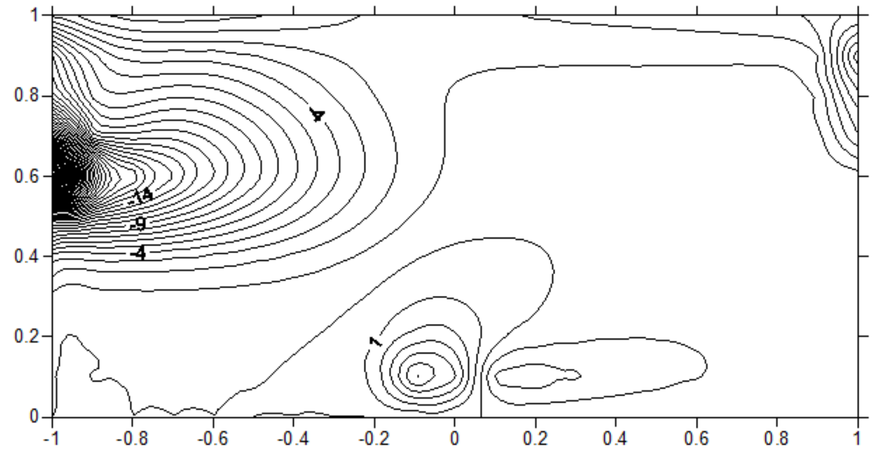

Figure 4.5: Vorticity field for orifice flow (ζ=0.3, Re=50, time=2.5)

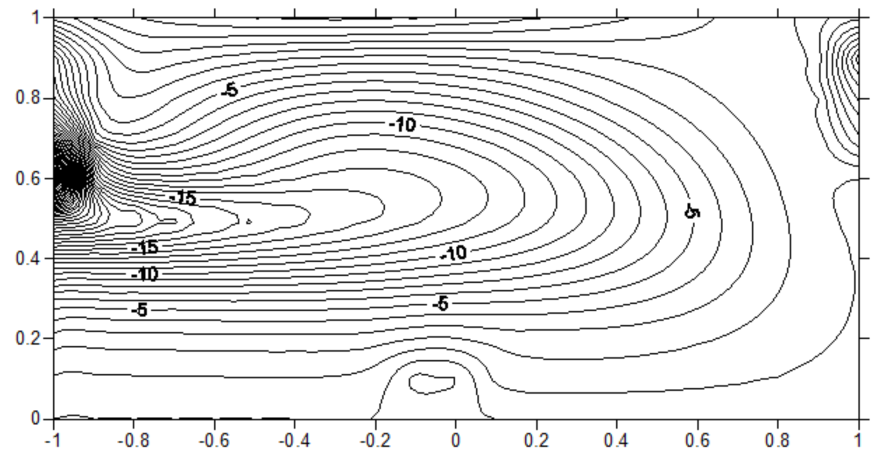

Figure 4.6: Flow streamline pattern (ζ=0.3, Re=50)

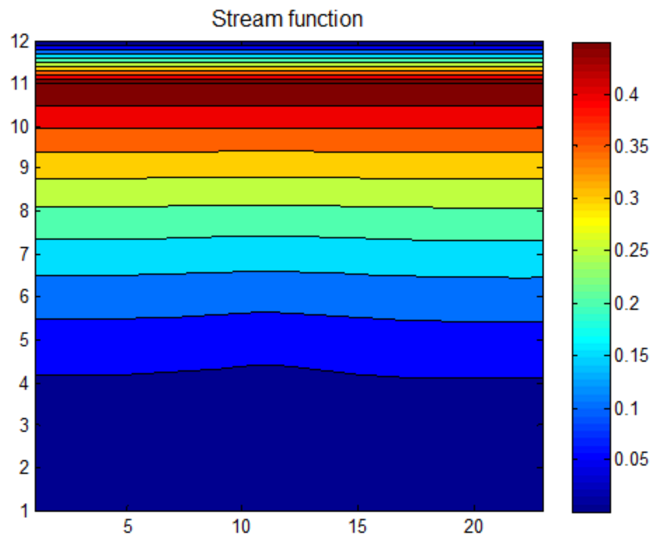

Figure 4.7: Flow streamline pattern (ζ=0.8, Re=50)

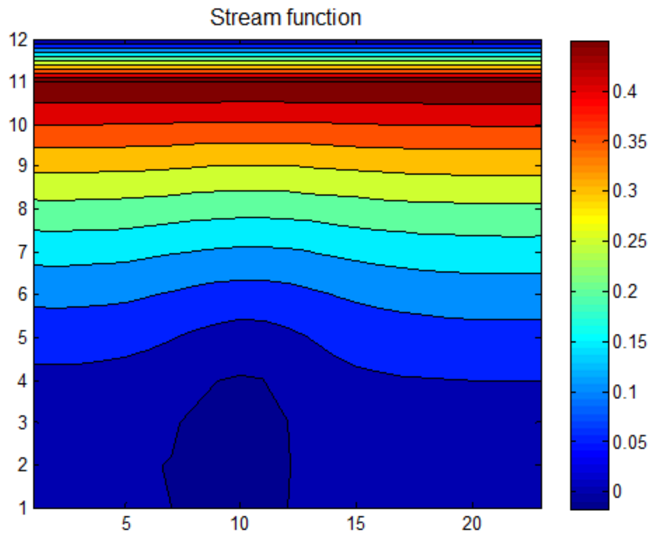

Figure 4.8: Axial velocity along tube axis (ζ=0.3, Re=50)

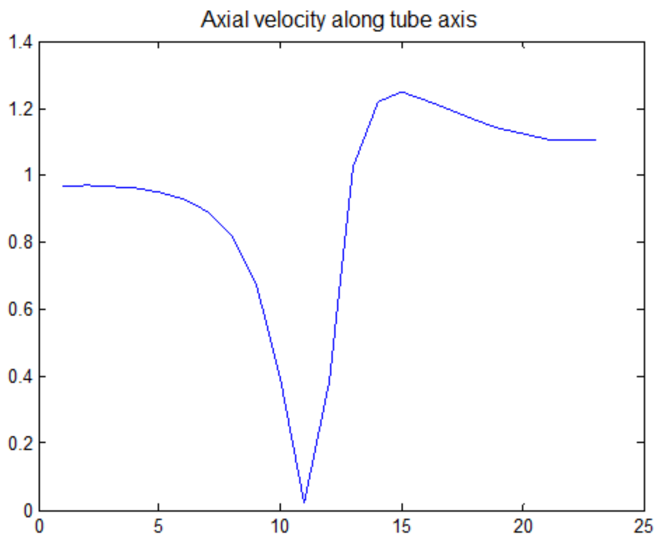

Figure 4.9: Axial velocity along tube axis (ζ=0.8, Re=50)

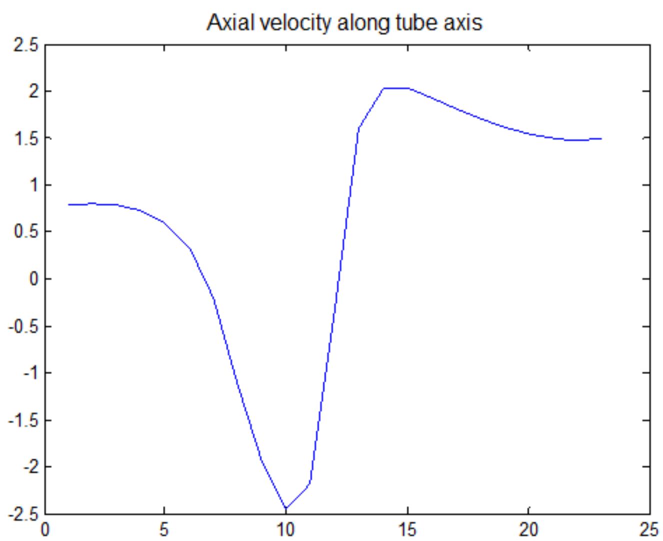

Figure 4.10: Vorticity contour (ζ=0.8, Re = 50)

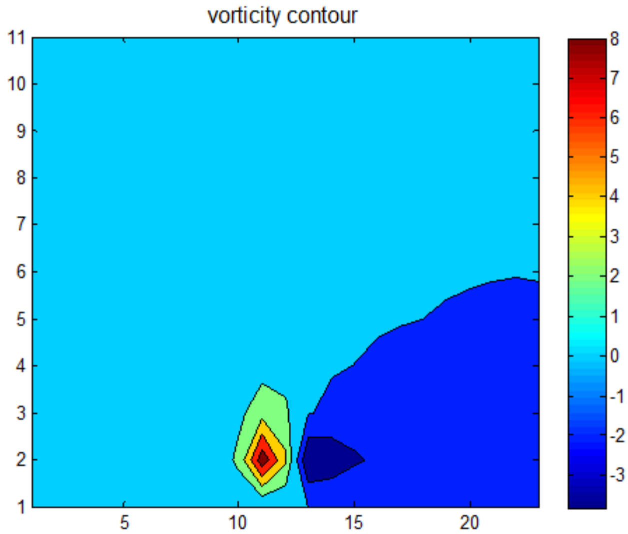

Figure 4.11: Vorticity contour (ζ=0.8, Re=60)

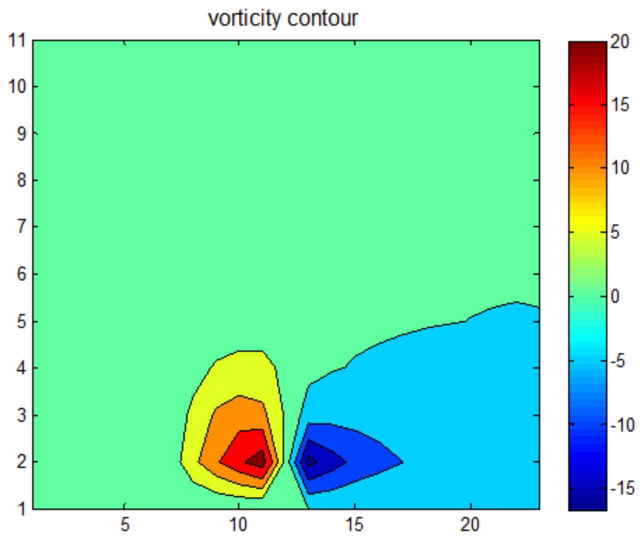

Figure 4.12: Horizontal velocity contour (ζ=0.8, Re=60)

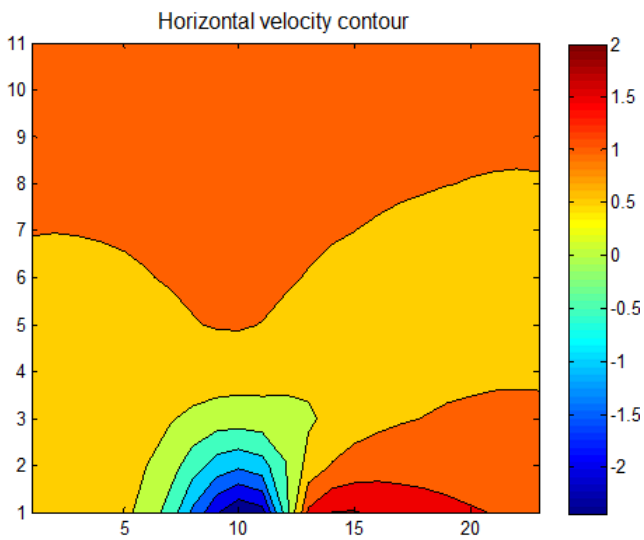

Figure 4.13: Horizontal velocity mesh(ζ=0.8, Re =60)

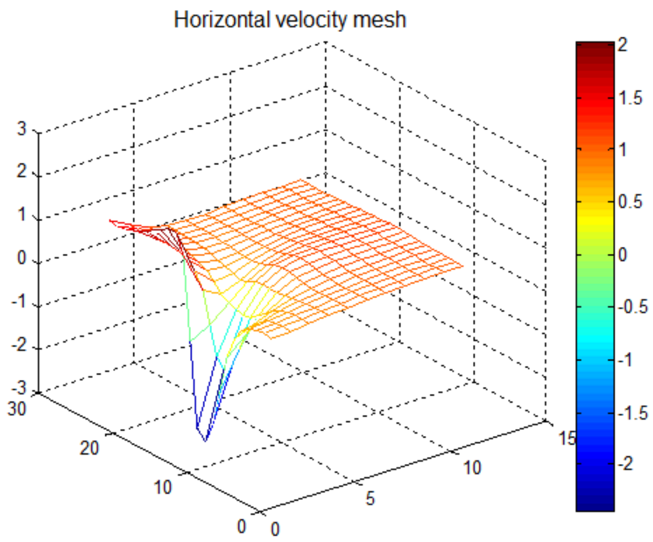

Figure 4.14: Vertical velocity contour (ζ=0.8, Re=60)

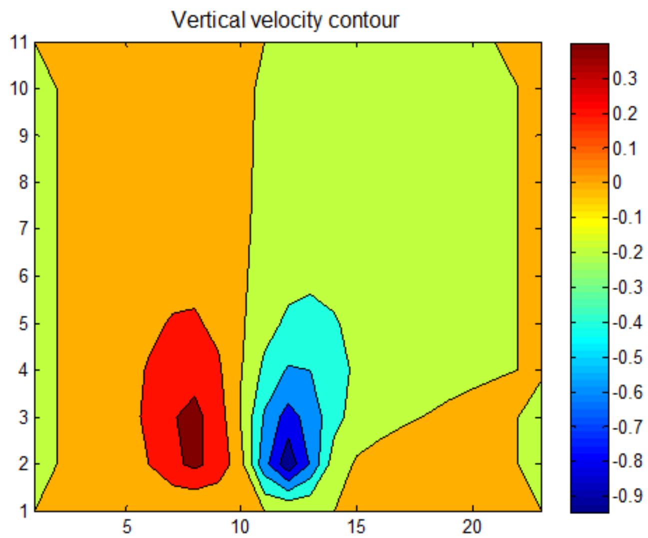

Figure 4.15: Vertical velocity mesh (ζ=0.8, Re=60)

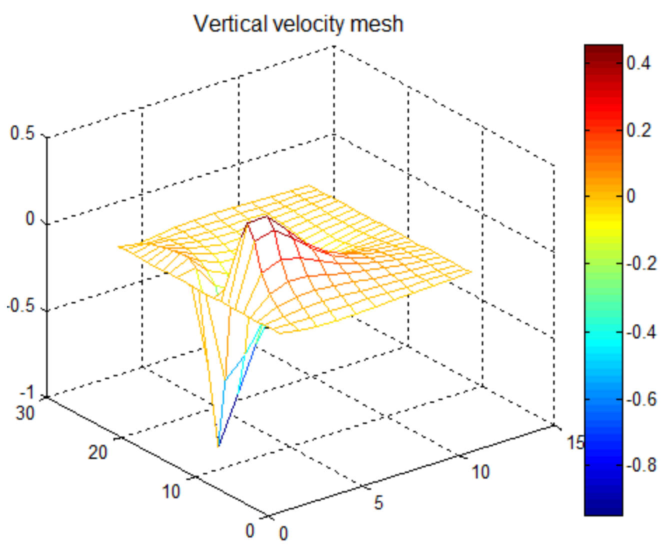

Figure 4.16: Current streamlines (ζ=0.3, Re=50)

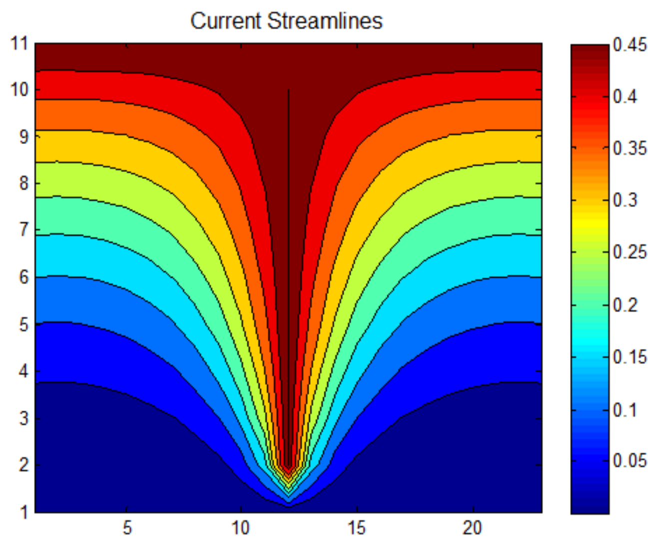
Current Streamlines
11
10
9
8
7
6
5
4
3
2
1
5
10
15
20
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05

Figure 4.17