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The Theory of Ultra - Groups and GAP Applications

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Keywords
Ultra-group, normal ultra-group, GAP programming language

Abstract — In this work, we introduce a system computational programming of the ultra-group notions with GAP, Groups, Algorithms and Programming. We have constructed some algorithms about ultra-groups and their substructures with GAP language. Also, we give the GAP algorithm about ultra-group homomorphism. So we have presented a GAP application for ultra-group theory.

Subject Classification (2020): 68W30, 20B05.

1. Introduction

The notion of ultra-groups over a group was introduced in [6] by Moghaddasi et al. In fact, an ultra-group is an algebraic structure whose underlying set is determined by a group and its subgroup. An ultra-group is a special subset of a group that has a binary operation that combines any two elements to generate a third element and a unary operation. Every group is an ultra-group, but the reverse is not always valid. The idea of an ultra-group was studied in [7],[12] and [13] with details.

By similarity with " computational group theory," we provide new GAP functions in this study. As an application, we have constructed some algorithms to create ultra-groups and subultra-groups. Also, the algorithm for checking the ultra-group homomorphism conditions as well as controlling an ultra-group structure and its various substructures has been presented. In [2], [1], [9] and [10], one can find several algorithms for many concepts which are implemented in the GAP packages.

2. Theory of Ultra-Groups

The concept of transversals was first introduced by Kurosh in [5], which is the foundation of the ultra-group concept.

We will recall this notion and some fundamental notions such as ultra-groups and their substructures from [6].

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Definition 2.1. Let G be a finite group. Assume M and H are two any subsets of group G. Obviously, $HM \neq MH$. (A, B) is a pair of subsets of group G and is named **transversal** if the equality $xy = x'y'$ implies $x = x'$ and $y = y'$ for $x, x' \in A, y, y' \in B$.

From this definition, it is easy to understand that a pair (H, M) of subgroups of G is transversal if and only if $H \bigcap M = \{e\}$.

Furthermore, if G is a group, H is a subgroup of G, and M is a subset of G, then we conclude that the pair (H, M) is a transversal if and only if $M \bigcap Hg$ contains **at most one element**, for all $g \in G$ [12].

We demonstrate the left and right quotient sets by

$$\begin{aligned} G/H &= \{xH | x \in G\} \\ H/G &= \{Hx | x \in G\} \end{aligned}$$

respectively. These sets are the partitions of G where H is the subgroup of G.

A transversal of a partition is a set with just one member from each part of the partition. Throughout the work, we will assume that a right transversal contains the identity of the group. In other words, $|M \bigcap Hg| = 1$ with all $g \in G$, subgroup H and a set M. Then $G = HM$. And so, we have $MH \subseteq G = HM$, [6].

Definition 2.2. Let G be a group with multiplication and H be a subgroup of G. A subset M of G is described as (right unitary) **complementary set** concerning the subgroup H, if for all elements $h \in H$ and $m \in M$ there exist the **unique** members $m' \in M$ and $h' \in H$ satisfying

$$h'm' = mh$$

and the identity of G in M. We denote m' and h' by m^h and ${}^m h$, respectively.

In a similar way, for all $m_1, m_2 \in M$ there is a unique $[m_1, m_2] \in M$ and $(m_1, m_2) \in H$ satisfying

$$m_1 m_2 = (m_1, m_2)[m_1, m_2]$$

For all $x \in M$, there exists $x^{-1} \in G$.

In $G = HM$, there exists $x^{[-1]} \in M$ and $x^{(-1)} \in H$ satisfying

$$x^{-1} = x^{(-1)} x^{[-1]}$$

[6].

Definition 2.3. A complementary set of H over group G namely ${}_H M$ is called a (right) ultra-group with a binary operation

$$\alpha :_H M \times_H M \longrightarrow_H M$$

and unary operation

$$\beta :_H M \longrightarrow_H M$$

which is defined as $\alpha(m_1, m_2) := [m_1, m_2]$, $\beta_h(m) := m^h$, for all $h \in H$, respectively.

Example 2.4. Let $G = S_3$ then $H = \{(1), (13)\}$ is a subgroup of G. With the right quotient set $H/G = \{\{(1), (13)\}, \{(12), (132)\}, \{(23), (123)\}\}$, we have four alternatives for M. For instance, if we choose the set $M = \{(1), (12), (23)\}$, we get the following multiplication tables:

α	(1)	(23)	(12)
(1)	(1)	(23)	(12)
(23)	(23)	(1)	(23)
(12)	(12)	(12)	(1)

β	(1)	(13)
(1)	(1)	(1)
(23)	(23)	(12)
(12)	(12)	(23)

(M, α, β) is an ultra-group of H over group G.

Example 2.5. Let $G = S_3$ then $H = \{(1), (13)\}$ is a subgroup of G. With the quotient set $H/G = \{\{(1), (13)\}, \{(12), (132)\}, \{(23), (123)\}\}$, we have four candidates for M. For instance if we choose the set $M = \{(1), (123), (132)\}$, we get the following multiplication tables:

α	(1)	(123)	(132)
(1)	(1)	(123)	(132)
(123)	(123)	(132)	(1)
(132)	(132)	(1)	(123)

β	(1)	(13)
(1)	(1)	(1)
(123)	(123)	(123)
(132)	(132)	(123)

(M, α, β) is an ultra-group of H over group G.

Example 2.6. Let D_9 be a dihedral group of order 18 and $H = < a^3 >$ its subgroup. We get 243 ultra-groups and two of them are the following:

$$M = \{e, a, b, a^2, ab, a^2b\}$$

α	e	a	b	a^2	ab	a^2b
e	e	a	b	a^2	ab	a^2b
a	a	a^2	ab	e	a^2b	b
b	b	a^2b	e	ab	a^2	a
a^2	a^2	e	a^2b	a	ab	ab
ab	ab	b	a	a^2b	b	a^2
a^2b	a^2b	ab	a^2	b	a	e

β	e	a^3	a^6
e	e	e	e
a	a	a	a
b	b	b	b
a^2	a^2	a^2	a^2
ab	ab	ab	ab
a^2b	a^2b	a^2b	a^2b

and

$$M = \{e, b, ab, a^2b, a^5, a^4\}$$

α	e	b	ab	a^2b	a^5	a^4
e	e	b	ab	a^2b	a^5	a^4
b	b	e	a^5	a^4	ab	a^2b
ab	ab	a^4	b	a^5	a^2b	b
a^2b	a^2b	a^5	a^4	e	b	ab
a^5	a^5	a^2b	b	ab	a^4	e
a^4	a^4	ab	a^2b	b	e	a^5

β	e	a^3	a^6
e	e	e	e
b	b	b	b
ab	ab	ab	ab
a^2b	a^2b	a^2b	a^2b
a^5	a^5	a^5	a^5
a^4	a^4	a^4	a^4

Definition 2.7. Let M be an ultra-group of H over G. A subset $U \subseteq M$ that includes the identity is defined as **subultra-group** of H over G, if U is closed under the operations α and β_h in the Definition 3.

It is clear that $\{e\}$ is a trivial subultra-group for all ultra-groups $_HM$ where e is the identity element of H.

Definition 2.8. Let $_HM$ be a ultra-group and A, B are the subsets of $_HM$. [A,B] is defined as the set

$$\{[x,y] | x \in A, y \in B\}$$

Furthermore, if A is a subultra-group $_HM$ and $y \in_H M$ so the subset $[A, y]$ is named a **right coset** of A in $_HM$ [6].

Definition 2.9. A N subultra-group of $_HM$ is named **normal** if

$$[N,[x,y]] = [x,[N,y]]$$

for all $x, y \in_H M$.

So we get the following features for normal ultra-groups:

- $[x,N] = [N,x]$, for every $x \in_H M$
- $[[N,x],[N,y]] = [N,[x,y]]$, for all $x, y \in_H M$.
- If $[N,y] = N$, then $y \in N$

[6].

Definition 2.10. Let $_{H_i}M_i$ be the ultra-group of H_i with G_i for $i = 1,2$. A mapping from $_{H_1}M_1$ to $_{H_2}M_2$ is an ultra-group homomorphism with

- a) $f([m_1, m_2]) = [f(m_1, m_2)]$
- b) $(f(m))^{\phi(h)} = f(m^h)$

  for ϕ is an group homomorphism from H_1 to H_2 and $m, m_1, m_2 \in M_1$, $h \in H_1$.

Example 2.11. Let $M_1 = \{e, a, b, a^2, ab, a^2b\}$ and $M_2 = \{e, b, ab, a^2b, a^5, a^4\}$ be as in example 2.6. Then there is an ultra-group isomorphism $f : M_1 \longrightarrow M_2$

$$f(x) = \begin{cases} a^4 & , \quad x = a \\ a^5 & , \quad x = a^2 \\ x & , \quad otherwise. \end{cases}$$

3. GAP Application

For computational discrete algebra, GAP is the most often used system. It is used to address problems involving symbolic computation. This type of computation illuminates number theory, combinatorics, and coding theory calculations in Mathematics and Computer Science.

GAP programming language;

- Free
- Open source
- Expandable seperate package library

It is published with GNU Public License. The kernel of the system is written in C programming language.The library of functions and additional packages are in a special language, also called GAP [11].

In the GAP system and extension packages, there are now 900K lines of GAP code and 360K lines of C code exist. The Small Groups library in these packages has been given in landmark computations with the **Millenium Project** in [4] which classifies all finite groups of order smaller than 2000.

Besche and Eick [2] used group isomorphism to classify small group libraries up to rank of 1000, with the exception of 512 and 768. With technological advancements over the years, some higher computers now calculate a rank of 2000 [3, 8].

The following are the language characteristics:

- Streams,
- Garbage collection,
- Flexible list and record data types,
- Built-in data types for key algebraic objects,
- Control structures akin to those found in Pascal.

  GAP is an **interactive environment** that includes features such as online help, break loops for debugging and profiling GAP programs, completion of tab, a graphical user interface for GAP, and other GAP interface programs created by users.

```
SmallGroup(n,k)
```

function makes a small group with a rank of n and group isomorphism class of k.

```
StructureDescription(G)
```

function gives the isomorphism class of group G.

In GAP [3] a polycyclic group(PC) is formed. It means that a group uses the polycyclic presentation for element arithmetic.

The Cayley Theorem is used in GAP to illustrate groups with permutation groups that are isomorphic to groups.

We define the functions that give ultra group structure and control some substructures in a short amount of time in this paper.

- The **ugroup** function is used to create an ultra-group object.
- The **isUltra** function determines whether or not an ultra-group exists.
- The subultra structure is controlled by the **iSubUltra** function.
- The **rightCoset** function creates the ultra-right group's coset.
- The **isNormalSubgroup** function determines whether or not an ultra-group is normal.
- With ultra-group, **bracket** function is employed for the alpha function.

The theoritical definition of the ultra-group structure is defined by the **Ugroup** function. With the subgroup H and the quotient group, it generates a suitable ultra-group. The alpha and beta function computations are listed in this function.

Take the symmetric group $G := S_3$ as an example. G has a subgroup called H. As a result, we get the possibilities to satify the ultra-group conditions. The first input demonstrated the group G, the second is subgroup H.

```
gap>G:=SymmetricGroup(3);;eG:=Elements(G);;
gap>Ugroup(G,Subgroup(G,[eG[6]]));
[[(),(2,3),(1,2)],[(),(2,3),(1,3,2)],[(),(1,2),(1,2,3)],[(),(1,2,3),(1,3,2)]]
```

Let take the main group S_4 symmetric group of order 4. We get 2048 ultra-group. Some are

```
gap>T:=SymmetricGroup(4);;eT:=Elements(T);;
gap>Ugroup(T,Subgroup(T,[eT[6]]));
[[ (), (3,4), (2,3), (1,2), (1,2)(3,4), (1,2,3), (1,3,2), (1,3), (1,3,4),
(1,4,3), (1,4), (1,4)(2,3) ],
[ (), (3,4), (2,3), (1,2), (1,2)(3,4), (1,2,3), (1,3,2), (1,3,4), (1,3)(2,4),
```

```
 (1,4,3), (1,4), (1,4)(2,3) ],
[ (), (3,4), (2,3), (1,2), (1,2)(3,4), (1,2,3), (1,3,4,2), (1,3), (1,3,2,4)
, (1,4,3), (1,4), (1,4)(2,3) ],
...
```

Another example of this function is the Dihedral group of order 18 also known as D_9. We get 243 ultra-groups and the computation tables arise from the $H =< a^3, b >$ subgroup which is the second parameter:

```
gap>G:=SmallGroup(18,1);;eG:=Elements(G);;
gap>Ugroup(G,Subgroup(G,[eG[4]]));
[[<identity> of ..., f1, f2, f3, f1*f2, f2^2, f2*f3, f1*f2^2, f2^2*f3 ],
[<identity> of ...,f1*f2,f2^2,f2*f3,f3^2,f1*f2^2,f1*f2*f3,f1*f3^2,f2^2*f3],
[<identity>of..,f2*f3,f3^2,f1*f2^2,f1*f2*f3,f1*f3^2,f2^2*f3,f2*f3^2,f1*f2^2*f3],
[<identity> of ...,f1*f2,f2^2,f2*f3,f3^2,f1*f2*f3,f1*f3^2,f2^2*f3,f2^2*f3^2],
[<identity> of..., f3^2, f1*f2*f3, f1*f3^2, f2*f3^2, f1*f2^2*f3, f1*f2*f3^2,
f2^2*f3^2, f1*f2^2*f3^2]
[<identity> of ..., f1*f2, f2^2, f3^2, f1*f2*f3, f1*f3^2, f1*f2*f3^2,
f2^2*f3^2, f1*f2^2*f3^2]
[<identity> of ..., f3^2, f1*f2^2, f1*f2*f3, f1*f3^2, f2*f3^2,
f1*f2^2*f3, f1*f2*f3^2, f1*f2^2*f3^2]
[<identity> of ..., f3, f1*f2, f3^2, f1*f2*f3, f1*f2^2*f3,
f1*f2*f3^2, f2^2*f3^2, f1*f2^2*f3^2]
[<identity> of ..., f1, f2, f3, f2^2, f2*f3, f1*f2^2, f2^2*f3, f2*f3^2],
...
```

Each list corresponds to the ultra-group M.

The **isUltra** control function is a control function. The boolean data type is what it responds to. Three parameters are required for this function. The first parameter M is the set over which we have control, whether or not this set is an ultra-group. The main group G last parameter and the subgroup H, the second list, are required.

```
gap>isUltra([(),(1,2),(2,3)],[(),(1,3)],SymmetricGroup(3));
true
```

A control function is the **iSubultra** function. It responds to the boolean data type. In this function, we need four parameters. S, the second parameter, is the set that we control whether this set is the subultra-group or not of the first parameter, the list M. We need the main group the last input G and the subgroup H, the third parameter.

```
gap>G:=SymmetricGroup(3);;eG:=Elements(G);;
gap>iSubultra([(),(1,2),(2,3)],[()],Subgroup(G,[eG[6]]),G);
true
```

The **Subultra** function defines subultra-groups of the M ultra-group with H over the G group. It generates the subultra-groups with three parameter, M, H, G.

```
gap>G:=SymmetricGroup(3);;eG:=Elements(G);;
gap>Subultra([(),(1,2),(2,3)],Subgroup(G,[eG[6]]),G);
[ [ () ], [ (), (2,3) ], [ (), (2,3), (1,2) ], [ (), (1,2) ] ],
[ [ () ], [ (), (2,3) ] ],[ [ () ], [ (), (1,2) ] ],
[ [ () ], [ (), (1,2,3), (1,3,2) ] ]
```

rightCoset function composes the right coset for an ultra-group like group version. It takes four parameters. The main group and ultra-group must be given. For A and B subultra-groups, we find $[A, B]$ set with the GAP function.

```
gap>rightCoset([(),(1,2),(2,3)],[()],[()],SymmetricGroup(3));
[(),(2,3)]
```

isNormalSubGrp function checks whether the second parameter is normal for the first parameter or not. For example in $G := S_3$ symmetric group

```
gap>isNormalsubGrp([(),(1,2),(2,3)],[(),(1,2),(2,3)],SymmetricGroup(3));
True
```

An ultra-group structure is a triple notation. A set M, an alpha function and a beta function. The alpha function is called with bracket notation. So, we use this function occasionally. In GAP,

```
gap>bracket((1,2),(2,3),[(),(1,2),(2,3)],[(),(1,3)]);
(1,2)
```

As we mentioned in the example 2.6 we build an ultra-group homomorphism. We have some outputs about ultra-group homomorphism control function.

```
gap>G:=SmallGroup(18,1);
<pc group of size 18 with 3 generators>
gap>eG:=Elements(G);
[<identity>of ...,f1,f2,f3, f1*f2, f1*f3, f2^2, f2*f3, f3^2, f1*f2^2,f1*f2*f3,
f1*f3^2, f2^2*f3, f2*f3^2, f1*f2^2*f3, f1*f2*f3^2, f2^2*f3^2, f1*f2^2*f3^2 ]
gap>H:=Subgroup(G,[eG[9]]);
Group([ f3^2 ])
gap>M:=Ugroup(G,H);;
```

```
gap>f:=GeneralMappingByElements(Domain(M[12]),Domain(M[76]),
gap>[Tuple([M[12][1],M[76][1]]),Tuple([M[12][2],M[76][2]]),Tuple([M[12][3],
gap>M[76][4]]),Tuple([M[12][4],M[76][6]]),Tuple([M[12][5],M[76][3]])
gap>,Tuple([M[12][6],M[76][5]])]);;
gap>IsUltraHom(f,G,G,H,H);
True
```

and the algorithm of ultra-group homomorphism is below.

An algorithm of the ultra-group homomorphism is located at the end of this.

Input: f, G_1, G_2, H_1, H_2, Homomorphism, Domain, Codomain
Output: true iff f is an ultra-group homomorphism
begin
  $M_1 \leftarrow$ the ultra-group of H_1 over G_1
  $M_2 \leftarrow$ the ultra-group of H_2 over G_2
  $l \leftarrow$ the parameter of H_2
  $\alpha_1 \leftarrow$ the equality of $l * f(m) * f(z)$
  $\alpha_2 \leftarrow$ the equality of $f(h * m * z)$
  $\beta_1 \leftarrow$ the equality of $l * f(m) * p(t)$
  $\beta_2 \leftarrow$ the equality of $f(h * m * t)$
  if α_1 *and* α_2 *are in ultra-groups* **then**
    if $\alpha_1 = \alpha_2$ **then**
      return *true*
    else
      return *false*
    end
  else
    return *false*
  end
  if β_1 *and* β_2 *are in ultra-groups* **then**
    if $\beta_1 = \beta_2$ **then**
      return *true*
    else
      return *false*
    end
  else
    return *false*
  end
end

A screen of the ultra-group homomorphism process is as follows.

```
f1---f3^2
Not in range of homomorphism
f1---f3^2
f3^2*f1*f3^2=f1SECOND CONDITION SUCCESS

 second cond. ended-------------
f2---f3
Not in range of homomorphism
f2---f3
Not in range of homomorphism
f2---f3
f3^2*f2*f3^2*f3=f2*f3^2SECOND CONDITION SUCCESS

 second cond. ended-------------
f2^2---f3
Not in range of homomorphism
f2^2---f3
Not in range of homomorphism
f2^2---f3
f3^2*f2^2*f3^2*f3=f2^2*f3^2SECOND CONDITION SUCCESS

 second cond. ended-------------
f1*f2*f3---f3^2
Not in range of homomorphism
f1*f2*f3---f3^2
Not in range of homomorphism
f1*f2*f3---f3^2
f3^2*f1*f2*f3*f3^2=f1*f2*f3SECOND CONDITION SUCCESS

 second cond. ended-------------
f1*f2^2*f3---f3^2
Not in range of homomorphism
f1*f2^2*f3---f3^2
Not in range of homomorphism
f1*f2^2*f3---f3^2
f3^2*f1*f2^2*f3*f3^2=f1*f2^2*f3SECOND CONDITION SUCCESS

 second cond. ended-------------
1
gap>
```

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On Vertex-Degree-Based Indices of Monogenic Semigroup Graphs

Seda Oğuz Ünal[1]

Keywords
Graphs,
Indices,
Monogenic semigroups

Abstract — Albertson and the reduced Sombor indices are vertex-degree-based graph invariants that given in [7] and [20], defined as

$$Alb(G) = \sum_{uv \in E(G)} |d_u - d_v|, \qquad SO_{red}(G) = \sum_{uv \in E(G)} \sqrt{(d_u - 1)^2 + (d_v - 1)^2},$$

respectively. In this work we show that a calculation of Albertson and reduced Sombor index which are vertex-degree-based topological indices over monogenic semigroup graphs.

Subject Classification (2020): 05C12, 05C50, 15A18, 15A36, 15A42.

1. Introduction and Preliminaries

Let $G = (V, E)$ denotes a simple graph in which vertex and edge sets are indicated by $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$, respectively. Throught this paper the degree of a vertex $v \in V$ will be stipulated by d_v and if the u and v vertices are connected, this will be specified by uv. See [40] for detailed information about graph theory. In [15], the authors introduced a new type of graph relevant to monogenic semigroups, called

monogenic semigroup graph. Initially, the authors identified a multiplicative monogenic semigroup that is finite and has one zero element as noted below

$$S_M = \left\{0, x, x^2, x^3, \ldots, x^n\right\}. \tag{1.1}$$

The researchers indicated the graph of a monogenic semigroup given in (1.1) with $\Gamma(S_M)$. The vertex set of $\Gamma(S_M)$ is $\left\{x, x^2, x^3, \ldots, x^n\right\}$ i.e. all elements in S_M except zero element and the necessary and the sufficent condition for any two different vertices, x^i and x^j in $\Gamma(S_M)$ to be linked to each other is that $i + j > n$. See [2–4] for detailed knowledge about a graph of a monogenic semigroup. Since graphs of monogenic semi-

group are inspired by zero divisor graphs, we will briefly mention about zero divisor graphs. Zero divisor graphs were primarily studied on commutative rings [13]. After this study, studies were also carried out on commutative and non-commutative rings [10–12]. After the studies on zero-divisor graphs on rings, researchers worked on zero-divisor graphs over semigroups [17, 18]. Topological invariants, which have been

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studied for many years in the field of chemistry [39], have recently attracted the attention of mathematicians. Topological indices create structural properties of molecules and equip us with data for industrial science, applied physics, environmental science, and toxicology [19]. The vertex-degree based Sombor index initially presented by Gutman in [20]. It was first used in chemistry [8, 9, 14, 24, 25, 35]. Subsequently fascinated the interest of mathematicians. [16, 23, 26, 32–34]. Network science has exploited the dynamic effect of modeling complex systems in biology and social technology [38]. There are also studies on the Sombor index related to military use [21]. Since its beggining (less than a year after its publication), the Sombor index has also been of interest to mathematicians. In this paper we will compute the Albertson and reduced Sombor index of a monogenic semigroup graph. In [7] the Albertson index given as

$$Alb(G) = \sum_{uv \in E(G)} |d_u - d_v| \tag{1.2}$$

The reduced Sombor index which is a vertex-degree-based index introduced in [20], stated as given below

$$SO_{red}(G) = \sum_{uv \in E(G)} \sqrt{(d_u - 1)^2 + (d_v - 1)^2} \tag{1.3}$$

Every vertex-degree-based topological index can be taken as a special case of a Sombor-type index [22]. Besides, for a real number r, we indicate by $\lfloor r \rfloor$ the greatest integer $\leq r$ and by $\lceil r \rceil$, the least integer $\geq r$. It is quite obvious that $r - 1 < \lfloor r \rfloor \leq r$ and $r \leq \lceil r \rceil < r + 1$. Furthermore, for a natural number n, we have

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n}{2} & if \quad \text{n} \quad is \quad even \\ \frac{n-1}{2} & if \quad \text{n} \quad is \quad odd \end{cases} \tag{1.4}$$

For further information about monogenic semigroup see [5, 6, 30, 31] and for the applications of some important topological indices in graph theory see [27–29]. In this study, we aim to calculate the Albertson and reduced Sombor index, which are vertex-degree-based indexes of a monogenic semigroup graph.

2. An Algorithm

In [2], the researchers developed an algorithm in relation to the neighborhood of vertices to facilitate the operations. We will consider this algorithm when calculating Albertson and reduced Sombor indices on monogenic semigroups graphs.

I_n: The vertex x^n is linked to every vertex x^{a_1} $(1 \leq a_1 \leq n-1)$ except itself.

I_{n-1}: The vertex x^{n-1} is linked to every vertex x^{a_2} $(2 \leq a_2 \leq n-2)$ except itself and the vertex x^n.

I_{n-2}: The vertex x^{n-2} is linked to every vertex x^{a_3} $(3 \leq a_3 \leq n-3)$ except itself and the vertices x^n and x^{n-1}.

Persisting the algorithm in this way and based on if the number n is odd or even, we get the result given below.

If n is even:

$I_{\frac{n}{2}+2}$: The vertex $x^{\frac{n}{2}+2}$ is linked not only to the vertices $x^{\frac{n}{2}-1}$, $x^{\frac{n}{2}}$ and $x^{\frac{n}{2}+1}$ also linked to the vertices x^n,

x^{n-1}, x^{n-2}, ..., $x^{\frac{n}{2}+3}$.

$I_{\frac{n}{2}+1}$: The vertex $x^{\frac{n}{2}+1}$ is linked not only to the single vertex $x^{\frac{n}{2}}$ also linked to the vertices x^{n}, x^{n-1}, x^{n-2}, ..., $x^{\frac{n}{2}+2}$.

If n is odd:

$I_{\frac{n+1}{2}}$: The vertex $x^{\frac{n+1}{2}+2}$ is linked not only to the vertices $x^{\frac{n+1}{2}-2}$, $x^{\frac{n+1}{2}-1}$, $x^{\frac{n+1}{2}}$ and $x^{\frac{n+1}{2}+1}$ also linked to the vertices x^{n}, x^{n-1}, x^{n-2}, ..., $x^{\frac{n+1}{2}+3}$.

$I_{\frac{n+1}{2}+1}$: The vertex $x^{\frac{n+1}{2}+1}$ is linked not only to the vertices $x^{\frac{n+1}{2}-1}$ and $x^{\frac{n+1}{2}}$ also linked to the vertices x^{n-1}, x^{n-2}, ..., $x^{\frac{n+1}{2}+2}$.

In Lemma 2.1, the degrees of vertices in $\Gamma(S_M)$ are indicated by $d_1, d_2, \ldots, d_n$. For more information about degree series see [1, 15] and citations in these papers. You see the proof of Lemma 2.1 in [15].

Lemma 2.1

$$d_1 = 1, \quad d_2 = 2, \quad \ldots \quad , d_{\lfloor \frac{n}{2} \rfloor} = \left\lfloor \frac{n}{2} \right\rfloor, \quad d_{\lfloor \frac{n}{2} \rfloor+1} = \left\lfloor \frac{n}{2} \right\rfloor, \quad d_{\lfloor \frac{n}{2} \rfloor+2} = \left\lfloor \frac{n}{2} \right\rfloor + 1, \quad \ldots \quad , d_n = n-1 \tag{2.1}$$

Remark 2.2 Note the repetitive terms given in Lemma 2.1 as follows,

$$d_{\lfloor \frac{n}{2} \rfloor} = \left\lfloor \frac{n}{2} \right\rfloor = d_{\lfloor \frac{n}{2} \rfloor+1}. \tag{2.2}$$

3. Calculating Albertson Index of $\Gamma(S_M)$

An accurate formula of Albertson index over a graph of a monogenic semigroup will be given in this part

Theorem 3.1. Let S_M denote a monogenic semigroup, as given in (1.1), the Albertson index of the monogenic semigroup graph $\Gamma(S_M)$ is stated as below

$$Alb(\Gamma(S_M)) = \begin{cases} \sum_{k=1}^{\frac{n}{2}-1} \sum_{i=k}^{n-k-1} |(n-k)-i| + \sum_{k=1}^{\frac{n}{2}} \left|(n-k) - \lfloor \frac{n}{2} \rfloor\right| & if \quad \text{n} \quad is \quad even \\ \sum_{k=1}^{\frac{n-1}{2}} \sum_{i=k}^{n-k-1} |(n-k)-i| + \sum_{k=1}^{\frac{n-1}{2}} \left|(n-k) - \lfloor \frac{n}{2} \rfloor\right| & if \quad \text{n} \quad is \quad odd \end{cases} \tag{3.1}$$

Proof.

Here our goal is to find a formula for $Alb(\Gamma(S_M))$ by utilising the algorithm given in Section 2. In addition, we will use (1.4), (2.1) equations and Remark 2.2 during our operations.

If n is odd:

$$\begin{aligned} [Alb]\,(\Gamma(S_M)) = {} & |d_n - d_1| + |d_n - d_2| + |d_n - d_3| + \ldots + |d_n - d_{n-2}| + |d_n - d_{n-1}| + \\ & + |d_{n-1} - d_2| + |d_{n-1} - d_3| + \ldots + |d_{n-1} - d_{n-2}| + \\ & + \ldots + \\ & + \left|d_{\frac{n+1}{2}+2} - d_{\frac{n+1}{2}-2}\right| + \left|d_{\frac{n+1}{2}+2} - d_{\frac{n+1}{2}-1}\right| + \left|d_{\frac{n+1}{2}+2} - d_{\frac{n+1}{2}}\right| + \left|d_{\frac{n+1}{2}+2} - d_{\frac{n+1}{2}+1}\right| \\ & + \left|d_{\frac{n+1}{2}+1} - d_{\frac{n+1}{2}-1}\right| + \left|d_{\frac{n+1}{2}+1} + d_{\frac{n+1}{2}}\right| \end{aligned} \tag{3.2}$$

Consequently, the Albertson index of $\Gamma(S_M)$ is written as given in the following

$$[Alb]\,(\Gamma(S_M) = \sum_{ij \in E(G)} \left|d_i - d_j\right| = [Alb]_n + [Alb]_{n-1} + \ldots + [Alb]_{\frac{n+1}{2}+2} + [Alb]_{\frac{n+1}{2}+1}. \tag{3.3}$$

Equality $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$ given in (1.4) is used while performing these operations (if n is odd).

$$\begin{aligned}[Alb]_n &= |(n-1)-1| + |(n-1)-2| + |(n-1)-3| + \ldots + \left|(n-1) - \left\lfloor \frac{n}{2} \right\rfloor\right| + \ldots + \\ &+ |(n-1)-(n-2)| + \left|(n-1) - \left\lfloor \frac{n}{2} \right\rfloor\right| \\ &= \sum_{i=1}^{n-2} |(n-1)-i| + \left|(n-1) - (\frac{n-1}{2})\right|\end{aligned} \tag{3.4}$$

In case analogous operations are applied to $[Alb]_{n-1}, \ldots [Alb]_{\frac{n+1}{2}+2}$ and $[Alb]_{\frac{n+1}{2}+1}$; we obtain

$$[Alb]_{n-1} = \sum_{i=2}^{n-3} |(n-2)-i| + \left|(n-2) - (\frac{n-1}{2})\right|, \tag{3.5}$$

$$[Alb]_{\frac{n+1}{2}+2} = \left|(\frac{n+3}{2}) - (\frac{n-3}{2})\right| + \left|(\frac{n+3}{2}) - (\frac{n-1}{2})\right| + \left|(\frac{n+3}{2}) - (\frac{n-1}{2})\right| + \left|(\frac{n+3}{2}) - (\frac{n+1}{2})\right| \tag{3.6}$$

and finally

$$[Alb]_{\frac{n+1}{2}+1} = \left|(\frac{n+1}{2}) - (\frac{n-1}{2})\right| + \left|(\frac{n+1}{2}) - (\frac{n-1}{2})\right|. \tag{3.7}$$

Hence

$$[Alb]_n + [Alb]_{n-1} + \ldots + [Alb]_{\frac{n+1}{2}+2} + [Alb]_{\frac{n+1}{2}+1} = \sum_{k=1}^{\frac{n-1}{2}} \sum_{i=k}^{n-k-1} |(n-k)-i| + \sum_{k=1}^{\frac{n-1}{2}} \left|(n-k) - (\frac{n-1}{2})\right| \tag{3.8}$$

If similar operations are performed in case n is even, the following sum is obtained

$$[Alb]_n + [Alb]_{n-1} + \ldots + [Alb]_{\frac{n}{2}+2} + [Alb]_{\frac{n}{2}+1} = \sum_{k=1}^{\frac{n}{2}-1} \sum_{i=k}^{n-k-1} |(n-k)-i| + \sum_{k=1}^{\frac{n}{2}} \left|(n-k) - (\frac{n}{2})\right|. \tag{3.9}$$

4. Calculating Reduced Sombor Index of $\Gamma(S_M)$

An accurate formula of replaced Sombor index over a graph of a monogenic semigroup will be given in this part.

Theorem 4.1. Let S_M denote a monogenic semigroup, as given in (1.1), the reduced Sombor index of the monogenic semigroup graph $\Gamma(S_M)$ is stated as below

$$SO_{red}(\Gamma(S_M)) = \begin{cases} \sum_{k=1}^{\frac{n}{2}-1} \sum_{i=k}^{n-k-1} \sqrt{(n-k-1)^2 + (i-1)^2} + \sum_{k=1}^{\frac{n}{2}} \sqrt{(n-k-1)^2 + \left(\frac{n}{2}-1\right)^2} & if \ \text{n} \ is \ even \\ \sum_{k=1}^{\frac{n-1}{2}} \sum_{i=k}^{n-k-1} \sqrt{(n-k-1)^2 + (i-1)^2} + \sum_{k=1}^{\frac{n-1}{2}} \sqrt{(n-k-1)^2 + \left(\frac{n-1}{2}-1\right)^2} & if \ \text{n} \ is \ odd \end{cases} \tag{4.1}$$

Proof.

Here our goal is to find an exact formula for $SO_{red}(\Gamma(S_M))$ by utilising the algorithm given in Section 2. Besides, we will utilise (1.4), (2.1) equations and Remark 2.2 during our operations. If n is odd:

$$
\begin{aligned}
[SO_{red}]\,(\Gamma(S_M) = & \sqrt{(d_n-1)^2+(d_1-1)^2}+\sqrt{(d_n-1)^2+(d_2-1)^2}+\sqrt{(d_n-1)^2+(d_3-1)^2}+\ldots\\
&+\sqrt{(d_n-1)^2+(d_{n-2}-1)^2}+\sqrt{(d_n-1)^2+(d_{n-1}-1)^2}+\sqrt{(d_{n-1}-1)^2+(d_2-1)^2}\\
&+\sqrt{(d_{n-1}-1)^2+(d_3-1)^2}+\ldots+\sqrt{(d_{n-1}-1)^2+(d_{n-2}-1)^2}+\ldots+\\
&+\sqrt{(d_{\frac{n+1}{2}+2}-1)^2+(d_{\frac{n+1}{2}-2}-1)^2}+\sqrt{(d_{\frac{n+1}{2}+2}-1)^2+(d_{\frac{n+1}{2}-1}-1)^2}+\sqrt{(d_{\frac{n+1}{2}+2}-1)^2+(d_{\frac{n+1}{2}}-1)^2}\\
&+\sqrt{(d_{\frac{n+1}{2}+2}-1)^2+(d_{\frac{n+1}{2}+1}-1)^2}+\sqrt{(d_{\frac{n+1}{2}+1}-1)^2+(d_{\frac{n+1}{2}-1}-1)^2}+\sqrt{(d_{\frac{n+1}{2}+1}-1)^2+(d_{\frac{n+1}{2}}-1)^2}
\end{aligned}
\tag{4.2}
$$

Consequently, the replaced Sombor index of $\Gamma(S_M)$ is written as stated below

$$
[SO_{red}]\,(\Gamma(S_M) = \sum_{ij\in E(G)}\sqrt{(d_i-1)^2+(d_j-1)^2} = [SO_{red}]_n+[SO_{red}]_{n-1}+\ldots+[SO_{red}]_{\frac{n+1}{2}+2}+[SO_{red}]_{\frac{n+1}{2}+1}
\tag{4.3}
$$

$\left\lfloor \frac{n}{2}\right\rfloor = \frac{n-1}{2}$ equality which is given in (1.4) is utilised while performing these operations (if n is odd).

$$
\begin{aligned}
[SO_{red}]_n = & \sqrt{((n-1)-1)^2+(1-1)^2}+\sqrt{((n-1)-1)^2+(2-1)^2}+\sqrt{((n-1)-1)^2+(3-1)^2}+\ldots\\
&+\sqrt{((n-1)-1)^2+\left\lfloor \frac{n}{2}\right\rfloor^2}+\ldots+\sqrt{((n-1)-1)^2+((n-2)-1)^2}+\sqrt{((n-1)-1)^2+\left\lfloor \frac{n}{2}\right\rfloor^2}\\
= & \sum_{i=1}^{n-2}\sqrt{((n-1)-1)^2+(i-1)^2}+\sqrt{((n-1)-1)^2+(\frac{n-1}{2}-1)^2}
\end{aligned}
\tag{4.4}
$$

In case analogous operations are applied to $[SO_{red}]_{n-1}$, we have the following

$$
[SO_{red}]_{n-1} = \sum_{i=2}^{n-3}\sqrt{((n-2)-1)^2+(i-1)^2}+\sqrt{((n-2)-1)^2+(\frac{n-1}{2}-1)^2},
\tag{4.5}
$$

$$
\begin{aligned}
[SO_{red}]_{\frac{n+1}{2}+2} = & \sqrt{(\frac{n+3}{2}-1)^2+(\frac{n-3}{2}-1)^2}\\
&+\sqrt{(\frac{n+3}{2}-1)^2+(\frac{n-1}{2}-1)^2}+\sqrt{(\frac{n+3}{2}-1)^2+(\frac{n-1}{2}-1)^2}+\sqrt{(\frac{n+3}{2}-1)^2+(\frac{n+1}{2}-1)^2}
\end{aligned}
\tag{4.6}
$$

and finally

$$
[SO_{red}]_{\frac{n+1}{2}+1} = \sqrt{(\frac{n+1}{2}-1)^2+(\frac{n1}{2}-1)^2}+\sqrt{(\frac{n+1}{2}-1)^2+(\frac{n-1}{2}-1)^2}.
\tag{4.7}
$$

Hence

$$
\begin{aligned}
[SO_{red}]_n+[SO_{red}]_{n-1}+\ldots+[SO_{red}]_{\frac{n+1}{2}+2}+[SO_{red}]_{\frac{n+1}{2}+1} = & \sum_{k=1}^{\frac{n-1}{2}}\sum_{i=k}^{n-k-1}\sqrt{(n-k-1)^2+(i-1)^2}\\
&+\sum_{k=1}^{\frac{n-1}{2}}\sqrt{(n-k-1)^2+(\frac{n-1}{2}-1)^2}
\end{aligned}
\tag{4.8}
$$

If similar operations are performed in case n is even, the following sum is obtained

$$[SO_{red}]_n + [SO_{red}]_{n-1} + \ldots + [SO_{red}]_{\frac{n}{2}+2} + [SO_{red}]_{\frac{n}{2}+1} = \sum_{k=1}^{\frac{n}{2}-1} \sum_{i=k}^{n-k-1} \sqrt{(n-k-1)^2 + (i-1)^2} + \sum_{k=1}^{\frac{n}{2}} \sqrt{(n-k-1)^2 + (\frac{n}{2}-1)^2}. \tag{4.9}$$

The following examples reinforce Theorem 3.1 and Theorem 4.1.

Example 4.2. We will compute the Albertson index over a graph of a monogenic semigroup S_M^4. S_M^4 monogenic semigroup and the graph of S_M^4 is given below.

$$S_M^4 = \{x, x^2, x^3, x^4\} \cup \{0\}$$

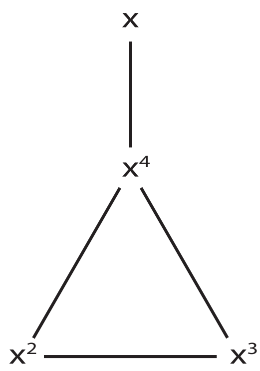

Figure 1. S_M^4 monogenic semigroup graph

$$\begin{aligned} Alb(\Gamma(S_M^4)) &= \sum_{k=1} \sum_{i=1}^{2} |(4-k)-i| + \sum_{k=1}^{2} \left|(4-k) - \left(\frac{4}{2}\right)\right| \\ &= |(4-1)-1| + |(4-1)-2| + |(4-1)-2| + |(4-2)-2| \quad = 2+1+1+0 = 4 \end{aligned} \tag{4.10}$$

The reduced Sombor index of $\Gamma(S_M^6)$ monogenic semigroup graph, is computed in the folowing example.

Example 4.3. We will compute the replaced Sombor index of $\Gamma(S_M^6)$ graph. S_M^6 monogenic semigroup and the graph of S_M^6 is given below:

$$S_M^6 = \{x, x^2, x^3, x^4, x^5, x^6\} \cup \{0\}.$$

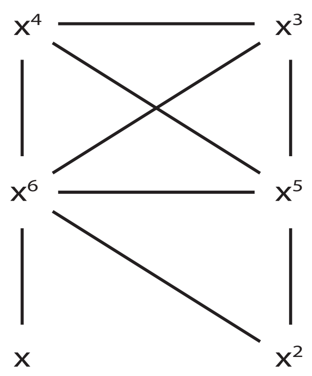

Figure 2. S_M^6 monogenic semigroup graph

$$\begin{aligned} SO_{red}(\Gamma(S_M^6)) &= \sum_{k=1}^{2}\sum_{i=k}^{5-k}\sqrt{(6-k-1)^2+(i-1)^2}+\sum_{k=1}^{3}\sqrt{(6-k-1)^2+(3-1)^2} \\ &= \sqrt{4^2}+\sqrt{4^2+1^2}+\sqrt{4^2+2^2}+\sqrt{4^2+3^2}+\sqrt{3^2+1^2}+\sqrt{3^2+2^2} \\ &+\sqrt{4^2+2^2}+\sqrt{3^2+2^2}+\sqrt{2^2+2^2} \\ &= \sqrt{20}+\sqrt{13}+\sqrt{8} \end{aligned} \tag{4.11}$$

As can be clearly seen, with the help of the given main theorems, the monogenic semigroup graphs of Albertson and replaced Sombor indices are easily computed.

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New Generalized Hypergeometric Functions

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Abstract — The classical Gauss hypergeometric function ${}_2F_1(\alpha,\beta,\gamma;z)$ and the Kumar confluent hypergeometric function ${}_1F_1(\alpha,\beta;z)$ are defined using a classical Pochammer symbol $(\alpha)_n$ and a factorial function. This research paper will present a two-parameter Pochhammer symbol $(\lambda,\mu)_n$ and discuss some of its properties such as recursive formulae and integral representation. In addition, the generalized Gauss and Kumar confluent hypergeometric functions are defined using the two-parameter Pochhammer symbol and a two-parameter factorial function $(m,\ j)!$ and some of the properties of the new generalized hypergeometric functions were also discussed.

Subject Classification (2020): 33B99, 33C05, 33C15, 33C20.

1. Introduction

The pochhammer symbol is named after the German mathematician Leo Pochhammer, defined as ashifted (rising) factorial [2] and given by

$$(\lambda)_n = \begin{cases} \lambda(\lambda+1)(\lambda+2)(\lambda+3)\cdots(\lambda+(n-1)), & n \in \square \\ 1, & n=0 \end{cases} \tag{1.1}$$

Rafael and Pariguan in [7] presented the definition of the pochhammer m -symbol as

$$(\lambda)_{n,\mu} = \lambda(\lambda+\mu)(\lambda+2\mu)(\lambda+3\mu)\cdots(\lambda+(n-1)\mu), \tag{1.2}$$

$$(\lambda \in \square,\ \mu \in \square \ \text{ and } \ n \in \square\ )$$

and introduced the m -analogue of the gamma function.

Remark 1.1. When $\mu = 1,$ (the classical Pochhammer symbol).

Srivastava in [11] generalized the Pochhammer symbol using the extended gamma function in [14] as

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$$(\lambda,\mu)_n=\begin{cases}\dfrac{\Gamma_\mu(\lambda+n)}{\Gamma(\lambda)}, & (\Re(\mu)>0;\lambda,n\in\square)\\ (\lambda)_n, & (\mu=0;\lambda,n\in\square)\end{cases} \tag{1.3}$$

A new generalization of the Pochhammer symbol in [11] was proposed by Sahin [14] as

$$(\lambda;\mathrm{r},\mathrm{s};\rho,\eta)_\mathrm{n}=\begin{cases}\dfrac{\Gamma_{\mathrm{r,s}}^{(\rho,\eta)}(\lambda+\mathrm{n})}{\Gamma(\lambda)}, & (\mathrm{Re(r)}>0,\mathrm{Re(s)}>0,\mathrm{Re}(\rho)>0,\mathrm{Re}(\eta)>0),\\ (\lambda)_\mathrm{n}, & (\mathrm{r}=1,\mathrm{s}=0,\rho=1,\eta=0),\end{cases} \tag{1.4}$$

Where $\Gamma_{r,s}^{(\rho,\eta)}$ is the generalized extended gamma function.

Sahai [18] generalized the Pochhammer symbol using an extended gamma function in [20] as

$$(\lambda;\mathrm{r},\rho,\eta)_\mathrm{n}=\frac{\Gamma_{\mathrm{r}}^{(\rho,\eta)}(\lambda+\mathrm{n})}{\Gamma(\lambda)}, \qquad (\mathrm{Re}(\rho)>0,\mathrm{Re}(\eta)>0,\mathrm{Re(r)}>0;\lambda,\mathrm{n}\in\square \tag{1.5}$$

Srivastava [16] introduced a generalized Pochhammer symbol from an extended gamma function in [14] as

$$(\lambda;\rho,\{\mathrm{k_v}\}_{\mathrm{v}\in\square_0})_\mathrm{n}=\frac{\Gamma_\rho^{\left(\{\mathrm{k_v}\}_{\mathrm{v}\in\square_0}\right)}(\lambda+\mathrm{n})}{\Gamma_\rho^{\left(\{\mathrm{k_v}\}_{\mathrm{v}\in\square_0}\right)}(\lambda)}, \qquad (\lambda,\mathrm{n}\in\square). \tag{1.6}$$

Another generalized Pochhammer symbol was defined by Safdar [21] using an extended gamma function in [22] as

$$(\lambda;\rho,\mu)_\mathrm{n}=\begin{cases}\dfrac{\Gamma_\rho(\lambda+\mathrm{n};\mu)}{\Gamma(\lambda)}, & (\Re(\rho)>0,\Re(\mu)>0;\lambda,\mathrm{n}\in\square),\\ (\lambda;\rho)_\mathrm{n}, & (\mu=1;\lambda,\mathrm{n}\in\square\setminus\{0\}),\end{cases} \tag{1.7}$$

A factorial function denoted by $(!)$ is given by

$$(\lambda)!=\begin{cases}\lambda(\lambda-1)(\lambda-2)(\lambda-3)\cdots3\cdot2\cdot1, & \lambda\in\square\\ 1, & \lambda=0\end{cases} \tag{1.8}$$

In 2014, Mubeen and Rehman [3] generalized the classical factorial function called (λ,μ)-factorials as

$$(\lambda,\mu)!=\lambda\mu(\lambda\mu-\mu)(\lambda\mu-2\mu)(\lambda\mu-3\mu)\cdots3\mu\cdot2\mu\cdot\mu, \qquad \lambda\in\square,\mu>0 \tag{1.9}$$

On simplifying the right-hand side of (1.8), we have

$$(\lambda,\mu)!=\mu^n\lambda!=\mu^n\Gamma(\lambda+1) \tag{1.10}$$

(1.10) is the relationship between the generalized factorial function $(\lambda,\mu)!$ and the gamma function.

Remark 1.2. When $\mu=1$, $(\lambda,\mu)!=(\lambda,1)!=(\lambda)!$ (the classical factorial function).

Other related literatures can be obtained [23, 24, 25 & 26].

Motivated by (1.9), the second part the paper will propose a new generalized Pochhammer symbol $(\lambda,\mu)_n$ and give some of its properties.

2. New Generalised Pochhammer Symbol

The $(\lambda,\mu)_n$ Pochhammer symbol is defined as

$$(\lambda,\mu)_n = \begin{cases} \lambda\mu(\lambda\mu+\mu)(\lambda\mu+2\mu)(\lambda\mu+3\mu)\cdots(\lambda\mu+(n-1)\mu), & \lambda,\mu\in\Box,\quad n\in\Box \\ 1, & n=0 \end{cases} \tag{2.1}$$

(2.1) can be simplified as

$$(\lambda,\mu)_n = \lambda\mu^n(\lambda+1)(\lambda+2)(\lambda+3)\cdots(\lambda+n-1) = \mu^n(\lambda)_n \tag{2.2}$$

Remark 2.1. When $\mu=1$ in (2.2), $(\lambda,\mu)_n=(\lambda,1)_n=(\lambda)_n$ (i.e. the classical Pochhammer symbol)

Theorem 2.1. The following formulas holds

$$(\lambda\mu,\mu)_n = \mu^n(\lambda\mu)_n \tag{2.3}$$

$$(\lambda+p,\mu)_n = \mu^n(\lambda+p)_n, \qquad p\in\Box,\ n\in\Box \tag{2.4}$$

$$((\lambda+q)\mu,\mu)_n = \mu^n((\lambda+q)\mu)_n, \quad q\in\Box,\ n\in\Box \tag{2.5}$$

From (2.2),

$$(\lambda,\mu)_n = \mu^n\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \tag{2.6}$$

(2.6) is the relationship between the two parameters Pochhammer symbol and the classical gamma function.

Proof To prove equations (2.3), (2.4) and (2.5), put $\lambda=\lambda\mu$, $\lambda=\lambda+p$ and $\lambda=(\lambda+q)\mu$ in (2.2) respectively.

The proof of (2.6) follows from the fact that $(\lambda)_n=\dfrac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$.

In the third part of the paper, we are going to state some recurrence formulae.

3. Main Results

Theorem 3.1. The following formula holds true

$$(\lambda,\mu)_{n+1} = (\lambda+n)(\lambda,\mu)_n \tag{3.1}$$

Proof

$$(\lambda,\mu)_{n+1} = \mu^n\lambda(\lambda+1)(\lambda+2)(\lambda+3)\cdots(\lambda+n-1)(\lambda+n)$$

Using (2.2) in the above equation, we obtained the desired result.

Theorem 3.2. The following formula holds true

$$(\lambda+1,\mu+1)_n-(\lambda,\mu)_n = (\lambda,\mu)_n\left[\frac{\lambda+n}{\lambda}\frac{(\mu+1)^n}{\mu^n}-1\right] \tag{3.2}$$

Proof

$$(\lambda+1,\mu+1)_n = (\mu+1)^n(\lambda+1)(\lambda+2)(\lambda+3)\cdots(\lambda+n-2)(\lambda+n)$$

Multiplying both sides of the equation by $\lambda\mu^n$ and dividing through by $(\mu+1)^n$, we get the required result.

Theorem 3.2.

$$(\lambda,\mu)_n = (\lambda+n-1)(\lambda,\mu)_{n-1} \tag{3.3}$$

Proof

$$(\lambda,\mu)_n = \mu^n \lambda(\lambda+1)(\lambda+2)(\lambda+3)\cdots(\lambda+n-2)(\lambda+n-1)$$

On using (2.2), we obtained the desired result.

Theorem 3.3.

$$(\lambda,\mu)_{m+n} = (\lambda,\mu)_m(\lambda+m,\mu+m)_n \tag{3.4}$$

Proof

$$\begin{aligned}(\lambda,\mu)_{m+n} &= \mu^{m+n}\lambda(\lambda+1)(\lambda+2)(\lambda+3)\cdots(\lambda+m+n-1)\\ &= \mu^{m+n}\lambda(\lambda+1)(\lambda+2)(\lambda+3)\cdots(\lambda+m-1)(\lambda+m)(\lambda+m+1)(\lambda+m+2)\cdots(\lambda+m+n-1)\\ &= (\lambda,\mu)_m(\lambda+m,\mu+m)_n.\end{aligned}$$

Theorem 3.4.

$$(\lambda-1,\mu-1)_{n+1} = \mu(\lambda+n-1)(\lambda-1,\mu-1)_n \tag{3.5}$$

Proof

$$\begin{aligned}(\lambda-1,\mu-1)_{n+1} &= (\mu-1)^{n+1}(\lambda-1)\lambda(\lambda+1)(\lambda+2)(\lambda+3)\cdots(\lambda+n-2)(\lambda+n-1)\\ &= \mu(\lambda+n-1)(\lambda-1,\mu-1)_n.\end{aligned}$$

Theorem 3.5.

$$(\lambda+1,\mu+1)_n = (\lambda+n)(\lambda+1,\mu+1)_{n-1} \tag{3.6}$$

Proof

$$(\lambda+1,\mu+1)_{n-1} = (\mu+1)^{n-1}(\lambda+1)(\lambda+2)(\lambda+3)\cdots(\lambda+n-1) \tag{3.7}$$

Multiplying both sides of (3.7) by $(\mu+1)(\lambda+n)$, we get the desired result.

Theorem 3.6.

$$(\lambda,\mu)_n = \frac{\lambda\mu^n}{(\lambda+n)(\mu+1)^n}(\lambda+1,\mu+1)_n \tag{3.8}$$

Proof

$$(\lambda+1,\mu+1)_n = (\mu+1)^n(\lambda+1)(\lambda+2)(\lambda+3)\cdots(\lambda+n-1)(\lambda+n)$$

Multiplying through by $\lambda\mu^n$ and dividing the result by $(\lambda+n)$, we get (3.8).

Theorem 3.7.

$$(\lambda,\mu)_n = \frac{\mu^n}{(\mu-m)^n}\frac{(\lambda-m,\mu-m)_{n+m}}{(\lambda-m,\mu-m)_m} \tag{3.9}$$

Proof

$$\frac{(\lambda-m,\mu-m)_{n+m}}{(\lambda-m,\mu-m)_m}=\frac{(\mu-m)^{n+m}(\lambda-m)(\lambda-m+1)(\lambda-m+2)\cdots(\lambda+n-1)}{(\mu-m)^m(\lambda-m)(\lambda-m+1)(\lambda-m+2)\cdots(\lambda-1)}$$

$$=(\mu-m)^n\,\lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1)$$

Multiplying both sides by μ^n , we've

$$\frac{\mu^n(\lambda-m,\mu-m)_{n+m}}{(\lambda-m,\mu-m)_m}=(\lambda,\mu)_n(\mu-m)^n$$

on dividing both sides by $(\mu-m)^n$, we get the desired result.

Theorem 3.8.

$$(\lambda,\mu)_n=\frac{\mu^n}{\mu^m(\mu+m)^{n-m}}(\lambda,\mu)_m(\lambda+m,\mu+m)_{n-m} \tag{3.10}$$

Proof

$$(\lambda,\mu)_m(\lambda+m,\mu+m)_{n-m}=\mu^m(\mu+m)^{n-m}\lambda(\lambda+1)(\lambda+2)\cdots(\lambda+m-1)(\lambda+m)(\lambda+m+1)(\lambda+m+2)\cdots(\lambda+n-1)$$

Multiplying both sides by μ^n , we've

$$\mu^n(\lambda,\mu)_m(\lambda+m,\mu+m)_{n-m}=\mu^m(\mu+m)^{n-m}\,\mu^n\lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1)$$

$$=\mu^m(\mu+m)^{n-m}(\lambda,\mu)_n$$

Dividing through by $\mu^m(\mu+m)^{n-m}$, we get the desired result.

Theorem 3.9.

$$(\lambda,\mu)_n=\frac{\mu^n}{(\mu+m)^n}\frac{\Gamma(\lambda+m)\Gamma(\lambda+n)}{\Gamma(\lambda)\Gamma(\lambda+m+n)}(\lambda+m,\mu+m)_n \tag{3.11}$$

Proof Taking the right-hand side of (3.11) and using (2.6), we get

$$\frac{\mu^n}{(\mu+m)^n}\frac{\Gamma(\lambda+m)\Gamma(\lambda+n)}{\Gamma(\lambda)\Gamma(\lambda+m+n)}(\lambda+m,\mu+m)_n=(\lambda,\mu)_n\frac{1}{(\lambda+m,\mu+m)_n}(\lambda+m,\mu+m)_n$$

$$=(\lambda,\mu)_n$$

Theorem 3.10.

$$(\lambda,\mu)_n=\frac{\mu^n}{(\mu-m)^n}\frac{\Gamma(\lambda-m)\Gamma(\lambda+n)}{\Gamma(\lambda)\Gamma(\lambda-m+n)}(\lambda-m,\mu-m)_n \tag{3.12}$$

Proof Taking the right-hand side of (3.12) and using (2.6), we get

$$\frac{\mu^n}{(\mu-m)^n}\frac{\Gamma(\lambda-m)\Gamma(\lambda+n)}{\Gamma(\lambda)\Gamma(\lambda+m+n)}(\lambda-m,\mu-m)_n=(\lambda,\mu)_n\frac{1}{(\lambda-m,\mu-m)_n}(\lambda-m,\mu-m)_n$$

$$.=(\lambda,\mu)_n$$

Corrolary 3.1. The following integral representation holds

$$(\lambda,\mu)_n = \frac{\mu^n}{\Gamma(\lambda)}\int_0^\infty t^{\lambda+n-1}e^{-t}dt \tag{3.13}$$

The hypergeometric functions is defined using a factorial function and a Pochhammer symbol. In the fourth part of this paper, we will define new generalized hypergeometric functions using (1.10) and (2.1).

4. Generalized Hypergeometric Functions

The new generalized Gauss and confluent hypergeometric functions are given by

$${}_pF_q\left[\begin{matrix}(\alpha_1,\beta),\alpha_2,\dots,\alpha_p\\ \delta_1,\delta_2,\dots,\delta_q\end{matrix};z\right] = \sum_{m=0}^{\infty}\frac{(\alpha_1,\beta)_m(\alpha_2)_m\cdots(\alpha_p)_m}{(\delta_1)_m(\delta_2)_m\cdots(\delta_q)_m}\frac{z^m}{(m,j)!} \tag{4.1}$$

In particular,

$${}_1F_1\left[(\alpha_1,\beta),\delta;z\right] = \sum_{m=0}^{\infty}\frac{(\alpha,\beta)_m}{(\delta)_m}\frac{z^m}{(m,j)!}, \tag{4.2}$$

And

$${}_2F_1\left[(\alpha_1,\beta),\gamma,\delta;z\right] = \sum_{m=0}^{\infty}\frac{(\alpha,\beta)_m(\gamma)_m}{(\delta)_m}\frac{z^m}{(m,j)!} \tag{4.3}$$

Theorem 4.1. The following integral representation holds true

$${}_pF_q\left[\begin{matrix}(\alpha_1,\beta),\alpha_2,\dots,\alpha_p\\ \delta_1,\delta_2,\dots,\delta_q\end{matrix};z\right] = \frac{\mu^n}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-t}\;{}_{p-1}F_q\left[\begin{matrix}\alpha_2,\alpha_3,\dots,\alpha_p\\ \delta_1,\delta_2,\dots,\delta_q\end{matrix};tz\right]dt, \tag{4.4}$$

Proof

$${}_pF_q\left[\begin{matrix}(\alpha_1,\beta),\alpha_2,\dots,\alpha_p\\ \delta_1,\delta_2,\dots,\delta_q\end{matrix};z\right] = \sum_{m=0}^{\infty}\frac{(\alpha_1,\beta)_m(\alpha_2)_m\cdots(\alpha_p)_m}{(\delta_1)_m(\delta_2)_m\cdots(\delta_q)_m}\frac{z^m}{(m,j)!}$$

$$= \sum_{m=0}^{\infty}\frac{\mu^n}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-t}\frac{(\alpha_2)_m(\alpha_3)_m\cdots(\alpha_p)_m}{(\delta_1)_m(\delta_2)_m\cdots(\delta_q)_m}\frac{(tz)^m}{(m,j)!}dt$$

$$= \frac{\mu^n}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-t}\sum_{m=0}^{\infty}\frac{(\alpha_2)_m(\alpha_3)_m\cdots(\alpha_p)_m}{(\delta_1)_m(\delta_2)_m\cdots(\delta_q)_m}\frac{(tz)^m}{(m,j)!}dt$$

$$= \frac{\mu^n}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-t}\;{}_{p-1}F_q\left[\begin{matrix}\alpha_2,\alpha_3,\dots,\alpha_p\\ \delta_1,\delta_2,\dots,\delta_q\end{matrix};tz\right]dt.$$

Theorem 4.2. The beta-type integral representation holds true

$$ {}_pF_q\left[\begin{matrix}(\alpha_1,\beta),\alpha_2,...,\alpha_p\\ \delta_1,\delta_2,...,\delta_q\end{matrix};z\right]=\frac{1}{B(\alpha_p,\delta_q-\alpha_p)}\int_0^1 t^{\alpha_p-1}(1-t)^{\delta_q-\alpha_p-1}\,{}_{p-1}F_{q-1}\left[\begin{matrix}(\alpha_1,\beta),\alpha_2,...,\alpha_{p-1}\\ \delta_1,\delta_2,...,\delta_{q-1}\end{matrix};tz\right]dt, \quad (4.5)$$

$$(\mathrm{Re}(\alpha_p)>0;\mathrm{Re}(\delta_q)>0;\mathrm{Re}(\beta)\geq 0)$$

Proof

$$ {}_pF_q\left[\begin{matrix}(\alpha_1,\beta),\alpha_2,...,\alpha_p\\ \delta_1,\delta_2,...,\delta_q\end{matrix};z\right]=\sum_{m=0}^{\infty}\frac{(\alpha_1,\beta)_m(\alpha_2)_m\cdots(\alpha_p)_m}{(\delta_1)_m(\delta_2)_m\cdots(\delta_q)_m}\frac{z^m}{(m,j)!}$$

$$=\frac{1}{B(\alpha_p,\delta_q-\alpha_p)}\sum_{m=0}^{\infty}\frac{(\alpha_1,\beta)_m(\alpha_2)_m\cdots(\alpha_{p-1})_m}{(\delta_1)_m(\delta_2)_m\cdots(\delta_{q-1})_m}\int_0^1 t^{\alpha_p+m-1}(1-t)^{\delta_q-\alpha_p-1}\frac{z^m}{(m,j)!}dt$$

$$=\frac{1}{B(\alpha_p,\delta_q-\alpha_p)}\int_0^1 t^{\alpha_p-1}(1-t)^{\delta_q-\alpha_p-1}\sum_{m=0}^{\infty}\frac{(\alpha_1,\beta)_m(\alpha_2)_m\cdots(\alpha_{p-1})_m}{(\delta_1)_m(\delta_2)_m\cdots(\delta_{q-1})_m}\frac{(tz)^m}{(m,j)!}dt$$

$$=\frac{1}{B(\alpha_p,\delta_q-\alpha_p)}\int_0^1 t^{\alpha_p-1}(1-t)^{\delta_q-\alpha_p-1}\,{}_{p-1}F_{q-1}\left[\begin{matrix}(\alpha_1,\beta),\alpha_2,...,\alpha_{p-1}\\ \delta_1,\delta_2,...,\delta_{q-1}\end{matrix};tz\right]dt.$$

Corollary 4.1. The following integral representations hold true

$$ {}_1F_1\left[(\alpha,\beta),\delta;z\right]=\frac{\mu^m}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-t}\,{}_0F_1\left[-,\delta;tz\right]dt, \quad (4.6)$$

$$ {}_2F_1\left[(\alpha,\beta),\delta;z\right]=\frac{\mu^m}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-t}\,{}_1F_1\left[\alpha,\delta;tz\right]dt, \quad (4.7)$$

$$ {}_1F_1\left[(\alpha,\beta),\delta;z\right]=\frac{1}{B(\alpha,\delta-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\delta-\alpha-1}\,{}_0F_0\left[-,-;tz\right]dt, \quad (4.8)$$

$$ {}_2F_1\left[(\alpha_1,\beta),\alpha_2,\delta;z\right]=\frac{1}{B(\alpha,\delta-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\delta-\alpha-1}\,{}_1F_0\left[(\alpha_1,\beta),-;tz\right]dt. \quad (4.9)$$

Theorem 4.3. The following derivative holds

$$\frac{d}{dz}\,{}_pF_q\left[\begin{matrix}(\alpha_1,\beta),\alpha_2,...,\alpha_p\\ \delta_1,\delta_2,...,\delta_q\end{matrix};z\right]=\frac{1}{j}\frac{(\alpha_1,\beta)_m(\alpha_2)_m\cdots(\alpha_p)_m}{(\delta_1)_m(\delta_2)_m\cdots(\delta_q)_m}\,{}_pF_q\left[\begin{matrix}(\alpha_1+1,\beta+1),\alpha_2,...,\alpha_p\\ \delta_1,\delta_2,...,\delta_q\end{matrix};z\right] \quad (4.10)$$

Proof

$$\frac{d}{dz}\,{}_pF_q\left[\begin{matrix}(\alpha_1,\beta),\alpha_2,\ldots,\alpha_p\\ \delta_1,\delta_2,\ldots,\delta_q\end{matrix};z\right]=\sum_{m=1}^{\infty}\frac{(\alpha_1,\beta)_m(\alpha_2)_m\cdots(\alpha_p)_m}{(\delta_1)_m(\delta_2)_m\cdots(\delta_q)_m}\frac{z^{m-1}}{j^m\Gamma(m)}$$

As $m\to m+1$,

$$\frac{d}{dz}\,{}_pF_q\left[\begin{matrix}(\alpha_1,\beta),\alpha_2,\ldots,\alpha_p\\ \delta_1,\delta_2,\ldots,\delta_q\end{matrix};z\right]=\sum_{m=0}^{\infty}\frac{(\alpha_1,\beta)_{m+1}(\alpha_2)_{m+1}\cdots(\alpha_p)_{m+1}}{(\delta_1)_{m+1}(\delta_2)_{m+1}\cdots(\delta_q)_{m+1}}\frac{z^{m}}{j^{m+1}\Gamma(m+1)}$$

Using (1.6), yields

$$\frac{d}{dz}\,{}_pF_q\left[\begin{matrix}(\alpha_1,\beta),\alpha_2,\ldots,\alpha_p\\ \delta_1,\delta_2,\ldots,\delta_q\end{matrix};z\right]=\frac{1}{j}\frac{(\alpha_1,\beta)(\alpha_2)\cdots(\alpha_p)}{(\delta_1)(\delta_2)\cdots(\delta_q)}\sum_{m=0}^{\infty}\frac{(\alpha_1,\beta)_m(\alpha_2)_m\cdots(\alpha_p)_m}{(\delta_1)_m(\delta_2)_m\cdots(\delta_q)_m}\frac{z^{m}}{(m,j)!}$$

Applying (4.1), we obtain the required result.

Corollary 4.2.

$$\frac{d}{dz}\left\{{}_pF_q\left[(\alpha,\beta),\lambda,\delta;z\right]\right\}=\frac{1}{j}\frac{\lambda(\alpha,\beta)}{\delta}\,{}_2F_1\left[(\alpha+1,\beta+1),\lambda+1,\delta+1;z\right], \tag{4.11}$$

and

$$\frac{d}{dz}\left\{{}_1F_1\left[(\alpha,\beta),\delta;z\right]\right\}=\frac{1}{j}\frac{(\alpha,\beta)}{\delta}\,{}_1F_1\left[(\alpha+1,\beta+1),\delta+1;z\right]. \tag{4.12}$$

Proof.

$${}_2F_1\left[(\alpha,\beta),\lambda,\delta;z\right]=\sum_{m=0}^{\infty}\frac{(\alpha,\beta)_m(\lambda)_m}{(\delta)_m}\frac{z^m}{(m,j)!}$$

$$=\sum_{m=0}^{\infty}\frac{(\alpha,\beta)_m(\lambda)_m}{(\delta)_m}\frac{mz^{m-1}}{(m,j)!}$$

As $m\to m+1$, we have

$$\frac{d}{dz}\,{}_2F_1\left[(\alpha,\beta),\lambda,\delta;z\right]=\sum_{m=0}^{\infty}\frac{(\alpha,\beta)_{m+1}(\lambda)_{m+1}}{(\delta)_{m+1}}\frac{z^m}{j^{m+1}\Gamma(m+1)}$$

$$=\frac{1}{j}\sum_{m=0}^{\infty}\frac{(\alpha,\beta)_{m+1}(\lambda)_{m+1}}{(\delta)_{m+1}}\frac{z^m}{j^{m}\Gamma(m+1)}$$

$$=\frac{1}{j}\frac{\lambda(\alpha,\beta)}{\delta}\sum_{m=0}^{\infty}\frac{(\alpha+1,\beta+1)_m(\lambda+1)_m}{(\delta+1)_m}\frac{z^m}{(m,j)!}$$

$$=\frac{1}{j}\frac{\lambda(\alpha,\beta)}{\delta}\,{}_2F_1[(\alpha+1,\beta+1),\lambda+1,\delta+1;z]$$

5. Families of Generating Functions Relations

In this section, we denote the following array of numbers $\frac{\lambda}{N}, \frac{\lambda+1}{N}, \ldots, \frac{\lambda+n-1}{N}$ by $\Delta(N,\lambda)$

Theorem 5.1. The following relation holds true

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_{r+N}F_s\left[\begin{matrix}\Delta(N,\lambda+n),(\alpha_1,\beta),\alpha_2,\ldots,\alpha_r \\ \delta_1,\delta_2,\ldots,\delta_s\end{matrix}; zj^N\right]$$

$$= {}_{r+N}F_s\left[\begin{matrix}\Delta(N,\lambda),(\alpha_1,\beta),\alpha_2,\ldots,\alpha_r \\ \delta_1,\delta_2,\ldots,\delta_s\end{matrix}; z\left(\frac{j}{1-t}\right)^N\right](1-t)^{-\lambda} \qquad (5.1)$$

Proof: Given that

$$\left[(1-z)\left(1-\frac{t}{1-z}\right)\right]^{-\lambda} = \left[(1-t)\left(1-\frac{z}{1-t}\right)\right]^{-\lambda}$$

Since

$$(1-z)^{-\lambda} = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} z^n \qquad (5.2)$$

Yields

$$(1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\left(\frac{t}{1-z}\right)^n = (1-t)^{-\lambda} \sum_{N=0}^{\infty} \frac{(\lambda)_N}{N!}\left(\frac{z}{1-t}\right)^N$$

Applying (5.2) again and (4.1), we get the desired result.

Theorem 5.2

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_{r+N}F_s\left[\begin{matrix}\Delta(N,-n),(\alpha_1,\beta),\alpha_2,\ldots,\alpha_r \\ \delta_1,\delta_2,\ldots,\delta_s\end{matrix}; zj^N\right]$$

$$= {}_{r+N}F_s\left[\begin{matrix}\Delta(N,\lambda),(\alpha_1,\beta),\alpha_2,\ldots,\alpha_r \\ \delta_1,\delta_2,\ldots,\delta_s\end{matrix}; z\left(\frac{-tj}{1-t}\right)^N\right](1-t)^{-\lambda} \qquad (5.3)$$

Proof: The proof of (5.3) is similar to (5.1). This can be obtained from the fact that

$$\left[(1-z)\left(1-\frac{-zt}{1-z}\right)\right]^{-\lambda} = \left[(1+zt)\left(1-\frac{z}{1+zt}\right)\right]^{-\lambda}$$

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An Approach Utilizing The Intuitionistic Fuzzy TOPSIS Method To Unmanned Air Vehicle Selection

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Keywords
Intuitionistic Fuzzy Sets,
Multi Criteria Decision Making,
TOPSIS,
Unmanned Air Vehicle(UAV)

Abstract — The intuitionistic fuzzy TOPSIS method is one of the popular multi-criteria decision making methods today, as it allows decision makers to reflect their views objectively. In this study, an intuitionistic fuzzy based decision making mechanism was created for the selection of UAVs, which have a very important place in today's military and civil sense. Experts in the field that is decision makers determined the criteria that are important in the selection of UAVs in this method. Afterward, they expressed their opinions about the UAVs to be evaluated according to these criteria, provided that each criterion is independent of each other. The most suitable UAV was selected among the target-oriented UAVs. The method used in the study and the mechanism established will shed light on many studies.

Subject Classification (2020): 03E72,90B50.

1. Introduction

UAV (Unmanned Aerial Vehicle) is air vehicles that are sent by a pilot on the ground and performed with remote control or that are automatically flown by uploading a previously made flight program. Very generally, it is collected in two main classes according to its technical features and usage purposes. According to their technical features; according to their weight, fuel/energy source, wing structure, automatic or remote control, etc. Moreover according to their intended use; military (reconnaissance and surveillance, target and weapon, attack, etc.) and civil (logistics, hobby, scientific and commercial) [19].
UAVs have played an active role in the tasks they have performed in the operational fields and as a developing technology with enormous potential, they have been indispensable in the execution of the duties of the navies. Unmanned aerial vehicles will also find use only if they gain an advantage over manned aircraft. Unmanned aerial vehicles operate in Dull, Dirty, Dangerous environments called 3D without endangering human life [22].

There are many studies on UAV and its selection in the relevant literature, such as UAVs sensors and applications for monitoring, selection of UAV for military fields, selection of UAV by using MCDM, selection

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using fuzzy Choquet integral,UAV selecting under group decision making, drone selection and evaluation using the interval-valued inferential fuzzy TOPSIS, algorithm in UAV formation network, classification of UAV vehicles, UAV landing, UAV history and legal status, electro-optical camera design for UAV, role of UAV, development of UAV, application of MCDM techniques in electro-optics and infrared sensor selection in UAV [1–4, 11, 11, 17, 19, 21–23, 25, 33].

Fuzzy logic, which reveals the feature of expressing the members even better with the rating method rather than just binary logic, was first defined by Zadeh ([40]). Furthermore, Atanassov described and developed intuitionistic fuzzy (IF) sets that are are a generalization of fuzzy sets ([5]). IF sets have shed light on many researchers because of their advantages such as membership degree and nonmembership degree, as well as expressing unstable states with hesitation degree. For a long time, multi-criteria decision making (MCDM) problems have been the focus of attention for all researchers. There are many MCDM methods defined so far ([24]). TOPSIS method is one of the MCDM methods. The TOPSIS method makes a ranking based on the positive ideal and negative ideal relationship [18]. In the IF TOPSIS method, this method is preferred because decision makers are free to express their ideas in linguistic terms. Many researchers have benefited from the TOPSIS method, fuzzy logic and intuitionistic fuzzy sets in theoretically and their application areas such as; supplier selection, renewable energy technologies, topology, algebra, statistics, controlled set, paper quality, education, mobile phone selection, etc. [7–10, 13–16, 20, 26, 27, 29–31, 34–36, 39]

2. Preliminaries

Definition 2.1. [5, 6] Let $X \neq \emptyset$. An intuitionistic fuzzy set A in X;

$$A = \left\{ \langle x, \mu_A(x), \nu_A(x) \rangle | \; x \in X \right\},$$

$$\mu_A(x), \nu_A(x), \pi_A(x) : X \to [0,1]$$

defined membership, nonmembership and hesitation degree of the element $x \in X$ respectively.

$$\mu_A(x) + \nu_A(x) + \pi_A(x) = 1.$$

IF TOPSIS model was presented an introduced by Rouyendegh (2015) ([28]). $A = A_1, A_2, \cdots, A_m$ is set of alternatives, $C = C_1, C_2, \cdots, C_n$ is set of criteria, and $L = l_1, l_2, \cdots, l_l$ is set of decision makers represents. The algorithm consists of seven steps as follows.

Step 1 The contribution of the decision-makers was determined thanks to IF numbers in Table 1 ([37]).

Table 1. Linguistic Terms for Rating DMs

Linguistic Terms	IFNs
Very Important (VI)	(0.80,0.10)
Important (I)	(0.50,0.20)
Medium (M)	(0.50,0.50)
Bad (B)	(0.30,0.50)
Very Bad (VB)	(0.20,0.70)

$Dl = [\mu l, \nu l, \pi l]$ is the IFN for lth DM ranking. As DMs express their opinions, their own weight of importance is assigned. It is expressed by the formula:

$$\lambda l = \frac{[\mu l + \pi l(\frac{\mu l}{\mu l + \nu l})]}{\sum_{l=1}^{k} [\mu l + \pi l(\frac{\mu l}{\mu l + \nu l})]} \tag{2.1}$$

$\lambda 1 \in [0,1]$ and $\sum_{l=1}^{k} \lambda l = 1$.

Step 2 The importance of criterion is represented as linguistic terms in Table 2.

Table 2. Linguistic Terms for Rating the Criterion

Linguistic Terms	IFNs
Very Important (VI)	(0.90,0.10)
Important (I)	(0.75,0.20)
Medium (M)	(0.50,0.45)
Unimportant (U)	(0.35,0.60)
Very Unimportant (VU)	(0.10,0.90)

The IF weighted averaging (IFWA) operator is used to calculate the weights of the criterion. The IFWA operator is developed by Xu (2007)[38]. According to linguistic terms in Table 2, the weight of criteria is calculated as:

$$\begin{aligned} w_j &= IFWAr_\lambda(w_j^{(1)}, w_j^{(2)}, \ldots, w_j^{(l)}) = \lambda_1 w_j^{(1)} \oplus \lambda_2 w_j^{(2)} \oplus, \ldots, \lambda_k w_j^{(k)} \\ &= \left[1 - \prod_{l=1}^{k} (1 - \mu_{ij}^{(l)})^{\lambda l}, (\prod_{l=1}^{k} \nu_{ij}^{(l)})^{\lambda l}, \prod_{l=1}^{k} (1 - \mu_{ij}^{(l)})^{\lambda l} - (\prod_{l=1}^{k} \nu_{ij}^{(l)})^{\lambda l} \right] \end{aligned} \tag{2.2}$$

Step 3 Using the linguistic terms in Table 3, the alternatives are evaluated individually for all criteria by each decision maker. At the end of this evaluation, Intuitionistic Fuzzy Decision Matrix (IFDM) is obtained.

Table 3. Linguistic Terms for Rating the Alternatives

Linguistic Terms	IFNs
Very Good (VG)	(1.00,0.00)
Good (G)	(0.85,0.05)
Medium Good (MG)	(0.70,0.20)
Fair (F)	(0.50,0.50)
Medium Poor (MP)	(0.40,0.50)
Poor (P)	(0.25,0.60)
Very Poor (VP)	(0.00,0.90)

Aggregated Intuitionistic Fuzzy Decision Matrix (AIFDM) is obtained as follows:

$R^{(l)} = (r_{ij}^{(l)})_{m*n}$is the IFDM of each DM.

$\lambda = \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$is the weight of the DM.

$$R = (r_{ij})_{m' \times n'}$$

$$\begin{aligned} r_{ij} &= IFWAr_\lambda(r_{ij}^{(1)}, r_{ij}^{(2)}, \ldots, r_{ij}^{(k)}) = \lambda_1 r_{ij}^{(1)} \oplus \lambda_2 r_{ij}^{(2)} \oplus, \ldots, \lambda_k r_{ij}^{(k)} \\ &= \left[1 - \prod_{l=1}^{k} (1-\mu_{ij}^{(l)})^{\lambda l}, (\prod_{l=1}^{k} \nu_{ij}^{(l)})^{\lambda l}, \prod_{l=1}^{k} (1-\mu_{ij}^{(l)})^{\lambda l} - (\prod_{l=1}^{k} \nu_{ij}^{(l)})^{\lambda l} \right] \end{aligned} \tag{2.3}$$

Step 4 S matrix is obtained as follows:

$$S = R \times W \tag{2.4}$$

$$R \otimes W = (\mu'_{ij}, \nu'_{ij}) = \{< x, \mu_{ij} \times \mu_j, \nu_{ij} + \nu_j - \nu_{ij} \times \nu_j >\} \tag{2.5}$$

Step 5 Positive and negative ideal solutions vary according to the criteria and alternatives. The ideal solution approach; the closer an alternative is to the positive, the farther from the negative, which represents the best alternative for the decision-maker. In this step, positive and negative ideal solutions are calculated. The IF positive and negative ideal solutions, A^+ and A^- respectively, in which J_1:benefit and J_2: cost criteria; are determined as follows:

$$A^+ = (r_1^{'*}, r_2^{'*}, \ldots, r_n^{'*}), r_j^{'*} = (\mu_j^{'*}, \nu_j^{'*}, \pi_j^{'*}), j = 1, 2, \ldots, n \tag{2.6}$$

$$A^- = (r_1^{'-}, r_2^{'-}, \ldots, r_n^{'-}), r_j^{'-} = (\mu_j^{'-}, \nu_j^{'-}, \pi_j^{'-}), j = 1, 2, \ldots, n \tag{2.7}$$

where

$$\mu_j^{'*} = \{(\max_i\{\mu'_{ij}\} j \in J_1), (\min_i\{\mu'_{ij}\} j \in J_2)\} \tag{2.8}$$

$$\nu_j^{'*} = \{(\min_i\{\nu'_{ij}\} j \in J_1), (\max_i\{\nu'_{ij}\} j \in J_2)\} \tag{2.9}$$

$$\mu_j^{'-} = \{(\min_i\{\mu'_{ij}\} j \in J_1), (\max_i\{\mu'_{ij}\} j \in J_2)\} \tag{2.10}$$

$$\nu_j^{'-} = \{(\max_i\{\nu'_{ij}\} j \in J_1), (\min_i\{\nu'_{ij}\} j \in J_2)\} \tag{2.11}$$

Step 6 The separation measures between the alternatives are determined. Many distance measures were defined on intuitionistic fuzzy sets ([32],[12]). In this step of the study, unlike other methods, the normalized Hamming measure was used. Studies have shown that the normalized Hamming measure is the most sensitive measure of distance compared to other distance measures. Therefore, in this study, the normalized Hamming distance measure will be used when calculating positive and negative ideal solutions. Through the positive and negative ideal solutions, S_i^+ and S_i^-, respectively, the separation measures of the alternatives are calculated.

$$S_i^+ = \frac{1}{2n}\sum_{i=1}^{n}[|\mu'_{ij} - \mu'^*_{ij}| + |\nu'_{ij} - \nu'^*_{ij}| + |\pi'_{ij} - \pi'^*_{ij}|] \tag{2.12}$$

$$S_i^- = \frac{1}{2n}\sum_{i=1}^{n}[|\mu'_{ij} - \mu'^-_{ij}| + |\nu'_{ij} - \nu'^{*-}_{ij}| + |\pi'_{ij} - \pi'^-_{ij}|] \tag{2.13}$$

Step 7 In the last step, the coefficient of closeness with respect to the positive and negative ideal solutions is calculated by the formula 2.14:

$$C_i^* = \frac{S_i^-}{S_i^+ + S_i^-}, \text{and} 0 \leq C_i^* \leq 1 \tag{2.14}$$

The resulting value is sorted from largest to smallest. A larger C_i^* value indicates better alternative. The hierarchy for the UAV selection decision mechanism is as follows:

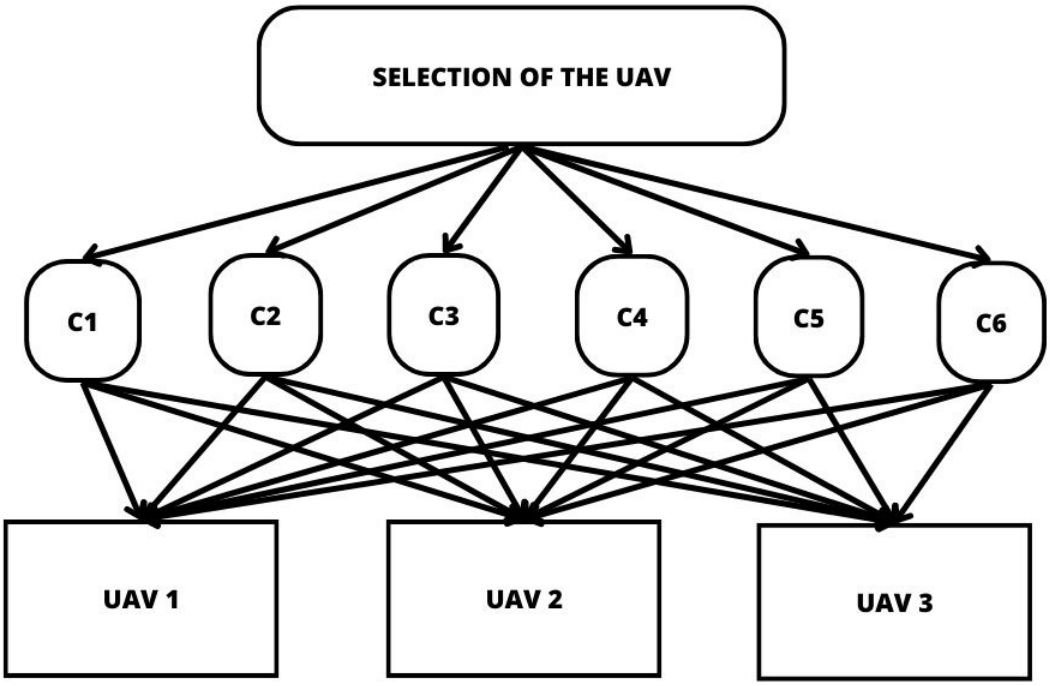

Figure 1. Hierarchy for the UAV selection decision mechanism

3. Selection of UAV Utilizing the Intuitionistic Fuzzy TOPSIS Method

UAVs play a very important role because of their benefits such as low fuel and flight costs, no risk of loss of life, less exposure to weather conditions, working at any time of the day, and scanning more areas. It is very important to determine purpose-oriented criteria when choosing a UAV. The criteria to be considered in the selection of UAVs were determined by the decision makers consisting of experts in the field of UAV. Afterwards, UAVs were evaluated according to the criteria determined by the decision makers. Decision makers first evaluated the criteria using linguistic terms and then evaluated the alternatives one by one independently for all criteria. $U = \{UAV_1, UAV_2, UAV_3\}$ is set of alternatives. Alternatives represent different sensors. $C = \{C_1, C_2, C_3, C_4, C_5, C_6\}$ is set of criteria. All criteria in this study were evaluated independently of each other. The classification of the criteria is as follows:

- C_1 : Performance
- C_2 : Cost
- C_3 : Power
- C_4 : Height
- C_5 : Durability
- C_6 : Weight

In this study, the opinions of 2 decision makers were consulted while using the intuitionistic fuzzy TOPSIS method. The importance of the contribution of decision makers;DM_1 is very important and DM_2 is important according to Table 1. Equation 2.1 is used when calculating the contributions of decision makers. As to; numerical values of DM_1, DM_2's importance weight are 0,554 and 0,446 respectively. Furthermore both decision makers specified the same linguistic terms when determining the importance of the criteria and showed in Table 4.

Table 4. Importance Weights of Criteria as to Decision Makers

Criteria	DM_1	DM_2
C_1	VI	VI
C_2	VI	VI
C_3	I	I
C_4	M	I
C_5	I	I
C_6	I	VI

According to the results obtained by using the values in the Table 4 and Equation 2.2; the weights of the criteria are shown in the Table 5. The importance of the alternatives for each criterion has been determined by the decision makers according to the linguistic expressions in Table 3 and has shown in Table 5.

Table 5. Values of Alternatives for Criteria

	DM_1						DM_2					
	C_1	C_2	C_3	C_4	C_5	C_6	C_1	C_2	C_3	C_4	C_5	C_6
UAV_1	MG	F	P	MG	MP	MP	MP	P	F	MG	MP	P
UAV_2	MG	MP	F	MG	MG	F	MP	F	MP	F	MP	MP
UAV_3	G	G	MG	G	G	MG	MG	MG	G	MG	G	MG

R matrix was created with the help of Equation 2.3. Afterwards, the S matrix was obtained with the help of the Equation 2.4 and the S matrix was shown in Table 6.
The positive ideal A^+ and negative ideal A^- solutions were calculated with the help of Equation 2.6 and shown in Table 7.

Table 6. S Matrix

	C_1	C_2	C_3	C_4	C_5	C_6
UAV_1	(0.532,0.371)	(0.361,0.588)	(0.281,0.643)	(0.443,0.451)	(0.300,0.600)	(0.281,0.609)
UAV_2	(0.532,0.371)	(0.402,0.550)	(0.343,0.600)	(0.394,0.521)	(0.444,0.441)	(0.382,0.573)
UAV_3	(0.716,0.183)	(0.716,0.183)	(0.585,0.286)	(0.504,0.377)	(0.638,0.240)	(0.584,0.317)

Table 7. The IF Positive and Negative Ideal Solution

	C_1	C_2	C_3	C_4	C_5	C_6
A^+	(0.72,0.18)	(0.72,0.18)	(0.58,0.29)	(0.51,0.37)	(0.64,0.24)	(0.58,0.32)
A^-	(0.53,0.37)	(0.36,0.58)	(0.28,0.64)	(0.39,0.52)	(0.30,0.60)	(0.28,0.60)

The separation measures S^+ and S^- of the alternatives calculated using the normalized Hamming measure and the closeness coefficient values were calculated in Table 8. In addition, the graphs of values were shown in Figure 2.

Table 8. Separation Measures and Closeness Coefficient Values

	S^+	S^-	C_i^*
UAV_1	0.2807	0.0115	0.0395
UAV_2	0.2445	0.0606	0.1986
UAV_3	0.0000	0.2923	1.0000

As a result of the evaluation made according to the opinions of the decision makers consisting of experts in the field with the intuitionistic fuzzy TOPSIS method, the ranking among the decision makers from the best to the worst is as follows:UAV_3-UAV_2-UAV_1 According to the result obtained in the decision making mechanism created, the best UAV is the UAV_3. It is recommended to select UAV_3 among the determined UAVs.

4. Conclusion and Suggestions

UAVs, which are non-pilot, remotely controlled by a pilot from the ground, or autonomously flying with various devices depending on the characteristics of the mission, have played an active role in the tasks they have performed in the operational fields and are developing technology with enormous potential. In addition, it has been indispensable in the execution of the duties of the navies. Using the intuitionistic fuzzy sets, membership, non-membership, and sensitivity degrees were all evaluated simultaneously. Thanks to the intuitionistic fuzzy TOPSIS method, the decision makers easily expressed their opinions in linguistic terms, which they had difficulty expressing with numerical values. In the study, 2 decision makers who are experts in their fields shared their views. Intuitionistic fuzzy TOPSIS method based decision making mechanism was created according to 6 criteria determined by the decision makers among a total of 3 UAVs. The most suitable UAV for the target was determined according to the decision making mechanism. Recently, the intuitionistic fuzzy TOPSIS method has attracted the attention of many researchers due to its

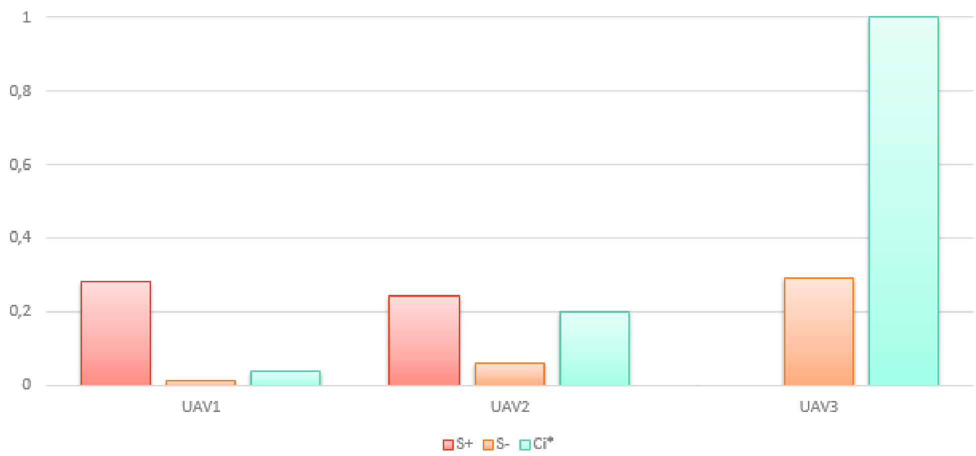

Figure 2. Graphic of values for the UAVs

advantages. UAV selection is a very important issue today. Instead of the intuitionistic fuzzy TOPSIS used in the study, evaluations may be made with different methods. The opinions of different experts may be consulted for the criteria. The range of UAVs to be evaluated may be expanded. This study, which will guide many researchers, has an important place for UAV selection. In addition to contributing to the literature in the future, it will give researchers a new perspective.

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Conformable Fractional Elzaki Decomposition Method of Conformable Fractional Space-Time Fractional Telegraph Equations

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Keywords:
Conformable fractional Elzaki decomposition method, Conformable time-fractional telegraph equation, Conformable Elzaki transform

Abstract — Conformable space-time fractional linear telegraph equations are examined using a new method known as conformable fractional Elzaki decomposition method. The suggested method combines the Adomian decomposition method with the conformable fractional Elzaki transform. It is found that numerical simulations confirm the effectiveness and reliability of the proposed method.

Subject Classification (2020): 65R10.

1. Introduction

The appearance of fractional calculus is based on a question that Leibniz asked L'Hospital on 30 September, 1695. Since 1695, the mathematicians have developed in fractional derivatives and produced derivatives of various orders. Recently, we have observed that fractional analysis allows an elegant modelling of a lot of interdisciplinary problems [1-7]. Until recently, the fractional derivative definitions such as Grunwald–Letnikov, Riesz, Riemann–Liouville, Caputo [2-3, 8] have been widely used in the solution methods to obtain the approximate solutions of differential equations. Since these derivative definitions include integral operators, the calculations are extremely challenging. Besides, analytical solutions usually can not be obtained in the models using these derivative definitions and to solve these equations scientists sometimes benefit from numerical methods.

Different fractional-order models are utilized in engineering and the applied sciences because these models provide a more accurate description of real-world scenarios. Various researchers have already utilized conformable fractional derivatives in numerous disciplines. [9]. The conformable fractional operator [3, 10-12] overcomes certain limitations of the existing fractional operators and provides traditional calculus with properties including the mean value theorem, the chain rule, the product of two functions, the derivative of the quotient of two functions, and Rolle's theorem.

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The telegraph equation has been improved by Oliver Heaviside in the 1880s, which defines the distance and time on an electric transmission line with voltage and current. The telegraph equation is usually implemented in the investigation of electric signals, as well as wave propagations in the wave phenomena and cable transmission line. The telegraph equation has a lot of applications in areas such as radio frequency, wireless signals, telephone lines, and microwave transmission [13]. Many numerical and analytical methods have been utilised to solve fractional-order telegraph equations, such as Laplace transform (LT) [14], homotopy perturbation method (HPM) [15], variational iteration method (VIM) [16] . Recently, Keskin and Oturanc has extended FRDM for FDEs, where they showed that FRDTM can simply obtain the exact solution for both linear and nonlinear FDEs [17-18]. In the literature, there are a lot of numerical and analytical methods such as conformable variational iteration method (C-VIM) [19], conformable fractional reduced differential transform method (CFRDTM) [19], conformable homotopy analysis method (C-HAM) [19], conformable fractional differential transform method (CFDTM) [20], conformable fractional adomian decomposition method (CFADM) [21] , and conformable modified homotopy perturbation method (CMHPM) [21]. The main motivation of writing this paper is to suggest a new method which is called conformable fractional Elzaki decomposition method (CFEDM) to obtain numerical solutions for the conformable time-fractional linear telegraph equations.

In this study, CFEDM is applied to solve the following types of the conformable time-fractional linear telegraph equations.

In this study, CFEDM is applied to solve the following types of the conformable time-fractional linear telegraph equations.

1) One-dimensional space-time conformable fractional telegraph equation is introduced by

$$\frac{\partial^{2\mu} w(x,t)}{\partial t^{2\mu}} + 2\alpha \frac{\partial^{\mu} w(x,t)}{\partial t^{\mu}} + \beta^2 w(x,t) = \frac{\partial^{2\vartheta} w(x,t)}{\partial t^{2\vartheta}} + h(x,t), 0 < \vartheta, \mu \leq 1. \tag{1}$$

with the initial and boundary conditions

$$w(x,0) = \Phi_1(x), w_t(x,0) = \Phi_2(x), w(0,t) = \Phi_3(t), w_x(0,t) = \Phi_4(t). \tag{2}$$

2) Two-dimensional conformable fractional-order telegraph equation is given by

$$\frac{\partial^{2\mu} w(x,y,t)}{\partial t^{2\mu}} + 2\alpha \frac{\partial^{\mu} w(x,y,t)}{\partial t^{\mu}} + \beta^2 w(x,y,t) = \frac{\partial^{2} w(x,y,t)}{\partial x^{2}} + \frac{\partial^{2} w(x,y,t)}{\partial y^{2}} + h(x,y,t),$$

$$0 < \mu \leq 1, \vartheta = 1. \tag{3}$$

with the initial and boundary conditions

$$w(x,y,0) = \xi_1(x,y), w_t(x,y,0) = \xi_2(x,y). \tag{4}$$

3) Three-dimensional conformable fractional-order telegraph equation is introduced by

$$\frac{\partial^{2\mu} w(x,y,z,t)}{\partial t^{2\mu}} + 2\alpha \frac{\partial^{\mu} w(x,y,z,t)}{\partial t^{\mu}} + \beta^2 w(x,y,z,t) = \frac{\partial^{2} w(x,y,z,t)}{\partial x^{2}} + \frac{\partial^{2} w(x,y,z,t)}{\partial y^{2}}$$

$$+ \frac{\partial^{2} w(x,y,z,t)}{\partial z^{2}} + h(x,y,z,t), 0 < \mu \leq 1, \vartheta = 1. \tag{5}$$

with the initial and boundary conditions

$$w(x,y,z,0) = \kappa_1(x,y,z), w_t(x,y,0) = \kappa_2(x,y,z). \tag{6}$$

In this study, the symbol D^{μ} represents the conformable fractional derivative operator.

2. Preliminaries

In this section, the definitions of conformable fractional calculus and Elzaki transform that should be utilized in the current study are presented.

Definition 1 [11-12, 22]. Given a function $f\colon[0,\infty)\to\mathbb{R}$. Then, the conformable fractional derivative of f order α is defined by

$$T_\alpha(f)(x)=\lim_{\varepsilon\to 0}\frac{f(x+\varepsilon x^{1-\alpha})-f(x)}{\varepsilon}, \tag{7}$$

for all $x>0, \alpha\in(0,1]$.

Theorem 1 [11, 23]. Let $\alpha\in(0,1]$ and f,g be α −differentiable at a point $x>0$. Then it is obtained as

$(i)\ T_\alpha(af+bg)=aT_\alpha(f)+bT_\alpha(g)$, for all $a,b\in\mathbb{R}$, (8)

$(ii) T_\alpha(x^p)=px^{p-1}$, for all $p\in\mathbb{R}$, (9)

$(iii)\ T_\alpha(\lambda)=0$, for all constant functions $f(t)=\lambda$, (10)

$(iv)\ T_\alpha(fg)=fT_\alpha(g)+gT_\alpha(f)$, (11)

$$(v)\ T_\alpha\left(\frac{f}{g}\right)=\frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}. \tag{12}$$

(vi) If f is differentiable, then $T_\alpha(f)(t)=t^{1-\alpha}\frac{d}{dt}f(t)$. (13)

Definition 2 [12]. Let f be an n −times differentiable at x. Then, the conformable fractional derivative of f order α is defined by:

$$T_\alpha(f)(x)=\lim_{\varepsilon\to 0}\frac{f^{([\alpha]-1)}\left(x+\varepsilon x^{([\alpha]-\alpha)}\right)-f^{([\alpha]-1)}(x)}{\varepsilon}, \tag{14}$$

for all $x>0, \alpha\in(n,n+1]$, $[\alpha]$ is the smallest integer greater than or equal to α.

Theorem 2 [12]. Let f be an n −times differentiable at x. Then

$$T_\alpha\big(f(x)\big)=x^{[\alpha]-\alpha}f^{[\alpha]}(x), \tag{15}$$

for all $x>0, \alpha\in(n,n+1]$.

Definition 3 [23]. The Mittag-Leffler function E_a is given as follows:

$$E_a(z)=\sum_{n=0}^{\infty}\frac{z^a}{\Gamma(na+1)}. \tag{16}$$

Definition 4 [12]. The conformable fractional exponential function is defined for every $t\geq 0$ by

$$E_\alpha(c,t) = exp\left(c\frac{t^\alpha}{\alpha}\right), \tag{17}$$

where $c \in \mathbb{R}$ and $0 < \alpha \leq 1$.

Definition 5 [24]. Let $0 < \alpha \leq 1$, $f: [0,\infty) \to \mathbb{R}$ be real valued function. The conformable fractional Elzaki transform of order α of f is defined by

$$E_\alpha[f(t)](v) = \int_0^\infty v\,E_\alpha\left[-\frac{1}{v}, t\right] f(t) d_\alpha t, v > 0. \tag{18}$$

The Elzaki transform for the conformable fractional-order derivative is described by

$$E_\alpha[T_\alpha f(t)](v) = \frac{1}{v}E_\alpha[f(t)](v) - vf(0). \tag{19}$$

Theorem 3. Let $F_\alpha[v] = E_\alpha[f(t)](\mathrm{v})$ exists for $v > 0$. Then, it is obtained as

1. If c is a constant, then

$$E_\alpha[c] = v^2, \tag{20}$$

2. If w is a constant, then

$$E_\alpha[t^w] = \alpha^{\frac{w}{\alpha}}\Gamma\left(1+\frac{w}{\alpha}\right)v^{2+\frac{w}{\alpha}}. \tag{21}$$

3. Conformable Fractional Elzaki Decomposition Method (CFEDM)

Now to present the fundamental idea of CFEDM, we consider the conformable fractional order nonlinear partial differential equation:

$$\frac{\partial^\mu u(x,t)}{\partial t^\mu} + Ru(x,t) + Nu(x,t) = f(x,t), t > 0, n-1 < \mu \leq n, \tag{22}$$

where R indicates the linear operator, N denotes the nonlinear operator, $f(x,t)$ symbolizes source term, and $\frac{\partial^\mu u(x,t)}{\partial t^\mu}$ is the conformable fractional derivative operator μ.

Now, by performing conformable Elzaki transform on Eq. (22) and using initial condition, we have

$$\frac{1}{v}E_\mu[u(x,t)] - vu(x,0) + E_\mu[Ru(x,t) + Nu(x,t)] = E_\mu[f(x,t)]. \tag{23}$$

If we simplify the Eq. (23), we get

$$E_\mu[u(x,t)] = v^2u(x,0) + vE_\mu[f(x,t)] - vE_\mu[Ru(x,t) + Nu(x,t)]. \tag{24}$$

On applying inverse conformable Elzaki transform to Eq. (24), we get

$$u(x,t) = H(x,t) - E_\mu{}^{-1}\{vE_\mu[Ru(x,t) + Nu(x,t)]\}, \tag{25}$$

where $H(x,t)$ is obtained from initial condition and non-homogeneous term. Now, assume that, the infinite series solution is of the form:

$$u(x,t)=\sum_{m=0}^{\infty}u_m(x,t). \tag{26}$$

By employing Eqs. (25)-(26), we have

$$\sum_{m=0}^{\infty}u_m(x,t)=H(x,t)-{E_\mu}^{-1}\left(vE_\mu\left[R\sum_{m=0}^{\infty}u_m(x,t)+\sum_{m=0}^{\infty}A_m\right]\right). \tag{27}$$

where A_m is the Adomian polynomial and which denotes the nonlinear term $Nu(x,t)$. By comparing both sides of Eq. (27), we get

$$u_0(x,t)=H(x,t), \tag{28}$$

$$u_1(x,t)=-{E_\mu}^{-1}\left(vE_\mu[Ru_0(x,t)+A_0]\right), \tag{29}$$

$$u_2(x,t)=-{E_\mu}^{-1}\left(vE_\mu[Ru_1(x,t)+A_1]\right), \tag{30}$$

$\vdots$

Similarly, we obtain the general recursive relation by

$$u_{m+1}(x,t)=-{E_\mu}^{-1}\left(vE_\mu[Ru_m(x,t)+A_m]\right), m\geq 1. \tag{31}$$

Finally, the approximate solution $u(x,t)$ is given by

$$u(x,t)=\sum_{m=0}^{\infty}u_m(x,t). \tag{32}$$

4. Convergence Analysis

Theorem 4.1. Let's assume that A is a Banach space. Then, the expansion result of $u(x,t)$ converges uncertainty; there becomes $\rho, 0 < \rho < 1$, so that $\|u_i(x,t)\| \leq \rho\|u_{i-1}(x,t)\|$, for all i ∈ N.

Proof. Consider the subsequent succession

$$H_i(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+\cdots+u_i(x,t). \tag{33}$$

It is vital to verify that successions of i-th partial sums $H_i(x,t)$ are a Cauchy series in Banach space. In this regard, we consider the following:

$$\|H_{i+1}(x,t)-H_i(x,t)\|\leq\|u_{i+1}(x,t)\|\leq\rho\|u_i(x,t)\|\leq\rho^2\|u_{i-1}(x,t)\|\leq\cdots\leq\rho^{i+1}\|u_0(x,t)\|. \tag{34}$$

For every $i,j \in N$, $i\leq j$, it is obtained as

$$\left\|H_i(x,t)-H_j(x,t)\right\|\leq\left\|H_{j+1}(x,t)-H_j(x,t)\right\|+\left\|H_{j+2}(x,t)-H_{j+1}(x,t)\right\|+\cdots$$

$$+\|H_i(x,t)-H_{i+1}(x,t)\|. \tag{35}$$

Using the triangle inequality, then the inequality (35) transforms into the inequality (36):

$$\left\|H_i(x,t)-H_j(x,t)\right\|\leq\left\|H_{j+1}(x,t)-H_j(x,t)\right\|+\left\|H_{j+2}(x,t)-H_{j+1}(x,t)\right\|$$

$$+\cdots+\|H_i(x,t)-H_{i+1}(x,t)\|. \tag{36}$$

The inequality (36) can be represented as follows:

$$\left\|H_i(x,t)-H_j(x,t)\right\| \le \rho^{j+1}\|u_0(x,t)\|+\rho^{j+2}\|u_0(x,t)\|+\cdots+\rho^{i}\|u_0(x,t)\|. \tag{37}$$

The simple calculation enables us to write the inequality (37) as

$$\left\|H_i(x,t)-H_j(x,t)\right\| \le \rho^{j+1}\left(1+\rho+\rho^2+\cdots+\rho^{i-j-1}\right)\|u_0(x,t)\|, \tag{38}$$

where $\left(\frac{1-\rho^{i-j}}{1-\rho}\right)=1+\rho+\rho^2+\cdots+\rho^{i-j-1}$.

Thus, inequality (38) is obtained as

$$\left\|H_i(x,t)-H_j(x,t)\right\| \le \rho^{j+1}\left(\frac{1-\rho^{i-j}}{1-\rho}\right)\|u_0(x,t)\|. \tag{39}$$

Hence it is acquired as $0<\rho<1$, and $1-\rho^{i-j}\le 1$.

Using inequality (39), we have

$$\left\|H_i(x,t)-H_j(x,t)\right\| \le \frac{\rho^{i+1}}{1-\rho}\|u_0(x,t)\|. \tag{40}$$

Since $u_0(x,t)$ is bounded, it is obtained as

$$\lim_{i,j\to\infty}\left\|H_i(x,t)-H_j(x,t)\right\|=0. \tag{41}$$

Thus, $\{H_i\}$ is a Cauchy series in Banach space. Hence, Eq. (32) converges.

5. Applications

Example 5.1 Consider the conformable time-fractional linear telegraph equation (CTFLTE) **[25]**

$$\frac{\partial^{2\mu}w(x,t)}{\partial t^{2\mu}}+2\frac{\partial^{\mu}w(x,t)}{\partial t^{\mu}}+w(x,t)=\frac{\partial^2 w(x,t)}{\partial x^2}, 0<\mu\le 1, t\ge 0, \tag{42}$$

with the initial condition

$$w(x,0)=e^x, w_t(x,0)=-2e^x. \tag{43}$$

Now, by performing conformable Elzaki transform on Eq. (42), then we get

$$\frac{1}{v^2}E_\mu\{w(x,t)\}-w(x,0)-vw_t(x,0)+E_\mu\left[2\frac{\partial^{\mu}w(x,t)}{\partial t^{\mu}}+w(x,t)-\frac{\partial^2 w(x,t)}{\partial x^2}\right]=0. \tag{44}$$

If we simplify the Eq. (44), then we have

$$E_\mu\{w(x,t)\}=v^2\, w(x,0)+v^3 w_t(x,0)-v^2 E_\mu\left[2\frac{\partial^{\mu}w(x,t)}{\partial t^{\mu}}+w(x,t)-\frac{\partial^2 w(x,t)}{\partial x^2}\right]. \tag{45}$$

Applying the inverse conformable Elzaki transform,

$$w(x,t)={E_\mu}^{-1}\left[v^2\, w(x,0)+v^3 w_t(x,0)-v^2 E_\mu\left[2\frac{\partial^{\mu}w(x,t)}{\partial t^{\mu}}+w(x,t)-\frac{\partial^2 w(x,t)}{\partial x^2}\right]\right]. \tag{46}$$

Using the ADM procedure, we obtain

$$w_0(x,t)={E_\mu}^{-1}[v^2\, w(x,0)+v^3 w_t(x,0)]={E_\mu}^{-1}[v^2\, e^x-2e^x v^3]= e^x-2e^x\frac{t^\mu}{\mu}, \tag{47}$$

$$w_{s+1}(x,t) = -E_{\mu}{}^{-1}\left[v^2 E_{\mu}\left[2\frac{\partial^{\mu} w_s(x,t)}{\partial t^{\mu}} + w_s(x,t) - \frac{\partial^2 w_s(x,t)}{\partial x^2}\right]\right], \qquad s = 0,1,2,\ldots \tag{48}$$

For $s = 0$ in Eq. (48), we obtain

$$w_1(x,t) = -E_{\mu}{}^{-1}\left[v^2 E_{\mu}\left[2\frac{\partial^{\mu} w_0(x,t)}{\partial t^{\mu}} + w_0(x,t) - \frac{\partial^2 w_0(x,t)}{\partial x^2}\right]\right], \tag{49}$$

$$w_1(x,t) = -E_{\mu}{}^{-1}\left[4e^x \mu^{\frac{\mu-1}{\mu}}\Gamma\left(1+\frac{\mu-1}{\mu}\right)v^{2+\frac{\mu-1}{\mu}+2}\right] = \frac{4e^x \mu^{\frac{\mu-1}{\mu}}\Gamma\left(1+\frac{\mu-1}{\mu}\right)t^{3\mu-1}}{\mu^{\frac{3\mu-1}{\mu}}\Gamma\left(1+\frac{3\mu-1}{\mu}\right)}. \tag{50}$$

We get the subsequent terms, recursively

$$w_2(x,t) = -E_{\mu}{}^{-1}\left[v^2 E_{\mu}\left[2\frac{\partial^{\mu} w_1(x,t)}{\partial t^{\mu}} + w_1(x,t) - \frac{\partial^2 w_1(x,t)}{\partial x^2}\right]\right]$$

$$= \frac{-8e^x \mu^{\frac{\mu-1}{\mu}}\Gamma\left(1+\frac{\mu-1}{\mu}\right)(3\mu-1)\mu^{\frac{3\mu-2}{\mu}}\Gamma\left(1+\frac{3\mu-2}{\mu}\right)t^{5\mu-2}}{\mu^{\frac{3\mu-1}{\mu}}\Gamma\left(1+\frac{3\mu-1}{\mu}\right)\mu^{\frac{5\mu-2}{\mu}}\Gamma\left(1+\frac{5\mu-2}{\mu}\right)}, \tag{51}$$

$$w_3(x,t) = -E_{\mu}{}^{-1}\left[v^2 E_{\mu}\left[2\frac{\partial^{\mu} w_2(x,t)}{\partial t^{\mu}} + w_2(x,t) - \frac{\partial^2 w_2(x,t)}{\partial x^2}\right]\right]$$

$$= \frac{16e^x \mu^{\frac{\mu-1}{\mu}}\Gamma\left(1+\frac{\mu-1}{\mu}\right)(3\mu-1)(5\mu-2)\mu^{\frac{3\mu-2}{\mu}}\Gamma\left(1+\frac{3\mu-2}{\mu}\right)}{\mu^{\frac{3\mu-1}{\mu}}\Gamma\left(1+\frac{3\mu-1}{\mu}\right)\mu^{\frac{5\mu-2}{\mu}}\Gamma\left(1+\frac{5\mu-2}{\mu}\right)},$$

$$\times \frac{\mu^{\frac{5\mu-3}{\mu}}\Gamma\left(1+\frac{5\mu-3}{\mu}\right)t^{7\mu-3}}{\mu^{\frac{7\mu-3}{\mu}}\Gamma\left(1+\frac{7\mu-3}{\mu}\right)}. \tag{52}$$

$$\vdots$$

Proceeding in a similar way, we obtain

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t) = w_0(x,t) + w_1(x,t) + w_2(x,t) + \cdots = e^x - 2e^x \frac{t^{\mu}}{\mu}$$

$$+ \frac{4e^x \mu^{\frac{\mu-1}{\mu}}\Gamma\left(1+\frac{\mu-1}{\mu}\right)t^{3\mu-1}}{\mu^{\frac{3\mu-1}{\mu}}\Gamma\left(1+\frac{3\mu-1}{\mu}\right)} - \frac{8e^x \mu^{\frac{\mu-1}{\mu}}\Gamma\left(1+\frac{\mu-1}{\mu}\right)(3\mu-1)\mu^{\frac{3\mu-2}{\mu}}\Gamma\left(1+\frac{3\mu-2}{\mu}\right)t^{5\mu-2}}{\mu^{\frac{3\mu-1}{\mu}}\Gamma\left(1+\frac{3\mu-1}{\mu}\right)\mu^{\frac{5\mu-2}{\mu}}\Gamma\left(1+\frac{5\mu-2}{\mu}\right)}$$

$$+ \frac{16e^x \mu^{\frac{\mu-1}{\mu}}\Gamma\left(1+\frac{\mu-1}{\mu}\right)(3\mu-1)(5\mu-2)\mu^{\frac{3\mu-2}{\mu}}\Gamma\left(1+\frac{3\mu-2}{\mu}\right)\mu^{\frac{5\mu-3}{\mu}}\Gamma\left(1+\frac{5\mu-3}{\mu}\right)t^{7\mu-3}}{\mu^{\frac{3\mu-1}{\mu}}\Gamma\left(1+\frac{3\mu-1}{\mu}\right)\mu^{\frac{5\mu-2}{\mu}}\Gamma\left(1+\frac{5\mu-2}{\mu}\right)\mu^{\frac{7\mu-3}{\mu}}\Gamma\left(1+\frac{7\mu-3}{\mu}\right)}$$

$$+ \cdots \tag{53}$$

Subsituting $\mu = 1$ in Eq. (53), then CFEDM solution is reduced as

$$w(x,t) = e^{x}\left[1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} - \cdots\right]. \quad (54)$$

This result is evaluated to the exact solution in a closed form:

$$w(x,t) = e^{x-2t}. \quad (55)$$

The CFEDM solutions of $w(x,t)$ is found to be in excellent agreement with the exact solution of problem. For more understanding the results for the variable $w(x,t)$ of Example 4.1 are plotted in Figure 1. In Figure 1, we observe that this solution is higher accuracy.

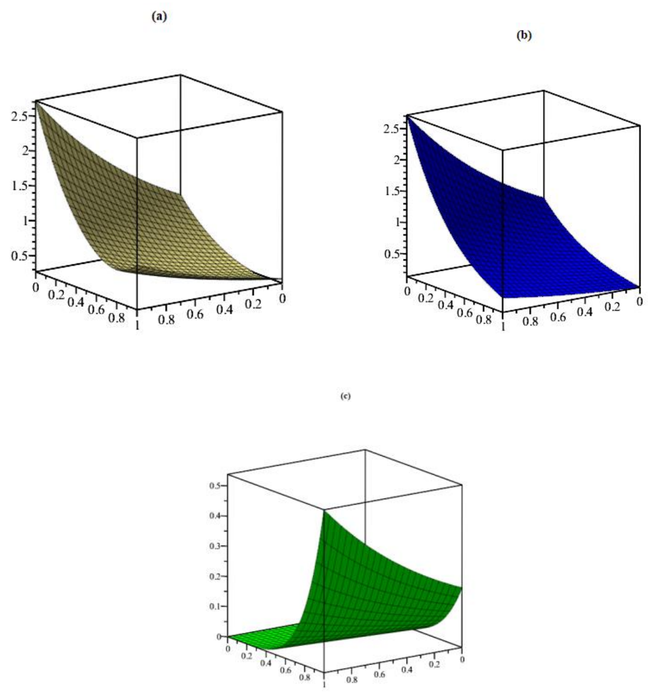

Fig. 1. (a) Nature of CFEDM solution for $w(x,t)$ **(b)** Nature of exact solution for $w(x,t)$ **(c)** Nature of absolute error=$|u_{exact} - u_{CFEDM}|$ in Ex. 5.1 at $\mu = 1$.

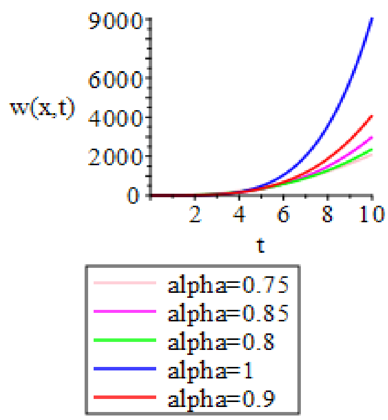

Fig. 2. Nature of CFEDM solution for $w(x,t)$ in Ex. 5.1 at $x = 0.5$ with distinct μ.

Table 1. Numerical solution of $w(x,t)$ for CTFLTE by CFEDM in Ex. 5.1 at with distinct values of x and t for diverse μ.

x	t	$\mu = 0.75$	$\mu = 0.8$	$\mu = 0.85$	$\mu = 0.9$	$\mu = 1$
0.1	0.001	1.3×10^{-2}	8.4×10^{-3}	5.0×10^{-3}	2.6×10^{-3}	2.9×10^{-16}
	0.002	2.0×10^{-2}	1.3×10^{-2}	8.5×10^{-3}	4.6×10^{-3}	9.4×10^{-15}
	0.003	2.6×10^{-2}	1.8×10^{-2}	1.1×10^{-2}	6.3×10^{-3}	7.1×10^{-14}
	0.004	3.1×10^{-2}	2.2×10^{-2}	1.4×10^{-2}	7.9×10^{-3}	3.0×10^{-13}
	0.005	3.5×10^{-2}	2.5×10^{-2}	1.6×10^{-2}	9.4×10^{-3}	9.1×10^{-13}
0.2	0.001	1.4×10^{-2}	9.3×10^{-2}	5.5×10^{-3}	2.9×10^{-3}	3.2×10^{-16}
	0.002	2.2×10^{-2}	1.5×10^{-2}	9.4×10^{-3}	5.1×10^{-3}	1.0×10^{-14}
	0.003	2.9×10^{-2}	2.0×10^{-2}	1.2×10^{-2}	7.0×10^{-3}	7.9×10^{-14}
	0.004	3.4×10^{-2}	2.4×10^{-2}	1.5×10^{-2}	8.8×10^{-3}	3.3×10^{-13}
	0.005	3.9×10^{-2}	2.8×10^{-2}	1.8×10^{-2}	1.0×10^{-2}	1.0×10^{-12}
0.3	0.001	1.6×10^{-2}	1.0×10^{-2}	0.6×10^{-3}	3.2×10^{-3}	3.5×10^{-16}
	0.002	2.5×10^{-2}	1.6×10^{-2}	1.0×10^{-2}	5.6×10^{-3}	1.1×10^{-14}
	0.003	3.2×10^{-2}	2.2×10^{-2}	1.4×10^{-2}	7.8×10^{-3}	8.7×10^{-14}
	0.004	3.8×10^{-2}	2.7×10^{-2}	1.7×10^{-2}	9.7×10^{-3}	3.6×10^{-13}
	0.005	4.3×10^{-2}	3.1×10^{-2}	2.0×10^{-2}	1.1×10^{-2}	1.1×10^{-12}
0.4	0.001	1.7×10^{-2}	1.1×10^{-2}	6.7×10^{-3}	3.5×10^{-3}	3.9×10^{-16}
	0.002	2.7×10^{-2}	1.8×10^{-2}	1.1×10^{-2}	6.2×10^{-3}	1.2×10^{-14}
	0.003	3.5×10^{-2}	2.4×10^{-2}	1.5×10^{-2}	8.6×10^{-3}	9.6×10^{-14}
	0.004	4.2×10^{-2}	3.0×10^{-2}	1.9×10^{-2}	1.0×10^{-2}	4.0×10^{-13}
	0.005	4.8×10^{-2}	3.4×10^{-2}	2.2×10^{-2}	1.2×10^{-2}	1.2×10^{-12}
0.5	0.001	1.9×10^{-2}	1.2×10^{-2}	7.5×10^{-3}	3.9×10^{-3}	4.3×10^{-16}
	0.002	3.0×10^{-2}	2.0×10^{-2}	1.2×10^{-2}	6.9×10^{-3}	1.4×10^{-14}
	0.003	3.9×10^{-2}	2.7×10^{-2}	1.7×10^{-2}	9.5×10^{-3}	1.0×10^{-13}
	0.004	4.7×10^{-2}	3.3×10^{-2}	2.1×10^{-2}	1.1×10^{-2}	4.4×10^{-13}
	0.005	5.3×10^{-2}	3.8×10^{-2}	2.4×10^{-2}	1.4×10^{-2}	1.3×10^{-12}

Example 5.2 Consider the conformable space-fractional linear telegraph equation (CSFLTE) [26-27]

$$\frac{\partial^{2\mu} w(x,t)}{\partial x^{2\mu}} = \frac{\partial^2 w(x,t)}{\partial t^2} + \frac{\partial w(x,t)}{\partial t} + w(x,t), 1 < \mu \leq 2, t \geq 0, \tag{56}$$

with the initial condition

$$w(0,t) = e^{-t}, w_x(0,t) = e^{-t}. \tag{57}$$

Now, by performing conformable Elzaki transform on Eq. (56), then we get

$$\frac{1}{v^2} E_\mu\{w(x,t)\} - w(0,t) - v w_x(0,t) = E_\mu \left[\frac{\partial^2 w(x,t)}{\partial t^2} + \frac{\partial w(x,t)}{\partial t} + w(x,t)\right]. \tag{58}$$

If we simplify the Eq. (58), then we have

$$E_\mu\{w(x,t)\} = v^2 w(0,t) + v^3 w_x(0,t) + v^2 E_\mu \left[\frac{\partial^2 w(x,t)}{\partial t^2} + \frac{\partial w(x,t)}{\partial t} + w(x,t)\right]. \tag{59}$$

Applying the inverse conformable Elzaki transform,

$$w(x,t) = E_\mu{}^{-1}\left[v^2\, w(0,t) + v^3 w_t(0,t) + v^2 E_\mu \left[\frac{\partial^2 w(x,t)}{\partial t^2} + \frac{\partial w(x,t)}{\partial t} + w(x,t)\right]\right]. \tag{60}$$

Using the ADM procedure, we obtain

$$w_0(x,t) = E_\mu{}^{-1}[v^2\, w(0,t) + v^3 w_t(0,t)] = E_\mu{}^{-1}[v^2\, e^{-t} + v^3 e^{-t}] = e^{-t} + e^{-t}\frac{x^\mu}{\mu}. \tag{61}$$

$$w_{s+1}(x,t) = E_\mu{}^{-1}\left[v^2 E_\mu \left[\frac{\partial^2 w_s(x,t)}{\partial t^2} + \frac{\partial w_s(x,t)}{\partial t} + w_s(x,t)\right]\right], \qquad s = 0,1,2,\ldots \tag{62}$$

For $s = 0$ in Eq. (62), we obtain

$$w_1(x,t) = E_\mu{}^{-1}\left[v^2 E_\mu \left[\frac{\partial^2 w_0(x,t)}{\partial t^2} + \frac{\partial w_0(x,t)}{\partial t} + w_0(x,t)\right]\right], \tag{63}$$

$$w_1(x,t) = E_\mu{}^{-1}[v^4\, e^{-t} + e^{-t} v^5] = e^{-t}\frac{x^{2\alpha}}{2!\,\alpha^2} + e^{-t}\frac{x^{3\alpha}}{3!\,\alpha^3}. \tag{64}$$

We get the subsequent terms, recursively

$$w_2(x,t) = E_\mu{}^{-1}\left[v^2 E_\mu \left[\frac{\partial^2 w_1(x,t)}{\partial t^2} + \frac{\partial w_1(x,t)}{\partial t} + w_1(x,t)\right]\right] = e^{-t}\frac{x^{4\alpha}}{4!\,\alpha^4} + e^{-t}\frac{x^{5\alpha}}{5!\,\alpha^5}, \tag{65}$$

$$w_3(x,t) = E_\mu{}^{-1}\left[v^2 E_\mu \left[\frac{\partial^2 w_2(x,t)}{\partial t^2} + \frac{\partial w_2(x,t)}{\partial t} + w_2(x,t)\right]\right] = e^{-t}\frac{x^{6\alpha}}{6!\,\alpha^6} + e^{-t}\frac{x^{7\alpha}}{7!\,\alpha^7}. \tag{66}$$

$$\vdots$$

Proceeding in a similar way, we obtain

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t) = w_0(x,t) + w_1(x,t) + w_2(x,t) + \cdots = e^{-t} + e^{-t}\frac{x^\mu}{\mu} + e^{-t}\frac{x^{2\alpha}}{2!\,\alpha^2}$$

$$+ e^{-t}\frac{x^{3\mu}}{3!\,\mu^3} + e^{-t}\frac{x^{4\mu}}{4!\,\mu^4} + e^{-t}\frac{x^{5\mu}}{5!\,\mu^5} + e^{-t}\frac{x^{6\mu}}{6!\,\mu^6} + e^{-t}\frac{x^{7\mu}}{7!\,\mu^7} + \cdots \tag{67}$$

Subsituting $\mu = 1$ in Eq. (67), then CFEDM solution is reduced as

$$w(x,t) = e^{-t}\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots\right]. \tag{68}$$

This result is evaluated to the exact solution in a closed form:

$$w(x,t) = e^{x-t}. \tag{69}$$

The CFEDM solutions of $w(x,t)$ is found to be in excellent agreement with the exact solution of problem. For more understanding the results for the variable $w(x,t)$ of Example 5.2 are plotted in Figure 3. In Figure 3, we conclude that this solution is higher accuracy.

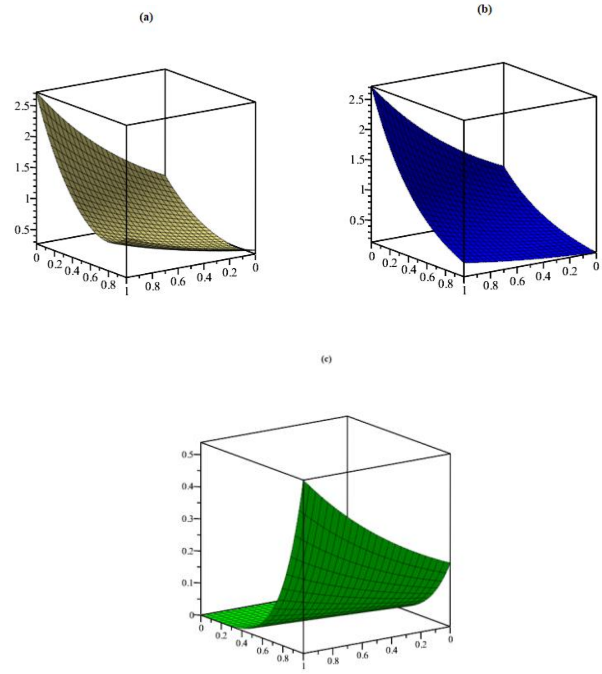

Fig. 3. (a) Nature of CFEDM solution for $w(x,t)$ **(b)** Nature of exact solution for $w(x,t)$ **(c)** Nature of absolute error=$|u_{exact} - u_{CFEDM}|$ in Ex. 5.2 at $\mu = 1$.

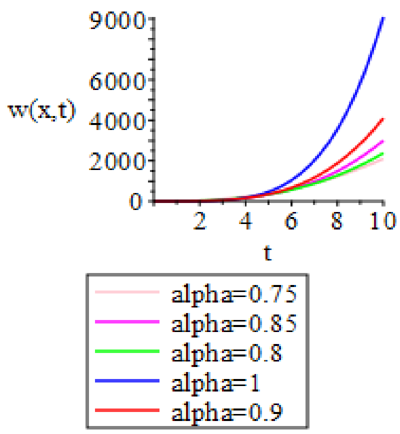

Fig. 4. Nature of CFEDM solution for $w(x,t)$ in Ex. 5.2 at $x = 0.5$ with distinct μ.

Table 2. Numerical solution of $w(x,t)$ for CSFLTE by CFEDM in Ex. 4.2 at with distinct values of x and t for diverse μ.

x	t	$\mu = 0.75$	$\mu = 0.8$	$\mu = 0.85$	$\mu = 0.9$	$\mu = 1$
0.1	0.001	1.3×10^{-2}	8.4×10^{-3}	5.0×10^{-3}	2.6×10^{-3}	2.9×10^{-16}
	0.002	2.0×10^{-2}	1.3×10^{-2}	8.5×10^{-3}	4.6×10^{-3}	9.4×10^{-15}
	0.003	2.6×10^{-2}	1.8×10^{-2}	1.1×10^{-2}	6.3×10^{-3}	7.1×10^{-14}
	0.004	3.1×10^{-2}	2.2×10^{-2}	1.4×10^{-2}	7.9×10^{-3}	3.0×10^{-13}
	0.005	3.5×10^{-2}	2.5×10^{-2}	1.6×10^{-2}	9.4×10^{-3}	9.1×10^{-13}
0.2	0.001	1.4×10^{-2}	9.3×10^{-2}	5.5×10^{-3}	2.9×10^{-3}	3.2×10^{-16}
	0.002	2.2×10^{-2}	1.5×10^{-2}	9.4×10^{-3}	5.1×10^{-3}	1.0×10^{-14}
	0.003	2.9×10^{-2}	2.0×10^{-2}	1.2×10^{-2}	7.0×10^{-3}	7.9×10^{-14}
	0.004	3.4×10^{-2}	2.4×10^{-2}	1.5×10^{-2}	8.8×10^{-3}	3.3×10^{-13}
	0.005	3.9×10^{-2}	2.8×10^{-2}	1.8×10^{-2}	1.0×10^{-2}	1.0×10^{-12}
0.3	0.001	1.6×10^{-2}	1.0×10^{-2}	0.6×10^{-3}	3.2×10^{-3}	3.5×10^{-16}
	0.002	2.5×10^{-2}	1.6×10^{-2}	1.0×10^{-2}	5.6×10^{-3}	1.1×10^{-14}
	0.003	3.2×10^{-2}	2.2×10^{-2}	1.4×10^{-2}	7.8×10^{-3}	8.7×10^{-14}
	0.004	3.8×10^{-2}	2.7×10^{-2}	1.7×10^{-2}	9.7×10^{-3}	3.6×10^{-13}
	0.005	4.3×10^{-2}	3.1×10^{-2}	2.0×10^{-2}	1.1×10^{-2}	1.1×10^{-12}
0.4	0.001	1.7×10^{-2}	1.1×10^{-2}	6.7×10^{-3}	3.5×10^{-3}	3.9×10^{-16}
	0.002	2.7×10^{-2}	1.8×10^{-2}	1.1×10^{-2}	6.2×10^{-3}	1.2×10^{-14}
	0.003	3.5×10^{-2}	2.4×10^{-2}	1.5×10^{-2}	8.6×10^{-3}	9.6×10^{-14}
	0.004	4.2×10^{-2}	3.0×10^{-2}	1.9×10^{-2}	1.0×10^{-2}	4.0×10^{-13}
	0.005	4.8×10^{-2}	3.4×10^{-2}	2.2×10^{-2}	1.2×10^{-2}	1.2×10^{-12}
0.5	0.001	1.9×10^{-2}	1.2×10^{-2}	7.5×10^{-3}	3.9×10^{-3}	4.3×10^{-16}
	0.002	3.0×10^{-2}	2.0×10^{-2}	1.2×10^{-2}	6.9×10^{-3}	1.4×10^{-14}
	0.003	3.9×10^{-2}	2.7×10^{-2}	1.7×10^{-2}	9.5×10^{-3}	1.0×10^{-13}
	0.004	4.7×10^{-2}	3.3×10^{-2}	2.1×10^{-2}	1.1×10^{-2}	4.4×10^{-13}
	0.005	5.3×10^{-2}	3.8×10^{-2}	2.4×10^{-2}	1.4×10^{-2}	1.3×10^{-12}

6. Discussion

In Figure 1, the behaviours of the exact solution, CFEDM solution and absolute error for Ex. 5.1 are plotted. Therefore, we observe that CFEDM solution is close to the exact solution. The numerical solutions for different μ values are evaluated in Table 1. From Table 1, it has been observed that the solutions get closer to the exact solution, when μ gets closer to 1. Also, especially for $\mu = 1$, it is concluded that the absolute error is extremely small in Table1. Similarly, the behaviours of the exact solution, CFEDM solution and absolute error for Ex. 5.2 are plotted in Figure 3. Therefore, we observe that CFEDM solution is close to the exact solution. The numerical solutions for different μ values are evaluated in Table 2. From Table 2, it has been observed that the solutions get closer to the exact solution, when μ gets closer to 1. Also, especially for $\mu = 1$, it is concluded that the absolute error is extremely small in Table 2. Additionally, 2D graphs of solutions of Ex. 5.1 and Ex. 5.2 for distinct μ values illustrate the behavior of CFEDM in Figures 2 and 5.

7. uld be given briefly. Besides, forward-looking suggestions and opinions related to the study results can be stated.

In the present framework, we profitably applied a new hybrid method, namely CFEDM to solve the conformable time-fractional linear telegraph equations. During the investigation, the obtained solutions are illustrated in terms of plots and tables with diverse values of space and time variables. We have observed that CFEDM is powerful, fast and efficient method to solve the conformable time-fractional linear telegraph equations.

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Completeness of the Category of Rack Crossed Modules

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Keywords
Crossed module,
Limit,
Rack

Abstract — In this paper, we prove that the category of rack crossed modules (with a fixed codomain) is finitely complete. In other words, we construct the product, pullback and equalizer objects in the category of crossed modules of racks. We therefore unify the group-theoretical analogy of the completeness property in the sense of the functor **Conj**: **Grp** → **Rack**.

Subject Classification (2020): 20N02, 18D05, 18A30.

1. Introduction

A rack is a set equipped with a binary operation satisfying two axioms that match to the second and third Reidemeister moves in knot theory. The most important and common rack which additionally satisfies an extra axiom analogues to the first Reidemeister move called quandle [11].

Crossed modules of groups are first introduced by Whitehead in [13] as models for homotopy 2-types. Afterwards, the notion of crossed module is also adapted to various algebraic structures such as algebras, Lie algebras, Hopf algebras; see [10] for more.

Crossed modules of racks are defined by Crans and Wagemann in [4]. They generalize the notion of crossed modules from the case of groups such that satisfying two Peiffer conditions as well. An interesting result of this notion is: the adjoint functors **As**: **Rack** → **Grp** and **Conj**: **Grp** → **Rack** between the categories of groups and racks are both preserving crossed module structures, see [4]. Therefore, one can consider them as (induced) functors between the category of group crossed modules **XGrp** and the category of rack crossed modules **XRack**; hence we get the following extended adjunction:

$$\mathrm{Hom}_{\mathrm{XGrp}}\big(\mathrm{As}^{\star}(\mathscr{X}),\mathscr{G}\big)\cong \mathrm{Hom}_{\mathrm{XRack}}\big(\mathscr{X},\mathrm{Conj}^{\star}(\mathscr{G})\big), \tag{1.1}$$

where $\mathscr{X}$ is a rack crossed module and $\mathscr{G}$ is a group crossed module [7].

A category $\mathscr{C}$ is said to be finitely complete if it has all (finite) limits. On the other hand, a category $\mathscr{C}$ has all finite limits iff one of the following conditions hold:

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- $\mathscr{C}$ has a terminal object and pullbacks,
- $\mathscr{C}$ has products and equalisers.

In this paper, we define such objects for rack crossed modules which will prove the completeness of the category of rack crossed modules. As another outcome, these constructions will be preserved under the functor **Conj** by using the properties of (1.1).

Many of these notions are examined for various algebraic structures such as (crossed modules and cat^1 objects of) groups, (associative) algebras, Lie algebras, etc. in [1–3, 5, 6, 8, 12].

2. Preliminaries

We recall some notions from [4, 9] which will be used in sequel.

2.1. Racks

Definition 2.1. A (right) rack consists of a set A equipped with a binary operation, satisfying:

$R1$) For each $a, a' \in A$, there exists a unique $a'' \in A$ such that:

$$a'' \lhd a = a',$$

$R2$) For all $a, a', a'' \in A$, we have:

$$\left(a \lhd a'\right) \lhd a'' = \left(a \lhd a''\right) \lhd \left(a' \lhd a''\right).$$

Definition 2.2. A pointed rack A is a rack equipped with a fixed element $1 \in A$ such that:

$$1 \lhd a = 1 \quad \text{and} \quad a \lhd 1 = a,$$

for all $a \in A$.

Remark 2.3. In this paper we only work with the pointed racks.

Definition 2.4. Let A and B be two racks. A rack morphism is a map:

$$f : A \to B$$

such that:

$$f\left(a \lhd a'\right) = f(a) \lhd f\left(a'\right) \qquad \left(\text{and } f(1) = 1\right)$$

for all $a, a' \in A$.

Thus we get the category of racks, denoted by **Rack**.

Some well-known examples of racks are given below:

1) If A is a group, one can define a rack (conjugation rack) with:

$$a \lhd a' = (a')^{-1} a a',$$

for all $a, a' \in A$. This yields a functor:

$$\mathbf{Conj} : \mathbf{Grp} \to \mathbf{Rack}$$

from the category of groups to the category of racks.

2) Another rack structure in a group A is defined by:

$$a \lhd a' = a' a^{-1} a',$$

for all $a, a' \in A$, which is called a core rack. But this is not functorial.

3) If A and B are two racks, then the set

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

which is the cartesian product of A and B, defines a rack structure with: $(a, a' \in A,\ b, b' \in B)$

$$(a, b) \lhd (a', b') = (a \lhd a', b \lhd b').$$

Notice that $A \times B$ is the product object in **Rack**.

Definition 2.5. Let A be a rack and B be a non empty subset of A . We say that B is a subrack of A if $b \lhd b' \in B$ for each $b, b' \in B$.

Definition 2.6. For a given rack A, a normal subrack N is a subrack if it further satisfies $n \lhd a \in N$, for all $n \in N$ and $a \in A$.

2.2. Rack Action

Definition 2.7. Let A be a rack and S be a set. We say that S is an A-set when there are bijections $(\cdot\, a) : S \to S$ for each $a \in A$ such that:

$$(s \cdot a) \cdot a' = (s \cdot a') \cdot (a \lhd a'),$$

for all $s \in S$ and $a' \in A$.

Definition 2.8. Let A be a rack and S be a A-set. We say that the hemi-semi-direct product $S \rtimes A$ is a rack with:

$$(s, a) \lhd (s', a') = (s \cdot a', a \lhd a')$$

for all $a, a' \in A$, $s, s' \in S$.

Definition 2.9. Let A, S be two racks. We say that S acts on A by automorphisms when there is a (right) rack action of S on A and:

$$(a \lhd a') \cdot s = (a \cdot s) \lhd (a' \cdot s)$$

for all $a, a' \in A$, $s \in S$.

3. Crossed Modules of Racks

Definition 3.1. A rack crossed module [4] is a rack morphism $\partial\colon A \to B$ together with a (right) rack action of B on A such that satisfying:

X1) $\partial(a \cdot b) = \partial(a) \lhd b$,

X2) $a \cdot \partial(a') = a \lhd a'$,

for all $a, a' \in A$ and $b \in B$. We denote any crossed module by (A, B, ∂).

If (A, B, ∂) and (A', B', ∂') are two rack crossed modules, a crossed module morphism:

$$(f_1, f_0) : (A, B, \partial) \to (A', B', \partial')$$

is a tuple which consists of rack morphisms $f_1 : A \to A'$, $f_0 : B \to B'$ such that making the following diagram commutative:

$$\begin{tikzcd} A \arrow[r, "\partial"] \arrow[d, "f_1"'] & B \arrow[d, "f_0"] \\ A' \arrow[r, "\partial'"'] & B' \end{tikzcd}$$

and:

$$f_1(a \cdot b) = f_1(a) \cdot f_0(b)$$

for all $a \in A$, $b \in B$.

Thus we get the category of rack crossed modules, denoted by **XRack**. We also have a full subcategory **XRack/B** where the codomain B is fixed for any rack crossed module.

Some examples of rack crossed modules are given below:

1) For any normal subrack N of S, the inclusion map $N \to S$ defines a rack crossed module with: $(n \in N, s \in S)$

$$n \cdot s = n \lhd s.$$

2) If (X, Y, μ) is a group crossed module, we obtain a rack crossed module by passing to the associated conjugation racks of X and Y as being:

$$\mathbf{Conj}(\mu)\colon \mathbf{Conj}(\mathbf{X}) \to \mathbf{Conj}(\mathbf{Y}).$$

3) If (X, Y, μ) and (X', Y', μ') are rack crossed modules, then:

$$(X \times X', Y \times Y', \mu \times \mu')$$

defines a rack crossed module where the action is defined in a natural way.

Definition 3.2. Let $\lambda\colon A \to C$ and $\theta\colon B \to C$ be two rack morphisms. The fiber product is the subrack of $A \times B$

defined by:

$$A \times_C B = \{(a,b) \mid \lambda(a) = \theta(b)\}.$$

From the categorical point of view, the fiber product is the equalizer of the two parallel rack morphisms:

$$A \times B \underset{\theta\circ\pi_2}{\overset{\lambda\circ\pi_1}{\rightrightarrows}} C\ .$$

Proposition 3.3. Let (A, C, λ) and (B, C, θ) be two rack crossed modules. The map:

$$\partial\colon A \times_C B \to C$$

defined by:

$$\partial(a,b) = \lambda(a) = \theta(b)$$

yields a crossed module $(A \times_C B, C, \partial)$ with the rack action:

$$\begin{array}{ccc} (A \times_C B) \times C & \to & A \times_C B \\ ((a,b), c) & \mapsto & (a,b)\cdot c = (a\cdot c, b\cdot c) \end{array}$$

Proposition 3.4. Let (A, C, μ) and (B, C, λ) be two rack crossed modules. Then we have the following natural rack crossed module morphisms:

$$\left(p_1, id_C\right) : (A \times_C B, C, \partial) \to \left(A, C, \mu\right),$$
$$\left(p_2, id_C\right) : (A \times_C B, C, \partial) \to (B, C, \lambda).$$

4. Some Categorical Constructions in XRack/C

Recall that, the trivial rack is the zero object in the category of racks. Moreover, for given two rack morphisms $f\colon A \to C$ and $g\colon B \to C$, we have the rack $A \times_C B$ which is the pullback. Therefore, we say that the category of racks **Rack** is finitely complete.

In this section, we give some categorical constructions for the case of rack crossed modules which will prove the completeness of **XRack/C**.

Theorem 4.1. The category **XRack/C** has products.

Proof.
Let (A, C, μ) and (B, C, λ) be two rack crossed modules. Define

$$\partial\colon A \times_C B \to C$$

where:

$$\partial(a,b) = \mu(a) = \lambda(b).$$

We already know from Proposition 3.3 that ∂ is a crossed module. Also we have crossed module morphisms $\left(p_1, id_C\right)$ and $\left(p_2, id_C\right)$ from Proposition 3.4. Now we need to check the universal property.

Let (P, C, α) be a crossed module with two crossed module morphisms:

$$\left(\epsilon, id_C\right) : (P, C, \alpha) \to \left(A, C, \mu\right),$$
$$\left(\delta, id_C\right) : (P, C, \alpha) \to (B, C, \lambda).$$

Then there must be a unique crossed module morphism:

$$\left(\phi, id_C\right) : (P, C, \alpha) \to (A \times_C B, C, \partial)$$

such that the diagram:

$$\begin{tikzcd}
 & (P, C, \alpha) \arrow[dl, "(\epsilon, id_C)"'] \arrow[d, dotted, "(\phi, id_C)"] \arrow[dr, "(\delta, id_C)"] & \\
\left(A, C, \mu\right) & (A \times_C B, C, \partial) \arrow[l, "(p_1, id_C)"] \arrow[r, "(p_2, id_C)"'] & (B, C, \lambda)
\end{tikzcd} \tag{4.1}$$

commutative. Define:

$$\phi\left(p\right) = \left(\epsilon\left(p\right), \delta\left(p\right)\right),$$

for all $p \in P$. By the diagram:

$$\begin{tikzcd}
P \arrow[r, "\alpha"] \arrow[d, "\phi"'] & C \arrow[d, "id_C"] \\
A \times_C B \arrow[r, "\partial"'] & C
\end{tikzcd}$$

$\left(\phi, id_C\right)$ becomes a crossed module morphism since:

$$\begin{aligned}
\phi\left(p \cdot c\right) &= \left(\epsilon\left(p \cdot c\right), \delta\left(p \cdot c\right)\right) \\
&= \left(\epsilon\left(p\right) \cdot id_C\left(c\right), \delta\left(p\right) \cdot id_C\left(c\right)\right) \\
&= \left(\epsilon\left(p\right), \delta\left(p\right)\right) \cdot id_C\left(c\right) \\
&= \phi\left(p\right) \cdot id_C\left(c\right),
\end{aligned}$$

and

$$\begin{aligned}
\partial\phi\left(p\right) &= \partial\left(\epsilon\left(p\right), \delta\left(p\right)\right) \\
&= \mu\left(\epsilon\left(p\right)\right) \\
&= id_C \alpha\left(p\right),
\end{aligned}$$

for all $p \in P$ and for all $c \in C$.

Furthermore diagram (4.1) is commutative because:

$$\begin{aligned} p_1\phi(p) &= p_1(\epsilon(p), \delta(p)) \\ &= \epsilon(p), \end{aligned}$$

$$\begin{aligned} p_2\phi(p) &= p_2(\epsilon(p), \delta(p)) \\ &= \delta(p), \end{aligned}$$

for all $p \in P$.

Let (ϕ', id_C) be a crossed module with:

$$\begin{aligned} (p_1, id_C)(\phi', id_C) &= (\epsilon, id_C), \\ (p_2, id_C)(\phi', id_C) &= (\delta, id_C). \end{aligned}$$

Define $(a,b) \in A \times_C B$ by $\phi'(p) = (a,b)$. Then we get:

$$\begin{aligned} p_1\phi'(p) = \epsilon(p) &\Leftrightarrow p_1(a,b) = \epsilon(p) \\ &\Leftrightarrow a = \epsilon(p), \\ p_2\phi'(p) = \delta(p) &\Leftrightarrow p_2(a,b) = \delta(p) \\ &\Leftrightarrow b = \delta(p) \end{aligned}$$

for all $p \in P$ that proves the uniqueness of ϕ by:

$$\phi'(p) = (a,b) = (\epsilon(p), \delta(p)) = \phi(p).$$

Theorem 4.2. The category **XRack/C** has pullbacks.

Proof.
Let $(f, id_C): (A, C, \mu) \to (D, C, \theta)$ and $(g, id_C): (B, C, \lambda) \to (D, C, \theta)$ be two crossed module morphisms. We already know from Proposition 3.3 that,

$$\partial: A \times_D B \to C$$

is a crossed module. Also we have crossed module morphisms (p_1, id_C) and (p_2, id_C) from Proposition 3.4. Then we get the diagram:

$$\begin{tikzcd} (A \times_D B, C, \partial) \arrow[r, "(p_2, id_C)"] \arrow[d, "(p_1, id_C)"'] & (B, C, \lambda) \arrow[d, "(g, id_C)"] \\ (A, C, \mu) \arrow[r, "(f, id_C)"'] & (D, C, \theta) \end{tikzcd}$$

which is commutative. Let (P,C,δ) be a crossed module with the following crossed module morphisms:

$$(\alpha, id_C) : (P,C,\delta) \to \left(A,C,\mu\right),$$
$$\left(\beta, id_C\right) : (P,C,\delta) \to (B,C,\lambda),$$

where:

$$\left(f, id_C\right)(\alpha, id_C) = \left(g, id_C\right)\left(\beta, id_C\right).$$

Then there must be a unique crossed module morphism:

$$\left(\phi, id_C\right) : (P,C,\delta) \to (A \times_D B, C, \partial)$$

such that the diagram:

$$\begin{tikzcd}
(P,C,\delta) \arrow[dr, dashed, "{(\phi,id_C)}"] \arrow[drr, bend left, "{(\beta,id_C)}"] \arrow[ddr, bend right, "{(\alpha,id_C)}"'] & & \\
& (A \times_D B, C, \partial) \arrow[r, "{(p_2,id_C)}"'] \arrow[d, "{(p_1,id_C)}"] & (B,C,\lambda) \arrow[d, "{(g,id_C)}"] \\
& \left(A,C,\mu\right) \arrow[r, "{(f,id_C)}"'] & (D,S,\theta).
\end{tikzcd} \tag{4.2}$$

commutes. Define

$$\phi(p) = (\alpha(p), \beta(p))$$

for all $p \in P$. By the diagram:

$$\begin{tikzcd}
P \arrow[r, "\delta"] \arrow[d, "\phi"'] & C \arrow[d, dash, "id_C"] \\
A \times_D C \arrow[r, "\partial"'] & C
\end{tikzcd}$$

(ϕ, id_C) becomes a crossed module morphism since:

$$\begin{aligned}
\phi\left(p \cdot c\right) &= \left(\alpha\left(p \cdot c\right), \beta\left(p \cdot c\right)\right) \\
&= \left(\alpha\left(p\right) \cdot id_C(c), \beta\left(p\right) \cdot id_C(c)\right) \\
&= (\alpha(p), \beta(p)) \cdot id_C(c) \\
&= \varphi(p) \cdot id_C(c),
\end{aligned}$$

and

$$\begin{aligned}\partial\phi(p) &= \partial(\alpha(p),\beta(p))\\ &= \mu\alpha\left(p\right)\\ &= id_C\delta\left(p\right),\end{aligned}$$

for all $p \in P$ and for all $c \in C$.

Furthermore we get:

$$\begin{aligned}p_1\phi\left(p\right) &= p_1\left(\alpha(p),\beta(p)\right)\\ &= \alpha(p),\end{aligned}$$

and

$$\begin{aligned}p_2\phi\left(p\right) &= p_2\left(\alpha(p),\beta(p)\right)\\ &= \beta(p),\end{aligned}$$

for all $p \in P$ that proves the commutativity of diagram (4.2).

Consider $\left(\phi', id_C\right)$ with the same property as $\left(\phi, id_C\right)$, i.e. the following conditions hold:

$$\begin{aligned}\left(p_1, id_C\right)\left(\phi', id_C\right) &= (\alpha, id_C),\\ \left(p_2, id_C\right)\left(\phi', id_C\right) &= \left(\beta, id_C\right).\end{aligned}$$

Define $(a,b) \in A \times_D B$ by $\phi'(p) = (a,b)$. We get:

$$\begin{aligned}p_1\phi'\left(p\right) = \alpha\left(p\right) &\Leftrightarrow p_1(a,b) = \alpha\left(p\right)\\ &\Leftrightarrow a = \alpha\left(p\right)\\ p_2\phi'\left(p\right) = \beta\left(p\right) &\Leftrightarrow p_2\left(a,b\right) = \beta\left(p\right)\\ &\Leftrightarrow b = \beta\left(p\right)\end{aligned}$$

for all $p \in P$ that proves the uniquness of ϕ by:

$$\begin{aligned}\phi'(p) &= (a,b)\\ &= \left(\alpha\left(p\right),\beta\left(p\right)\right)\\ &= \phi(p).\end{aligned}$$

Theorem 4.3. The category **XRack/C** has equalizers.

Proof.

Let we have two parallel crossed module morphisms:

$$\left(A,C,\mu\right)\underset{(g,id_C)}{\overset{(f,id_C)}{\rightrightarrows}}(B,C,\lambda).$$

Define:

$$P=\left\{a\in A\mid f(a)=g(a)\right\}.$$

P is a subrack of A, since:

$$\begin{aligned}f\left(a\lhd a'\right)&=f(a)\lhd f\left(a'\right)\\&=g(a)\lhd g\left(a'\right)\\&=g\left(a\lhd a'\right),\end{aligned}$$

for all $a,a'\in P$.

Define $\partial:P\to C$ by $\partial(a)=\mu(a)$, for all $a\in P$. Here ∂ becomes a rack morphism since:

$$\begin{aligned}\partial\left(a\lhd a'\right)&=\mu\left(a\lhd a'\right)\\&=\mu(a)\lhd\mu\left(a'\right)\\&=\partial(a)\lhd\partial\left(a'\right),\end{aligned}$$

for all $a,a'\in P$.

Moreover, (P,C,∂) is a rack crossed module since:

XM1)

$$\begin{aligned}\partial(a\cdot c)&=\mu(a\cdot c)\\&=\mu(a)\lhd c\\&=\partial(a)\lhd c,\end{aligned}$$

XM2)

$$\begin{aligned}a\cdot\partial\left(a'\right)&=a\cdot\mu\left(a'\right)\\&=a\lhd a',\end{aligned}$$

for all $a,a'\in P$ and $c\in C$.

The tuple:

$$(i,id_C):(P,C,\partial)\to\left(A,C,\mu\right)$$

where i is the inclusion map is a crossed module morphism since:

$$\begin{aligned} i(a\cdot c) &= a\cdot c \\ &= i(a)\cdot c \\ &= i(a)\cdot id_C(c), \end{aligned}$$

and

$$\begin{aligned} \mu i(a) &= \mu(a) \\ &= \partial(a) \\ &= id_C\partial(a), \end{aligned}$$

for all $a\in P$ and $c\in C$.

Furthermore, we get:

$$\begin{aligned} (fi)(a) &= f(i(a)) \\ &= f(a) \\ &= g(a) \\ &= g(i(a)) \\ &= (gi)(a), \end{aligned}$$

for all $a\in P$ that proves the commutativity of diagram:

$$(P,C,\partial)\xhookrightarrow{(i,id_C)}(A,C,\mu)\underset{(g,id_C)}{\overset{(f,id_C)}{\rightrightarrows}}(B,C,\lambda).$$

Let

$$(h,id_C):(Q,C,\delta)\to(A,C,\mu)$$

be any crossed module morphism such that the diagram:

$$(Q,C,\delta)\xrightarrow{(h,id_C)}(A,C,\mu)\underset{(g,id_C)}{\overset{(f,id_C)}{\rightrightarrows}}(B,C,\lambda)$$

commutes. Then there must be a unique crossed module morphism:

$$(\varphi,id_C):(Q,C,\delta)\to(P,C,\partial),$$

such that the diagram:

$$\begin{tikzcd}
(P,C,\partial) \arrow[r, hook, "(i,id_C)"] & \left(A,C,\mu\right) \arrow[r, shift left, "(f,id_C)"] \arrow[r, shift right, "(g,id_C)"'] & (B,C,\lambda) \\
(Q,C,\delta) \arrow[u, dotted, "(\varphi,id_C)"] \arrow[ur, "(h,id_C)"'] & &
\end{tikzcd} \tag{4.3}$$

commutes. We can say that $h\left(q\right) \in P$ since:

$$f\left(h\left(q\right)\right) = g\left(h\left(q\right)\right),$$

for all $q \in Q$. Define φ by $\varphi\left(q\right) = h\left(q\right)$ for all $q \in Q$. Then we get:

$$\begin{aligned} i\varphi(q) &= ih(q) \\ &= h(q), \end{aligned}$$

for all $q \in Q$ proves the commutativity of (4.3).

Consider $\left(\varphi', id_C\right)$ with the same property as $\left(\varphi, id_C\right)$, i.e. the following condition hold:

$$(i, id_C)\left(\varphi', id_C\right) = (h, id_C)$$

Define $q \in Q$ by $\varphi'\left(q\right) = a$. We get:

$$\begin{aligned} i\varphi'\left(q\right) = h\left(q\right) &\Leftrightarrow i\left(q\right) = h\left(q\right) \\ &\Leftrightarrow a = h\left(q\right) \end{aligned}$$

for all $q \in Q$ that proves the uniquness of φ by:

$$\begin{aligned} \varphi'\left(q\right) &= a \\ &= \varphi\left(q\right). \end{aligned}$$

Therefore, we have proved the following:

Theorem 4.4. The category of rack crossed modules **XRack/C** is (finitely) complete.

5. Conclusion

We already know that, we have the adjunction:

$$\mathrm{Hom}_{\mathrm{XGrp}}\left(\mathrm{As}^{\star}(\mathscr{X}), \mathscr{G}\right) \cong \mathrm{Hom}_{\mathrm{XRack}}\left(\mathscr{X}, \mathrm{Conj}^{\star}(\mathscr{G})\right),$$

between the category of rack crossed modules and the category of group crossed modules. As a result of this adjunction, we can say that the functor $\mathrm{Conj}^{\star}$ preserve limits and $\mathrm{As}^{\star}$ preserve colimits. Therefore, all the constructions given in the previous section which are the certain cases of limits are preserved under the functor Conj*.

For instance, let $\mathscr{A}$ and $\mathscr{B}$ be two group crossed modules with the same codomain. Their product is the crossed module $\mathscr{D}$ which is defined by using the fiber product of groups. By using the adjunction above, we can say that the product of rack crossed modules Conj$^\star(\mathscr{A})$ and Conj$^\star(\mathscr{B})$ is Conj$^\star(\mathscr{D})$. As we mentioned above, not only the product object, we can also give the similar properties for all of the notions we have defined.

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