# C O M M U N I C A T I O N S

FACULTY OF SCIENCES
UNIVERSITY OF ANKARA

DE LA FACULTE DES SCIENCES
DE L'UNIVERSITE D'ANKARA

**Series A1: Mathematics and Statistics**

**VOLUME: 73** **Number: 1** **YEAR: 2024**

Faculty of Sciences, Ankara University
06100 Beşevler, Ankara-Turkey

ISSN 1303-5991 e-ISSN 2618-6470

# C O M M U N I C A T I O N S

FACULTY OF SCIENCES
UNIVERSITY OF ANKARA

DE LA FACULTE DES SCIENCES
DE L'UNIVERSITE D'ANKARA

Series A1: Mathematics and Statistics

**Volume: 73** **Number: 1** **Year: 2024**





This Journal is published four issues in a year by the Faculty of Sciences, University of Ankara. Articles and any other material published in this journal represent the opinions of the author(s) and should not be construed to reflect the opinions of the Editor(s) and the Publisher(s).



**Correspondence Address:**
COMMUNICATIONS EDITORIAL OFFICE
Ankara University, Faculty of Sciences,
06100 Tandoğan, ANKARA – TURKEY
**Tel: (90) 312-2126720 Fax: (90) 312-2235000**
e-mail: commun@science.ankara.edu.tr
http://communications.science.ankara.edu.tr/index.php?series=A1



**Print:**
Ankara University Press
İncitaş Sokak No:10 06510 Beşevler
ANKARA – TURKEY
Tel: (90) 312-2136655

# C O M M U N I C A T I O N S

FACULTY OF SCIENCES
UNIVERSITY OF ANKARA

DE LA FACULTE DES SCIENCES
DE L'UNIVERSITE D'ANKARA

**Series A1: Mathematics and Statistics**

**VOLUME: 73** **Number: 1** **YEAR: 2024**

Faculty of Sciences, Ankara University
06100 Beşevler, Ankara-Turkey

ISSN 1303-5991 e-ISSN 2618-6470

# C O M M U N I C A T I O N S

FACULTY OF SCIENCES
UNIVERSITY OF ANKARA

DE LA FACULTE DES SCIENCES DE
L'UNIVERSITE D'ANKARA

Series A1: Mathematics and Statistics

**Volume: 73** **Number: 1** **Year: 2024**


## Research Articles


Bilender ALLAHVERDİEV, Hüseyin TUNA, Fractional Dirac systems with Mittag-Leffler kernel...... 1
Merve ÖZKAN, Murat ÖZDEMİR, İsa YİLDİRİM, On θ-convex contractive mappings with application to integral equations...... 13
Jiarul HOQUE, Shyamapada MODAK, A stronger form of locally closed set and its homeomorphic image...... 25
Sure KÖME, Zeynep KUMTAS, Generalized bivariate conditional Fibonacci and Lucas hybrinomials...... 37
Ferhat TAŞ, Trajectory curves and surfaces: A new perspective via projective geometric algebra...... 64
Yasin ALTİNİSİK, Emel ÇANKAYA, Exponentiated generalized Ramos-Louzada distribution with properties and applications...... 76
Gutti V. R. BABU, Madugula VINOD KUMAR, PPF Dependent common fixed points of generalized weakly contractive type multi-valued mappings...... 104
Pranay GOSWAMI, Murli Manohar GOUR, Som Prakash GOYAL, Lokesh Kumar YADAV, Majorization property for certain classes of analytic functions associated with general operator...... 122
Semih YILMAZ, Betül ERDOĞAN, Cholesky algorithm of a Lucas type matrix...... 131
Zeynep YAPTI ÖZKURT, Normal automorphisms of free metabelian Leibniz algebras...... 147
Rüya ÜSTER, Affine mappings and multipliers for weighted Orlicz spaces over an affine group $R_+ \times R$...... 153
Aykut HAS, Beyhan YILMAZ, Translation surfaces generating with some partner curve...... 165
Busra AL, Mustafa ALKAN, Compositions of integers and Fibonacci numbers...... 178
Yogeeta NARWAL, Savita RATHEE, Fractional order mathematical modeling of lumpy skin disease...... 192
Meltem ALTUN ÖZARSLAN, Semiregular, semiperfect and semipotent matrix rings relative to an ideal...... 211
Nour El Houda NESBA, Mohammed BEGGAS, Multigrid methods for non coercive variational inequalities...... 222
Joseph Thomas EGHWERIDO, Lawrence NZEI, A weighted Gompertz-G family of distributions for reliability and lifetime data analysis...... 235
Ahu ERCAN, Fractional approach for Dirac operator involving M-truncated derivative...... 259
Şeyma TÜLÜCE DEMİRAY, Uğur BAYRAKCI, Vehpi YILDIRIM, Application of the GKM to some nonlinear partial equations...... 274
Esra KARAOĞLU, On the stability analysis of a fractional order epidemic model including the general forms of nonlinear incidence and treatment function...... 285

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 1–12 (2024)
DOI:10.31801/cfsuasmas.1298907
ISSN 1303-5991 E-ISSN 2618-6470

*Research Article; Received: May 18, 2023; Accepted: September 25, 2023*

# FRACTIONAL DIRAC SYSTEMS WITH MITTAG–LEFFLER KERNEL

Bilender P. ALLAHVERDİEV[1] and Hüseyin TUNA[2]

[1]Department of Mathematics, Khazar University, AZ1096 Baku, AZERBAIJAN
[2]Department of Mathematics, Mehmet Akif Ersoy University, 15030 Burdur, TÜRKİYE
[1,2]Research Center of Econophysics, UNEC-Azerbaijan State University of Economics, Baku, AZERBAIJAN

Abstract. In this paper, we study some fractional Dirac-type systems with the Mittag–Leffler kernel. We extend the basic spectral properties of the ordinary Dirac system to the Dirac-type systems with the Mittag–Leffler kernel. First, this problem was handled in a continuous form. The self-adjointness of the operator produced by this system, the reality of its eigenvalues, and the orthogonality of the eigenfunctions have been investigated. Later, similar results were obtained by considering the discrete state.

## 1. Introduction

In recent years, the subject of fractional differential equations has become very popular among mathematicians. The investigation of all kinds of problems in the theory of differential equations under the framework of fractional has revealed a very wide field of study. The Dirac equation, which is one of the important equations in the history of physics, should also be investigated. Although fractional Sturm–Liouville problems have been investigated a lot, research on fractional Dirac equivalents is less. Contributing to the gap in this area in the literature is the main motivation of this research.

There are many types of fractional derivatives. One of them is the one based on the Mittag–Leffler function. Atangana and Baleanu introduced a new fractional derivative with the Mittag–Leffler kernel [4]. In [1], Abdeljawad and Belanau defined integration with the part formula using the right fractional derivative and the right fractional integral corresponding to the Mittag–Leffler kernel. In [5], the

2020 *Mathematics Subject Classification.* 34A08, 26A33, 34L40.
*Keywords.* Fractional differential equations, Mittag–Leffler kernel, Dirac systems.
[1] bilenderpasaoglu@gmail.com; 0000-0002-9315-4652
[2] hustuna@gmail.com-Corresponding author; 0000-0001-7240-8687.

authors studied the discrete versions of these fractional derivatives. In [12], Mert et al. studied fractional Sturm–Liouville operators with the Mittag–Leffler kernels. With the help of the Laplace transform, Ercan is obtained the representation of solutions for fractional Dirac system with the Mittag–Leffler kernel ( [8]). Yalçınkaya handled some Dirac systems with exponential kernel in [13]. In [7], the authors studied a fractional Sturm–Liouville problem with exponential and Mittag-Leffler kernels.

In this study, we will investigate this type of fractional version of the Dirac system. Some basic features will be obtained for such systems. In the first chapter, the basic concepts and theorems that will be used in the study are given. In the following sections, the Dirac system with the Mittag–Leffler kernel in a continuous and discrete cases is discussed. This type of fractional Dirac system turns into the classical Dirac system by taking $\alpha \to 1$. It is transformed into a Riemann-Liouville type fractional Dirac system with a Laplace transform method. In this way, we examine these two systems under a single system. According to the knowledge of the authors, since there is no study on this subject in the literature, it will contribute to researchers working on this subject.

## 2. Preliminaries

This section covers the definitions and properties of fractional derivatives with the Mittag–Leffler kernel.

**Definition 1.** *( [1]) Let $u \in H^1(a,b)$ ( the usual Sobolev space ), $a < b,\ \alpha \in [0,1]$. Then the definition of the left Caputo fractional derivative with the Mittag–Leffler kernel is given by*

$$ {}_{a}^{ABC}D^{\alpha}u\left(\xi\right) = \frac{B(\alpha)}{1-\alpha}\int_a^{\xi} E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(\xi-t)^{\alpha}\right) d\left(u\left(t\right)\right), \tag{1}$$

*where $B(\alpha) > 0$ is a normalization function with $B(0) = B(1) = 1$;*

$$E_{\alpha,\beta}\left(t\right) = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma\left(\alpha k+\beta\right)}, \tag{2}$$

*and $E_{\alpha}\left(t\right) = E_{\alpha,1}\left(t\right)$. The convergence condition of infinite series (2) is $\operatorname{Re}\alpha > 0$ and $\operatorname{Re}\beta > 0$ ( [9]). Similarly, the left Riemann–Liouville fractional derivative with the Mittag–Leffler kernel has the following form*

$$ {}_{a}^{ABR}D^{\alpha}u\left(\xi\right) = \frac{B(\alpha)}{1-\alpha}\frac{d}{d\xi}\int_a^{\xi} E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(\xi-t)^{\alpha}\right) u(t)dt. \tag{3}$$

*The associated fractional integral is given by*

$$ {}_{a}^{AB}I^{\alpha}u\left(\xi\right) = \frac{1-\alpha}{B(\alpha)}u(\xi) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^{\xi}\left(\xi-t\right)^{\alpha-1}u(t)dt.$$

*The right Caputo fractional derivative with the Mittag–Leffler kernel is given by*

$$ {}^{ABC}D_b^{\alpha} u(\xi) = -\frac{B(\alpha)}{1-\alpha} \int_{\xi}^{b} E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(t-\xi)^{\alpha}\right) u(t)dt, \tag{4} $$

*and the right Riemann–Liouville derivative with the Mittag–Leffler kernel is defined by the formula*

$$ {}^{ABR}D_b^{\alpha} u(\xi) = -\frac{B(\alpha)}{1-\alpha}\frac{d}{d\xi} \int_{\xi}^{b} E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(t-\xi)^{\alpha}\right) u(t)dt. $$

*Moreover, the corresponding fractional integral is given by*

$$ {}^{AB}I_b^{\alpha} u(\xi) = \frac{1-\alpha}{B(\alpha)} u(\xi) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{\xi}^{b} (t-\xi)^{\alpha-1} u(t)dt. $$

**Proposition 1.** *( [1]) Let $\alpha > 0,\ p \geq 1,\ q \geq 1,$ and $\frac{1}{p}+\frac{1}{q} \leq 1+\alpha$ ($p \neq 1$ and $q \neq 1$ when $\frac{1}{p}+\frac{1}{q} = 1+\alpha$).*

(1) If $u \in L_p(a.b)$ and $v \in L_q(a.b)$, then

$$ \int_a^b u(\xi) {}_a^{AB}I^{\alpha} v(\xi) d\xi = \int_a^b v(\xi) {}^{AB}I_b^{\alpha} u(\xi) d\xi. $$

(2) If $u \in {}^{AB} I_b^{\alpha}(L_p)$ and $v \in {}_a^{AB} I^{\alpha}(L_q)$, then

$$ \int_a^b u(\xi) {}_a^{ABR}D^{\alpha} v(\xi) d\xi = \int_a^b v(\xi) {}^{ABR}D_b^{\alpha} f(\xi) d\xi, $$

where

$$ {}^{AB}I_b^{\alpha}(L_p) = \left\{u : u = {}^{AB} I_b^{\alpha} v, v \in L_p(a.b)\right\}, $$

and

$$ {}_a^{AB}I^{\alpha}(L_q) = \left\{u : u = {}_a^{AB} I^{\alpha} v, v \in L_q(a.b)\right\}. $$

**Theorem 1.** *( [1]) Let $u, v \in H^1(a,b),\ a < b$ and $\alpha \in (0,1)$. Then we have*

(1)

$$ \int_a^b u(\xi) {}_a^{ABC}D^{\alpha} v(\xi) d\xi = \int_a^b v(\xi) {}^{ABR}D_b^{\alpha} u(\xi) d\xi $$

$$ + \frac{B(\alpha)}{1-\alpha} v(\xi) E_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-} u(\xi)\Big|_a^b, $$

where

$$ E_{\alpha,\beta,w,b^-} u(\xi) = \int_{\xi}^{b} (t-\xi)^{\beta-1} E_{\alpha,\beta}\left(w(t-\xi)^{\alpha}\right) u(t)\, dt,\ \xi < b. $$

2.
$$\int_a^b u(\xi)^{ABC}D_b^{\alpha}v(\xi)d\xi = \int_a^b v(\xi)_a^{ABR}D^{\alpha}u(\xi)d\xi$$

$$-\frac{B(\alpha)}{1-\alpha}v(b)E_{\alpha,1,\frac{-\alpha}{1-\alpha},a^+}u(b)$$

$$+\frac{B(\alpha)}{1-\alpha}v(a)E_{\alpha,1,\frac{-\alpha}{1-\alpha},a^+}u(a),$$

where
$$E_{\alpha,\beta,w,a^+}u(\xi) = \int_a^{\xi}(\xi-t)^{\beta-1}E_{\alpha,\beta}\left(w(\xi-t)^{\alpha}\right)u(t)\,dt,\ \xi > a.$$

Let
$$\mathbb{N}_a = \{a, a+1, a+2, ...\},$$

$$_b\mathbb{N} = \{..., b-2, b-1, b\},$$

$$\mathbb{N}_{a,b} = \{a, a+1, a+2, ..., b\},$$

where $a, b \in \mathbb{R}$ and $b-a$ is a positive integer.

**Definition 2.** *( [5, 6, 10]) Let $u : \mathbb{N}_a \to \mathbb{R}$ and $\alpha \in (0, 1/2)$. Then the nabla discrete left Caputo difference with the Mittag–Leffler kernel is defined by*

$$_a^{ABC}\nabla^{\alpha}u(\xi) = \frac{B(\alpha)}{1-\alpha}\sum_{i=a+1}^{\xi}\nabla_i u(i)\,E_{\overline{\alpha}}(\frac{-\alpha}{1-\alpha}, \xi-\rho(i)),\ \xi \in \mathbb{N}_{a+1},$$

*and the left Riemannn–Liouville one by*

$$_a^{ABR}\nabla^{\alpha}u(\xi) = \frac{B(\alpha)}{1-\alpha}\nabla_{\xi}\sum_{i=a+1}^{\xi}u(i)\,E_{\overline{\alpha}}(\frac{-\alpha}{1-\alpha}, \xi-\rho(i)),\ \xi \in \mathbb{N}_{a+1},$$

*where $\rho(i) = i-1$; and the discrete the Mittag–Leffler kernel is defined by the formula*

$$E_{\overline{\alpha}}(\lambda, z) = \sum_{i=0}^{\infty}\lambda^i\frac{z^{\overline{i\alpha}}}{\Gamma(i\alpha+1)},$$

*where $z^{\overline{i\alpha}} = \prod_{i=0}^{i\alpha-1}(t+i)$, $z^{\overline{0}} = 1$, $t \in \mathbb{R}$. Moreover, the associated fractional sum function*

$$_a^{AB}\nabla^{-\alpha}u(\xi) = \frac{1-\alpha}{B(\alpha)}u(\xi) + \frac{\alpha}{B(\alpha)}\nabla_a^{-\alpha}u(\xi),\ \xi \in \mathbb{N}_{a+1},$$

*where*

$$\nabla_a^{-\alpha}u(\xi)=\frac{1}{\Gamma(\alpha)}\sum_{i=a+1}^{\xi}(\xi-\rho(i))^{\overline{\alpha-1}}u(i),\ \xi\in\mathbb{N}_{a+1}\ \textit{(see [2,3])}.$$

**Definition 3.** *( [5]) Let $u :_b \mathbb{N}\to\mathbb{R}$ and $\alpha\in(0,1/2)$. Then the nabla discrete right Caputo difference with the Mittag–Leffler kernel is defined by*

$$^{ABC}\nabla_b^{\alpha}u(\xi)=\frac{-B(\alpha)}{1-\alpha}\sum_{i=\xi}^{b-1}\Delta u\,(i)\,E_{\overline{\alpha}}(\frac{-\alpha}{1-\alpha},i-\rho(\xi)),\ \xi\in_{b-1}\mathbb{N},$$

*and the right Reimann–Liouville one by*

$$^{ABR}\nabla_b^{\alpha}u(\xi)=\frac{-B(\alpha)}{1-\alpha}\Delta_\xi\sum_{i=\xi}^{b-1}u\,(i)\,E_{\overline{\alpha}}(\frac{-\alpha}{1-\alpha},i-\rho(\xi)),\ \xi\in_{b-1}\mathbb{N}.$$

*Further, the associated fractional sum is defined by*

$$^{AB}\nabla_b^{-\alpha}u(\xi)=\frac{1-\alpha}{B(\alpha)}u(\xi)+\frac{\alpha}{B(\alpha)}\nabla_b^{-\alpha}u(\xi),\ \xi\in_{b-1}\mathbb{N},$$

*where*

$$\nabla_b^{-\alpha}u(\xi)=\frac{1}{\Gamma(\alpha)}\sum_{i=\xi}^{b-1}(i-\rho(\xi))^{\overline{\alpha-1}}u(i),\ \xi\in_{b-1}\mathbb{N}\ \textit{( see [2,3])}.$$

**Theorem 2.** *( [5]) Let $u,v:\mathbb{N}_{a,b}\to\mathbb{R}$ and $\alpha\in(0,1/2)$. Then we have*

$$\sum_{\xi=a+1}^{b-1}v(\xi)_a^{AB}\nabla^{-\alpha}u(\xi)=\sum_{\xi=a+1}^{b-1}u(\xi)^{AB}\nabla_b^{-\alpha}v(\xi),$$

$$\sum_{\xi=a+1}^{b-1}v(\xi)_a^{ABR}\nabla^{\alpha}u(\xi)=\sum_{\xi=a+1}^{b-1}u(\xi)^{ABR}\nabla_b^{\alpha}v(\xi),$$

*and*

$$\begin{aligned}\sum_{\xi=a+1}^{b-1}u(\xi)_a^{ABC}\nabla^{\alpha}v(\xi)&=\sum_{\xi=a+1}^{b-1}v(\xi-1)^{ABR}\nabla_b^{\alpha}u(\xi-1)\\&+v(b-1)\frac{B(\alpha)}{1-\alpha}E^{1}_{\overline{\alpha,1},\frac{-\alpha}{1-\alpha},b^-}u(b-1)\\&-v(a)\frac{B(\alpha)}{1-\alpha}E^{1}_{\overline{\alpha,1},\frac{-\alpha}{1-\alpha},b^-}u(a),\end{aligned}$$

*where*

$$E^{1}_{\overline{\rho,\mu},w,b^{-}}u(\xi)=\sum_{a=\xi}^{b-1}\left(a-\rho\left(\xi\right)\right)^{\overline{\mu-1}}E_{\overline{\rho,\mu}}\left(w,a-\rho\left(\xi\right)\right)u(\xi),\ \xi\in_{b}\mathbb{N}.$$

## 3. The Continuous Case

Let us consider the below continuous fractional Dirac system

$$Lu:=Bu+Qu=\lambda u,\ a\leq x\leq b<\infty, \tag{5}$$

where

$$Q=\begin{pmatrix} p & 0 \\ 0 & r \end{pmatrix},\ u:=\begin{pmatrix} u_1 \\ u_2 \end{pmatrix},\ B=\begin{pmatrix} 0 & {}^{ABC}_{a}D^{\alpha} \\ {}^{ABR}D^{\alpha}_{b} & 0 \end{pmatrix},$$

$\alpha\in(0,1)$, $\lambda\in\mathbb{C}$; $p,r\in C\left[a,b\right]$; $p\left(x\right)>0,\ r\left(x\right)>0,\ \forall x\in\left[a,b\right]$. We also consider the following boundary conditions

$$\varkappa_{11}E^{1}_{\overline{\alpha,1},\frac{-\alpha}{1-\alpha},b^{-}}u_1\left(a\right)+\varkappa_{12}u_2\left(a\right)=0, \tag{6}$$

$$\varkappa_{21}E^{1}_{\overline{\alpha,1},\frac{-\alpha}{1-\alpha},b^{-}}u_1\left(b\right)+\varkappa_{22}u_2\left(b\right)=0, \tag{7}$$

with $\varkappa_{11}^2+\varkappa_{12}^2\neq 0$ and $\varkappa_{21}^2+\varkappa_{22}^2\neq 0$.

Now let's define the inner product suitable for this system. Let $L^2((a,b);\mathbb{R}^2)$ denotes the Hilbert space with the following inner product

$$(u,v):=\int_a^b u_1v_1dx+\int_a^b u_2v_2dx, \tag{8}$$

where

$$u=\begin{pmatrix} u_1 \\ u_2 \end{pmatrix},\ v=\begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

$u_i$ and $v_i$ $(i=1,2)$ are real-valued continuous functions defined on $\left[a,b\right]$.

**Theorem 3.** *The operator $L$ defined by (5)-(7) is formally self-adjoint on $L^2\left((a,b);\mathbb{R}^2\right)$.*

*Proof.* Using (8), we get

$$\begin{aligned}(Lu,v)-(u,Lv)&=\int_a^b\left({}^{ABC}_{a}D^{\alpha}u_2+p\left(x\right)u_1\right)v_1dx\\&+\int_a^b\left({}^{ABR}D^{\alpha}_{b}u_1+r\left(x\right)u_2\right)v_2dx\\&-\int_a^b u_1\left({}^{ABC}_{a}D^{\alpha}v_2+p\left(x\right)v_1\right)dx\end{aligned}$$

$$-\int_a^b u_2\left({}^{ABR}D_b^{\alpha}v_1+r\left(x\right)v_2\right)dx$$

$$=\int_a^b\left({}_a^{ABC}D^{\alpha}u_2v_1\right)dx+\int_a^b {}^{ABR}D_b^{\alpha}u_1v_2dx$$

$$-\int_a^b u_1({}_a^{ABC}D^{\alpha}v_2)dx-\int_a^b u_2\left({}^{ABR}D_b^{\alpha}v_1\right)dx,$$

where $u,v\in L^2\left((a,b)\,;\mathbb{R}^2\right)$. From Proposition 1 and Theorem 1, we obtain

$$(Lu,v)-(u,Lv)=[u,v]_b-[u,v]_a \tag{9}$$

where

$$[u,v]_x=v_2(x)\frac{B(\alpha)}{1-\alpha}E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}u_1(x)-u_2(x)\frac{B(\alpha)}{1-\alpha}E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}v_1(x).$$

By conditions (6)-(7), we get the desired result. □

**Corollary 1.** *The eigenvalues of Eq. (5) subject to the boundary conditions (6)-(7) are real. The eigenfunctions corresponding to different eigenvalues of the system (5)-(7) are orthogonal.*

Let us define the Wronskian of $u$ and $v$ by

$$W\left(u,v\right)(x)=\left(\frac{B(\alpha)}{1-\alpha}E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}u_1\left(x\right)\right)v_2\left(x\right)-\left(\frac{B(\alpha)}{1-\alpha}E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}v_1\left(x\right)\right)u_2\left(x\right),$$

where

$$u=\begin{pmatrix}u_1\\u_2\end{pmatrix},\ v=\begin{pmatrix}v_1\\v_2\end{pmatrix}\in L^2((a,b);\mathbb{R}^2).$$

**Theorem 4.** *Let $v_1$ and $v_2$ be two solutions of Eq. (5). Then $W(v_1,v_2)$ is independent of $x$.*

*Proof.* By (9), we obtain

$$(\lambda v_1,v_2)-(v_1,\lambda v_2)=[v_1,v_2]_b-[v_1,v_2]_a\,,$$

since $Lv_1=\lambda v_1$ and $Lv_2=\lambda v_2$. Hence

$$[v_1,v_2]_b=[v_1,v_2]_a=W(v_1,v_2)(a).$$

□

**Theorem 5.** *Any two solutions of the Eq. (5) are linearly dependent if and only if their Wronskian is zero.*

*Proof.* Assume $v_1$ and $v_2$ be two linearly dependent solutions of Eq. (5). Then there exists a constant $\eta > 0$ such that $v_1 = \eta v_2$. Hence

$$W(v_1, v_2)(x) = \begin{vmatrix} v_{11}(x) & \frac{B(\alpha)}{1-\alpha} E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-} v_{12}(x) \\ v_{21}(x) & \frac{B(\alpha)}{1-\alpha} E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-} v_{22}(x) \end{vmatrix}$$

$$= \begin{vmatrix} \eta v_{21}(x) & \eta \frac{B(\alpha)}{1-\alpha} E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-} v_{22}(x) \\ v_{21}(x) & \frac{B(\alpha)}{1-\alpha} E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-} v_{22}(x) \end{vmatrix} = 0.$$

On the other hand, if the Wronskian $W(v_1, v_2)(x)$ is zero for some $x$ in $[a, b]$, then we obtain

$$v_1 = \eta v_2$$

i.e., $v_1$ and $v_2$ are linearly dependent on $[a, b]$. □

Let us now give an example to illustrate our results.

**Example 1.** *If we take $\alpha \to 1^-$ in (5), we obtain the ordinary Dirac system ( [11]) defined as*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{du}{dx} + Qu = \lambda u, \ \ a \leq x \leq b < \infty,$$

*where*

$$Q = \begin{pmatrix} p & 0 \\ 0 & r \end{pmatrix} \text{ and } u := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

*In fact, for $\alpha \in (0, 1]$, the ABR and ABC fractional operators become well-defined due to the Mittag-Leffler kernel (2) doesn't have a convergence problem.*

## 4. The Discrete Case

Let us consider the nabla discrete fractional Dirac systems

$$L_1 u = Cu + Qu = \lambda u, \ \ x \in \mathbb{N}_{a,b-1}, \tag{10}$$

$$Q = \begin{pmatrix} p & 0 \\ 0 & r \end{pmatrix}, \ u := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \ C = \begin{pmatrix} 0 & {}^{ABC}_{\ a}\nabla^{\alpha} \\ {}^{ABR}\nabla^{\alpha}_{b} & 0 \end{pmatrix},$$

where $\alpha \in (0, 1/2)$, $\lambda \in \mathbb{C}$; $p$ and $r$ are real-valued functions on $\mathbb{N}_{a,b-1}$; $p(x) > 0, r(x) > 0, \ \forall x \in \mathbb{N}_{a,b-1}$. We consider the following conditions

$$\varkappa_{11} \left( {}^{ABR}\nabla^{\alpha}_{b} - \frac{B(\alpha)}{1-\alpha} E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-} \right) u_1(a) + \varkappa_{12} u_2(a) = 0, \tag{11}$$

$$\varkappa_{21} \left( {}^{ABR}\nabla^{\alpha}_{b} - \frac{B(\alpha)}{1-\alpha} E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-} \right) u_1(b-1) + \varkappa_{22} u_2(b-1) = 0, \tag{12}$$

where $\varkappa_{11}^2 + \varkappa_{12}^2 \neq 0$ and $\varkappa_{21}^2 + \varkappa_{22}^2 \neq 0$.

Let $L^2_{\nabla}(\mathbb{N}_{a,b-1};\mathbb{R}^2)$ denotes the Hilbert space with the following inner product

$$\langle u,v\rangle := \sum_{x=a+1}^{b-1} u_1(x)v_1(x) + \sum_{x=a+1}^{b-1} u_2(x)v_2(x),$$

where

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},\ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

$u_i$ and $v_i$ $(i=1,2)$ are real-valued functions defined on $\mathbb{N}_{a,b-1}$.

**Theorem 6.** *The operator $L_1$ defined by (10)-(12) is formally self-adjoint on $L^2_{\nabla}(\mathbb{N}_{a,b-1};\mathbb{R}^2)$.*

*Proof.* Let $u,v\in L^2_{\nabla}(\mathbb{N}_{a,b-1};\mathbb{R}^2)$. Then we see that

$$\begin{aligned}
\langle L_1u,v\rangle - \langle u,L_1v\rangle &= \sum_{x=a+1}^{b-1} \left({}^{ABC}_{a}\nabla^{\alpha}u_2 + p\left(x\right)u_1\right)v_1 + \sum_{x=a+1}^{b-1}\left({}^{ABR}\nabla^{\alpha}_{b}u_1 + r\left(x\right)u_2\right)v_2 \\
&- \sum_{x=a+1}^{b-1} u_1\left({}^{ABC}_{a}\nabla^{\alpha}v_2 + p\left(x\right)v_1\right) - \sum_{x=a+1}^{b-1} u_2\left({}^{ABR}\nabla^{\alpha}_{b}\ v_1 + r\left(x\right)v_2\right) \\
&= \sum_{x=a+1}^{b-1} {}^{ABC}_{a}\nabla^{\alpha}u_2v_1 + \sum_{x=a+1}^{b-1} p(x)u_1(x)v_1(x) + \sum_{x=a+1}^{b-1} {}^{ABR}\nabla^{\alpha}_{b}\ u_1v_2 \\
&+ \sum_{x=a+1}^{b-1} r(x)u_2(x)v_2(x) - \sum_{x=a+1}^{b-1} u_1({}^{ABC}_{a}\nabla^{\alpha}v_2) - \sum_{x=a+1}^{b-1} p(x)u_1(x)v_1(x) \\
&- \sum_{x=a+1}^{b-1} u_2\left({}^{ABR}\nabla^{\alpha}_{b}\ v_1\right) - \sum_{x=a+1}^{b-1} r(x)u_2(x)v_2(x) \\
&= \sum_{x=a+1}^{b-1} {}^{ABC}_{a}\nabla^{\alpha}u_2\ v_1 + \sum_{x=a+1}^{b-1} {}^{ABR}\nabla^{\alpha}_{b}\ u_1v_2 \\
&- \sum_{x=a+1}^{b-1} u_1({}^{ABC}_{a}\nabla^{\alpha}v_2) - \sum_{x=a+1}^{b-1} u_2\left({}^{ABR}\nabla^{\alpha}_{b}\ v_1\right) \\
&= \sum_{x=a+1}^{b-1} u_2\left(x-1\right)^{ABR}\nabla^{\alpha}_{b}\ v_1\left(x-1\right) + \frac{B(\alpha)}{1-\alpha}u_2(x)E^{1}_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}v_1(x)\Big|_a^{b-1} \\
&+ \sum_{x=a+1}^{b-1}\left({}^{ABR}\nabla^{\alpha}_{b}u_1\right)v_2 - \sum_{x=a+1}^{b-1} v_2\left(x-1\right)^{CFR}\nabla^{\alpha}_{b}u_1\left(x-1\right) \\
&- \frac{B(\alpha)}{1-\alpha}v_2(x)E^{1}_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}u_1(x)\Big|_a^{b-1} - \sum_{x=a+1}^{b-1} u_2\left({}^{ABR}\nabla^{\alpha}_{b}\ v_1\right)
\end{aligned}$$

$$= u_2(a)^{ABR}\nabla_b^\alpha\; v_1(a) - u_2(b-1)^{ABR}\nabla_b^\alpha\; v_1(b-1)$$

$$+ \frac{B(\alpha)}{1-\alpha} u_2(b-1) E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-} v_1(b-1) - \frac{B(\alpha)}{1-\alpha} u_2(a) E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-} v_1(a)$$

$$+ v_2(b-1)^{ABR}\nabla_b^\alpha\; u_1(b-1) - v_2(a)^{ABR}\nabla_b^\alpha\; u_1(a)$$

$$+ \frac{B(\alpha)}{1-\alpha} v_2(a) E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-} u_1(a) - \frac{B(\alpha)}{1-\alpha} v_2(b-1) E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-} u_1(b-1)$$

$$= z_2(b-1)\left({}^{CFR}\nabla_b^\alpha - \frac{B(\alpha)}{1-\alpha} E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}\right) y_1(b-1)$$

$$- y_2(b-1)\left({}^{CFR}\nabla_b^\alpha - \frac{B(\alpha)}{1-\alpha} E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}\right) z_1(b-1)$$

$$- \left[\begin{array}{c} v_2(a)\left({}^{ABR}\nabla_b^\alpha - \frac{B(\alpha)}{1-\alpha} E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}\right) u_1(a) \\ -u_2(a)\left({}^{ABR}\nabla_b^\alpha - \frac{B(\alpha)}{1-\alpha} E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}\right) v_1\;(a) \end{array}\right].$$

It follows from (11) and (12) that

$$\langle L_1 u, v\rangle - \langle u, L_1 v\rangle = 0.$$

□

**Corollary 2.** *All eigenvalues of the problem (10)-(12) are real. Eigenfunctions corresponding to different eigenvalues are orthogonal.*

**Theorem 7.** *Let*

$$W(u,v)(x) = \left|\begin{array}{cc} \left({}^{ABR}\nabla_b^\alpha \;-\frac{B(\alpha)}{1-\alpha} E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}\right) u_1(x) & u_2(x) \\ \left({}^{ABR}\nabla_b^\alpha \;-\frac{B(\alpha)}{1-\alpha} E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-}\right) v_1(x) & v_2(x) \end{array}\right|,$$

*where*

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},\; v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in L^2_\nabla(\mathbb{N}_{a,b-1};\mathbb{R}^2)$$

*and let $\theta_1$ and $\theta_2$ be two solutions of Eq. (5). Then $W(\theta_1,\theta_2)$ is independent of $x$. Moreover, any two linearly independent solutions $\varphi_1,\varphi_2$ of Eq. (10) are linearly dependent if and only if $W(\varphi_1,\varphi_2) = 0$.*

*Proof.* The proof is as in Theorem 4 and Theorem 5. □

## 5. Conclusion

In this work, we have considered some fractional Dirac systems with Mittag–Leffler kernel. Firstly, a continuous fractional Dirac system with Mittag–Leffler kernel is studied. Its spectral properties are investigated. Later, the nabla discrete fractional Dirac system with Mittag–Leffler kernel is constructed. Similar properties are studied. Since Dirac systems have an important place in quantum physics, the properties of such systems are studied intensively. In this context, investigating fractional Dirac systems with Mittag–Leffler kernel will contribute to researchers working in this field. In the future, Green's function can be created for this system and eigenfunction expansions can be investigated.

**Author Contribution Statements** All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

**Declaration of Competing Interests** This work does not have any conflict of interest.

## References

[1] Abdeljawad, T., Baleanu, D., Integration by parts and its applications of a new non-local fractional derivative with Mittag–Leffler nonsingular kernel, *J. Nonlinear Sci. Appl.*, 10(3) (2017), 1098-1107. http://dx.doi.org/10.22436/jnsa.010.03.20

[2] Abdeljawad, T., On Delta and Nabla Caputo fractional differences and dual identities, *Discrete Dyn. Nat. Soc.*, 2013 (2013), Art. ID 406910, 1-12. https://doi.org/10.1155/2013/406910

[3] Abdeljawad, T., Dual identities in fractional difference calculus within Riemann, *Adv. Differ. Equ.*, 2013 (2013), 1-16. https://doi.org/10.1186/1687-1847-2013-36

[4] Atangana, A., Baleanu, D., New fractional derivatives with non-local and non-singular kernel: Theory and application to heat transfer model, *Therm. Sci.*, 20 (2016), 763-769. https://doi.org/10.2298/TSCI160111018A

[5] Abdeljawad, T., Baleanu, D., Discrete fractional differences with nonsingular discrete Mittag–Leffler kernels, *Adv. Differ. Equ.*, 2016(232) (2016), 1-18. https://doi.org/10.1186/s13662-016-0949-5

[6] Atici, F. M., Eloe, P. W., Discrete fractional calculus with the nabla operator, *Electron. J. Qual. Theory Differ. Equ.*, Spec. Ed. I , 2009 (2009), 1-12.

[7] Bas, E., Ozarslan, R., Baleanu, D., Ercan, A., Comparative simulations for solutions of fractional Sturm–Liouville problems with non-singular operators, *Adv. Diff. Equ.*, 2018(350) (2018). https://doi.org/10.1186/s13662-018-1803-8

[8] Ercan, A., On the fractional Dirac systems with non-singular operators, *Thermal science*, 23(6) 2019, 2159-2168. https://doi.org/10.2298/TSCI190810405E

[9] Erdelyi, A., Book Reviews: Higher Transcendental Functions. Vol. III. Based in Part on Notes Left by Harry Bateman. Science, 122, 290, 1955.

[10] Goodrich, C., Peterson, A. C., Discrete Fractional Calculus, Springer, Cham, 2015. https://doi.org/10.1007/978-3-319-25562-0

[11] Levitan, B. M., Sargsjan, I. S., Sturm–Liouville and Dirac operators. Mathematics and its Applications (Soviet Series), Kluwer Academic Publishers Group, Dordrecht, 1991 (translated from the Russian).
[12] Mert, R., Abdeljawad, T., Peterson, A., A Sturm–Liouville approach for continuous and discrete Mittag–Leffler kernel fractional operators, *Discr. Contin. Dynam. System, Series S*, 14(7) (2021), 2417-2434. https://doi.org/10.3934/dcdss.2020171
[13] Yalçınkaya, Y., Some fractional Dirac systems, *Turkish J. Math.*, 47 (2023), 110–122. doi:10.55730/1300-0098.3349

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 13–24 (2024)
DOI:10.31801/cfsuasmas.1302945
ISSN 1303-5991 E-ISSN 2618-6470

*Research Article; Received: May 26, 2023; Accepted: September 17, 2023*

# ON $\theta-$CONVEX CONTRACTIVE MAPPINGS WITH APPLICATION TO INTEGRAL EQUATIONS

Merve ÖZKAN[1], Murat ÖZDEMİR[2] and İsa YILDIRIM[3]

[1,2,3]Department of Mathematics, Faculty of Science, Atatürk University
25240 Erzurum, Turkey

Abstract. The fundamental goal of our paper is to study $\theta-$convex contractive mappings in metric spaces. We demonstrate some fixed point results for such mappings. Also, we give an application to integral equations of our results. Consequently, our results encompass numerous generalizations of the Banach contraction principle on metric space.

## 1. Introduction and Preliminaries

Banach [1] initially gave the Banach contraction principle which is an outstanding result in fixed point theory. Due to its significance, over the years, abounding researchers extended and generalized this contraction in many ways.

The notion of almost contraction was introduced by Berinde [2]. Also almost contraction was compared with other contractions and Berinde [2], [3], [4] demonstrated some fixed point theorems related to almost contraction.

Firstly Jleli [5] gave an attractive contraction called $\theta-$contraction and researched the uniqueness and existence of these mappings in complete metric spaces. After Jleli's first article [5], some different fixed point theorems were introduced Jleli [6], Hussain [7] and Imdad [8] by changing and relaxing the conditions of $\mho$.

In recent years, a remarkable generalization of the Banach contraction principle is the theorem by Istratescu [9]. Again, Istarescu studied convex contractions in [9], [10], [11]. Since Istratescu's fixed point theorems, many authors studied numerous generalizations and applications of the result of Istratescu (see [12]- [24]).

2020 *Mathematics Subject Classification.* 47H10.
*Keywords.* Integral equation, fixed point, convex contraction, $\theta-$contraction.
[1] merve.ozkan@atauni.edu.tr-Corresponding author; 0000-0002-7213-4070
[2] mozdemir@atauni.edu.tr; 0000-0002-4928-3115
[3] isayildirim@atauni.edu.tr; 0000-0001-6165-716X.

Ciric [25] used the concept of orbitally continuous for proving the uniqueness and existence of the fixed point mappings. Afterwards, Bisht [15] proved some fixed point theorems by replacing the continuity condition with orbital continuity.

Merging the ideas of Istratescu [9] and Jleli [5], we introduce a generalization of convex type contractions. The goal of our paper is to introduce generalized $\theta-$convex contractive mappings and to demonstrate some fixed point theorems. Theorems that have been demonstrated in our paper are generalizations of a variety of results in the literature.

Now, at first we mention some fundemental definitions and notions related to our work.

$F(h) = \{t \in W : ht = t\}$ is fixed point of $h$.

Bisht [15] gave the following definition instead of continuity condition to be used their theorems.

**Definition 1.** *[15] Let $(W, \varrho)$ be a metric space and $h$ be a self mapping on $W$. We say that $h$ is orbitally continuous at a point $u \in W$ if $\lim\limits_{j\to\infty} h^{n_j} t = u$ implies that $\lim\limits_{j\to\infty} h^{n_j} t = hu$.*

Berinde [2], [3], [4] gave the concepts of almost contraction, multivalued almost contraction and the continuity of almost contractions.

**Definition 2.** *[2] Let $(W, \varrho)$ be a metric space and $h$ be a self mapping on W. $h$ is called an almost contraction if there exists a constant $\zeta \in (0, 1)$ and $L \geq 0$ such that*

$$\varrho(ht, hs) \leq \zeta\varrho(t, s) + L\varrho(s, ht)$$

*for all $t, s \in W$.*

Firstly, Jleli [5] gave the concept of $\theta-$contraction mappings and the following family.

Let $\mho$ denotes the set of all mappings $\theta : (0, \infty) \to (1, \infty)$ which hold the following conditions:

(1) $\theta$ is strictly increasing;
(2) for all sequence $\{\eta_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(\eta_n) = 1$ if and only if $\lim_{n\to\infty} \eta_n = 0$;
(3) there exist $\ell \in (0, \infty]$ and $r \in (0, 1)$ such that $\lim_{n\to\infty} \frac{\theta(\eta)-1}{(\eta)^r} = \ell$.

$\Upsilon$ be the set of nondecreasing functions $\varsigma : [0, +\infty) \to [0, +\infty)$ such that $\sum\limits_{j=1}^{+\infty} \varsigma^j(\eta) < +\infty$ for each $\eta > 0$, where $\varsigma^j$ is the $j-$th iterate of $\varsigma$.

**Remark 1.** *Each function $\varsigma \in \Upsilon$ satisfies $\lim_{n\to\infty} \varsigma^n(\eta) = 0$ and $\varsigma(\eta) < \eta$ for all $\eta > 0$.*

Firstly, Jleli [5] gave the definition of $\theta-$contraction as follows.

**Definition 3.** *Let $(W, \varrho)$ be a metric space and $h: W \to W$ be a self-mapping. Then $h$ is called $\theta-$contraction if there exist $\kappa \in (0,1)$ such that*

$$\theta(\varrho(ht, hs)) \leq [\theta(\varrho(t, s))]^{\kappa}$$

*for all $t, s \in W$, with $ht \neq hs$.*

Istratescu [9], [10] gave the following definitions.

**Definition 4.** *Let $(W, \varrho)$ be a metric space and $h: W \to W$ be a self-mapping. Then $h$ is called convex contraction of order 2 if there exist $d_1, d_2 \in (0,1)$ such that $d_1 + d_2 < 1$ and*

$$\varrho(h^2 t, h^2 s) \leq d_1 \varrho(ht, hs) + d_2 \varrho(t, s)$$

*for all $t, s \in W$.*

**Definition 5.** *Let $(W, \varrho)$ be a metric space and $h: W \to W$ be a self-mapping. Then $h$ is called two-sided convex contraction mappings if there exist $d_1, d_2, d_3, d_4 \in (0,1)$ such that $d_1 + d_2 + d_3 + d_4 < 1$ and*

$$\varrho(h^2 t, h^2 s) \leq d_1 \varrho(t, ht) + d_2 \varrho(ht, h^2 t) + d_3 \varrho(s, hs) + d_4 \varrho(hs, h^2 s)$$

*for all $t, s \in W$.*

## 2. Main Results

In this chapter, we give concept of generalized $\theta-$convex contractions in metric spaces. We demonstrate some fixed point results for such contractions on metric spaces. The following Theorem's hypothesis are basically weaker than the set of contraction type mappings.

Now, we will give the definition of generalized $\theta-$convex contractive mappings.

**Definition 6.** *Let $(W, \varrho)$ be a metric space and $h: W \to W$ be a self-mapping. Then $h$ is called generalized $\theta-$convex contraction if there exist $L \geq 0$, $\varsigma \in \Upsilon$ and $\kappa \in (0,1)$ such that*

$$\varrho(h^2 t, h^2 s) > 0 \Rightarrow \theta\left(\varrho(h^2 t, h^2 s)\right) \leq [\theta\left(\varsigma(M_I(t, s))\right)]^{\kappa} + L N_I(t, s) \qquad (1)$$

*where $\theta \in \mho$ and*

$$\begin{aligned} M_I(t, s) &= \max\left\{\varrho(t, s), \varrho(ht, hs), \varrho(t, ht), \varrho(s, hs), \varrho(ht, h^2 t), \varrho(hs, h^2 s)\right\}, \\ N_I(t, s) &= \min\left\{\varrho(t, ht), \varrho(s, hs), \varrho(t, hs), \varrho(s, ht), \varrho(ht, h^2 t), \varrho(hs, h^2 s)\right\}, \end{aligned}$$

*for all $t, s \in W$.*

**Remark 2.** *Every convex contraction of order 2 and two-sided convex contraction are a generalized $\theta-$convex contraction. Also, every $\theta-$contraction is a generalized $\theta-$convex contraction. But the reverse doesn't have to be true.*

Since, our novel class of contractive type mappings is more general, it will be more advantageous to work using this new class.

The following theorem is our first result related to generalized $\theta-$convex contractive mappings.

**Theorem 1.** *Let $(W,\varrho)$ be a complete metric space and $h : W \to W$ be a generalized $\theta-$convex contraction. If $h$ is either orbitally continuous on $W$ or $h$ is continuous, then $h$ has a unique fixed point.*

*Proof.* Starting at the point $t_0 \in W$, the sequence $\{t_n\}$ is constructed by $t_n = ht_{n-1} = h^n t_0$, $n \geq 1$. If $t_{n_0+1} = t_{n_0}$ for any $n_0 \in \mathbb{N} \cup \{0\}$, then it is clear that, $t_{n_0}$ is a fixed point of $h$. Consequently, assume that $t_{n_0+1} \neq t_{n_0}$ for all $n_0 \in \mathbb{N} \cup \{0\}$. Setting $m = \max\{\varrho(t_0,t_1), \varrho(t_1,t_2)\}$. First of all, we show that $\{\varrho(t_n,t_{n+1})\}$ is a strictly nonincreasing sequence in $W$. Since $h$ is a generalized $\theta-$convex contraction, using Remark 1 and from the first axiom of $\theta$, we have

$$\begin{aligned}
\theta(\varrho(t_2,t_3)) &= \theta\left(\varrho\left(h^2t_0,h^2t_1\right)\right)\\
&\leq \left[\theta\left(\varsigma\left(\max\left\{\begin{array}{c}\varrho(t_0,t_1), \varrho(ht_0,ht_1), \varrho(t_0,ht_0),\\ \varrho\left(ht_0,h^2t_0\right), \varrho(t_1,ht_1), \varrho\left(ht_1,h^2t_1\right)\end{array}\right\}\right)\right)\right]^{\kappa}\\
&\quad +L\min\left\{\begin{array}{c}\varrho(t_0,ht_0), \varrho(t_1,ht_1), \varrho(t_0,ht_1),\\ \varrho(t_1,ht_0), \varrho\left(ht_0,h^2t_0\right), \varrho\left(ht_1,h^2t_1\right)\end{array}\right\}\\
&= \left[\theta\left(\varrho\left(\max\left\{\varrho(t_0,t_1), \varrho(t_1,t_2), \varrho(t_2,t_3)\right\}\right)\right)\right]^{\kappa}\\
&\leq \left[\theta\left(\max\left\{m, \varrho(t_2,t_3)\right\}\right)\right]^{\kappa}.
\end{aligned}$$

If $\max\{m,\varrho(t_2,t_3)\} = \varrho(t_2,t_3)$, then we have

$$\theta(\varrho(t_2,t_3)) \leq [\theta(\varrho(t_2,t_3))]^{\kappa}.$$

If we take ln two both sides of the inquality, then we have

$$\ln\theta(\varrho(t_2,t_3)) \leq \kappa\ln[\theta(\varrho(t_2,t_3))]$$

which is a contradiction. Hence, we get

$$\max\{m,\varrho(t_2,t_3)\} = m = \max\{\varrho(t_0,t_1), \varrho(t_1,t_2)\}.$$

Since $\varsigma(\eta) < \eta$ for all $\eta > 0$, we have

$$\begin{aligned}
\theta(\varrho(t_3,t_4)) &\leq \left[\theta\left(\varsigma\left(\max\left\{\begin{array}{c}\varrho(t_1,t_2), \varrho(ht_1,ht_2), \varrho(t_1,ht_1),\\ \varrho(ht_1,h^2t_1), \varrho(t_2,ht_2), \varrho\left(h_2,h^2t_2\right)\end{array}\right\}\right)\right)\right]^{\kappa}\\
&\quad +L\min\left\{\begin{array}{c}\varrho(t_1,ht_1), \varrho(t_2,ht_2), \varrho(t_1,ht_2),\\ \varrho(t_2,ht_1), \varrho(ht_1,h^2t_1), \varrho\left(ht_2,h^2t_2\right)\end{array}\right\}\\
&\leq \left[\theta\left(\max\left\{\varrho(t_1,t_2), \varrho(t_2,t_3), \varrho(t_3,t_4)\right\}\right)\right]^{\kappa}.
\end{aligned}$$

If $\max\{\varrho(t_1,t_2), \varrho(t_2,t_3), \varrho(t_3,t_4)\} = \varrho(t_3,t_4)$, then we obtain

$$\theta(\varrho(t_3,t_4)) \leq [\theta(\varrho(t_3,t_4))]^{\kappa}.$$

If we take ln two both sides of the inequality, then we have

$$\ln\theta(\varrho(t_3,t_4)) \leq \kappa\ln[\theta(\varrho(t_3,t_4))].$$

This is one more contradiction, from which it is concluded that $\max\{\varrho(t_1,t_2), \varrho(t_2,t_3)\} > \varrho(t_3,t_4)$. Thus, $m > \varrho(t_2,t_3) > \varrho(t_3,t_4)$. Hence, by induction one can get

$\{\varrho(t_n,t_{n+1})\}$ is a strictly nonincreasing sequence in $W$. This implies that

$$\begin{aligned}
\theta(\varrho(t_n,t_{n+1})) &\leq \left[\theta\left(\varsigma\left(\max\left\{\begin{array}{c}\varrho(t_{n-2},t_{n-1}),\varrho(t_{n-1},t_n),\varrho(t_{n-2},t_{n-1}),\\ \varrho(t_{n-1},t_n),\varrho(t_{n-1},t_n),\varrho(t_n,t_{n+1})\end{array}\right\}\right)\right)\right]^{\kappa}\\
&\quad +L\min\left\{\begin{array}{c}\varrho(t_{n-2},t_{n-1}),\varrho(t_{n-1},t_n),\varrho(t_{n-2},t_n),\\ \varrho(t_{n-1},t_{n-1}),\varrho(t_{n-1},t_n),\varrho(t_n,t_{n+1})\end{array}\right\}\\
&\leq [\theta(\max\{\varrho(t_{n-2},t_{n-1}),\varrho(t_{n-1},t_n),\varrho(t_n,t_{n+1})\})]^{\kappa}.
\end{aligned}$$

If $\max\{\varrho(t_{n-2},t_{n-1}),\varrho(t_{n-1},t_n),\varrho(t_n,t_{n+1})\}=\varrho(t_n,t_{n+1})$ then we get

$$\theta(\varrho(t_n,t_{n+1}))\leq[\theta(\varrho(t_n,t_{n+1}))]^{\kappa},$$

which is once again contradiction. Therefore, we have

$$\theta(\varrho(t_n,t_{n+1}))\leq[\theta(\max\{\varrho(t_{n-2},t_{n-1}),\varrho(t_{n-1},t_n)\})]^{\kappa}$$

and

$$\begin{aligned}
\theta(\varrho(t_n,t_{n+1})) &\leq [\theta(\varrho(t_{n-1},t_n))]^{\kappa}\\
&\leq [\theta(\varrho(t_{n-2},t_{n-1}))]^{\kappa^2}\\
&\quad \vdots\\
&\leq [\theta(m)]^{\kappa^l},
\end{aligned}$$

whenever $l=2n$ or $l=2n+1$, for $l\geq1$. Hence, we have

$$1\leq\theta(\varrho(t_n,t_{n+1}))\leq[\theta(m)]^{\kappa^l},\text{ for all } l\geq1. \tag{2}$$

Letting $n\to\infty$, following two cases arise.

Case 1. $1\leq\theta(\varrho(t_n,t_{n+1}))\leq[\theta(m)]^{\kappa^n}$, for all $n\geq2$ and $n$ is even.

Case 2. $1\leq\theta(\varrho(t_n,t_{n+1}))\leq[\theta(m)]^{\kappa^{n-1}}$, for all $n\geq3$ and $n$ is odd.

From Case 1 and Case 2 we get $\lim\limits_{n\to\infty}\theta(\varrho(t_n,t_{n+1}))=1$. By the second axiom of $\theta$, we get $\lim\limits_{n\to\infty}\varrho(t_n,t_{n+1})=0$. From the third axiom of $\theta$, there exist $\ell\in(0,\infty]$ and $r\in(0,1)$

$$\lim_{n\to\infty}\frac{\theta(\varrho(t_n,t_{n+1}))-1}{[\varrho(t_n,t_{n+1})]^r}=\ell.$$

Assume that $\ell<\infty$ and $\top=\frac{\ell}{2}>0$. From the limit definition, there exists $n_0\in\mathbb{N}$ such that

$$\left|\frac{\theta(\varrho(t_n,t_{n+1}))-1}{[\varrho(t_n,t_{n+1})]^r}-\ell\right|\leq\top\text{ for all } n\geq n_0$$

which implies that

$$\frac{\theta\varrho(t_n,t_{n+1})-1}{[\varrho(t_n,t_{n+1})]^r}\leq\ell-\top=\top\text{ for all } n\geq n_0.$$

Therefore, we have

$$n[\varrho(t_n,t_{n+1})]^r\leq\Bbbk n[\theta(\varrho(t_n,t_{n+1}))-1]\ \text{ for all } n\geq n_0$$

where $\mathbb{k} = \frac{1}{\top}$. Assume that $\top > 0$ is an arbitrary number and $\ell = \infty$. From the limit definition, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\theta\left(\varrho\left(t_n, t_{n+1}\right)\right) - 1}{\left[\varrho\left(t_n, t_{n+1}\right)\right]^r} \geq \top \text{ for all } n \geq n_0$$

which implies that

$$n\left[\varrho\left(t_n, t_{n+1}\right)\right]^r \leq \mathbb{k} n\left[\theta\left(\varrho\left(t_n, t_{n+1}\right)\right) - 1\right] \text{ for all } n \geq n_0 \tag{3}$$

where $\mathbb{k} = \frac{1}{\top}$. Therefore, in two cases there exists $n \geq n_0$ and $\mathbb{k} > 0$ such that (2.3) is satisfied. Using (2.2), we get

$$n\left[\varrho\left(t_n, t_{n+1}\right)\right]^r \leq \mathbb{k} n\left(\left[\theta\left(m\right)\right]^{k^l} - 1\right) \text{ for all } l \geq 2n_0 + 1 \text{ or } l \geq 2n_0.$$

Letting $n \to \infty$, we get $\lim\limits_{n\to\infty} n\left[\varrho\left(t_n, t_{n+1}\right)\right]^r = 0$. Hence, there exists $n_1 \in \mathbb{N}$ such that

$$\varrho\left(t_n, t_{n+1}\right) \leq \frac{1}{n^{\frac{1}{r}}} \text{ for all } n \geq n_1.$$

Now, we will demostrate that $\{t_n\}$ is a Cauchy sequence. For all $p > q \geq n_1$, we get

$$\begin{aligned}
\varrho\left(t_p, t_q\right) &\leq \varrho\left(t_p, t_{p-1}\right) + \varrho\left(t_{p-1}, t_{p-2}\right) + \cdots + \varrho\left(t_{q+1}, t_q\right) \\
&\leq \sum_{j=q}^{p-1} \varrho\left(t_j, t_{j+1}\right) \\
&< \sum_{j=q}^{\infty} \varrho\left(t_j, t_{j+1}\right) \\
&\leq \sum_{j=q}^{\infty} \frac{1}{j^{\frac{1}{r}}}.
\end{aligned}$$

Since $\sum_{j=q}^{\infty} \frac{1}{j^{\frac{1}{r}}}$ is convergent, $\lim\limits_{p,q\to\infty} \varrho\left(t_p, t_q\right) = 0$. Hence, we get that $\{t_n\}$ is a Cauchy sequence in $W$. Since $(W, \varrho)$ is a complete metric space, there exists $u \in W$ such that $t_n \to u$. Assume that $h$ is continuous. Since $t_n \to u \in W$ and $W$ is complete metric space, we get

$$\varrho\left(u, hu\right) = \lim_{n\to\infty} \varrho\left(t_n, ht_n\right) = \lim_{n\to\infty} \varrho\left(t_n, t_{n+1}\right) = 0.$$

Therefore $u \in F\left(h\right)$. Again, assume that $h$ is orbitally continuous on $W$, then

$$t_{n+1} = ht_n = h\left(h^n t_0\right) \to hu \text{ as } n \to \infty.$$

Since $W$ is complete metric space, $hu = u$ that is $u \in F\left(h\right)$. Now, assume that $u$ and $v$ are arbitrary two fixed point of $h$. Then we get

$$\theta\left(\varrho\left(u, v\right)\right) = \theta\left(\varrho\left(h^2 u, h^2 v\right)\right) \leq \left[\theta\left(\varsigma\left(M_I\left(u, v\right)\right)\right)\right]^{\kappa} + LN_I\left(u, v\right)$$

$$\begin{aligned}
&\leq \left[\theta\left(\varsigma\left(\max\left\{\begin{array}{c} \varrho(u,v), \varrho(hu,hv), \varrho(u,hu), \varrho(v,hv), \\ \varrho\left(hu,h^2u\right), \varrho\left(hv,h^2v\right)\end{array}\right\}\right)\right)\right]^{\kappa} \\
&\quad +L\min\left\{\begin{array}{c} \varrho(u,hu), \varrho(v,hv), \varrho(u,hv), \varrho(v,hu), \\ \varrho\left(hu,h^2u\right), \varrho\left(hv,h^2v\right)\end{array}\right\} \\
&\leq [\theta(\varrho(u,v))]^{\kappa}.
\end{aligned}$$

Thus we get

$$\theta(\varrho(u,v)) \leq [\theta(\varrho(u,v))]^{\kappa}.$$

If we take ln two both sides of the inquality, then we obtain

$$\ln\theta(\varrho(u,v)) \leq \kappa\ln\theta(\varrho(u,v)).$$

Since $\kappa \in (0,1)$, it is a contradiction. Hence $u = v$, that is, $h$ has a unique fixed point in $W$. □

Now, we shall give an example to illustrate the generality of Theorem 1.

**Example 1.** *Let $(W,\varrho)$ be a metric space, $h$ be a self mapping on $W$ and $\theta(t) = e^{\sqrt{t}}$ for $t > 0$, that is, $\theta \in \mho$. Assume that $h$ is a convex contraction of type$-2$ for all $t, s \in W$ with $\varrho(h^2t, h^2s) > 0$, $B = \sum_{j=1}^{6} d_j < 1$ and $d_j \geq 0$ for all $j = 1, 2, \ldots, 6$.*

$$\begin{aligned}
\varrho(h^2t,h^2s) &\leq d_1\varrho(t,s) + d_2\varrho(ht,hs) + d_3\varrho(t,ht) + d_4\varrho(s,hs) \\
&\quad +d_5\varrho(ht,h^2t) + d_6\varrho(hs,h^2s) \\
&\leq \sum_{j=1}^{6} d_j \max\left\{\varrho(t,s), \varrho(ht,hs), \varrho(t,ht), \varrho(s,hs), \varrho(ht,h^2t), \varrho(hs,h^2s)\right\} \\
&\leq B\max\left\{\varrho(t,s), \varrho(ht,hs), \varrho(t,ht), \varrho(s,hs), \varrho(ht,h^2t), \varrho(hs,h^2s)\right\},
\end{aligned}$$

*where $t, s \in W$ with $\varrho(h^2t, h^2s) > 0$. We obtain that*

$$\varrho\left(h^2t,h^2s\right) \leq BM_I(t,s).$$

*Taking $\varsigma(t) = B^{\frac{1}{2}}t$, we have*

$$e^{\sqrt{\rho(h^2t,h^2s)}} \leq e^{B^{\frac{1}{4}}\sqrt{M_I(t,s)}} = \left[e^{\sqrt{\varrho(M_I(t,s))}}\right]^{\kappa}$$

*where $\kappa = B^{\frac{1}{4}}$. Since $\theta(t) = e^{\sqrt{t}}$ for $t > 0$, we deduce that*

$$\begin{aligned}
\theta\left(\varrho(h^2t,h^2s)\right) &\leq [\theta(\varsigma(M_I(t,s)))]^{\kappa} \\
&\leq [\theta(\varsigma(M_I(t,s)))]^{\kappa} + LN_I(t,s),
\end{aligned}$$

*where $L \geq 0$. This shows that, $h$ is a generalized $\theta-$convex contractive mapping.*

**Remark 3.** *Above example show that our contraction condition generalizes Istratescu's contraction conditions [9], [10].*

**Definition 7.** *Let $(W, \varrho)$ be a metric space. A self-mapping $h : W \to W$ is called an almost $\theta-$convex contraction if there exist $L \geq 0$ and $\kappa \in (0, 1)$ such that*

$$\varrho(h^2t, h^2s) > 0 \Rightarrow \theta(\varrho(h^2t, h^2s) \leq [\theta(M_I(t, s))]^{\kappa} + LN_I(t, s)$$

*where $\theta \in \mho$ and*

$$\begin{aligned} M_I(t, s) &= \max\left\{\varrho(t, s), \varrho(ht, hs), \varrho(t, ht), \varrho(s, hs), \varrho(ht, h^2t), \varrho(hs, h^2s)\right\}, \\ N_I(t, s) &= \min\left\{\varrho(t, ht), \varrho(s, hs), \varrho(t, hs), \varrho(s, ht), \varrho(ht, h^2t), \varrho(hs, h^2s)\right\}, \end{aligned}$$

*for all $t, s \in W$.*

Definition 6 and Definition 7 generalize and merge the results derived by Jleli [5] and Istratescu [9], [10], and some other connected results in the literature. Also, our novel contractions can be considered as an attracted generalization of Darbo's fixed point problem [26], [27].

**Corollary 1.** *Let $(W, \varrho)$ be a complete metric space and $h : W \to W$ be an almost $\theta-$convex contraction. If $h$ is either orbitally continuous on $W$ or $h$ is continuous, then $h$ has a unique fixed point that is $u = hu$, $u \in W$.*

If we take $L = 0$ in Theorem 1, then we obtain the following corollary.

**Corollary 2.** *Let $(W, \varrho)$ be a metric space and $h : W \to W$ be a self-mapping. If there exist $\varsigma \in \Upsilon$ and $\kappa \in (0, 1)$ such that*

$$\varrho(h^2t, h^2s) > 0 \Rightarrow \theta(\varrho(h^2t, h^2s)) \leq [\theta(\varsigma(M_I(t, s)))]^{\kappa} \tag{4}$$

*where $\theta \in \mho$ and*

$$M_I(t, s) = \max\left\{\varrho(t, s), \varrho(ht, hs), \varrho(t, ht), \varrho(s, hs), \varrho(ht, h^2t), \varrho(hs, h^2s)\right\},$$

*for all $t, s \in W$. Also, assume that $h$ is either orbitally continuous on $W$ or $h$ is continuous, then $u = hu$, $u \in W$.*

By taking $L = 0$ and not considering $\varsigma \in \Upsilon$ in Theorem 1, we deduce the following corollary.

**Corollary 3.** *Let $(W, \varrho)$ be a metric space and a self-mapping $h$ on $W$. If there exist $\kappa \in (0, 1)$ such that*

$$\varrho(h^2t, h^2s) > 0 \Rightarrow \theta(\varrho(h^2t, h^2s)) \leq [\theta(M_I(t, s))]^{\kappa} \tag{5}$$

*where $\theta \in \mho$ and*

$$M_I(t, s) = \max\left\{\varrho(t, s), \varrho(ht, hs), \varrho(t, ht), \varrho(s, hs), \varrho(ht, h^2t), \varrho(hs, h^2s)\right\},$$

*for all $t, s \in W$. Also, assume that $h$ is either orbitally continuous on $W$ or $h$ is continuous, then $u = hu$, $u \in W$.*

We get the following results as shown in Example 1.

**Corollary 4.** *Let $(W,\varrho)$ be a metric space and a self-mapping $h$ on $W$. For all $t,s\in W$,*

$$\varrho(h^2t,h^2s)\le B\max\left\{\varrho(t,s),\varrho(ht,hs),\varrho(t,ht),\varrho(s,hs),\varrho(ht,h^2t),\varrho(hs,h^2s)\right\}$$

*where $B\in[0,1)$. Also, assume that $h$ is either orbitally continuous on $W$ or $h$ is continuous, then $u=hu$, $u\in W$.*

**Corollary 5.** *Let $(W,\varrho)$ be a metric space and $h$ is a convex contraction of type$-2$ on $W$. Also, assume that $h$ is either orbitally continuous on $W$ or $h$ is continuous, then $u=hu$, $u\in W$.*

## 3. Application

Now, we give an application of our result for nonlinear integral equations.

$$t(u)=\vartheta(u)+\int_e^f K(u,v,t(v))\,dv \tag{6}$$

where $e,f\in\mathbb{R}$, $C[e,f]=\{h:[e,f]\to\mathbb{R}$ continuous functions$\}$, $t\in C([e,f],\mathbb{R})$, $K:[e,f]\times[e,f]\times\mathbb{R}\to\mathbb{R}$ and $\vartheta:[e,f]\to\mathbb{R}$.

**Theorem 2.** *Consider the integral equation* (3.1). *Assume that the following conditions satisfy:*

(i) $K:[e,f]\times[e,f]\times\mathbb{R}\to\mathbb{R}$ and $\vartheta:[e,f]\to\mathbb{R}$ are continuous functions;
(ii) there exists $\gamma\in[0,1)$ such that

$$|K(u,v,ht(v))-K(u,v,hs(v))|\le\gamma\frac{\max\left\{\begin{array}{c}|t(v)-s(v)|,|ht(v)-hs(v)|,\\ |t(v)-ht(v)|,\left|ht(v)-h^2t(v)\right|,\\ |s(v)-hs(v)|,\left|hs(v)-h^2s(v)\right|\end{array}\right\}}{f-e}$$

for all $t,s\in C([e,f],\mathbb{R})$ and $u,v\in[e,f]$.

Then nonlinear integral equation (3.1) has a unique solution.

*Proof.* $W=C[e,f]$, $\varrho(h,g)=|h-g|=\max_{t\in[e,f]}|ht-gt|$, for all $h,g\in W$, and $(W,\varrho)$ is a complete metric space. $h:W\to W$ be a continuous operator defined by

$$ht(u)=\vartheta(u)+\int_e^f K(u,v,t(v))\,dv.$$

Starting at the point $t_0\in W$, the sequence $\{t_n\}$ is constructed by $t_n=ht_{n-1}=h^nt_0$, $n\ge1$. From (3.1), we get

$$t_{n+1}=ht_n(u)=\vartheta(u)+\int_e^f K(u,v,t_n(v))\,dv.$$

Now, we will demonstrate that $h$ is a generalized $\theta-$convex contractive mapping. We can write

$$\begin{aligned}\left|h^2t(u)-h^2s\left(u\right)\right| &= \left|\int_e^f K\left(u,v,ht\left(v\right)\right)dv-\int_e^f K\left(u,v,hs\left(v\right)\right)dv\right| \\ &\leq \int_e^f \left|K\left(u,v,ht\left(v\right)\right)-K\left(u,v,hs\left(v\right)\right)\right|dv \\ &\leq \frac{\gamma}{f-e}\int_e^f \max\left\{\begin{array}{c}\left|t(v)-s\left(v\right)\right|,\left|ht(v)-hs\left(v\right)\right|, \\ \left|t(v)-ht\left(v\right)\right|,\left|ht(v)-h^2t\left(v\right)\right|, \\ \left|s(v)-hs\left(v\right)\right|,\left|hs(v)-h^2s\left(v\right)\right|\end{array}\right\}dv\end{aligned}$$

and

$$\begin{aligned}\varrho\left(h^2t,h^2s\right) &= \max_{u\in[e,f]}\left|h^2t(u)-h^2s\left(u\right)\right| \\ &\leq \frac{\gamma}{f-e}\max_{u\in[e,f]}\int_e^f \max\left\{\begin{array}{c}\left|t(v)-s\left(v\right)\right|,\left|ht(v)-hs\left(v\right)\right|, \\ \left|t(v)-ht\left(v\right)\right|,\left|ht(v)-h^2t\left(v\right)\right|, \\ \left|s(v)-hs\left(v\right)\right|,\left|hs(v)-h^2s\left(v\right)\right|\end{array}\right\}dv \\ &\leq \frac{\gamma}{f-e}\max\left[\max_{c\in[e,f]}\left\{\begin{array}{c}\left|t(c)-s\left(c\right)\right|,\left|ht(c)-hs\left(c\right)\right|, \\ \left|t(c)-ht\left(c\right)\right|,\left|ht(c)-h^2t\left(c\right)\right|, \\ \left|s(c)-hs\left(c\right)\right|,\left|hs(c)-h^2s\left(c\right)\right|\end{array}\right\}\right]\int_e^f dv \\ &\leq \gamma\max\left\{\varrho(t,s),\varrho(ht,hs),\varrho(t,ht),\varrho(s,hs),\varrho(ht,h^2t),\varrho(hs,h^2s)\right\} \\ &\leq \gamma M_I(t,s).\end{aligned}$$

Thus

$$\varrho\left(h^2t,h^2s\right)\leq\gamma M_I(t,s).$$

Define $\theta(t)=e^{\sqrt{t}}$ for $t>0$ and $\varsigma(t)=\gamma^{\frac{1}{2}}t$. We have

$$e^{\sqrt{\varrho(h^2t,h^2s)}}\leq e^{\gamma^{\frac{1}{4}}\sqrt{M_I(t,s)}}=\left[e^{\sqrt{\varrho(M_I(t,s))}}\right]^{\kappa}$$

where $\kappa=\gamma^{\frac{1}{4}}$. Thus, we get

$$\theta\left(\varrho(h^2t,h^2s\right)\leq\left[\theta\left(\varsigma\left(M_I(t,s)\right)\right)\right]^{\kappa}+LN_I(t,s)$$

where $L\geq 0$. This shows that, $h$ is a generalized $\theta-$convex contractive mapping. That is, the conditions of Theorem 1 are hold. Thus, $h$ has a unique fixed point in $W$, and so, the nonlinear integral equation (3.1) has a unique solution. □

## 4. Conclusion

We present generalized $\theta-$convex contractive mappings in this paper. This contractive condition not only extends several existing contraction definitions but also merge some existing contractions. Afterward, we investigate the existence of a fixed point for our novel type contraction, we state some consequences. Our results generalize and merge the results derived by Istratescu [9], [10] and Jleli [5], and some other connected results in the literature. Our new contraction can be considered as an interesting generalization of Darbo's fixed point problem [26], [27]. As well as the corollaries in this paper, to underline the novelty of our given results, we show an example that shows that Theorem 1 is a genuine generalization of Istratescu's results [9]. Moreover, as a possible application, we applied our main results to study the existence of a solution for a nonlinear integral equation. The new concept allows for further studies and applications. By choosing the appropriate auxiliary function such as simulation function and others, one can get several more results. Also, one can get the analogue of our result in the set-up of cyclic mappings.

**Authors Contribution Statement** The authors contributed equally to this work. All authors read and approved the final copy of this paper.

**Declaration of Competing Interests** Authors state no conflict of interest.

## References


[1] Banach, S., Sur les opérationes dans les ensembles abstraits et leur application aux équation intégrale, *Fundam. Math.*, 3(1) (1922), 133-181.

[2] Berinde, V., Pacurar, M., Fixed points and continuity of almost contractions, *Fixed Point Theory,* 9(1) (2008), 23–34.

[3] Berinde, V., Approximating fixed points of weak contraction using the Picard iteration, *Nonlinear Anal. Forum*, 9 (2004), 43–53.

[4] Berinde, M., Berinde, V., On general class of multivalued weakly Picard mappings, *J. Nonlinear Sci. Appl.,* 326(2) (2007), 772–782. https://doi.org/10.1016/j.jmaa.2006.03.016

[5] Jleli, M., Samet, B., A new generalization of the Banach contractive principle, *J. Inequal. Appl.*, 38 (2014). https://doi.org/10.1186/1029-242X-2014-38

[6] Jleli, M., Karapınar, E., Samet, B., Further generalizations of the Banach contraction principle, *J. Inequal. Appl.*, 439 (2014). https://doi.org/10.1186/1029-242X-2014-439

[7] Hussain, N., Parvaneh, V., Samet, B., Vetro, C., Some fixed point theorems for generalized contractive mappings in complete metric spaces, *Fixed Point Theory Appl.,* 17 (2015). https://doi.org/10.1186/s13663-015-0433-z

[8] Imdad, M., Alfaqih, W. M., Khan, I. A., Weak $\theta-$contractions and some fixed point results with applications to fractal theory, *Adv. Differ. Equ.*, 439 (2018). https://doi.org/10.1186/s13662-018-1900-8

[9] Istratescu, V. I., Some fixed point theorems for convex contraction mappings and convex non-expansive mapping, *Libertas Mathematica*, 1 (1981), 151–163.

[10] Istratescu, V. I., Some fixed point theorems for convex contraction mappings and mappings with convex diminishing diameters-I, *Ann. Mat. Pura Appl.*, 130 (1982), 89–104. https://doi.org/10.1007/BF01761490

[11] Istratescu, V. I., Some fixed point theorems for convex contraction mappings and mappings with convex diminishing diameters-II, *Ann. Mat. Pura Appl.,* 134 (1983), 327–362. https://doi.org/10.1007/BF01773511

[12] Alghamdi, M. A., Alnafei, S. H., Radenovic, S., Shahzad, N., Fixed point theorems for convex contraction mappings on cone metric spaces, *Math. Comput. Model.*, 54 (2011), no. 9-10, 2020–2026. https://doi.org/10.1016/j.mcm.2011.05.010

[13] Ampadu, C. K. B., On the analogue of the convex contraction mapping theorem for tricyclic convex contraction mapping of order 2 in b-metric space, *J. Global Research in Math. Archives*, 4(6) (2017), 1–5.

[14] Ampadu, C. K. B., Some fixed point theory results for convex contraction mapping of order 2, *JP J. Fixed Point Theory Appl.,* 12(2-3) (2017), 81–130.

[15] Bisht, R. K., Hussain, N., A note on convex contraction mappings and discontinuity at fixed point, *J. Math. Anal.*, 8(4) (2017), 90-96.

[16] Bisht, R. K., Rakocevic, V., Fixed points of convex and generalized convex contractions, *Rendiconti Del Circolo Matematico Di Palermo Series 2*, 69 (2020), 21-28. https://doi.org/10.1007/s12215-018-0386-2

[17] Georgescu, F., IFS consisting of generalized convex contractions, *An. Stiint. Univ. Ovidius Constant.,* 25(1) (2017), 77–86, https://doi.org/10.1515/auom-2017-0007

[18] Khan, M. S., Singh, Y. M., Maniu, G., Postolache, M., On generalized convex contractions of type$-2$ in b-metric and $2-$metric spaces, *J. Nonlinear Sci. Appl.,* 10 (2017), 2902–2913. https://doi.org/10.22436/jnsa.010.06.05

[19] Khan, M. S., Singh, Y. M., Maniu, G., Postolache, M., On $(\alpha, p)-$convex contraction and asymptotic regularity, *J. Math. Comput. Sci.,* 18 (2018), 132–145. https://doi.org/10.22436/jmcs.018.02.01

[20] Karapınar, E., Fulga, A., Petruşel, A., On Istratescu type contractions in $b-$metric spaces, *Mathematics,* 8(3) (2020), 388. https://doi.org/10.3390/math8030388

[21] Latif, A., Sintunavarat, W., Ninsri, A., Approximate fixed point theorems for partial generalized convex contraction mappings in $a-$complete metric spaces, *Taiwanese J. Math.,* 19(1) (2015), 315–333. https://doi.org/10.11650/tjm.19.2015.4746

[22] Miandarag, M. A., Postolache, M., Rezapour, S., Approximate fixed points of generalizes convex contractions, *Fixed Point Theory Appl.,* 255 (2013), 1–8. https://doi.org/ 10.1186/1687-1812-2013-255

[23] Miculescu, R., Mihail, A., A generalization of Istratescu's fixed piont theorem for convex contractions, *Fixed Point Theory,* 18(2) (2017), 689-702. https://doi.org/10.24193/fpt-ro.2017.2.55

[24] Singh, Y. M., Khan, M. S., Kang, S. M., $F$-Convex contraction via admissible mapping and related fixed point theorems with an application, *Mathematics,* 6(6) (2018), 105. https://doi.org/10.3390/math6060105

[25] Ciric, L. B., Generalized contractions and fixed point theorems, *Publ. Inst. Math.*, 12(26) (1971), 19–26, https://doi.org/10.2307/2039517

[26] Gabeleh, M.,Vetro, C., A best proximity point approach to existence of solutions for a system of ordinary differential equations, *Bull. Belg. Math. Soc.,* 26(4) (2019), 493-503. https://doi.org/10.36045/bbms/1576206350

[27] Asadi, M., Gabeleh M., Vetro, C., A new approach to the generalization of Darbo's fixed point problem by using simulation functions with application to integral equations, *Results Math.,* 74(86) (2019), 1-15. https://doi.org/10.1007/s00025-019-1010-2

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 25–36 (2024)
DOI:10.31801/cfsuasmas.1186168
ISSN 1303-5991 E-ISSN 2618-6470

*Research Article; Received: October 8, 2022; Accepted: September 30, 2023*

# A STRONGER FORM OF LOCALLY CLOSED SET AND ITS HOMEOMORPHIC IMAGE

Jiarul HOQUE[1] and Shyamapada MODAK[2]

[1]Department of Science & Humanities, Gangarampur Govt. Polytechnic, Gangarampur, INDIA
[2]Department of Mathematics, University of Gour Banga, Malda, INDIA

Abstract. Through this paper, via the operators $(\cdot)^\star$ and $\Psi$, we presented notion of $\star$-Locally set in an ideal topological space $\zeta_{\mathbb{I}}$ as a new stronger form of locally closed set, and considered relations with various existing weak form of locally closed set. Preservations of direct images as well as inverse images of $(\cdot)^\star$, $\Psi$, $\star$-perfect and various weak forms of locally closed set including $\star$-Locally closed set are important investigating part. Besides, we pointed out that consideration of 'bijectivity' in Lemma 3.1 of [24] is sufficient, and the Lemma 3.3 of [24] is wrong. We demonstrated two modifications of the last one.

## 1. Introduction

Locally closed set and its study is not a new idea in topology. This notion was disclosed by Bourbaki [3], and after that it has been extensively studied by a good number of mathematicians (see [7,12,20,21]). This study has been interesting because it generalizes both open and closed sets. But the study of a locally closed set relative to an ideal (see [13]) is a new idea, and this has been introduced through this paper. The authors Jeyanthi *et al.* [12] and the author Dontchev [6] have studied locally closed sets in terms of ideal, but these locally closed sets differ somewhat from the current one.

We now consider some preliminary concepts from literature for developing the paper.

Consider a topological space $(\mathbf{Z}, \mathbb{T})$ (henceforth, in this paper we shall denote it by $\zeta$), and suppose $\mathbb{I}$ is an ideal on $\mathbf{Z}$. The set-valued map $(\cdot)^\star : \wp(\mathbf{Z}) \to \wp(\mathbf{Z})$ associated by the formula '$H^\star = \{a \in \mathbf{Z} : G_a \cap H \notin \mathbb{I}$ for every $G_a \in \mathbb{T}_a\}$ for every $H \subseteq \mathbf{Z}$' is designated as the local function [11] w.r.t. the ideal $\mathbb{I}$ and the topology $\mathbb{T}$,

---

2020 *Mathematics Subject Classification.* 54D10, 54C10, 54D9, 54C99.
*Keywords.* Ideal, local function, $\Psi$-operator, locally closed set, homeomorphism.
[1] jiarul8435@gmail.com-Corresponding author; 0000-0003-1055-9820
[2] spmodak2000@yahoo.co.in; 0000-0002-0226-2392.

where $\mathbb{T}_a = \{G \in \mathbb{T} : a \in G\}$ and $\wp(\mathbf{Z})$ stands for power set of $\mathbf{Z}$. Other notations used instead of $H^\star$ are $H^\star(\mathbb{I}, \mathbb{T})$ and $H^\star(\mathbb{I})$. For the trivial ideals $\{\varnothing\}$ and $\wp(\mathbf{Z})$, values of $(\cdot)^\star$ are $H^\star(\{\varnothing\}) = \mathrm{Cl}(H)$ (closure operator) and $H^\star(\wp(\mathbf{Z})) = \varnothing$ (zero operator), respectively. An interesting ideal on $\mathbf{Z}$ is $\mathbb{I}_n$ consisting of all nowhere dense sets of $\zeta$, and $H^\star(\mathbb{I}_n) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(H)))$ (see [11]), where 'Int' stands for interior operator. Further, for the ideals $\mathbb{I}_f = \{I \subseteq \mathbf{Z} : I \text{ is finite}\}$ and $\mathbb{I}_c = \{I \subseteq \mathbf{Z} : I \text{ is countable}\}$, $H^\star(\mathbb{I}_f) = H^\omega$ (collection of all $\omega$-accumulation point of $H$) and $H^\star(\mathbb{I}_c) = H^{cd}$ (collection of all condensation point of $H$) (see [11]). Thus one can think the local function $(\cdot)^\star$ as a generalization of closure operator.

An important set-operator familiar to researchers as a complement of the local function $(\cdot)^\star : \wp(\mathbf{Z}) \to \wp(\mathbf{Z})$ is $\Psi : \wp(\mathbf{Z}) \to \wp(\mathbf{Z})$, and its value acting on $H \subseteq \mathbf{Z}$ is calculated by the formula $\Psi(H) = \mathbf{Z} \setminus (\mathbf{Z} \setminus H)^\star$ [22]. Note that $(\cdot)^\star$ (resp., $\Psi$) is not necessarily a closure (resp., interior) operator. However, the operator $\mathrm{Cl}^\star : \wp(\mathbf{Z}) \to \wp(\mathbf{Z})$ given by the formula $\mathrm{Cl}^\star(H) = H \cup H^\star$ determines Kuratowski's closure operator [2, 11, 13, 27], and henceforth $\mathbf{Z}$ gets a new topology, named $\star$-topology [1,2,8–10,16,23], induced by $\mathrm{Cl}^\star$. Let's name this topology as $\mathbb{T}^\star$. Clearly, $\mathbb{T} \subseteq \mathbb{T}^\star$ (see [11]). The interior operator of the space $\zeta^\star = (\mathbf{Z}, \mathbb{T}^\star)$ is given by $Int^\star(H) = \mathbf{Z} \setminus \mathrm{Cl}^\star(\mathbf{Z} \setminus H)$.

Moreover, if $H \subseteq H^\star$, then $H$ is known as $\star$-dense in itself [10], and if $H = H^\star$, then $H$ is termed as $\star$-perfect [10].

## 2. $L^\star$ Operator

We are beginning this section with an example to draw interest to the fact that through idealizing a space $\zeta$ by way of a proper ideal $\mathbb{I}$ (i.e., $\mathbf{Z} \notin \mathbb{I}$), one can find an $H \subseteq \mathbf{Z}$ for which $H^\star$ intersects $\Psi(H)$ i.e., the assertion '$K^\star \cap \Psi(K) = \varnothing$ for every $K \subseteq \mathbf{Z}$' need no longer be correct. The notations $\zeta_{\mathbb{I}}$ and $\zeta^\star_{\mathbb{I}}$ will be used to recognize respectively the triplets $(\mathbf{Z}, \mathbb{T}, \mathbb{I})$ and $(\mathbf{Z}, \mathbb{T}^\star, \mathbb{I})$, ideal topological spaces, in this write-up.

**Example 1.** *Consider $\mathbb{T} = \{\varnothing, \{\ell_1\}, \mathbf{Z}\}$ and $\mathbb{I} = \{\varnothing, \{\ell_2\}\}$ on $\mathbf{Z} = \{\ell_1, \ell_2, \ell_3\}$. Then for $H = \{\ell_1, \ell_2\}$, $H^\star = \mathbf{Z}$, $\Psi(H) = \{\ell_1\}$ and $H^\star \cap \Psi(H) \neq \varnothing$.*

**Definition 1.** *We define the $L^\star$ operator on $\zeta_{\mathbb{I}}$ as a set-valued map $L^\star : \wp(\mathbf{Z}) \to \wp(\mathbf{Z})$ by the equation $L^\star(H) = H^\star \cap \Psi(H)$ for every $H \subseteq \mathbf{Z}$.*

**Remark 1.** *As, we know from [11] that $H^\star(\mathbb{I}, \mathbb{T}) = H^\star(\mathbb{I}, \mathbb{T}^\star)$, so $L^\star$ values of every $H \subseteq \mathbf{Z}$ w.r.t. $\zeta_{\mathbb{I}}$ and $\zeta^\star_{\mathbb{I}}$ are same.*

We shall now discuss the value of $L^\star(H)$ for different ideals on a topological space.

- $\mathbb{I} = \{\varnothing\}$ implies $L^\star(H) = \mathrm{Cl}(H) \cap (\mathbf{Z} \setminus \mathrm{Cl}(\mathbf{Z} \setminus H)) = \mathrm{Cl}(H) \cap \mathrm{Int}(H) = \mathrm{Int}(H)$.
- $\mathbb{I} = \wp(\mathbf{Z})$ implies $L^\star(H) = \varnothing \cap \Psi(H) = \varnothing$.
- $\mathbb{I} = \mathbb{I}_n$ implies $L^\star(H) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(H))) \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(H))) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(H)))$.

- $\mathbb{I} = \mathbb{I}_f$ implies $L^\star(H) = H^\omega \cap (\mathbf{Z} \setminus (\mathbf{Z} \setminus H)^\omega) \subseteq H^\omega$.
- $\mathbb{I} = \mathbb{I}_c$ implies $L^\star(H) = H^{cd} \cap (\mathbf{Z} \setminus (\mathbf{Z} \setminus H)^{cd}) \subseteq H^{cd}$.

Study of the $L^\star$ operator will be therefore fascinating if we are deal with a non-trivial ideal (non-trivial means other than $\{\varnothing\}$ and $\wp(\mathbf{Z})$).

**Theorem 1.** *For $H$, $K \subseteq \mathbf{Z}$, the followings are true in $\zeta_{\mathbb{I}}$:*

(1) $L^\star(\varnothing) = \varnothing$,
(2) $L^\star(\mathbf{Z}) = \mathbf{Z}^\star$,
(3) $L^\star(\mathbf{Z}) = \mathbf{Z}$ *if and only if* $\mathbb{I} \cap \mathbb{T} = \{\varnothing\}$,
(4) $L^\star(H) = \Psi(H) \setminus \Psi(\mathbf{Z} \setminus H)$,
(5) $L^\star(H) = H^\star \setminus (\mathbf{Z} \setminus H)^\star$,
(6) $\mathbf{Z} \setminus L^\star(H) = (\mathbf{Z} \setminus H)^\star \cup (\mathbf{Z} \setminus H^\star)$,
(7) $L^\star(\mathbf{Z} \setminus H) = \mathbf{Z} \setminus (\Psi(H) \cup H^\star)$,
(8) *For* $H \subseteq K$, $L^\star(H) \subseteq L^\star(K)$,
(9) $L^\star(H) \cup L^\star(K) \subseteq L^\star(H \cup K)$,
(10) $L^\star(H \cap K) \subseteq L^\star(H) \cap L^\star(K)$,
(11) $L^\star(H) \subseteq H^\star$,
(12) $L^\star(H) \subseteq \Psi(H)$,
(13) $H \cap L^\star(H) = H^\star \cap \mathrm{Int}^\star(H)$,
(14) $H \cap L^\star(H) \subseteq \mathrm{Int}^\star(H)$,
(15) $L^\star(H) \subseteq H^\star \subseteq \mathrm{Cl}^\star(H) \subseteq \mathrm{Cl}(H)$,
(16) *For* $H \in \mathbb{T}^\star$, $H \cap H^\star \subseteq L^\star(H) \subseteq H^\star$,
(17) *For* $H \in \mathbb{T}$, $H \cap H^\star \subseteq L^\star(H) \subseteq H^\star$,
(18) *For a regular open* $H$ *[25]*, $L^\star(H) = H \cap H^\star$,
(19) $\mathrm{Int}(L^\star(H)) = \Psi(H) \cap \mathrm{Int}(H^\star)$,
(20) $\mathrm{Int}^\star(L^\star(H)) \supseteq \Psi(H) \cap \mathrm{Int}^\star(H^\star)$,
(21) $\mathrm{Cl}(L^\star(H)) \subseteq \mathrm{Cl}(\Psi(H)) \cap H^\star$,
(22) $\mathrm{Cl}^\star(L^\star(H)) \subseteq \mathrm{Cl}^\star(\Psi(H)) \cap H^\star$,
(23) $\mathrm{Int}^\star(H^\star) \cap \Psi(H) \subseteq \mathrm{Int}^\star(L^\star(H)) \subseteq \mathrm{Cl}^\star(L^\star(H)) \subseteq H^\star \cap \mathrm{Cl}^\star(\Psi(H))$,
(24) *For a $\star$-perfect set* $H$, $L^\star(H) = H \cap \Psi(H) = \mathrm{Int}^\star(H)$,
(25) *For a $\star$-dense in itself set* $H$, $L^\star(H) \supseteq \mathrm{Int}^\star(H)$.

*Proof.* (1) $L^\star(\varnothing) = \varnothing^\star \cap \Psi(\varnothing) = \varnothing$.
(2) $L^\star(\mathbf{Z}) = \mathbf{Z}^\star \cap \Psi(\mathbf{Z}) = \mathbf{Z}^\star \cap \mathbf{Z} = \mathbf{Z}^\star$.
(3) Follows from the fact $\mathbf{Z}^\star = \mathbf{Z}$ if and only if $\mathbb{I} \cap \mathbb{T} = \{\varnothing\}$.
(4) $L^\star(H) = H^\star \cap \Psi(H) = (\mathbf{Z} \setminus \Psi(\mathbf{Z} \setminus H)) \cap \Psi(H) = \Psi(H) \setminus \Psi(\mathbf{Z} \setminus H)$.
(5) $L^\star(H) = H^\star \cap \Psi(H) = H^\star \cap (\mathbf{Z} \setminus (\mathbf{Z} \setminus H)^\star) = H^\star \setminus (\mathbf{Z} \setminus H)^\star$.
(6) $\mathbf{Z} \setminus L^\star(H) = \mathbf{Z} \setminus (H^\star \cap \Psi(H)) = (\mathbf{Z} \setminus H^\star) \cup (\mathbf{Z} \setminus \Psi(H)) = (\mathbf{Z} \setminus H^\star) \cup (\mathbf{Z} \setminus H)^\star$.
(7) Obvious.
(8) Obvious.
(9) Follows from 8.
(10) Follows from 8.
(11) Obvious.

(12) Obvious.
(13) $H \cap L^\star(H) = H \cap (H^\star \cap \Psi(H)) = H^\star \cap \mathrm{Int}^\star(H)$.
(14) From 10, $L^\star(H) \subseteq \Psi(H)$. Therefore, $H \cap L^\star(H) \subseteq H \cap \Psi(H) = \mathrm{Int}^\star(H)$.
(15) Obvious from the fact $H^\star \subseteq H^\star \cup H = \mathrm{Cl}^\star(H) \subseteq \mathrm{Cl}(H)$.
(16) $H \in \mathbb{T}$ implies $H \subseteq \Psi(H)$. Now $L^\star(H) = H^\star \cap \Psi(H)$ implies $H^\star \cap H \subseteq L^\star(H)$.
(17) Obvious from the fact $\mathbb{T} \subseteq \mathbb{T}^\star$.
(18) Since $H$ is regular open, so $H = \Psi(H)$ [2,8,18]. Now, $L^\star(H) = H^\star \cap \Psi(H) = H^\star \cap H$.
(19) $\mathrm{Int}(L^\star(H)) = \mathrm{Int}(\Psi(H) \cap H^\star) = \mathrm{Int}(\Psi(H)) \cap \mathrm{Int}(H^\star) = \Psi(H) \cap \mathrm{Int}(H^\star)$.
(20) $\mathrm{Int}^\star(L^\star(H)) = \mathrm{Int}^\star(H^\star \cap \Psi(H)) = [H^\star \cap \Psi(H)] \cap \Psi[H^\star \cap \Psi(H)] = [H^\star \cap \Psi(H)] \cap [\Psi(H^\star) \cap \Psi(\Psi(H))] \supseteq [H^\star \cap \Psi(H)] \cap [\Psi(H^\star) \cap \Psi(H)] = [H^\star \cap \Psi(H^\star)] \cap \Psi(H) = \mathrm{Int}^\star(H^\star) \cap \Psi(H)$.
(21) Similar to 19.
(22) Similar to 19.
(23) Follows from 20.
(24) Trivial.
(25) Trivial.

□

Inequality of the result (9) of Theorem 1 is highlighted in next example.

**Example 2.** *Take $\mathbf{Z} = \mathbb{R}$ (set of reals) with usual topology and $\mathbb{I} = \{\varnothing\}$. Pick $H = [0, 2021)$ and $K = [2021, 2022)$. Then $L^\star(H) = \mathrm{Int}(H) = (0, 2021)$, $L^\star(K) = \mathrm{Int}(K) = (2021, 2022)$ and $L^\star(H \cup K) = L^\star([0, 2022)) = (0, 2022)$. Evidently, $L^\star(H) \cup L^\star(K) \neq L^\star(H \cup K)$.*

**Theorem 2.** *Suppose $\mathbb{I}$ is an ideal on $\zeta$ and $H \subseteq \mathbf{Z}$. If $a \in L^\star(H)$, then there exists at least one $K_a \in \mathbb{T}_a$ such that $K_a \notin \mathbb{I}$ but $K_a \setminus H \in \mathbb{I}$.*

*Proof.* $a \in L^\star(H)$ gives $a \in H^\star$ but $a \notin (\mathbf{Z} \setminus H)^\star$. Now, $a \notin (\mathbf{Z} \setminus H)^\star$ assures the existence of a $K_a \in \mathbb{T}_a$ such that $K_a \cap (\mathbf{Z} \setminus H) = K_a \setminus H \in \mathbb{I}$. On the other hand, $a \in H^\star$ tells that $K_a \cap H \notin \mathbb{I}$. This directs that $K_a \notin \mathbb{I}$, since $\mathbb{I}$ is an ideal. Hence, $K_a \notin \mathbb{I}$ but $K_a \setminus H \in \mathbb{I}$, as aimed. □

We talk about the validation of the converse part of Theorem 2 in next example.

**Example 3.** *Take $\mathbb{T} = \{\varnothing, \{\ell_1\}, \{\ell_2\}, \mathbf{Z}\}$ and $\mathbb{I} = \{\varnothing, \{\ell_1\}\}$ on $\mathbf{Z} = \{\ell_1, \ell_2\}$. Let $H = \{\ell_2\}$. Then $H^\star = \{\ell_2\}$ and $\Psi(H) = \mathbf{Z}$ and hence $L^\star(H) = \{\ell_2\}$. Now, pick up the point $\ell_1$ and choose $K_{\ell_1} = \mathbf{Z} \in \mathbb{T}_{\ell_1}$. Evidently, $K_{\ell_1} \notin \mathbb{I}$, $K_{\ell_1} \setminus H = \{\ell_1\} \in \mathbb{I}$ but $\ell_1 \notin L^\star(H)$. Therefore, the reverse direction of Theorem 2 will usually not work.*

## 3. $\star$-Locally Closed Sets

**Definition 2.** *We call an $H \subseteq \mathbf{Z}$ as $\star$-Locally closed in $\zeta_{\mathbb{I}}$ if there is a $K \subseteq \mathbf{Z}$ such that $H = L^\star(K)$, and use the symbol $L^\star(\zeta_{\mathbb{I}})$ to mean $\{H \subseteq \mathbf{Z} : H$ is $\star$-Locally closed$\}$.*

**Example 4.** *Topologize* $\mathbf{Z} = \mathbb{R}$ *by considering* $\mathbb{T} = \{\varnothing, \mathbb{Q}, \mathbb{R}\}$ *and* $\mathbb{I} = \wp(\mathbb{Q})$, *where* $\mathbb{Q}$ *is the set of all rationals. Then for any* $H \subseteq \mathbf{Z}$,

$$H^\star = \begin{cases} \varnothing, & \text{if } H \cap (\mathbb{R} \setminus \mathbb{Q}) = \varnothing \\ \mathbb{R} \setminus \mathbb{Q}, & \text{if } H \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \varnothing. \end{cases}$$

*Take* $L = \mathbb{R} \setminus \mathbb{Q}$. *We observe that* $L = L^\star \cap \Psi(L)$. *So,* $\mathbb{R} \setminus \mathbb{Q}$ *is a* $\star$*-Locally closed set.*

**Example 5.** *Consider* $\zeta_{\mathbb{I}}$ *discussed in Example 1, and take* $H = \{\ell_1\}$, $K = \{\ell_1, \ell_2\}$. *Since* $H = K^\star \cap \Psi(K)$, *so* $H$ *is* $\star$*-Locally closed in* $\zeta_{\mathbb{I}}$.

**Definition 3.** *An* $L \subseteq \mathbf{Z}$ *of a space* $\zeta$ *is familiar with the name locally closed [7] (resp., semi-locally closed [26],* $\lambda$*-locally closed [20]) if we can give the form* $L = H \cap K$, *where* $H$ *is open (resp., semi-open [14],* $\lambda$*-open [20]) and* $K$ *is closed (resp., semi-closed, closed).*

**Definition 4.** *An* $L \subseteq \mathbf{Z}$ *is addressed as* $\mathbb{I}$*-locally closed [6] (resp., semi-*$\mathbb{I}$*-locally closed [12]) if we can present* $L$ *as* $L = H \cap K$, *where* $H \in \mathbb{T}$ *and* $K$ *is* $\star$*-perfect (resp.,* $L = H \cap L^\star$, *where* $H$ *is semi-open). An equivalent definition of* $L$ *to be* $\mathbb{I}$*-locally closed is* $L = H \cap L^\star$, *where* $H \in \mathbb{T}$ *(see [12]).*

**Remark 2.** *As we know from [11],* $H^\star$ *is closed, and from [22],* $\Psi(H)$ *is open, it is derived that* $\star$*-Locally closed sets are locally closed. For reverse direction, we consider next example.*

**Example 6.** *Take* $\mathbb{T} = \{\varnothing, \{\ell_1\}, \{\ell_2\}, \{\ell_4\}, \{\ell_1, \ell_2\}, \{\ell_1, \ell_4\}, \{\ell_2, \ell_4\}, \{\ell_1, \ell_2, \ell_4\}, \mathbf{Z}\}$ *and* $\mathbb{I} = \{\varnothing, \{\ell_1\}, \{\ell_3\}, \{\ell_1, \ell_3\}\}$ *on* $\mathbf{Z} = \{\ell_1, \ell_2, \ell_3, \ell_4\}$. *Different values of* $K \subseteq \mathbf{Z}$ *under the operators* Cl, Int, $(\cdot)^\star$ *and* $\Psi$ *are considered in TABLE 1.*

TABLE 1. Values of $K \subseteq \mathbf{Z}$ under various operators

| $K$ | $\mathrm{Cl}(K)$ | $\mathrm{Int}(K)$ | $\mathrm{Cl}(\mathrm{Int}(K))$ | $K^\star$ | $\Psi(K)$ | $L^\star(K)$ |
|---|---|---|---|---|---|---|
| $\varnothing$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $\{\ell_1\}$ | $\varnothing$ |
| $\{\ell_1\}$ | $\{\ell_1, \ell_3\}$ | $\{\ell_1\}$ | $\{\ell_1, \ell_3\}$ | $\varnothing$ | $\{\ell_1\}$ | $\varnothing$ |
| $\{\ell_2\}$ | $\{\ell_2, \ell_3\}$ | $\{\ell_2\}$ | $\{\ell_2, \ell_3\}$ | $\{\ell_2, \ell_3\}$ | $\{\ell_1, \ell_2\}$ | $\{\ell_2\}$ |
| $\{\ell_3\}$ | $\{\ell_3\}$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $\{\ell_1\}$ | $\varnothing$ |
| $\{\ell_4\}$ | $\{\ell_3, \ell_4\}$ | $\{\ell_4\}$ | $\{\ell_3, \ell_4\}$ | $\{\ell_3, \ell_4\}$ | $\{\ell_1, \ell_4\}$ | $\{\ell_4\}$ |
| $\{\ell_1, \ell_2\}$ | $\{\ell_1, \ell_2, \ell_3\}$ | $\{\ell_1, \ell_2\}$ | $\{\ell_1, \ell_2, \ell_3\}$ | $\{\ell_2, \ell_3\}$ | $\{\ell_1, \ell_2\}$ | $\{\ell_2\}$ |
| $\{\ell_1, \ell_3\}$ | $\{\ell_1, \ell_3\}$ | $\{\ell_1\}$ | $\{\ell_1, \ell_3\}$ | $\varnothing$ | $\{\ell_1\}$ | $\varnothing$ |
| $\{\ell_1, \ell_4\}$ | $\{\ell_1, \ell_3, \ell_4\}$ | $\{\ell_1, \ell_4\}$ | $\{\ell_1, \ell_3, \ell_4\}$ | $\{\ell_3, \ell_4\}$ | $\{\ell_1, \ell_4\}$ | $\{\ell_4\}$ |
| $\{\ell_2, \ell_3\}$ | $\{\ell_2, \ell_3\}$ | $\{\ell_2\}$ | $\{\ell_2, \ell_3\}$ | $\{\ell_2, \ell_3\}$ | $\{\ell_1, \ell_2\}$ | $\{\ell_2\}$ |
| $\{\ell_2, \ell_4\}$ | $\{\ell_2, \ell_3, \ell_4\}$ | $\{\ell_2, \ell_4\}$ | $\{\ell_2, \ell_3, \ell_4\}$ | $\{\ell_2, \ell_3, \ell_4\}$ | $\mathbf{Z}$ | $\{\ell_2, \ell_3, \ell_4\}$ |
| $\{\ell_3, \ell_4\}$ | $\{\ell_3, \ell_4\}$ | $\{\ell_4\}$ | $\{\ell_3, \ell_4\}$ | $\{\ell_3, \ell_4\}$ | $\{\ell_1, \ell_4\}$ | $\{\ell_4\}$ |
| $\{\ell_1, \ell_2, \ell_3\}$ | $\{\ell_1, \ell_2, \ell_3\}$ | $\{\ell_1, \ell_2\}$ | $\{\ell_1, \ell_2, \ell_3\}$ | $\{\ell_2, \ell_3\}$ | $\{\ell_1, \ell_2\}$ | $\{\ell_2\}$ |
| $\{\ell_1, \ell_2, \ell_4\}$ | $\mathbf{Z}$ | $\{\ell_1, \ell_2, \ell_4\}$ | $\mathbf{Z}$ | $\{\ell_2, \ell_3, \ell_4\}$ | $\mathbf{Z}$ | $\{\ell_2, \ell_3, \ell_4\}$ |
| $\{\ell_1, \ell_3, \ell_4\}$ | $\{\ell_1, \ell_3, \ell_4\}$ | $\{\ell_1, \ell_4\}$ | $\{\ell_1, \ell_3, \ell_4\}$ | $\{\ell_3, \ell_4\}$ | $\{\ell_1, \ell_4\}$ | $\ell_4$ |
| $\{\ell_2, \ell_3, \ell_4\}$ | $\{\ell_2, \ell_3, \ell_4\}$ | $\{\ell_2, \ell_4\}$ | $\{\ell_2, \ell_3, \ell_4\}$ | $\{\ell_2, \ell_3, \ell_4\}$ | $\mathbf{Z}$ | $\{\ell_2, \ell_3, \ell_4\}$ |
| $\mathbf{Z}$ | $\mathbf{Z}$ | $\mathbf{Z}$ | $\mathbf{Z}$ | $\{\ell_2, \ell_3, \ell_4\}$ | $\mathbf{Z}$ | $\{\ell_2, \ell_3, \ell_4\}$ |

*We observe that $\{\ell_3\}$ is locally closed but not $\star$-Locally closed. Also, $\{\ell_2\}$ is $\star$-Locally closed but not $\star$-perfect whereas $\{\ell_2, \ell_3\}$ is $\star$-perfect but not $\star$-Locally closed. Further, $\{\ell_3, \ell_4\}$ is $\mathbb{I}$-locally closed but not $\star$-Locally closed; $\{\ell_2, \ell_4\}$ is semi-$\mathbb{I}$-locally closed but not $\star$-Locally closed. Here, $\star$-Locally closed sets are precisely $\varnothing$, $\{\ell_2\}$, $\{\ell_4\}$ and $\{\ell_2, \ell_3, \ell_4\}$, and these are also $\mathbb{I}$-locally closed and hence, they are semi-$\star$-locally closed (as we know from [12] that $\mathbb{I}$-locally closed implies semi-$\mathbb{I}$-locally closed). Because $\{\ell_4\}$ is $\star$-Locally closed is locally closed and hence, $\lambda$-locally closed (since locally closed implies $\lambda$-locally closed [20]), whereas $\{d\}$ in Example 2.3 of [20] $\lambda$-locally closed but not $\star$-Locally closed.*

Following diagram will provide a transparent idea regarding different local versions of sets just discussed above:

$$\begin{tikzcd}
 & \lambda - \textit{locally closed} & \\
\star - \textit{Locally closed} \arrow[r, Rightarrow] & \textit{locally closed} \arrow[u, Rightarrow] \arrow[r, Rightarrow] & \textit{semi} - \textit{locally closed} \\
\star - \textit{perfect} \arrow[r, Rightarrow] & \mathbb{I} - \textit{locally closed} \arrow[u, Rightarrow] \arrow[r, Rightarrow] & \textit{semi} - \mathbb{I} - \textit{locally closed} \arrow[u, Rightarrow]
\end{tikzcd}$$

FIGURE 1. Implication Diagram

**Theorem 3.** *If $H$ be $\star$-dense in itself and $\star$-Locally closed in $\zeta_{\mathbb{I}}$, then $H$ is $\mathbb{I}$-locally closed.*

*Proof.* Straightforward. □

**Corollary 1.** *If $H$ be $\star$-dense in itself and $\star$-Locally closed in $\zeta_{\mathbb{I}}$, then $H$ is semi-$\mathbb{I}$-locally closed.*

**Theorem 4.** *An $L \subseteq \mathbf{Z}$ is $\star$-Locally closed in $\zeta_{\mathbb{I}}$ if and only if $L = H^\star \setminus (\mathbf{Z} \setminus H)^\star$ for some $H \subseteq \mathbf{Z}$.*

*Proof.* Immediate from Theorem 1(5). □

**Theorem 5.** *An $L \subseteq \mathbf{Z}$ is $\star$-Locally closed in $\zeta_{\mathbb{I}}$ if and only if $L = \Psi(H) \setminus \Psi(\mathbf{Z} \setminus H)$ for some $H \subseteq \mathbf{Z}$.*

*Proof.* Immediate from Theorem 1(4). □

**Theorem 6.** *An $L \subseteq \mathbf{Z}$ is $\star$-Locally closed in $\zeta_{\mathbb{I}}$ if and only if $\mathbf{Z} \setminus L = (\mathbf{Z} \setminus H)^\star \cup (\mathbf{Z} \setminus H^\star)$ for some $H \subseteq \mathbf{Z}$.*

*Proof.* Obvious from Theorem 1(6). □

It is known that in $\zeta$, open as well as closed sets are locally closed whereas in $\zeta_{\mathbb{I}}$, this occurrence need not longer be true in case of $\star$-Locally closedness. For this purpose, consider the next example.

**Example 7.** *Think about Example 3, and pick $\{\ell_1\}$, a clopen set. Since no $H \subseteq \mathbf{Z}$ satisfies $\{\ell_1\} = H^\star \cap \Psi(H)$, $\{\ell_1\}$ is not $\star$-Locally closed in $\zeta_{\mathbb{I}}$.*

**Theorem 7.** *If $\mathbb{I} \cap \mathbb{T} = \{\varnothing\}$, then every regular open set is $\star$-Locally closed in $\zeta_{\mathbb{I}}$.*

*Proof.* Pick a regular open set $H$. So $H = \Psi(H)$. Now, $\mathbb{I} \cap \mathbb{T} = \{\varnothing\}$ yields $H \subseteq H^\star$. Evidently, $H^\star \cap \Psi(H) = H$. This allows that $H \in L^\star(\zeta_{\mathbb{I}})$. □

**Example 8.** *Following facts are observed in a $\zeta_{\mathbb{I}}$:*

- *In Example 3, $\{\ell_2\}$ is $\star$-Locally closed but its complement $\{\ell_1\}$ is not.*
- *In Example 3, $\{\ell_2\}$ is $\star$-Locally closed but its super set $\{\ell_2, \ell_3\}$ is not.*
- *In Example 6, $\{\ell_2, \ell_3, \ell_4\}$ is $\star$-Locally closed but its subset $\{\ell_2, \ell_3\}$ is not.*
- *In Example 6, for the subset $\{\ell_2, \ell_3, \ell_4\}$, $L^\star(\{\ell_2, \ell_3, \ell_4\})$ is not open.*
- *In Example 6, for the subset $\{\ell_4\}$, $L^\star(\{\ell_4\})$ is not closed.*
- *In Example 6, $\{\ell_2\}$ and $\{\ell_4\}$ are $\star$-closed but their union $\{\ell_2, \ell_4\}$ is not.*

**Remark 3.** *From above example, we say that the compilation $L^\star(\zeta_{\mathbb{I}})$ usually does not form a topology, boolean algebra, generalized topology [15], ideal, filter [4] and grill [5, 17].*

## 4. Homeomorphisms

Though this entire section, an ideal $\mathbb{I}$ is considered as proper, $\vartheta$ as $(\mathbf{W}, \mathbb{O})$ and $\vartheta_{\curlyvee(\mathbb{I})}$ as $(\mathbf{W}, \mathbb{O}, \curlyvee(\mathbb{I}))$.

**Lemma 1.** *[24] If an ideal $\mathbb{I}$ on $\mathbf{Z}$ be proper and $\curlyvee : \mathbf{Z} \to \mathbf{W}$ bijective, then the ideal $\curlyvee(\mathbb{I}) = \{\curlyvee(I) : I \in \mathbb{I}\}$ is proper on $\mathbf{W}$.*

Below, we now disclose that 'bijectivity' of $\curlyvee$ in Lemma 1 is sufficient to carry a (proper) ideal to a (proper) ideal.

**Lemma 2.** *Suppose $\curlyvee : \mathbf{Z} \to \mathbf{W}$ is a map, and $\mathbb{I}$ an ideal on $\mathbf{Z}$. Then $\curlyvee(\mathbb{I})$ defined in Lemma 1 is an ideal on $\mathbf{W}$. Moreover, injectivity of $\curlyvee$ preserves 'properness' of $\mathbb{I}$.*

*Proof.* Firstly, $\varnothing \in \mathbb{I}$ (since an ideal) implies $\curlyvee(\varnothing) \in \curlyvee(\mathbb{I})$. But $\curlyvee(\varnothing) = \varnothing$. So, $\varnothing \in \curlyvee(\mathbb{I})$. Secondly, pick $E_1$, $E_2 \in \curlyvee(\mathbb{I})$. Then, by the definition of $\curlyvee(\mathbb{I})$, choose $I_1$, $I_2 \in \mathbb{I}$ such that $E_1 = \curlyvee(I_1)$ and $E_2 = \curlyvee(I_2)$. Now, $E_1 \cup E_2 = \curlyvee(I_1) \cup \curlyvee(I_2) = \curlyvee(I_1 \cup I_2) = \curlyvee(I_3)$, where $I_3 = I_1 \cup I_2 \in \mathbb{I}$ (since $\mathbb{I}$ is ideal). This permits that $E_1 \cup E_2 \in \curlyvee(\mathbb{I})$. Lastly, take $F_1 \subseteq F_2$ and $F_2 \in \curlyvee(\mathbb{I})$. So, there is an $I \in \mathbb{I}$ such that $F_2 = \curlyvee(I)$. Now, $F_1 \subseteq \curlyvee(I) = \{\curlyvee(u) : u \in I\}$ knocks us to construct an $I_0 \subseteq \mathbf{Z}$ as: 'Pick those $u \in I$ whose images under $\curlyvee$ goes to $F_1$, and keep such $u$ in $I_0$'. Thus, $I_0 = \{u \in I : \curlyvee(u) \in F_1\}$. Clearly, $\curlyvee(I_0) = F_1$ and $I_0 \subseteq I$. Because $\mathbb{I}$ is an ideal, $I \in \mathbb{I}$ implies $I_0 \in \mathbb{I}$. This again implies $\curlyvee(I_0) \in \curlyvee(\mathbb{I})$ i.e., $F_1 \in \curlyvee(\mathbb{I})$. Thus, we

finally present that $\Upsilon(\mathbb{I})$ is an ideal on $\mathbf{W}$.
For second part, suppose $\mathbb{I}$ is proper and $\Upsilon$ injective. Claim: $\Upsilon(\mathbb{I})$ is proper i.e., $\mathbf{W} \notin \Upsilon(\mathbb{I})$. If not, there exists $I \in \mathbb{I}$ such that $\Upsilon(I) = \mathbf{W}$. Now, $I \subseteq \mathbf{Z}$ implies $\mathbf{W} = \Upsilon(I) \subseteq \Upsilon(\mathbf{Z}) \subseteq \mathbf{W}$ whence $\Upsilon(I) = \Upsilon(\mathbf{Z})$. This yields $\Upsilon^{-1}(\Upsilon(I)) = \Upsilon^{-1}(\Upsilon(\mathbf{Z}))$ implies $I = \mathbf{Z}$ (since $\Upsilon$ is injective). So, $\mathbf{Z} \in \mathbb{I}$, a contradiction. □

As consequences of the Lemma 1 we have following:

**Theorem 8.** *Let $\Upsilon : \zeta_{\mathbb{I}} \to \vartheta$ is a homeomorphism. Then, for every $H \subseteq \mathbf{Z}$, we have*

(1) $\Upsilon[H^\star(\mathbb{I})] = [\Upsilon(H)]^\star(\Upsilon(\mathbb{I}))$,
(2) $\Upsilon[\Psi(H)(\mathbb{I})] = \Psi[\Upsilon(H)](\Upsilon(\mathbb{I}))$.

*Proof.* (1) Assume $v \notin [\Upsilon(H)]^\star(\Upsilon(\mathbb{I}))$. Pick an $E \in \mathbb{O}$ such that $v \in E$ and $E \cap \Upsilon(H) \in \Upsilon(\mathbb{I})$. Draw an $I \in \mathbb{I}$ such that $\Upsilon(I) = E \cap \Upsilon(H)$. Because $\Upsilon$ is injective, $\Upsilon^{-1}(E) \cap H = \Upsilon^{-1}(E) \cap \Upsilon^{-1}(\Upsilon(H)) = \Upsilon^{-1}(E \cap \Upsilon(H)) = \Upsilon^{-1}(\Upsilon(I)) = I \in \mathbb{I}$, where $\Upsilon^{-1}(E) \in \mathbb{T}_{\Upsilon^{-1}(v)}$ (by continuity of $\Upsilon$). This tells that $\Upsilon^{-1}(v) \notin H^\star(\mathbb{I})$, and we have $v \notin \Upsilon[H^\star(\mathbb{I})]$. So, $\Upsilon[H^\star(\mathbb{I})] \subseteq [\Upsilon(H)]^\star(\Upsilon(\mathbb{I}))$. Reversely, pick $u \in \mathbf{W}$ such that $u \notin \Upsilon[H^\star(\mathbb{I})]$. Then, $\Upsilon^{-1}(u) \notin H^\star(\mathbb{I})$. There is $G \in \mathbb{T}_{\Upsilon^{-1}(u)}$ such that $G \cap H \in \mathbb{I}$. So, $\Upsilon(G) \cap \Upsilon(H) = \Upsilon(G \cap H) \in \Upsilon(\mathbb{I})$, where $\Upsilon(G) \in \mathbb{O}_u$. This highlights that $u \notin [\Upsilon(H)]^\star(\Upsilon(\mathbb{I}))$. Therefore, $[\Upsilon(H)]^\star(\Upsilon(\mathbb{I})) \subseteq \Upsilon[H^\star(\mathbb{I})]$. Hence, the result.
(2) $\Upsilon[\Psi(H)(\mathbb{I})] = \Upsilon[\mathbf{Z} \setminus (\mathbf{Z} \setminus H)^\star(\mathbb{I})] = \mathbf{W} \setminus \Upsilon[(\mathbf{Z} \setminus H)^\star(\mathbb{I})] = \mathbf{W} \setminus [\Upsilon(\mathbf{Z} \setminus H)]^\star(\Upsilon(\mathbb{I}))$ (by first part) $= \mathbf{W} \setminus [\mathbf{W} \setminus \Upsilon(H)]^\star(\Upsilon(\mathbb{I})) = \Psi[\Upsilon(H)](\Upsilon(\mathbb{I}))$.
□

**Theorem 9.** *For a homeomorphism $\Upsilon : \zeta_{\mathbb{I}} \to \vartheta_{\Upsilon(\mathbb{I})}$, followings are well fulfilled:*

(1) *if $H$ be $\star$-perfect in $\zeta_{\mathbb{I}}$, then $\Upsilon(H)$ is $\star$-perfect in $\vartheta_{\Upsilon(\mathbb{I})}$,*
(2) *if $H$ be $\mathbb{I}$-locally closed in $\zeta_{\mathbb{I}}$, then $\Upsilon(H)$ is $\Upsilon(\mathbb{I})$-locally closed in $\vartheta_{\Upsilon(\mathbb{I})}$,*
(3) *if $H$ be semi-$\mathbb{I}$-locally closed in $\zeta_{\mathbb{I}}$, then $\Upsilon(H)$ is semi-$\Upsilon(\mathbb{I})$-locally closed in $\vartheta_{\Upsilon(\mathbb{I})}$.*

*Proof.* First two results are straightforward from Theorem 8 (1), and third one follows from Theorem 8 (1) and the fact that '$E$ is semi-open implies $\Upsilon(E)$ is semi-open'. □

For more homeomorphic image regarding $(\cdot)^\star$ and $\Psi$ operators interested readers can see [19].

**Theorem 10.** *For a homeomorphism $\Upsilon : \zeta_{\mathbb{I}} \to \vartheta_{\Upsilon(\mathbb{I})}$ and for $H \subseteq \mathbf{Z}$, we have*

(1) $\Upsilon[L^\star(H)(\mathbb{I})] = L^\star[\Upsilon(H)](\Upsilon(\mathbb{I}))$,
(2) $H \in L^\star(\zeta_{\mathbb{I}})$ *implies* $\Upsilon(H) \in L^\star(\vartheta_{\Upsilon(\mathbb{I})})$.

*Proof.* First one is derived from Theorem 8, and second one is a consequence of first part. □

**Lemma 3.** *[24] If an ideal $\mathbb{J}$ on $\mathbf{W}$ be proper and $\Upsilon : \mathbf{Z} \to \mathbf{W}$ surjective, then the ideal $\Upsilon^{-1}(\mathbb{J}) := \{\Upsilon^{-1}(J) : J \in \mathbb{J}\}$ is proper on $\mathbf{Z}$.*

Below, by presenting a sophisticated counterexample, we will show the Lemma 3 is wrong.

**Example 9.** *Consider the map $\Upsilon : \mathbb{Z} \to \mathbb{N} \cup \{0\}$ as $x \mapsto |x|$. Here, $\mathbb{Z}$ and $\mathbb{N}$ denote the set of all integers and the set of all positive integers, respectively, and $|\cdot|$ is the modulus function. Note that $\Upsilon$ is surjective. Consider the subset $O$ of all odd positive integers, and take $\mathbb{J} = \wp(O)$. Then, $\mathbb{J}$ is a proper ideal on $\mathbb{N} \cup \{0\}$. Now, $\{1\} \in \mathbb{J}$ implies $\Upsilon^{-1}(\{1\}) = \{-1, +1\} \in \Upsilon^{-1}(\mathbb{J})$. Though $\{-1\} \subseteq \{-1, +1\}$, $\{-1\} \notin \Upsilon^{-1}(\mathbb{J})$. Thus, $\Upsilon^{-1}(\mathbb{J})$ is not an ideal on $\mathbb{Z}$.*

A modification of Lemma 3 is presented below:

**Lemma 4.** *Let $\Upsilon : \mathbf{Z} \to \mathbf{W}$ be a map, and $\mathbb{J}$ an ideal on $\mathbf{W}$. Then*

$$\Upsilon^{\leftarrow}(\mathbb{J}) := \{E \subseteq \mathbf{Z} : E \subseteq \Upsilon^{-1}(J),\ J \in \mathbb{J}\}$$

*is an ideal on $\mathbf{Z}$. In addition, surjectivity of $\Upsilon$ preserves 'properness' of $\mathbb{J}$.*

*Proof.* Firstly, $\varnothing \subseteq \Upsilon^{-1}(\varnothing)$, where $\varnothing \in \mathbb{J}$ (since an ideal) implies $\varnothing \in \Upsilon^{\leftarrow}(\mathbb{J})$. Secondly, take $E_1 \subseteq E_2$ and $E_2 \in \Upsilon^{\leftarrow}(\mathbb{J})$. There is a $J \in \mathbb{J}$ such that $E_2 \subseteq \Upsilon^{-1}(J)$, and so, $E_1 \subseteq \Upsilon^{-1}(J)$ implies that $E_1 \in \Upsilon^{\leftarrow}(\mathbb{J})$. Thirdly, consider $E_1$, $E_2 \in \Upsilon^{\leftarrow}(\mathbb{J})$. Then, pick $J_1$, $J_2 \in \mathbb{J}$ such that $E_1 \subseteq \Upsilon^{-1}(J_1)$ and $E_2 \subseteq \Upsilon^{-1}(J_2)$. Now, $E_1 \cup E_2 \subseteq \Upsilon^{-1}(J_1) \cup \Upsilon^{-1}(J_2) = \Upsilon^{-1}(J_1 \cup J_2)$, where $J_1 \cup J_2 \in \mathbb{J}$ (since $\mathbb{J}$ is an ideal). Therefore, $E_1 \cup E_2 \in \Upsilon^{\leftarrow}(\mathbb{J})$. Thus, we demonstrate that $\Upsilon^{\leftarrow}(\mathbb{J})$ is an ideal on $\mathbf{Z}$.
For second part, consider $\Upsilon$ is surjective and $\mathbb{J}$ proper. Claim: $\Upsilon^{\leftarrow}(\mathbb{J})$ is proper. If not so, $\mathbf{Z} \in \Upsilon^{\leftarrow}(\mathbb{J})$. Choose $J \in \mathbb{J}$ such that $\mathbf{Z} \subseteq \Upsilon^{-1}(J)$. Because $\Upsilon$ is surjective, $\mathbf{W} = \Upsilon(\mathbf{Z}) \subseteq \Upsilon(\Upsilon^{-1}(J)) = J \subseteq \mathbf{W}$ implies $\mathbf{W} = J \in \mathbb{J}$, a contradiction. □

We demonstrate another modification of Lemma 3 in next corollary:

**Corollary 2.** *If $\Upsilon$ be bijective, then $\Upsilon^{-1}(\mathbb{J})$ of Lemma 3 coincides with $\Upsilon^{\leftarrow}(\mathbb{J})$, and hence, becomes an ideal.*

*Proof.* It is transparent from the fact 'for each $J \in \mathbb{J}$, $\Upsilon^{-1}(J) \subseteq \Upsilon^{-1}(J)$' that $\Upsilon^{-1}(\mathbb{J}) \subseteq \Upsilon^{\leftarrow}(\mathbb{J})$. For backward part, let's pick an $E \in \Upsilon^{\leftarrow}(\mathbb{J})$. Then, $E \subseteq \Upsilon^{-1}(J)$ for some $J \in \mathbb{J}$. Because $\Upsilon$ is surjective, $\Upsilon(E) \subseteq \Upsilon(\Upsilon^{-1}(J)) = J$ implies $\Upsilon(E) \in \mathbb{J}$. Because $\Upsilon$ is injective, $E = \Upsilon^{-1}(\Upsilon(E)) \in \Upsilon^{-1}(\mathbb{J})$. Thus, $\Upsilon^{\leftarrow}(\mathbb{J}) \subseteq \Upsilon^{-1}(\mathbb{J})$, as aimed. □

As an application of Corollary 2, we have following important result:

**Theorem 11.** *For a homeomorphism $\Upsilon : \zeta_{\Upsilon^{-1}(\mathbb{J})} \to \vartheta_{\mathbb{J}}$, and for $K \subseteq \mathbf{W}$, we have*

1. $\Upsilon^{-1}[K^{\star}(\mathbb{J})] = [\Upsilon^{-1}(K)]^{\star}(\Upsilon^{-1}(\mathbb{J}))$,
2. $\Upsilon^{-1}[\Psi(K)(\mathbb{J})] = \Psi[\Upsilon^{-1}(K)](\Upsilon^{-1}(\mathbb{J}))$.

*Proof.* (1) Assume $u \notin [\Upsilon^{-1}(K)]^{\star}(\Upsilon^{-1}(\mathbb{J}))$. Select an $E \in \mathbb{T}_u$ for which $E \cap \Upsilon^{-1}(K) \in \Upsilon^{-1}(\mathbb{J})$. Draw a $J \in \mathbb{J}$ such that $E \cap \Upsilon^{-1}(K) = \Upsilon^{-1}(J)$. Because $\Upsilon$ is bijective, $\Upsilon(E) \cap K = \Upsilon(E) \cap \Upsilon(\Upsilon^{-1}(K)) = \Upsilon(E \cap \Upsilon^{-1}(K)) = \Upsilon(\Upsilon^{-1}(J)) = J \in \mathbb{J}$, where continuity of $\Upsilon^{-1}$ implies $\Upsilon(E) \in \mathbb{O}_{\Upsilon(u)}$. This states that $\Upsilon(u) \notin K^{\star}(\mathbb{J})$, and this again implies $u \notin \Upsilon^{-1}[K^{\star}(\mathbb{J})]$. Therefore, $\Upsilon^{-1}[K^{\star}(\mathbb{J})] \subseteq [\Upsilon^{-1}(K)]^{\star}(\Upsilon^{-1}(\mathbb{J}))$. For reverse part, pick $v \notin \Upsilon^{-1}[K^{\star}(\mathbb{J})]$. Then, $\Upsilon(v) \notin K^{\star}(\mathbb{J})$. Choose $F \in \mathbb{O}_{\Upsilon(v)}$ such that $F \cap K \in \mathbb{J}$. Continuity of $\Upsilon$ assures $\Upsilon^{-1}(F) \in \mathbb{T}_v$, and $\Upsilon^{-1}(F) \cap \Upsilon^{-1}(K) = \Upsilon^{-1}(F \cap K) \in \Upsilon^{-1}(\mathbb{J})$. This indicates $v \notin [\Upsilon^{-1}(K)]^{\star}(\Upsilon^{-1}(\mathbb{J}))$, and consequently $[\Upsilon^{-1}(K)]^{\star}(\Upsilon^{-1}(\mathbb{J})) \subseteq \Upsilon^{-1}[K^{\star}(\mathbb{J})]$.

(2) $\Upsilon^{-1}[\Psi(K)(\mathbb{J})] = \Upsilon^{-1}[\mathbf{W} \setminus (\mathbf{W} \setminus K)^{\star}(\mathbb{J})] = \mathbf{Z} \setminus \Upsilon^{-1}[(\mathbf{W} \setminus K)^{\star}(\mathbb{J})] = \mathbf{Z} \setminus [\Upsilon^{-1}(\mathbf{W} \setminus K)]^{\star}(\Upsilon^{-1}(\mathbb{J}))$ (by first part) $= \mathbf{Z} \setminus [\mathbf{Z} \setminus \Upsilon^{-1}(K)]^{\star}(\Upsilon^{-1}(\mathbb{J})) = \Psi[\Upsilon^{-1}(K)](\Upsilon^{-1}(\mathbb{J}))$. □

**Theorem 12.** *For a homeomorphism $\Upsilon : \zeta_{\Upsilon^{-1}(\mathbb{J})} \to \vartheta_{\mathbb{J}}$, followings are well fulfilled:*

(1) *if $K$ be $\star$-perfect in $\vartheta_{\mathbb{J}}$, then $\Upsilon^{-1}(K)$ is $\star$-perfect in $\zeta_{\Upsilon^{-1}(\mathbb{J})}$,*
(2) *if $K$ be $\mathbb{J}$-locally closed in $\vartheta_{\mathbb{J}}$, then $\Upsilon^{-1}(K)$ is $\Upsilon^{-1}(\mathbb{J})$-locally closed in $\zeta_{\Upsilon^{-1}(\mathbb{J})}$,*
(3) *if $K$ be semi-$\mathbb{J}$-locally closed in $\vartheta_{\mathbb{J}}$, then $\Upsilon^{-1}(K)$ is semi-$\Upsilon^{-1}(\mathbb{J})$-locally closed in $\zeta_{\Upsilon^{-1}(\mathbb{J})}$.*

*Proof.* First two results are straightforward from Theorem 11 (1), and third one follows from Theorem 11 (1) and the fact that '$F$ is semi-open implies $\Upsilon^{-1}(F)$ is semi-open'. □

**Theorem 13.** *For a homeomorphism $\Upsilon : \zeta_{\Upsilon^{-1}(\mathbb{J})} \to \vartheta_{\mathbb{J}}$, and for $K \subseteq \mathbf{W}$, we have*

(1) $\Upsilon^{-1}[L^{\star}(K)(\mathbb{J})] = L^{\star}[\Upsilon^{-1}(K)](\Upsilon^{-1}(\mathbb{J}))$,
(2) $K \in L^{\star}(\vartheta_{\mathbb{J}})$ *implies* $\Upsilon^{-1}(K) \in L^{\star}(\zeta_{\Upsilon^{-1}(\mathbb{J})})$.

*Proof.* First one is derived from Theorem 11, and second one is a consequence of first part. □

## 5. Conclusion

Kuratowski's local function '$(\cdot)^{\star}$' is a generalized operator of the classic closure operator 'Cl', and '$\Psi$' operator is a generalized operator of the classic interior operator 'Int'. On the other side, one can think Bourbaki's locally closed sets are applications of the operators 'Cl' and 'Int'. Replacing these classic operators by the updated generalized operator '$(\cdot)^{\star}$' and '$\Psi$', we derived a new version of locally closed set, and named $\star$-Locally closed. Example 6 and FIGURE 1 show that our $\star$-Locally closed version is a stronger form of locally closed set.

**Author Contribution Statements** Both the authors Jiarul Hoque and Shyamapada Modak have equal contribution in preparing this research article.

**Declaration of Competing Interests** The authors declare that there is no conflict of interest.

**Acknowledgements** The authors would like to thank the reviewers for their valuable suggestions and comments which helped to improve the paper.

## References

[1] Al-Omari, A., Al-Saadi, H., A topology via $\omega$-local functions in ideal spaces, *Mathematica*, 60(83) (2018), 103–110. https://doi.org/10.24193/mathcluj.2018.2.01
[2] Bandhopadhyay, C., Modak, S., A new topology via $\Psi$-operator, *Proc. Nat. Acad. Sci. India*, 76(A)(IV) (2006), 317–320.
[3] Bourbaki, N., Elements of Mathematics, General Topology, Addison-Wesley, Reading, Massachusetts, 1966.
[4] Bourbaki, N., General Topology, Chapter 1-4, Springer, 1989.
[5] Choquet, G., Sur les notions de filter et grill, *C.R. Acad. Sci. Paris*, 224 (1947), 171—173.
[6] Dontchev, J., Idealization of Ganster-Reilly decomposition theorems, arXIV:math. Gn/9901017v1 [math.GN], 1999. https://doi.org/10.48550/arXiv.math/9901017
[7] Ganster, M., Reilly, I.L., Locally closed sets and LC-continuous functions, *Int. J. Math. Math. Sci.*, 12(3) (1989), 417–424.
[8] Hamlett, T.R., Janković, D., Ideals in topological spaces and the set operator $\Psi$, *Boll. Uni. Mat. Ital.*, 7(4)(B) (1990), 863–874.
[9] Hashimoto, H., On the $\star$-topology and its applications, *Fund. Math.*, 91 (1976), 5–10.
[10] Hayashi, E., Topologies defined by local properties, *Math. Ann.*, 156 (1964), 205–215.
[11] Janković, D., Hamlett, T.R., New topologies from old via ideals, *Amer. Math. Monthly*, 97 (1990), 295–310. https://doi.org/10.2307/2324512
[12] Jeyanthi, V., Devi, V.R., Sivaraj, D., Subsets of ideal topological spaces, *Acta Math. Hungar.*, 114 (2007), 117–131. https://doi.org/10.1007/s10474-006-0517-7
[13] Kuratowski, K., Topology, Vol. I, Academic Press, New York, 1966.
[14] Levine, N., Semi-open sets and semi continuity in topological spaces, *Amer. Math. Monthly*, 70(1) (1963), 36–41. https://doi.org/10.2307/2312781
[15] Lugojan, S., Generalized topology, *Stud. Cerc. Mat.*, 34 (1982), 348–360 .
[16] Modak, S., Some new topologies on ideal topological spaces, *Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci.*, 82(3) (2012), 233–243. https://doi.org/10.1007/s40010-012-0039-3
[17] Modak, S., Minimal spaces with a mathematical structure, *J. Assoc. Arab Univ. Basic Appl. Sci.*, 22 (2017), 98–101. https://doi.org/10.1016/j.jaubas.2016.05.005
[18] Modak, S., Bandyopadhyay, C., A note on $\psi$-operator, *Bull. Malyas. Math. Sci. Soc.*, 30(1) (2007), 43–48.
[19] Modak, S., Hoque, J., Selim, Sk., Homeomorphic image of some kernels, *Çankaya Univ. J. Sci. Eng.*, 17(1) (2020), 52-62.
[20] Modak, S., Noiri, T., Some generalizations of locally closed sets, *Iran. J. Math. Sci. Inform.*, 14 (2019), 159–165. https://doi.org/10.7508/ijmsi.2019.01.014
[21] Modak, S., Noiri, T., Remarks on locally closed set, *Acta Comment. Univ. Tartu. Math.*, 22(1) (2018), 57–64. http://dx.doi.org/10.12697/ACUTM.2018.22.06
[22] Natkaniec, T., On $\mathcal{I}$-continuity and $\mathcal{I}$-semicontinuity points, *Math. Slovaca*, 36(3) (1986), 297–312. http://dml.cz/dmlcz/128786

[23] Samuel, P., A topology formed from a given topology and ideal, *J. Lond. Math. Soc.*, 10 (1975), 409–416. https://doi.org/10.1112/jlms/s2-10.4.409
[24] Selim, Sk., Noiri, T., Modak, S., Ideals and the associated filters on topological spaces, *Euras. Bull. Math.*, 2(3) (2019), 80–85.
[25] Stone, M.H., Applications of the theory of Boolean rings to general topology, *Trans. Amer. Math. Soc.*, 41 (1937), 375–481. https://doi.org/10.2307/1989788
[26] Sundaram, P., Balachandran, K., Semi generalized locally closed sets in topological spaces, preprint.
[27] Vaidyanathswamy, R., Set Topology, Chelsea Publishing Co., New York, 1960.

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 37–63 (2024)
DOI:10.31801/cfsuasmas.1249576
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

*Research Article; Received: February 9, 2023; Accepted: September 19, 2023*

# GENERALIZED BIVARIATE CONDITIONAL FIBONACCI AND LUCAS HYBRINOMIALS

Sure KÖME[1] and Zeynep KUMTAS DALLAROĞLU[2]

[1,2]Department of Mathematics, Nevşehir Hacı Bektaş Veli University, Nevşehir, TÜRKİYE

Abstract. The Hybrid numbers are generalizations of complex, hyperbolic and dual numbers. In recent years, studies related with hybrid numbers have been increased significantly. In this paper, we introduce the generalized bivariate conditional Fibonacci and Lucas hybrinomials. Also, we present the Binet formula, generating functions, some significant identities, Catalan's identities and Cassini's identities of the generalized bivariate conditional Fibonacci and Lucas hybrinomials. Finally, we give more general results compared to the previous works.

## 1. Introduction

The Fibonacci and Lucas numbers are defined by

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n \geqslant 2 \end{cases} \text{ and } L_n = \begin{cases} 2 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ L_{n-1} + L_{n-2} & \text{if } n \geqslant 2 \end{cases}, \quad (1)$$

respectively. For more information about the Fibonacci and Lucas numbers, we refer to book [9]. Until now, there have been interesting generalizations and applications of the Fibonacci and Lucas numbers [5–7, 12, 16]. For example, Falcon and Plaza found the general $k-$Fibonacci sequence $\{F_{k,n}\}_{n=0}^{\infty}$ by studying the recursive application of two geometrical transformations used in the well-known $4-$triangle longest-edge (4TLE) partition [7]. Furthermore, Edson and Yayenie [6] proposed the bi-periodic Fibonacci sequence. Also they gave generating function, the generalized Binet formula and some basic identities for $q_n$. By analogy to the studies [6] and [16], Bilgici [5] defined the bi-periodic Lucas numbers and he gave generating functions, the Binet formulas and some special identities for these sequences. Later,

2020 *Mathematics Subject Classification.* 05A19, 11B37.
*Keywords.* Bivariate conditional polynomials, hybrid numbers, Binet formula's, generating function, Catalan's identities, Cassini's identities.
[1] surekome@gmail.com-Corresponding author; 0000-0002-3558-0557
[2] mathzeynep2111@gmail.com; 0000-0002-7575-7644.

Yılmaz et al. [18] presented generalized of Fibonacci and Lucas polynomials. Also they obtained some new algebraic properties on these numbers and polynomials. Yazlık et al. introduced a novel extension of the Fibonacci and Lucas $p-$numbers and demonstrated that these generalized Fibonacci and Lucas $p-$sequences can be simplified into various other number sequences [17]. Ait-Amrane and Belbachir presented the bi-periodic $r-$Fibonacci sequence and its related family of companion sequences. They also explored the bi-periodic $r-$Lucas sequence of type $s$, where $s$ ranges from 1 to $r$, extending the classical Fibonacci and Lucas sequences. [1]. Belbachir and Bencherif [4] have generalized to bivariate polynomials of the Fibonacci and Lucas, properties obtained for Chebyshev polynomials. Ait-Amrane et al. presented a novel extension of hybrid polynomials, which combine elements of both Fibonacci and Lucas polynomials and studied various fundamental characteristics of these polynomials, including recurrence relations, generating functions, Binet formulas, summation formulas, and a matrix representation [2]. Panwar and Singh [11] introduced a generalized bivariate Fibonacci-Like polynomials sequence. Bala and Verma [15] presented the generalized Bivariate bi-periodic Fibonacci polynomials.

For any nonzero real numbers $a, b, c$ and $d$, the generalization of bivariate bi-periodic Fibonacci polynomial is defined as [15],

$$B_n(x,y) = \begin{cases} axB_{n-1}(x,y) + cyB_{n-2}(x,y), & \text{if } n \text{ is even} \\ bxB_{n-1}(x,y) + dyB_{n-2}(x,y), & \text{if } n \text{ is odd} \end{cases}, n \geq 2 \tag{2}$$

where $B_0(x,y) = 0, B_1(x,y) = 1$. Also, the authors obtained Catalan's identity, Cassini's identity, d'Ocagne identity and Gelin Cesaro identity along with Generating function and Binet's formula for the bivariate bi-periodic Fibonacci polynomial. The authors presented the generating function of the bivariate bi-periodic Fibonacci polynomial as:

$$G(t) = \frac{t + axt^2 - cyt^3}{1 - (abx^2 + (c+d)y)t^2 + cdy^2t^4}. \tag{3}$$

Moreover, they obtained Binet's formula for the bivariate bi-periodic Fibonacci polynomial as:

$$B_n(x,y) = \frac{(ax)^{1-\xi(n)}}{(abx^2)^{\lfloor \frac{n}{2} \rfloor}} \left( \frac{\beta_1^{\lfloor \frac{n}{2} \rfloor}(\beta_1 + (d-c)y)^{n-\lfloor \frac{n}{2} \rfloor} - \beta_2^{\lfloor \frac{n}{2} \rfloor}(\beta_2 + (d-c)y)^{n-\lfloor \frac{n}{2} \rfloor}}{\beta_1 - \beta_2} \right). \tag{4}$$

Then, Bala and Verma [3] defined the bivariate bi-periodic Lucas polynomials as follows:

For any nonzero real numbers $a_1$ and $a_2$, the generalization of bivariate bi-periodic Lucas polynomial is defined as [3],

$$l_n(x,y) = \begin{cases} a_1xl_{n-1}(x,y) + yl_{n-2}(x,y), & \text{if } n \text{ is even} \\ a_2xl_{n-1}(x,y) + yl_{n-2}(x,y), & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2 \tag{5}$$

where, $l_0(x,y) = 2, l_1(x,y) = a_2x$.

Özdemir [10] introduced the hybrid numbers as a new generalization of complex, hyperbolic and dual numbers. The set of hybrid numbers, denoted by $\mathbb{K}$, is defined as

$$\mathbb{K}=\left\{a+b\mathbf{i}+c\boldsymbol{\varepsilon}+d\mathbf{h}:\quad a,b,c,d\in\mathbb{R},\mathbf{i}^2=-1,\boldsymbol{\varepsilon}^2=0,\mathbf{h}^2=1,\mathbf{ih}=-\mathbf{hi}=\boldsymbol{\varepsilon}+\mathbf{i}\right\}. \quad (6)$$

The following table presents products of $\mathbf{i}$, $\boldsymbol{\varepsilon}$, and $\mathbf{h}$.

Table 1. Products of $\mathbf{i}$, $\boldsymbol{\varepsilon}$, and $\mathbf{h}$

| $\times$ | $1$ | $\mathbf{i}$ | $\boldsymbol{\varepsilon}$ | $\mathbf{h}$ |
|---|---|---|---|---|
| $1$ | $1$ | $\mathbf{i}$ | $\boldsymbol{\varepsilon}$ | $\mathbf{h}$ |
| $\mathbf{i}$ | $\mathbf{i}$ | $-1$ | $1-\mathbf{h}$ | $\boldsymbol{\varepsilon}+\mathbf{i}$ |
| $\boldsymbol{\varepsilon}$ | $\boldsymbol{\varepsilon}$ | $\mathbf{h}+1$ | $0$ | $-\boldsymbol{\varepsilon}$ |
| $\mathbf{h}$ | $\mathbf{h}$ | $-\boldsymbol{\varepsilon}-\mathbf{i}$ | $\boldsymbol{\varepsilon}$ | $1$ |

This table shows that the multiplication operation in the hybrid numbers is not commutative, but associative. Liana [13] presented the special kind of hybrid numbers, namely Horadam hybrid numbers. Then, Kızılateş [8] obtained a new generalization of Fibonacci hybrid and Lucas hybrid numbers. He gave some algebraic properties of $q-$Fibonacci hybrid numbers and the $q-$Lucas hybrid numbers. Finally, Liana and Wloch [14] introduced the Fibonacci and Lucas hybrinomials, which can be considered as a generalization of the Fibonacci hybrid numbers and the Lucas hybrid numbers. Sevgi [12] defined the generalized Lucas hybrinomials with two variables. Also, he obtained the Binet formula, generating function and some properties for the generalized Lucas hybrinomials.

In the light of the above–cited recent works, some natural questions are that: can we define the bivariate conditional Fibonacci and Lucas Hybrinomials? Moreover, can we find the generating function, Binet formulas and some important identities for the bivariate conditional Fibonacci and Lucas Hybrinomials? In this study, we will investigate the answer to these questions.

This paper is structured in four section. First section includes preliminaries and literature review. In the second section, we define bivariate conditional Fibonacci hybrinomials and we give generating functions, Binet formulas and some important identities of these hybrinomials. In the third section, we discuss bivariate conditional Lucas polynomials and the bivariate conditional Lucas hybrinomials.

## 2. Generalized Bivariate Conditional Fibonacci Hybrinomials

In this section we give some identities of the generalized bivariate conditional Fibonacci hybrinomials. The next definition presents the bivariate conditional Fibonacci Hybrinomials.

**Definition 1.** *For any variables $x,y$ and nonzero real numbers $a,b,c$ and $d$, we have*

$$BH_n(x,y)=B_n(x,y)+\boldsymbol{i}B_{n+1}(x,y)+\varepsilon B_{n+2}(x,y)+\boldsymbol{h}B_{n+3}(x,y), \quad (7)$$

*where $B_n(x,y)$ was given in (2) and the initial conditions are $BH_0(x,y) = \boldsymbol{i} + \varepsilon ax + \boldsymbol{h}(abx^2+dy)$ and $BH_1(x,y) = 1 + \boldsymbol{i}ax + \varepsilon(abx^2+dy) + \boldsymbol{h}(a^2bx^3+adxy+acxy)$.*

We can see from the following table that the generalized bivariate conditional Fibonacci hybrinomials are the generalization of many works for different values of $a, b, c$ and $d$.

TABLE 2. The generalized bivariate conditional Fibonacci hybrinomials

| $a$ | $b$ | $c$ | $d$ | ***Generalized Bivariate Conditional Fibonacci Hybrinomials*** |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | *Bivariate Fibonacci Hybrinomials* |
| $a$ | $b$ | 1 | 1 | *Bivariate Conditional Fibonacci Hybrinomials* |
| 2 | 2 | 1 | 1 | *Bivariate Pell Hybrinomials* |
| 1 | 1 | 2 | 2 | *Bivariate Jacobsthal Hybrinomials* |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

**Lemma 1.** *For the generalized bivariate conditional Fibonacci hybrinomials $\{BH_n(x,y)\}_{n=0}^{\infty}$, we have*

$$\begin{aligned}
BH_{2n}(x,y) &= \left(abx^2+(c+d)y\right)BH_{2n-2}(x,y) - cdy^2 BH_{2n-4}(x,y)\\
BH_{2n+1}(x,y) &= \left(abx^2+(c+d)y\right)BH_{2n-1}(x,y) - cdy^2 BH_{2n-3}(x,y).
\end{aligned}$$

*Proof.* By using the definition of the generalized bivariate conditional Fibonacci hybrinomials, we obtain

$$\begin{aligned}
BH_{2n}(x,y) &= B_{2n}(x,y) + \mathbf{i}B_{2n+1}(x,y) + \varepsilon B_{2n+2}(x,y) + \mathbf{h}B_{2n+3}(x,y)\\
&= (axB_{2n-1}(x,y) + cyB_{2n-2}(x,y)) + \mathbf{i}\,(bxB_{2n}(x,y) + dyB_{2n-1}(x,y))\\
&\quad + \varepsilon(axB_{2n+1}(x,y) + cyB_{2n}(x,y))\\
&\quad + \mathbf{h}(bxB_{2n+2}(x,y) + dyB_{2n+1}(x,y))\\
&= [ax\,(bxB_{2n-2}(x,y) + dyB_{2n-3}(x,y)) + cyB_{2n-2}(x,y)]\\
&\quad + \mathbf{i}\,[bx\,(axB_{2n-1}(x,y) + cyB_{2n-2}(x,y)) + dyB_{2n-1}(x,y)]\\
&\quad + \varepsilon\,[ax\,(bxB_{2n}(x,y) + dyB_{2n-1}(x,y)) + cyB_{2n}(x,y))]\\
&\quad + \mathbf{h}\,[bx\,(axB_{2n+1}(x,y) + cyB_{2n}(x,y)) + dyB_{2n+1}(x,y)]\\
&= [\left(abx^2+cy\right)B_{2n-2}(x,y) + dy\,(axB_{2n-3}(x,y))]\\
&\quad + \mathbf{i}[\left(abx^2+dy\right)B_{2n-1}(x,y) + cy\,(bxB_{2n-2}(x,y))]\\
&\quad + \varepsilon[\left(abx^2+cy\right)B_{2n}(x,y) + dy\,(axB_{2n-1}(x,y))]\\
&\quad + \mathbf{h}[\left(abx^2+dy\right)B_{2n+1}(x,y) + cy\,(bxB_{2n}(x,y))]
\end{aligned}$$

$$
\begin{aligned}
&= [\left(abx^2 + cy\right) B_{2n-2}(x,y) + dy\left(B_{2n-2}(x,y) - cyB_{2n-4}(x,y)\right)] \\
&\quad + \mathbf{i}[\left(abx^2 + dy\right) B_{2n-1}(x,y) + cy\left(B_{2n-1}(x,y) - dyB_{2n-3}(x,y)\right)] \\
&\quad + \varepsilon[\left(abx^2 + cy\right) B_{2n}(x,y) + dy\left(B_{2n}(x,y) - cyB_{2n-2}(x,y)\right)] \\
&\quad + \mathbf{h}[\left(abx^2 + dy\right) B_{2n+1}(x,y) + cy\left(B_{2n+1}(x,y) - dyB_{2n-1}(x,y)\right)] \\
&= [\left(abx^2 + (c+d)\,y\right) B_{2n-2}(x,y) - cdy^2 B_{2n-4}(x,y)] \\
&\quad + \mathbf{i}[\left(abx^2 + (c+d)\,y\right) B_{2n-1}(x,y) - cdy^2 B_{2n-3}(x,y)] \\
&\quad + \varepsilon[\left(abx^2 + (c+d)\,y\right) B_{2n}(x,y) - cdy^2 B_{2n-2}(x,y)] \\
&\quad + \mathbf{h}[\left(abx^2 + (c+d)\,y\right) B_{2n+1}(x,y) - cdy^2 B_{2n-1}(x,y)] \\
&= \left(abx^2 + (c+d)\,y\right)[B_{2n-2}(x,y) + \mathbf{i}B_{2n-1}(x,y) + \varepsilon B_{2n}(x,y) + \mathbf{h}B_{2n+1}(x,y)] \\
&\quad - cdy^2[B_{2n-4}(x,y) + \mathbf{i}B_{2n-3}(x,y) + \varepsilon B_{2n-2}(x,y) + \mathbf{h}B_{2n-1}(x,y)] \\
&= \left(abx^2 + (c+d)\,y\right) BH_{2n-2}(x,y) - cdy^2 BH_{2n-4}(x,y).
\end{aligned}
$$

Similar to the above steps, we can obtain

$$BH_{2n+1}(x,y) = \left(abx^2 + (c+d)y\right) BH_{2n-1}(x,y) - cdy^2 BH_{2n-3}(x,y).$$

Thus, the proof is completed. □

Next, we give the generating function of the bivariate conditional Fibonacci hybrinomial $BH_n(x,y)$.

**Theorem 1.** *The generating function for the bivariate conditional Fibonacci hybrinomial $BH_n(x,y)$ is*

$$
\begin{aligned}
\mathfrak{G}(t) = \sum_{n=0}^{\infty} BH_n(x,y)t^n &= \frac{BH_0(x,y) + BH_1(x,y)t}{1 - (abx^2 + (c+d)\,y)\,t^2 + cdy^2t^4} \\
&\quad + \frac{\left[BH_2(x,y) - \left(abx^2 + (c+d)y\right) BH_0(x,y)\right] t^2}{1 - (abx^2 + (c+d)\,y)\,t^2 + cdy^2t^4} \\
&\quad + \frac{\left[BH_3(x,y) - \left(abx^2 + (c+d)\,y\right) BH_1(x,y)\right] t^3}{1 - (abx^2 + (c+d)\,y)\,t^2 + cdy^2t^4}.
\end{aligned} \tag{8}
$$

*Proof.* We define

$$
\begin{aligned}
\mathfrak{G}_0(t) &= \sum_{n=0}^{\infty} BH_{2n}(x,y)t^{2n} \\
\mathfrak{G}_1(t) &= \sum_{n=0}^{\infty} BH_{2n+1}(x,y)t^{2n+1}.
\end{aligned}
$$

So that

$$\mathfrak{G}(t) = \mathfrak{G}_0(t) + \mathfrak{G}_1(t).$$

We have

$$\begin{aligned}
\mathfrak{G}_0(t) &= \sum_{n=0}^{\infty} BH_{2n}(x,y)t^{2n} \\
&= \sum_{n=0}^{\infty} BH_{2n}(x,y)t^{2n} = BH_0(x,y)t^0 + BH_2(x,y)t^2 + \sum_{n=2}^{\infty} BH_{2n}(x,y)t^{2n} \\
&= BH_0(x,y) + BH_2(x,y)t^2 \\
&\quad + \sum_{n=2}^{\infty} \left[\left(abx^2+(c+d)y\right) BH_{2n-2}(x,y) - cdy^2 BH_{2n-4}(x,y)\right] t^{2n} \\
&= BH_0(x,y) + BH_2(x,y)t^2 + \left(abx^2+(c+d)y\right) t^2 \sum_{n=2}^{\infty} BH_{2n-2}(x,y)t^{2n-2} \\
&\quad - cdy^2t^4 \sum_{n=2}^{\infty} BH_{2n-4}(x,y)t^{2n-4} \\
&= BH_0(x,y) + BH_2(x,y)t^2 \\
&\quad + \left(abx^2+(c+d)y\right) t^2 \\
&\qquad \times \left[\sum_{n=2}^{\infty} BH_{2n-2}(x,y)t^{2n-2} + BH_0(x,y)t^0 - BH_0(x,y)t^0\right] \\
&\quad - cdy^2t^4\mathfrak{G}_0(t) \\
&= BH_0(x,y) + BH_2(x,y)t^2 + \left(abx^2+(c+d)y\right) t^2\mathfrak{G}_0(t) \\
&\quad - \left(abx^2+(c+d)y\right) t^2 BH_0(x,y) - cdy^2t^4\mathfrak{G}_0(t).
\end{aligned}$$

Thus, we get

$$\mathfrak{G}_0(t) = \frac{BH_0(x,y) + \left(BH_2(x,y) - \left(abx^2+(c+d)y\right) BH_0(x,y)\right) t^2}{1-\left(abx^2+(c+d)y\right) t^2 + cdy^2t^4}. \tag{9}$$

Similarly, we find

$$\begin{aligned}
\mathfrak{G}_1(t) &= \sum_{n=0}^{\infty} BH_{2n+1}(x,y)t^{2n+1} \\
&= \sum_{n=0}^{\infty} BH_{2n+1}(x,y)t^{2n+1} \\
&= BH_1(x,y)t + BH_3(x,y)t^3 + \sum_{n=2}^{\infty} BH_{2n+1}(x,y)t^{2n+1}
\end{aligned}$$

$$
\begin{aligned}
&= BH_1(x,y)t + BH_3(x,y)t^3 \\
&\quad + \sum_{n=2}^{\infty} \left[\left(abx^2 + (c+d)y\right) BH_{2n-1}(x,y) - cdy^2 BH_{2n-3}(x,y)\right] t^{2n+1} \\
&= BH_1(x,y)t + BH_3(x,y)t^3 + \left(abx^2 + (c+d)y\right) t^2 \sum_{n=2}^{\infty} BH_{2n-1}(x,y)t^{2n-1} \\
&\quad - cdy^2 t^4 \sum_{n=2}^{\infty} BH_{2n-3}(x,y)t^{2n-3} \\
&= BH_1(x,y)t + BH_3(x,y)t^3 \\
&\quad + \left(abx^2 + (c+d)y\right) t^2 \left[\sum_{n=2}^{\infty} BH_{2n-1}(x,y)t^{2n-1} + BH_1(x,y)t - BH_1(x,y)t\right] \\
&\quad - cdy^2 t^4 \mathfrak{G}_1(t) \\
&= BH_1(x,y)t + BH_3(x,y)t^3 + \left(abx^2 + (c+d)y\right) t^2 \mathfrak{G}_1(t) \\
&\quad - \left(abx^2 + (c+d)y\right) t^3 BH_1(x,y) - cdy^2 t^4 \mathfrak{G}_1(t).
\end{aligned}
$$

Therefore, we get

$$
\mathfrak{G}_1(t) = \frac{BH_1(x,y)t + \left(BH_3(x,y) - \left(abx^2 + (c+d)y\right) BH_1(x,y)\right) t^3}{1 - (abx^2 + (c+d)y)\, t^2 + cdy^2 t^4}. \tag{10}
$$

By virtue of (9) and (10), we can obtain

$$
\begin{aligned}
\mathfrak{G}(t) &= \mathfrak{G}_0(t) + \mathfrak{G}_1(t) \\
&= \sum_{n=0}^{\infty} BH_n(x,y)t^n \\
&= \frac{BH_0(x,y) + BH_1(x,y)t + \left[BH_2(x,y) - \left(abx^2 + (c+d)y\right) BH_0(x,y)\right] t^2}{1 - \left(abx^2 + (c+d)\, y\right) t^2 + cdy^2 t^4} \\
&\qquad + \frac{\left[BH_3(x,y) - \left(abx^2 + (c+d)\, y\right) BH_1(x,y)\right] t^3}{1 - \left(abx^2 + (c+d)\, y\right) t^2 + cdy^2 t^4}.
\end{aligned}
$$

Hence, the proof is completed. □

Now we give the Binet formula of the bivariate conditional Fibonacci hybrinomial $BH_n(x,y)$.

**Theorem 2.** *The $n^{th}$ term of the generalized bivariate conditional Fibonacci hybrinomial $BH_n(x,y)$ is*

$$
BH_n(x,y) = \frac{\widehat{\alpha}_{\xi(n)} \beta_1^{\lfloor \frac{n}{2} \rfloor} \left(\beta_1 + (d-c)y\right)^{\lfloor \frac{n}{2} \rfloor + \xi(n)} - \widehat{\gamma}_{\xi(n)} \beta_2^{\lfloor \frac{n}{2} \rfloor} \left(\beta_2 + (d-c)y\right)^{\lfloor \frac{n}{2} \rfloor + \xi(n)}}{(abx^2)^{\lfloor \frac{n}{2} \rfloor} (\beta_1 - \beta_2)}. \tag{11}
$$

*where* $\beta_1$ *and* $\beta_2$ *roots of the characteristic equation* $\lambda^2 - (abx^2 + (c-d)y)\lambda - abdx^2 y = 0$*. Also,*

$$\begin{aligned}\widehat{\alpha}_{\xi(n)} &= (ax)^{\xi(n+1)} + \boldsymbol{i}\frac{(ax)^{\xi(n)}\,\beta_1^{\xi(n)}}{(abx^2)^{\xi(n)}}\left(\beta_1 + (d-c)y\right)^{\xi(n+1)} \\ &\quad + \varepsilon\frac{(ax)^{\xi(n+1)}\,\beta_1}{(abx^2)}\left(\beta_1 + (d-c)y\right) \\ &\quad + \boldsymbol{h}\frac{(ax)^{\xi(n)}\,\beta_1^{\xi(n)+1}}{(abx^2)^{\xi(n)+1}}\left(\beta_1 + (d-c)y\right)^{\xi(n+1)+1}\end{aligned}$$

*and*

$$\begin{aligned}\widehat{\gamma}_{\xi(n)} &= (ax)^{\xi(n+1)} + \boldsymbol{i}\frac{(ax)^{\xi(n)}\,\beta_2^{\xi(n)}}{(abx^2)^{\xi(n)}}\left(\beta_2 + (d-c)y\right)^{\xi(n+1)} \\ &\quad + \varepsilon\frac{(ax)^{\xi(n+1)}\,\beta_2}{(abx^2)}\left(\beta_2 + (d-c)y\right) \\ &\quad + \boldsymbol{h}\frac{(ax)^{\xi(n)}\,\beta_2^{\xi(n)+1}}{(abx^2)^{\xi(n)+1}}\left(\beta_2 + (d-c)y\right)^{\xi(n+1)+1}.\end{aligned}$$

*Proof.* We use the following properties throughout the proof:

- $\beta_1 + \beta_2 = abx^2 + (c-d)y$
- $\beta_1 \cdot \beta_2 = -abdx^2 y$
- $(\beta_1 + dy)(\beta_2 + dy) = cdy^2$
- $(\beta_1 + dy)\left(abx^2\right) = \beta_1\left(\beta_1 + (d-c)y\right)$
- $(\beta_2 + dy)\left(abx^2\right) = \beta_2\left(\beta_2 + (d-c)y\right)$.

Note that $\beta_1(x,y) = \beta_1$ and $\beta_2(x,y) = \beta_2$. By using (4), we have

$$\begin{aligned}BH_{2n}(x,y) &= B_{2n}(x,y) + \mathbf{i}B_{2n+1}(x,y) + \varepsilon B_{2n+2}(x,y) + \mathbf{h}B_{2n+3}(x,y) \\ &= \frac{(ax)}{(abx^2)^n}\left[\frac{\beta_1^n\left(\beta_1 + (d-c)y\right)^n - \beta_2^n\left(\beta_2 + (d-c)y\right)^n}{\beta_1 - \beta_2}\right] \\ &\quad + \mathbf{i}\frac{1}{(abx^2)^n}\left[\frac{\beta_1^n\left(\beta_1 + (d-c)y\right)^{n+1} - \beta_2^n\left(\beta_2 + (d-c)y\right)^{n+1}}{\beta_1 - \beta_2}\right] \\ &\quad + \varepsilon\frac{(ax)}{(abx^2)^{n+1}}\left[\frac{\beta_1^{n+1}\left(\beta_1 + (d-c)y\right)^{n+1} - \beta_2^{n+1}\left(\beta_2 + (d-c)y\right)^{n+1}}{\beta_1 - \beta_2}\right] \\ &\quad + \mathbf{h}\frac{1}{(abx^2)^{n+1}}\left[\frac{\beta_1^{n+1}\left(\beta_1 + (d-c)y\right)^{n+2} - \beta_2^{n+1}\left(\beta_2 + (d-c)y\right)^{n+2}}{\beta_1 - \beta_2}\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{\beta_1^n \left(\beta_1 + (d-c)y\right)^n}{\left(abx^2\right)^n \left(\beta_1 - \beta_2\right)} \\
&\quad \times \left[ax + \mathbf{i}\left(\beta_1 + (d-c)y\right) + \varepsilon \frac{ax}{abx^2}\beta_1\left(\beta_1 + (d-c)y\right) + \mathbf{h}\frac{1}{abx^2}\beta_1\left(\beta_1 + (d-c)y\right)^2\right] \\
&\quad - \frac{\beta_2^n \left(\beta_2 + (d-c)y\right)^n}{\left(abx^2\right)^n \left(\beta_1 - \beta_2\right)} \\
&\quad \times \left[ax + \mathbf{i}\left(\beta_2 + (d-c)y\right) + \varepsilon \frac{ax}{abx^2}\beta_2\left(\beta_2 + (d-c)y\right) + \mathbf{h}\frac{1}{abx^2}\beta_2\left(\beta_2 + (d-c)y\right)^2\right]
\end{aligned}
$$

Here, we choose the $\widehat{\alpha}_0$ and $\widehat{\gamma}_0$ as follows:

$$
\begin{aligned}
\widehat{\alpha}_0 &= \left[ax + \mathbf{i}\left(\beta_1 + (d-c)y\right) + \varepsilon \frac{ax}{abx^2}\beta_1\left(\beta_1 + (d-c)y\right) + \mathbf{h}\frac{1}{abx^2}\beta_1\left(\beta_1 + (d-c)y\right)^2\right] \\
\widehat{\gamma}_0 &= \left[ax + \mathbf{i}\left(\beta_2 + (d-c)y\right) + \varepsilon \frac{ax}{abx^2}\beta_2\left(\beta_2 + (d-c)y\right) + \mathbf{h}\frac{1}{abx^2}\beta_2\left(\beta_2 + (d-c)y\right)^2\right].
\end{aligned}
$$

Finally, the following equation is obtained:

$$
BH_{2n}(x,y) = \frac{\widehat{\alpha}_0 \beta_1^n \left(\beta_1 + (d-c)y\right)^n - \widehat{\gamma}_0 \beta_2^n \left(\beta_2 + (d-c)y\right)^n}{\left(abx^2\right)^n \left(\beta_1 - \beta_2\right)}. \tag{12}
$$

In a similar way, by using (4), we have

$$
\begin{aligned}
BH_{2n+1}(x,y) &= B_{2n+1}(x,y) + \mathbf{i}B_{2n+2}(x,y) + \varepsilon B_{2n+3}(x,y) + \mathbf{h}B_{2n+4}(x,y) \\
&= \frac{1}{\left(abx^2\right)^n}\left[\frac{\beta_1^n \left(\beta_1 + (d-c)y\right)^{n+1} - \beta_2^n \left(\beta_2 + (d-c)y\right)^{n+1}}{\beta_1 - \beta_2}\right] \\
&\quad + \mathbf{i}\frac{(ax)}{\left(abx^2\right)^{n+1}}\left[\frac{\beta_1^{n+1} \left(\beta_1 + (d-c)y\right)^{n+1} - \beta_2^{n+1} \left(\beta_2 + (d-c)y\right)^{n+1}}{\beta_1 - \beta_2}\right] \\
&\quad + \varepsilon\frac{1}{\left(abx^2\right)^{n+1}}\left[\frac{\beta_1^{n+1} \left(\beta_1 + (d-c)y\right)^{n+2} - \beta_2^{n+1} \left(\beta_2 + (d-c)y\right)^{n+2}}{\beta_1 - \beta_2}\right] \\
&\quad + \mathbf{h}\frac{(ax)}{\left(abx^2\right)^{n+2}}\left[\frac{\beta_1^{n+2} \left(\beta_1 + (d-c)y\right)^{n+2} - \beta_2^{n+2} \left(\beta_2 + (d-c)y\right)^{n+2}}{\beta_1 - \beta_2}\right] \\
&= \frac{\beta_1^n \left(\beta_1 + (d-c)y\right)^{n+1}}{\left(abx^2\right)^n \left(\beta_1 - \beta_2\right)} \\
&\quad \times \left[1 + \mathbf{i}\frac{ax}{abx^2}\beta_1 + \varepsilon\frac{1}{abx^2}\beta_1\left(\beta_1 + (d-c)y\right) + \mathbf{h}\frac{ax}{(abx^2)^2}\beta_1^2\left(\beta_1 + (d-c)y\right)\right] \\
&\quad - \frac{\beta_2^n \left(\beta_2 + (d-c)y\right)^{n+1}}{\left(abx^2\right)^n \left(\beta_2 - \beta_2\right)} \\
&\quad \times \left[1 + \mathbf{i}\frac{ax}{abx^2}\beta_2 + \varepsilon\frac{1}{abx^2}\beta_2\left(\beta_2 + (d-c)y\right) + \mathbf{h}\frac{ax}{(abx^2)^2}\beta_2^2\left(\beta_2 + (d-c)y\right)\right].
\end{aligned}
$$

Here, we choose the $\widehat{\alpha}_1$ and $\widehat{\gamma}_1$ as follows;

$$\widehat{\alpha}_1=\left[1+\mathbf{i}\frac{ax}{abx^2}\beta_1+\varepsilon\frac{1}{abx^2}\beta_1\left(\beta_1+(d-c)y\right)+\mathbf{h}\frac{ax}{(abx^2)^2}\beta_1^2\left(\beta_1+(d-c)y\right)\right]$$
$$\widehat{\gamma}_1=\left[1+\mathbf{i}\frac{ax}{abx^2}\beta_2+\varepsilon\frac{1}{abx^2}\beta_2\left(\beta_2+(d-c)y\right)+\mathbf{h}\frac{ax}{(abx^2)^2}\beta_2^2\left(\beta_2+(d-c)y\right)\right].$$

Finally, the following equation is obtained.

$$BH_{2n+1}(x,y)=\frac{\widehat{\alpha}_1\beta_1^n\left(\beta_1+(d-c)y\right)^{n+1}-\widehat{\gamma}_1\beta_2^n\left(\beta_2+(d-c)y\right)^{n+1}}{\left(abx^2\right)^n\left(\beta_1-\beta_2\right)} \tag{13}$$

By virtue of (12) and (13), we can obtain the following equation.

$$BH_n(x,y)=\frac{\widehat{\alpha}_{\xi(n)}\beta_1^{\lfloor\frac{n}{2}\rfloor}\left(\beta_1+(d-c)y\right)^{\lfloor\frac{n}{2}\rfloor+\xi(n)}-\widehat{\gamma}_{\xi(n)}\beta_2^{\lfloor\frac{n}{2}\rfloor}\left(\beta_2+(d-c)y\right)^{\lfloor\frac{n}{2}\rfloor+\xi(n)}}{(abx^2)^{\lfloor\frac{n}{2}\rfloor}\left(\beta_1-\beta_2\right)}.$$

where $\beta_1$ and $\beta_2$ roots of the characteristic equation $\lambda^2-(abx^2+(c-d)y)\lambda-abdx^2y=0$. Also,

$$\begin{aligned}\widehat{\alpha}_{\xi(n)}&=(ax)^{\xi(n+1)}+\mathbf{i}\frac{(ax)^{\xi(n)}\beta_1^{\xi(n)}}{(abx^2)^{\xi(n)}}\left(\beta_1+(d-c)y\right)^{\xi(n+1)}\\&+\varepsilon\frac{(ax)^{\xi(n+1)}\beta_1}{(abx^2)}\left(\beta_1+(d-c)y\right)\\&+\mathbf{h}\frac{(ax)^{\xi(n)}\beta_1^{\xi(n)+1}}{(abx^2)^{\xi(n)+1}}\left(\beta_1+(d-c)y\right)^{\xi(n+1)+1}\end{aligned}$$

and

$$\begin{aligned}\widehat{\gamma}_{\xi(n)}&=(ax)^{\xi(n+1)}+\mathbf{i}\frac{(ax)^{\xi(n)}\beta_2^{\xi(n)}}{(abx^2)^{\xi(n)}}\left(\beta_2+(d-c)y\right)^{\xi(n+1)}\\&+\varepsilon\frac{(ax)^{\xi(n+1)}\beta_2}{(abx^2)}\left(\beta_2+(d-c)y\right)\\&+\mathbf{h}\frac{(ax)^{\xi(n)}\beta_2^{\xi(n)+1}}{(abx^2)^{\xi(n)+1}}\left(\beta_2+(d-c)y\right)^{\xi(n+1)+1}.\end{aligned}$$

□

Now, we give the Catalan's identity of the bivariate conditional Fibonacci hybrinomial $BH_n(x,y)$.

**Theorem 3.** *For any integers $n$ and $r$ and $n \geqslant r \geqslant 0$, we have*

$$
\begin{aligned}
& BH_{2(n+r)+\xi(i)}(x,y)BH_{2(n-r)+\xi(i)}(x,y) - \left(BH_{2n+\xi(i)}(x,y)\right)^2 \\
&= \frac{1}{(abx^2)^{2n}(\beta_1-\beta_2)^2} \\
&\quad \times \left[\widehat{\alpha}_{\xi(i)}\widehat{\gamma}_{\xi(i)}{\beta_1}^n{\beta_2}^n(\beta_1+(d-c)y)^{n+\xi(i)}(\beta_2+(d-c)y)^{n+\xi(i)}\left[1-\left(\frac{\beta_1(\beta_1+(d-c)y)}{\beta_2(\beta_2+(d-c)y)}\right)^r\right]\right] \\
&\quad + \frac{1}{(abx^2)^{2n}(\beta_1-\beta_2)^2} \\
&\quad \times \left[\widehat{\gamma}_{\xi(i)}\widehat{\alpha}_{\xi(i)}{\beta_2}^n{\beta_1}^n(\beta_2+(d-c)y)^{n+\xi(i)}(\beta_1+(d-c)y)^{n+\xi(i)}\left[1-\left(\frac{\beta_2(\beta_2+(d-c)y)}{\beta_2(\beta_2+(d-c)y)}\right)^r\right]\right],
\end{aligned}
$$

*where $\widehat{\alpha}_{\xi(i)}$ and $\widehat{\gamma}_{\xi(i)}$ are defined in Theorem(2) and $i \in \{0,1\}$.*

*Proof.* In order to prove Catalan's identity, we will examine in two different cases.
Case $i=0$:

$$
\begin{aligned}
BH_{2(n+r)}(x,y) &= \frac{\widehat{\alpha}_{\xi(2n+2r)}\beta_1^{\lfloor\frac{2n+2r}{2}\rfloor}(\beta_1+(d-c)y)^{\lfloor\frac{2n+2r}{2}\rfloor+\xi(2n+2r)}}{(abx^2)^{\lfloor\frac{2n+2r}{2}\rfloor}(\beta_1-\beta_2)} \\
&\quad - \frac{\widehat{\gamma}_{\xi(2n+2r)}\beta_2^{\lfloor\frac{2n+2r}{2}\rfloor}(\beta_2+(d-c)y)^{\lfloor\frac{2n+2r}{2}\rfloor+\xi(2n+2r)}}{(abx^2)^{\lfloor\frac{2n+2r}{2}\rfloor}(\beta_1-\beta_2)} \\
&= \frac{\widehat{\alpha}_0\beta_1^{n+r}(\beta_1+(d-c)y)^{n+r} - \widehat{\gamma}_0\beta_2^{n+r}(\beta_2+(d-c)y)^{n+r}}{(abx^2)^{n+r}(\beta_1-\beta_2)}
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
BH_{2(n-r)}(x,y) &= \frac{\widehat{\alpha}_{\xi(2n-2r)}\beta_1^{\lfloor\frac{2n-2r}{2}\rfloor}(\beta_1+(d-c)y)^{\lfloor\frac{2n-2r}{2}\rfloor+\xi(2n-2r)}}{(abx^2)^{\lfloor\frac{2n-2r}{2}\rfloor}(\beta_1-\beta_2)} \\
&\quad - \frac{\widehat{\gamma}_{\xi(2n-2r)}\beta_2^{\lfloor\frac{2n-2r}{2}\rfloor}(\beta_2+(d-c)y)^{\lfloor\frac{2n-2r}{2}\rfloor+\xi(2n-2r)}}{(abx^2)^{\lfloor\frac{2n-2r}{2}\rfloor}(\beta_1-\beta_2)} \\
&= \frac{\widehat{\alpha}_0\beta_1^{n-r}(\beta_1+(d-c)y)^{n-r} - \widehat{\gamma}_0\beta_2^{n-r}(\beta_2+(d-c)y)^{n-r}}{(abx^2)^{n-r}(\beta_1-\beta_2)}
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
BH_{2n}(x,y) &= \frac{\widehat{\alpha}_{\xi(2n)}\beta_1^{\lfloor\frac{2n}{2}\rfloor}(\beta_1+(d-c)y)^{\lfloor\frac{2n}{2}\rfloor+\xi(2n-2r)}}{(abx^2)^{\lfloor\frac{2n}{2}\rfloor}(\beta_1-\beta_2)} \\
&\quad - \frac{\widehat{\gamma}_{\xi(2n)}\beta_2^{\lfloor\frac{2n}{2}\rfloor}(\beta_2+(d-c)y)^{\lfloor\frac{2n}{2}\rfloor+\xi(2n)}}{(abx^2)^{\lfloor\frac{2n}{2}\rfloor}(\beta_1-\beta_2)} \\
&= \frac{\widehat{\alpha}_0\beta_1^{n}(\beta_1+(d-c)y)^{n} - \widehat{\gamma}_0\beta_2^{n}(\beta_2+(d-c)y)^{n}}{(abx^2)^{n}(\beta_1-\beta_2)}.
\end{aligned} \tag{16}
$$

By virtue of (14), (15) and (16), we have

$$\begin{aligned}
&BH_{2(n+r)}(x,y)BH_{2(n-r)}(x,y)-(BH_{2n}(x,y))^2\\
&=\frac{1}{(abx^2)^{2n}(\beta_1-\beta_2)^2}\\
&\quad\times\left[\widehat{\alpha}_0\widehat{\gamma}_0{\beta_1}^n{\beta_2}^n(\beta_1+(d-c)y)^n(\beta_2+(d-c)y)^n\left[1-\left(\frac{\beta_1(\beta_1+(d-c)y)}{\beta_2(\beta_2+(d-c)y)}\right)^r\right]\right]\\
&\quad+\frac{1}{(abx^2)^{2n}(\beta_1-\beta_2)^2}\\
&\quad\times\left[\widehat{\gamma}_0\widehat{\alpha}_0{\beta_2}^n{\beta_1}^n(\beta_2+(d-c)y)^n(\beta_1+(d-c)y)^n\left[1-\left(\frac{\beta_2(\beta_2+(d-c)y)}{\beta_2(\beta_2+(d-c)y)}\right)^r\right]\right].
\end{aligned}$$

Case $i=1$

$$BH_{2(n+r)+1}(x,y)=\frac{\widehat{\alpha}_1\beta_1^{n+r}(\beta_1+(d-c)y)^{n+r+1}-\widehat{\gamma}_1\beta_2^{n+r}(\beta_2+(d-c)y)^{n+r+1}}{(abx^2)^{n+r}(\beta_1-\beta_2)} \tag{17}$$

$$BH_{2(n-r)+1}(x,y)=\frac{\widehat{\alpha}_1\beta_1^{n-r}(\beta_1+(d-c)y)^{n-r+1}-\widehat{\gamma}_1\beta_2^{n-r}(\beta_2+(d-c)y)^{n-r+1}}{(abx^2)^{n-r}(\beta_1-\beta_2)} \tag{18}$$

$$BH_{2n+1}(x,y)=\frac{\widehat{\alpha}_1\beta_1^{n}(\beta_1+(d-c)y)^{n+1}-\widehat{\gamma}_1\beta_2^{n}(\beta_2+(d-c)y)^{n+1}}{(abx^2)^{n}(\beta_1-\beta_2)}. \tag{19}$$

By virtue of (17), (18) and (19), we have

$$\begin{aligned}
&BH_{2(n+r)+1}(x,y)BH_{2(n-r)+1}(x,y)-(BH_{2n+1}(x,y))^2\\
&=\frac{1}{(abx^2)^{2n}(\beta_1-\beta_2)^2}\\
&\quad\times\left[\widehat{\alpha}_1\widehat{\gamma}_1{\beta_1}^n{\beta_2}^n(\beta_1+(d-c)y)^{n+1}(\beta_2+(d-c)y)^{n+1}\left[1-\left(\frac{\beta_1(\beta_1+(d-c)y)}{\beta_2(\beta_2+(d-c)y)}\right)^r\right]\right]\\
&\quad+\frac{1}{(abx^2)^{2n}(\beta_1-\beta_2)^2}\\
&\quad\times\left[\widehat{\gamma}_1\widehat{\alpha}_1{\beta_2}^n{\beta_1}^n(\beta_2+(d-c)y)^{n+1}(\beta_1+(d-c)y)^{n+1}\left[1-\left(\frac{\beta_2(\beta_2+(d-c)y)}{\beta_2(\beta_2+(d-c)y)}\right)^r\right]\right].
\end{aligned}$$

Finally, we get

$$\begin{aligned}
&BH_{2(n+r)+\xi(i)}(x,y)BH_{2(n-r)+\xi(i)}(x,y)-\left(BH_{2n+\xi(i)}(x,y)\right)^2\\
&=\frac{1}{(abx^2)^{2n}(\beta_1-\beta_2)^2}\\
&\quad\times\left[\widehat{\alpha}_{\xi(i)}\widehat{\gamma}_{\xi(i)}{\beta_1}^n{\beta_2}^n(\beta_1+(d-c)y)^{n+\xi(i)}(\beta_2+(d-c)y)^{n+\xi(i)}\left[1-\left(\frac{\beta_1(\beta_1+(d-c)y)}{\beta_2(\beta_2+(d-c)y)}\right)^r\right]\right]\\
&\quad+\frac{1}{(abx^2)^{2n}(\beta_1-\beta_2)^2}\\
&\quad\times\left[\widehat{\gamma}_{\xi(i)}\widehat{\alpha}_{\xi(i)}{\beta_2}^n{\beta_1}^n(\beta_2+(d-c)y)^{n+\xi(i)}(\beta_1+(d-c)y)^{n+\xi(i)}\left[1-\left(\frac{\beta_2(\beta_2+(d-c)y)}{\beta_2(\beta_2+(d-c)y)}\right)^r\right]\right].
\end{aligned}$$

□

Now, we give the Cassini's identity of the bivariate conditional Fibonacci hybrinomial $BH_n(x,y)$.

**Corollary 1.** *For $n \geq 0$, we get*

$$
\begin{aligned}
&BH_{2(n+1)+\xi(i)}(x,y)BH_{2(n-1)+\xi(i)}(x,y) - \left(BH_{2n+\xi(i)}(x,y)\right)^2 \\
&= \frac{1}{\left(abx^2\right)^{2n}\left(\beta_1-\beta_2\right)^2} \\
&\quad \times \left[\widehat{\alpha}_{\xi(i)}\widehat{\gamma}_{\xi(i)}{\beta_1}^n{\beta_2}^n\left(\beta_1+(d-c)y\right)^{n+\xi(i)}\left(\beta_2+(d-c)y\right)^{n+\xi(i)}\left[1-\left(\frac{\beta_1\left(\beta_1+(d-c)y\right)}{\beta_2\left(\beta_2+(d-c)y\right)}\right)\right]\right] \\
&\quad + \frac{1}{\left(abx^2\right)^{2n}\left(\beta_1-\beta_2\right)^2} \\
&\quad \times \left[\widehat{\gamma}_{\xi(i)}\widehat{\alpha}_{\xi(i)}{\beta_2}^n{\beta_1}^n\left(\beta_2+(d-c)y\right)^{n+\xi(i)}\left(\beta_1+(d-c)y\right)^{n+\xi(i)}\left[1-\left(\frac{\beta_2\left(\beta_2+(d-c)y\right)}{\beta_2\left(\beta_2+(d-c)y\right)}\right)\right]\right].
\end{aligned}
$$

*where $\widehat{\alpha}_{\xi(i)}$ and $\widehat{\gamma}_{\xi(i)}$ are defined in Theorem(2) and $i \in \{0,1\}$.*

*Proof.* Taking $r=1$ in Catalan's identity the proof is completed. □

## 3. Generalized Bivariate Conditional Lucas Hybrinomials

In this section we give some identities of the generalized bivariate conditional Lucas hybrinomials. We start with the following definition.

**Definition 2.** *For any four numbers $a, b, c$ and $d$ belonging to $\mathbb{R}-\{0\}$, the generalization of bivariate conditional Fibonacci polynomial is defined as,*

$$
L_n(x,y) = \begin{cases} bxL_{n-1}(x,y) + dyL_{n-2}(x,y), & \text{if } n \text{ is even} \\ axL_{n-1}(x,y) + cyL_{n-2}(x,y), & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2, \tag{20}
$$

*where $L_0(x,y) = 2, L_1(x,y) = ax$.*

**Lemma 2.** *For the generalized bivariate conditional Lucas polynomials $\{L_n(x,y)\}_{n=0}^{\infty}$, we have*

$$
\begin{aligned}
L_{2n}(x,y) &= \left(abx^2+(c+d)y\right)L_{2n-2}(x,y) - cdy^2L_{2n-4}(x,y) \\
L_{2n+1}(x,y) &= \left(abx^2+(c+d)y\right)L_{2n-1}(x,y) - cdy^2L_{2n-3}(x,y).
\end{aligned}
$$

*Proof.* By using the definition of the generalized bivariate conditional Lucas polynomials, we have

$$
\begin{aligned}
L_{2n}(x,y) &= \left(bxL_{2n-1}(x,y) + dyL_{2n-2}(x,y)\right) \\
&= \left[bx\left(axL_{2n-2}(x,y) + cyL_{2n-3}(x,y)\right) + dyL_{2n-2}(x,y)\right] \\
&= \left[\left(abx^2+dy\right)L_{2n-2}(x,y) + cy\left(bxL_{2n-3}(x,y)\right)\right] \\
&= \left[\left(abx^2+dy\right)L_{2n-2}(x,y) + cy\left(L_{2n-2}(x,y) - dyL_{2n-4}(x,y)\right)\right] \\
&= \left[\left(abx^2+(c+d)\,y\right)L_{2n-2}(x,y) - cdy^2L_{2n-4}(x,y)\right].
\end{aligned}
$$

Similar to above steps, we can obtain

$$
L_{2n+1}(x,y) = \left(abx^2+(c+d)y\right)L_{2n-1}(x,y) - cdy^2L_{2n-3}(x,y).
$$

Thus, the proof is completed. □

Next we give the generating function for the bivariate conditional Lucas polynomial $L_n(x,y)$.

**Theorem 4.** *The generating function for the bivariate conditional Lucas polynomial $L_n(x,y)$ is*

$$E(t)=\sum_{n=0}^{\infty}L_n(x,y)t^n=\frac{2+axt-\left(abx^2+2cy\right)t^2+adxyt^3}{1-\left(abx^2+(c+d)\,y\right)t^2+cdy^2t^4}. \tag{21}$$

*Proof.* We define

$$E_0(t)=\sum_{n=0}^{\infty}L_{2n}(x,y)t^{2n}$$
$$E_1(t)=\sum_{n=0}^{\infty}L_{2n+1}(x,y)t^{2n+1}.$$

So that

$$E(t)=E_0(t)+E_1(t).$$

We have

$$\begin{aligned}
E_0(t)&=\sum_{n=0}^{\infty}L_{2n}(x,y)t^{2n}\\
&=\sum_{n=0}^{\infty}L_{2n}(x,y)t^{2n}=L_0(x,y)t^0+L_2(x,y)t^2+\sum_{n=2}^{\infty}L_{2n}(x,y)t^{2n}\\
&=L_0(x,y)+L_2(x,y)t^2\\
&\quad+\sum_{n=2}^{\infty}\left[\left(abx^2+(c+d)y\right)L_{2n-2}(x,y)-cdy^2L_{2n-4}(x,y)\right]t^{2n}\\
&=L_0(x,y)+L_2(x,y)t^2+\left(abx^2+(c+d)y\right)t^2\sum_{n=2}^{\infty}L_{2n-2}(x,y)t^{2n-2}\\
&\quad-cdy^2t^4\sum_{n=2}^{\infty}L_{2n-4}(x,y)t^{2n-4}\\
&=2+\left(abx^2+2dy\right)t^2\\
&\quad+\left(abx^2+(c+d)y\right)t^2\left[\sum_{n=2}^{\infty}L_{2n-2}(x,y)t^{2n-2}+L_0(x,y)t^0-L_0(x,y)t^0\right]\\
&\quad-cdy^2t^4E_0(t)
\end{aligned}$$

$$\begin{aligned}
&= 2 + \left(abx^2 + 2dy\right) t^2 + \left(abx^2 + (c+d)y\right) t^2 E_0(t) \\
&\quad - 2\left(abx^2 + (c+d)y\right) t^2 - cdy^2 t^4 E_0(t) \\
E_0(t)&[1 - \left(abx^2 + (c+d)y\right) t^2 + cdy^2 t^4\big] = 2 - \left(abx^2 + 2cy\right) t^2.
\end{aligned}$$

Thus, we get

$$E_0(t) = \frac{2 - \left(abx^2 + 2cy\right) t^2}{1 - \left(abx^2 + (c+d)y\right) t^2 + cdy^2 t^4}. \tag{22}$$

Similarly, we find

$$\begin{aligned}
E_1(t) &= \sum_{n=0}^{\infty} L_{2n+1}(x,y) t^{2n+1} \\
&= \sum_{n=0}^{\infty} L_{2n+1}(x,y) t^{2n+1} = L_1(x,y)t^1 + L_3(x,y)t^3 + \sum_{n=2}^{\infty} L_{2n+1}(x,y) t^{2n+1} \\
&= L_1(x,y)t + L_3(x,y)t^3 \\
&\quad + \sum_{n=2}^{\infty} \left[\left(abx^2 + (c+d)y\right) L_{2n-1}(x,y) - cdy^2 L_{2n-3}(x,y)\right] t^{2n+1} \\
&= L_1(x,y)t + L_3(x,y)t^3 \\
&\quad + \left(abx^2 + (c+d)y\right) t^2 \sum_{n=2}^{\infty} L_{2n-1}(x,y) t^{2n-1} \\
&\quad - cdy^2 t^4 \sum_{n=2}^{\infty} L_{2n-3}(x,y) t^{2n-3} \\
&= axt + \left(a^2bx^3 + 2adxy + acxy\right) t^3 \\
&\quad + \left(abx^2 + (c+d)y\right) t^2 \left[\sum_{n=2}^{\infty} L_{2n-1}(x,y) t^{2n-1} + L_1(x,y)t - L_1(x,y)t\right] \\
&\quad - cdy^2 t^4 E_1(t) \\
&= axt + \left(a^2bx^3 + 2adxy + acxy\right) t^3 + \left(abx^2 + (c+d)y\right) t^2 E_1(t) \\
&\quad - ax\left(abx^2 + (c+d)y\right) t^3 - cdy^2 t^4 E_1(t) \\
E_1(t)&[1 - \left(abx^2 + (c+d)y\right) t^2 + cdy^2 t^4\big] = axt + adxyt^3.
\end{aligned}$$

Therefore, we get

$$E_1(t) = \frac{axt + adxyt^3}{1 - \left(abx^2 + (c+d)y\right) t^2 + cdy^2 t^4}. \tag{23}$$

By virtue of (22) and (23), we can obtain

$$\begin{aligned} E(t) &= E_0(t) + E_1(t) \\ &= \sum_{n=0}^{\infty} L_n(x,y)t^n = \frac{2 + axt - \left(abx^2 + 2cy\right)t^2 + adxyt^3}{1 - \left(abx^2 + (c+d)\,y\right)t^2 + cdy^2t^4}. \end{aligned}$$

Hence, the proof is completed. □

Now we give the Binet formula of the bivariate conditional Lucas polynomial $L_n(x,y)$.

**Theorem 5.** *The $n^{th}$ term of the generalized of bivariate conditional Lucas polynomial $L_n(x,y)$ is*

$$\begin{aligned} L_n(x,y) = &\frac{(-ax)^{\xi(n)}}{\beta_1 - \beta_2} \\ &\times \Bigg[ \left(\xi(n+1)\beta_1 + (-1)^{\xi(n+1)}\beta_2\right)(\beta_2 + dy)^{\lfloor \frac{n}{2} \rfloor} \\ &\quad + \left((-1)^{\xi(n+1)}(2)^{\xi(n)}dy + \xi(n+1)cy\right)(\beta_2 + dy)^{\lfloor \frac{n}{2} \rfloor} \\ &\quad - \left(\xi(n+1)\beta_2 + (-1)^{\xi(n+1)}\beta_1\right)(\beta_1 + dy)^{\lfloor \frac{n}{2} \rfloor} \\ &\quad - \left((-1)^{\xi(n+1)}(2)^{\xi(n)}dy + \xi(n+1)cy\right)(\beta_1 + dy)^{\lfloor \frac{n}{2} \rfloor} \Bigg] \end{aligned} \tag{24}$$

*where $\beta_1$ and $\beta_2$ roots of the characteristic equation $\lambda^2 - (abx^2 + (c-d)y)\lambda - abdx^2y = 0$.*

*Proof.* We use the following properties throughout the proof:

- $\beta_1 + \beta_2 = abx^2 + (c-d)y$
- $\beta_1 \cdot \beta_2 = -abdx^2y$
- $(\beta_1 + dy)(\beta_2 + dy) = cdy^2$
- $(\beta_1 + dy)\left(abx^2\right) = \beta_1\left(\beta_1 + (d-c)y\right)$
- $(\beta_2 + dy)\left(abx^2\right) = \beta_2\left(\beta_2 + (d-c)y\right)$

Note that $\beta_1(x,y) = \beta_1$ and $\beta_2(x,y) = \beta_2$. Since $\frac{\beta_1+dy}{cdy^2}$ and $\frac{\beta_2+dy}{cdy^2}$ are roots of

$$1 - (abx^2 + (c+d)y)t^2 + cdy^2t^4 = 0.$$

If we assume

$$E_0(t) = \sum_{n=0}^{\infty} L_{2n}(x,y)t^{2n}$$

$$E_1(t) = \sum_{n=0}^{\infty} L_{2n+1}(x,y)t^{2n+1}.$$

Then,

$$E(t) = E_0(t) + E_1(t).$$

By using Maclaurin's series expansion

$$\frac{Az - B}{z^2 - C} = \sum_{n=0}^{\infty} BC^{-n-1} Z^{2n} - \sum_{n=0}^{\infty} AC^{-n-1} Z^{2n+1}$$

and above-mentioned identities, we simplify both $E_0(t)$ and $E_1(t)$ as follows:

$$\begin{aligned}
E_0(t) &= \frac{1}{cdy^2\left(\beta_1 - \beta_2\right)} \left[ \frac{2cdy^2 - \left(abx^2 + 2cy\right) \cdot \left(\beta_1 + dy\right)}{t^2 - \left(\frac{\beta_1 + dy}{cdy^2}\right)} \right. \\
&\qquad \left. - \frac{2cdy^2 - \left(abx^2 + 2cy\right)\left(\beta_2 + y\right)}{t^2 - \left(\frac{\beta_2 + dy}{cdy^2}\right)} \right] \\
&= \frac{1}{cdy^2\left(\beta_1 - \beta_2\right)} \sum_{n=0}^{\infty} \left[ \left((abx^2 + 2cy)(\beta_1 + dy) - 2cdy^2\right) \left(\frac{\beta_1 + dy}{cdy^2}\right)^{-n-1} \right] t^{2n} \\
&\quad - \frac{1}{cdy^2\left(\beta_1 - \beta_2\right)} \sum_{n=0}^{\infty} \left[ \left((abx^2 + 2cy)(\beta_2 + dy) - 2cdy^2\right) \left(\frac{\beta_2 + dy}{cdy^2}\right)^{-n-1} \right] t^{2n} \\
&= \frac{1}{cdy^2\left(\beta_1 - \beta_2\right)} \sum_{n=0}^{\infty} \left[ \left( (abx^2 + 2cy)(\beta_1 + dy)(\beta_2 + dy) \right. \right. \\
&\qquad \left. \left. - 2cdy^2(\beta_2 + dy) \right) \left(\beta_2 + dy\right)^n \right] t^{2n} \\
&\quad - \frac{1}{cdy^2\left(\beta_1 - \beta_2\right)} \sum_{n=0}^{\infty} \left[ \left( (abx^2 + 2cy)(\beta_2 + dy)(\beta_1 + dy) \right. \right. \\
&\qquad \left. \left. - 2cdy^2(\beta_1 + dy) \right) \left(\beta_1 + dy\right)^n \right] t^{2n} \\
&= \frac{1}{\left(\beta_1 - \beta_2\right)} \sum_{n=0}^{\infty} \left[ \left(abx^2 - 2\beta_2 + 2cy - 2dy\right) \left(\beta_2 + dy\right)^n \right. \\
&\qquad \left. - \left(abx^2 - 2\beta_1 + 2cy - 2dy\right) \left(\beta_1 + dy\right)^n \right] t^{2n} \\
&= \frac{1}{\left(\beta_1 - \beta_2\right)} \sum_{n=0}^{\infty} \left[ \left(\beta_1 - \beta_2 - (d - c)y\right) \left(\beta_2 + dy\right)^n \right. \\
&\qquad \left. - \left(\beta_2 - \beta_1 - (d - c)y\right) \left(\beta_1 + dy\right)^n \right] t^{2n}.
\end{aligned}$$

We solve $E_1(t)$ with the same approach used in $E_0(t)$ and we get the value of

$$E_1(t) = \frac{-ax}{(\beta_1 - \beta_2)} \sum_{n=0}^{\infty} \left[ (\beta_2 + 2dy)(\beta_2 + dy)^n - (\beta_1 + 2dy)(\beta_1 + dy)^n \right] t^{2n+1}.$$

We know that $E(t) = E_0(t) + E_1(t)$. So we find

$$\begin{aligned} E(t) = \sum_{n=0}^{\infty} \frac{(-ax)^{\xi(n)}}{\beta_1 - \beta_2} \Bigg[ \Bigg( & \xi(n+1)\beta_1 + (-1)^{\xi(n+1)}\beta_2 \\ & + (-1)^{\xi(n+1)}(2)^{\xi(n)}dy + \xi(n+1)cy \Bigg) (\beta_2 + dy)^{\lfloor \frac{n}{2} \rfloor} \\ - \Bigg( & \xi(n+1)\beta_2 + (-1)^{\xi(n+1)}\beta_1 \\ & + (-1)^{\xi(n+1)}(2)^{\xi(n)}dy + \xi(n+1)cy \Bigg) (\beta_1 + dy)^{\lfloor \frac{n}{2} \rfloor} \Bigg]. \end{aligned}$$

Thus,

$$\begin{aligned} L_n(x,y) = \frac{(-ax)^{\xi(n)}}{\beta_1 - \beta_2} \Bigg[ \Bigg( & \xi(n+1)\beta_1 + (-1)^{\xi(n+1)}\beta_2 \\ & + (-1)^{\xi(n+1)}(2)^{\xi(n)}dy + \xi(n+1)cy \Bigg) (\beta_2 + dy)^{\lfloor \frac{n}{2} \rfloor} \\ - \Bigg( & \xi(n+1)\beta_2 + (-1)^{\xi(n+1)}\beta_1 \\ & + (-1)^{\xi(n+1)}(2)^{\xi(n)}dy + \xi(n+1)cy \Bigg) (\beta_1 + dy)^{\lfloor \frac{n}{2} \rfloor} \Bigg]. \end{aligned}$$

□

In the following definition, we give bivariate conditional Lucas Hybrinomials.

**Definition 3.** *For any variable $x, y$ and nonzero real numbers $a, b, c$ and $d$, we have*

$$LH_n(x,y) = L_n(x,y) + \boldsymbol{i}L_{n+1}(x,y) + \varepsilon L_{n+2}(x,y) + \boldsymbol{h}L_{n+3}(x,y) \tag{25}$$

*where $L_n(x,y)$ was given in (20) and the initial conditions are with $LH_0(x,y) = 2 + \boldsymbol{i}ax + \varepsilon(abx^2 + 2dy) + \boldsymbol{h}(a^2bx^3 + 2adxy + acxy)$ and $LH_1(x,y) = ax + \boldsymbol{i}(abx^2 + 2dy) + +\varepsilon(a^2bx^3 + 2adxy + acxy) + \boldsymbol{h}(a^2b^2x^4 + 2bcdx^2y + abcx^2y + abdx^2y + 2d^2y^2)$.*

We can see from the following table that the generalized bivariate conditional Fibonacci hybrinomials are the generalization of many works for different values of $a, b, c$ and $d$.

TABLE 3. The generalized bivariate conditional Lucas hybrinomials

| ***a*** | ***b*** | ***c*** | ***d*** | ***Generalized Bivariate Conditional Lucas Hybrinomials*** |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | *Bivariate Lucas Hybrinomials* |
| $a$ | $b$ | 1 | 1 | *Bivariate Conditional Lucas Hybrinomials* |
| 2 | 2 | 1 | 1 | *Bivariate Pell Lucas Hybrinomials* |
| 1 | 1 | 2 | 2 | *Bivariate Jacobsthal Lucas Hybrinomials* |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

**Lemma 3.** *For the generalized bivariate conditional Lucas hybrinomials* $\{LH_n(x,y)\}_{n=0}^{\infty}$*, we have*

$$LH_{2n}(x,y) = \left(abx^2 + (c+d)y\right) LH_{2n-2}(x,y) - cdy^2 LH_{2n-4}(x,y)$$

$$LH_{2n+1}(x,y) = \left(abx^2 + (c+d)y\right) LH_{2n-1}(x,y) - cdy^2 LH_{2n-3}(x,y).$$

*Proof.* By using the definition of the generalized bivariate conditional Lucas hybrinomials, we obtain

$$\begin{aligned}
LH_{2n}(x,y) &= L_{2n}(x,y) + \mathbf{i}L_{2n+1}(x,y) + \varepsilon L_{2n+2}(x,y) + \mathbf{h}L_{2n+3}(x,y) \\
&= (bxL_{2n-1}(x,y) + dyL_{2n-2}(x,y)) + \mathbf{i}\,(axL_{2n}(x,y) + cyL_{2n-1}(x,y)) \\
&\quad + \varepsilon(bxL_{2n+1}(x,y) + dyL_{2n}(x,y)) + \mathbf{h}(axL_{2n+2}(x,y) + cyL_{2n+1}(x,y)) \\
&= [bx\,(axL_{2n-2}(x,y) + cyL_{2n-3}(x,y)) + dyL_{2n-2}(x,y)] \\
&\quad + \mathbf{i}\,[ax\,(bxL_{2n-1}(x,y) + dyL_{2n-2}(x,y)) + cyL_{2n-1}(x,y)] \\
&\quad + \varepsilon\,[bx\,(axL_{2n}(x,y) + cyL_{2n-1}(x,y)) + dyL_{2n}(x,y))] \\
&\quad + \mathbf{h}\,[ax\,(bxL_{2n+1}(x,y) + dyL_{2n}(x,y)) + cyL_{2n+1}(x,y)] \\
&= [\left(abx^2 + dy\right) L_{2n-2}(x,y) + cy\,(bxL_{2n-3}(x,y))] \\
&\quad + \mathbf{i}[\left(abx^2 + cy\right) L_{2n-1}(x,y) + dy\,(axL_{2n-2}(x,y))] \\
&\quad + \varepsilon[\left(abx^2 + dy\right) L_{2n}(x,y) + cy\,(bxL_{2n-1}(x,y))] \\
&\quad + \mathbf{h}[\left(abx^2 + cy\right) L_{2n+1}(x,y) + dy\,(axL_{2n}(x,y))] \\
&= [\left(abx^2 + dy\right) L_{2n-2}(x,y) + cy\,(L_{2n-2}(x,y) - dyL_{2n-4}(x,y))] \\
&\quad + \mathbf{i}[\left(abx^2 + cy\right) L_{2n-1}(x,y) + dy\,(L_{2n-1}(x,y) - cyL_{2n-3}(x,y))] \\
&\quad + \varepsilon[\left(abx^2 + dy\right) L_{2n}(x,y) + cy\,(L_{2n}(x,y) - dyL_{2n-2}(x,y))] \\
&\quad + \mathbf{h}[\left(abx^2 + cy\right) L_{2n+1}(x,y) + dy\,(L_{2n+1}(x,y) - cyL_{2n-1}(x,y))] \\
&= [\left(abx^2 + (c+d)\,y\right) L_{2n-2}(x,y) - cdy^2 L_{2n-4}(x,y)] \\
&\quad + \mathbf{i}[\left(abx^2 + (c+d)\,y\right) L_{2n-1}(x,y) - cdy^2 L_{2n-3}(x,y)] \\
&\quad + \varepsilon[\left(abx^2 + (c+d)\,y\right) L_{2n}(x,y) - cdy^2 L_{2n-2}(x,y)] \\
&\quad + \mathbf{h}[\left(abx^2 + (c+d)\,y\right) L_{2n+1}(x,y) - cdy^2 L_{2n-1}(x,y)]
\end{aligned}$$

$$
\begin{aligned}
&= \left(abx^2 + (c+d)\,y\right)[L_{2n-2}(x,y) + \mathbf{i}L_{2n-1}(x,y) + \varepsilon L_{2n}(x,y) + \mathbf{h}L_{2n+1}(x,y)] \\
&\quad - cdy^2[L_{2n-4}(x,y) + \mathbf{i}L_{2n-3}(x,y) + \varepsilon L_{2n-2}(x,y) + \mathbf{h}L_{2n-1}(x,y)] \\
&= \left(abx^2 + (c+d)\,y\right) LH_{2n-2}(x,y) - cdy^2 LH_{2n-4}(x,y)
\end{aligned}
$$

Similar to above, we can obtain

$$
LH_{2n+1}(x,y) = \left(abx^2 + (c+d)y\right) LH_{2n-1}(x,y) - cdy^2 LH_{2n-3}(x,y).
$$

Thus, the proof is completed. □

Next we give the generating function of the bivariate conditional Lucas hybrinomial $LH_n(x,y)$.

**Theorem 6.** *The generating function for the bivariate conditional Lucas hybrinomial $LH_n(x,y)$ is*

$$
\begin{aligned}
\Omega(t) &= \sum_{n=0}^{\infty} LH_n(x,y)t^n \\
&= \frac{LH_0(x,y) + LH_1(x,y)t + \left[LH_2(x,y) - \left(abx^2 + (c+d)y\right) LH_0(x,y)\right] t^2}{1 - \left(abx^2 + (c+d)\,y\right) t^2 + cdy^2t^4} \\
&\quad + \frac{\left[LH_3(x,y) - \left(abx^2 + (c+d)\,y\right) LH_1(x,y)\right] t^3}{1 - \left(abx^2 + (c+d)\,y\right) t^2 + cdy^2t^4}.
\end{aligned}
$$

*Proof.* We define

$$
\begin{aligned}
\Omega_0(t) &= \sum_{n=0}^{\infty} LH_{2n}(x,y)t^{2n} \\
\Omega_1(t) &= \sum_{n=0}^{\infty} LH_{2n+1}(x,y)t^{2n+1}.
\end{aligned}
$$

So that

$$
\Omega(t) = \Omega_0(t) + \Omega_1(t).
$$

We have

$$\begin{aligned}
\Omega_0(t) &= \sum_{n=0}^{\infty} LH_{2n}(x,y)t^{2n} \\
&= \sum_{n=0}^{\infty} LH_{2n}(x,y)t^{2n} = LH_0(x,y)t^0 + LH_2(x,y)t^2 + \sum_{n=2}^{\infty} LH_{2n}(x,y)t^{2n} \\
&= LH_0(x,y) + LH_2(x,y)t^2 \\
&\quad + \sum_{n=2}^{\infty} \left[\left(abx^2+(c+d)y\right) LH_{2n-2}(x,y) - cdy^2 LH_{2n-4}(x,y)\right] t^{2n} \\
&= LH_0(x,y) + LH_2(x,y)t^2 + \left(abx^2+(c+d)y\right) t^2 \sum_{n=2}^{\infty} LH_{2n-2}(x,y)t^{2n-2} \\
&\quad - cdy^2 t^4 \sum_{n=2}^{\infty} LH_{2n-4}(x,y)t^{2n-4} \\
&= LH_0(x,y) + LH_2(x,y)t^2 \\
&\quad + \left(abx^2+(c+d)y\right) t^2 \left[\sum_{n=2}^{\infty} LH_{2n-2}(x,y)t^{2n-2} + LH_0(x,y)t^0 - LH_0(x,y)t^0\right] \\
&\quad - cdy^2 t^4 \Omega_0(t) \\
&= LH_0(x,y) + LH_2(x,y)t^2 + \left(abx^2+(c+d)y\right) t^2 \Omega_0(t) \\
&\quad - \left(abx^2+(c+d)y\right) t^2 LH_0(x,y) - cdy^2 t^4 \Omega_0(t).
\end{aligned}$$

Thus, we get

$$\Omega_0(t) = \frac{LH_0(x,y) + \left(LH_2(x,y) - \left(abx^2+(c+d)y\right) LH_0(x,y)\right) t^2}{1 - \left(abx^2+(c+d)y\right) t^2 + cdy^2 t^4}. \tag{26}$$

Similarly, we find

$$\begin{aligned}
\Omega_1(t) &= \sum_{n=0}^{\infty} LH_{2n+1}(x,y)t^{2n+1} \\
&= \sum_{n=0}^{\infty} LH_{2n+1}(x,y)t^{2n+1} = LH_1(x,y)t + LH_3(x,y)t^3 \\
&\quad + \sum_{n=2}^{\infty} LH_{2n+1}(x,y)t^{2n+1} \\
&= LH_1(x,y)t + LH_3(x,y)t^3 \\
&\quad + \sum_{n=2}^{\infty} \left[\left(abx^2+(c+d)y\right) LH_{2n-1}(x,y) - cdy^2 LH_{2n-3}(x,y)\right] t^{2n+1}
\end{aligned}$$

$$
\begin{aligned}
&= LH_1(x,y)t + LH_3(x,y)t^3 \\
&\quad + \left(abx^2 + (c+d)y\right) t^2 \sum_{n=2}^{\infty} LH_{2n-1}(x,y)t^{2n-1} - cdy^2t^4 \sum_{n=2}^{\infty} LH_{2n-3}(x,y)t^{2n-3} \\
&= LH_1(x,y)t + LH_3(x,y)t^3 \\
&\quad + \left(abx^2 + (c+d)y\right) t^2 \left[\sum_{n=2}^{\infty} LH_{2n-1}(x,y)t^{2n-1} + LH_1(x,y)t - LH_1(x,y)t\right] \\
&\quad - cdy^2t^4 \Omega_1(t) \\
&= LH_1(x,y)t + LH_3(x,y)t^3 + \left(abx^2 + (c+d)y\right) t^2 \Omega_1(t) \\
&\quad - \left(abx^2 + (c+d)y\right) t^3 LH_1(x,y) - cdy^2t^4 \Omega_1(t).
\end{aligned}
$$

Therefore, we get

$$
\Omega_1(t) = \frac{LH_1(x,y)t + \left(LH_3(x,y) - \left(abx^2 + (c+d)y\right) LH_1(x,y)\right) t^3}{1 - \left(abx^2 + (c+d)y\right) t^2 + cdy^2t^4}. \tag{27}
$$

By virtue of (26) and (27), we can obtain

$$
\begin{aligned}
\Omega(t) &= \Omega_0(t) + \Omega_1(t) \\
&= \sum_{n=0}^{\infty} LH_n(x,y)t^n \\
&= \frac{LH_0(x,y) + LH_1(x,y)t + \left[LH_2(x,y) - \left(abx^2 + (c+d)y\right) LH_0(x,y)\right] t^2}{1 - \left(abx^2 + (c+d)\, y\right) t^2 + cdy^2t^4} \\
&\quad + \frac{\left[LH_3(x,y) - \left(abx^2 + (c+d)\, y\right) LH_1(x,y)\right] t^3}{1 - \left(abx^2 + (c+d)\, y\right) t^2 + cdy^2t^4}.
\end{aligned}
$$

Hence, the proof is completed. □

Now we give the Binet formula of the bivariate conditional Lucas hybrinomial $LH_n(x,y)$.

**Theorem 7.** *The $n^{th}$ term of the generalized of bivariate conditional Lucas hybrinomial $LH_n(x,y)$ is*

$$
LH_n(x,y) = \frac{\widehat{\omega}_{\xi(n)} \left(\beta_2 + dy\right)^{\lfloor \frac{n}{2} \rfloor} - \widehat{\sigma}_{\xi(n)} \left(\beta_1 + dy\right)^{\lfloor \frac{n}{2} \rfloor}}{\beta_1 - \beta_2}. \tag{28}
$$

*where* $\beta_1$ *and* $\beta_2$ *roots of the characteristic equation* $\lambda^2-(abx^2+(c-d)y)\lambda-abdx^2y=0$. *Also,*

$$\begin{aligned}
\widehat{\omega}_{\xi(n)} &= (-ax)^{\xi(n)}\left(\xi(n+1)\beta_1+(-1)^{\xi(n+1)}\beta_2+(-1)^{\xi(n+1)}(2)^{\xi(n)}dy+\xi(n+1)cy\right)\\
&\quad+\boldsymbol{i}\,(-ax)^{\xi(n+1)}\left(\xi(n)\beta_1+(-1)^{\xi(n)}\beta_2+(-1)^{\xi(n)}(2)^{\xi(n+1)}dy+\xi(n)cy\right)(\beta_2+dy)^{\xi(n)}\\
&\quad+\varepsilon\,(-ax)^{\xi(n)}\left(\xi(n+1)\beta_1+(-1)^{\xi(n+1)}\beta_2+(-1)^{\xi(n+1)}(2)^{\xi(n)}dy+\xi(n+1)cy\right)(\beta_2+dy)\\
&\quad+\boldsymbol{h}\,(-ax)^{\xi(n+1)}\left(\xi(n)\beta_1+(-1)^{\xi(n)}\beta_2+(-1)^{\xi(n)}(2)^{\xi(n+1)}dy+\xi(n)cy\right)(\beta_2+dy)^{\xi(n)+1}
\end{aligned}$$

*and*

$$\begin{aligned}
\widehat{\sigma}_{\xi(n)} &= (-ax)^{\xi(n)}\left(\xi(n+1)\beta_2+(-1)^{\xi(n+1)}\beta_1+(-1)^{\xi(n+1)}(2)^{\xi(n)}dy+\xi(n+1)cy\right)\\
&\quad+\boldsymbol{i}\,(-ax)^{\xi(n+1)}\left(\xi(n)\beta_2+(-1)^{\xi(n)}\beta_1+(-1)^{\xi(n)}(2)^{\xi(n+1)}dy+\xi(n)cy\right)(\beta_1+dy)^{\xi(n)}\\
&\quad+\varepsilon\,(-ax)^{\xi(n)}\left(\xi(n+1)\beta_2+(-1)^{\xi(n+1)}\beta_1+(-1)^{\xi(n+1)}(2)^{\xi(n)}dy+\xi(n+1)cy\right)(\beta_1+dy)\\
&\quad+\boldsymbol{h}\,(-ax)^{\xi(n+1)}\left(\xi(n)\beta_2+(-1)^{\xi(n)}\beta_1+(-1)^{\xi(n)}(2)^{\xi(n+1)}dy+\xi(n)cy\right)(\beta_1+dy)^{\xi(n)+1}.
\end{aligned}$$

*Proof.* Firstly, by using (24), we have

$$\begin{aligned}
LH_{2n}(x,y) &= L_{2n}(x,y)+\mathbf{i}L_{2n+1}(x,y)+\varepsilon L_{2n+2}(x,y)+\mathbf{h}L_{2n+3}(x,y)\\
&= \frac{(\beta_2+dy)^n}{\beta_1-\beta_2}\Big[\,(\beta_1-\beta_2-dy+cy)+\mathbf{i}(-ax)\,(\beta_2+2dy)\\
&\qquad+\varepsilon\,(\beta_1-\beta_2-dy+cy)\,(\beta_2+dy)\\
&\qquad+\mathbf{h}(-ax)\,(\beta_2+2dy)\,(\beta_2+dy)\,\Big]\\
&\quad-\frac{(\beta_1+dy)^n}{\beta_1-\beta_2}\Big[\,(\beta_2-\beta_1-dy+cy)+\mathbf{i}(-ax)\,(\beta_1+2dy)\\
&\qquad+\varepsilon\,(\beta_2-\beta_1-dy+cy)\,(\beta_1+dy)\\
&\qquad+\mathbf{h}(-ax)\,(\beta_1+2dy)\,(\beta_1+dy)\,\Big].
\end{aligned}$$

Here, we choose the $\widehat{\omega}_0$ and $\widehat{\sigma}_0$ as follows;

$$\begin{aligned}
\widehat{\omega}_0 &= \Big[\,(\beta_1-\beta_2-dy+cy)+\mathbf{i}(-ax)\,(\beta_2+2dy)+\varepsilon\,(\beta_1-\beta_2-dy+cy)\,(\beta_2+dy)\\
&\quad+\mathbf{h}(-ax)\,(\beta_2+2dy)\,(\beta_2+dy)\,\Big]\\
\widehat{\sigma}_0 &= \Big[\,(\beta_2-\beta_1-dy+cy)+\mathbf{i}(-ax)\,(\beta_1+2dy)+\varepsilon\,(\beta_2-\beta_1-dy+cy)\,(\beta_1+dy)\\
&\quad+\mathbf{h}(-ax)\,(\beta_1+2dy)\,(\beta_1+dy)\,\Big].
\end{aligned}$$

Finally, the following equation is obtained:

$$LH_{2n}(x,y)=\frac{\widehat{\omega}_0\,(\beta_2+dy)^n-\widehat{\sigma}_0\,(\beta_1+dy)^n}{\beta_1-\beta_2}. \tag{29}$$

In similar way, by using (24), we have

$$\begin{aligned}
LH_{2n+1}(x,y) &= L_{2n+1}(x,y)+\mathbf{i}L_{2n+2}(x,y)+\varepsilon L_{2n+3}(x,y)+\mathbf{h}L_{2n+4}(x,y)\\
&= \frac{(\beta_2+dy)^n}{\beta_1-\beta_2}\big[(-ax)(\beta_2+2dy)+\mathbf{i}(\beta_1-\beta_2-dy+cy)(\beta_2+dy)\\
&\qquad +\varepsilon(-ax)(\beta_2+2dy)(\beta_2+dy)\\
&\qquad +\mathbf{h}(\beta_1-\beta_2-dy+cy)(\beta_2+dy)^2\big]\\
&\quad -\frac{(\beta_1+dy)^n}{\beta_1-\beta_2}\big[(-ax)(\beta_1+2dy)+\mathbf{i}(\beta_2-\beta_1-dy+cy)(\beta_1+dy)\\
&\qquad +\varepsilon(-ax)(\beta_1+2dy)(\beta_1+dy)\\
&\qquad +\mathbf{h}(\beta_2-\beta_1-dy+cy)(\beta_1+dy)^2\big].
\end{aligned}$$

Here, we choose the $\widehat{\omega}_1$ and $\widehat{\sigma}_1$ as follows;

$$\begin{aligned}
\widehat{\omega}_1 &= \big[(-ax)(\beta_2+2dy)+\mathbf{i}(\beta_1-\beta_2-dy+cy)(\beta_2+dy)\\
&\quad +\varepsilon(-ax)(\beta_2+2dy)(\beta_2+dy)+\mathbf{h}(\beta_1-\beta_2-dy+cy)(\beta_2+dy)^2\big]\\
\widehat{\sigma}_1 &= \big[(-ax)(\beta_1+2dy)+\mathbf{i}(\beta_2-\beta_1-dy+cy)(\beta_1+dy)\\
&\quad +\varepsilon(-ax)(\beta_1+2dy)(\beta_1+dy)+\mathbf{h}(\beta_2-\beta_1-dy+cy)(\beta_1+dy)^2\big].
\end{aligned}$$

Finally, the following equation is obtained.

$$LH_{2n+1}(x,y)=\frac{\widehat{\omega}_1(\beta_2+dy)^n-\widehat{\sigma}_1(\beta_1+dy)^n}{\beta_1-\beta_2}. \tag{30}$$

By virtue of (29) and (30), we can obtain the following equation

$$LH_n(x,y)=\frac{\widehat{\omega}_{\xi(n)}(\beta_2+dy)^{\lfloor\frac{n}{2}\rfloor}-\widehat{\sigma}_{\xi(n)}(\beta_1+dy)^{\lfloor\frac{n}{2}\rfloor}}{\beta_1-\beta_2}.$$

where $\beta_1$ and $\beta_2$ roots of the characteristic equation $\lambda^2-(abx^2+(c-d)y)\lambda-abdx^2y=0$. □

Now, we give the Catalan's identity of the bivariate conditional Lucas hybrinomial $LH_n(x,y)$.

**Theorem 8.** *For any integers $n$ and $r$ and $n\geqslant r\geqslant 0$, $r\geq 0$, we have*

$$\begin{aligned}
&LH_{2(n+r)+\xi(i)}(x,y)LH_{2(n-r)+\xi(i)}(x,y)-\left(LH_{2n+\xi(i)}(x,y)\right)^2\\
&\qquad=\frac{\widehat{\omega}_{\xi(i)}\widehat{\sigma}_{\xi(i)}(\beta_2+dy)^n(\beta_1+dy)^n\left[1-\left(\frac{\beta_2+dy}{\beta_1+dy}\right)^r\right]}{(\beta_1-\beta_2)^2}\\
&\qquad\quad+\frac{\widehat{\sigma}_{\xi(i)}\widehat{\omega}_{\xi(i)}(\beta_1+dy)^n(\beta_2+dy)^n\left[1-\left(\frac{\beta_1+dy}{\beta_2+dy}\right)^r\right]}{(\beta_1-\beta_2)^2}.
\end{aligned}$$

*where $\widehat{\omega}_{\xi(i)}$ and $\widehat{\sigma}_{\xi(i)}$ are defined in Theorem(7) and $i\in\{0,1\}$.*

*Proof.* In order to prove Catalan's identity, we will examine two different cases.
Case $i=0$:

$$\begin{aligned} LH_{2(n+r)}(x,y) &= \frac{\widehat{\omega}_{\xi(2n+2r)}\left(\beta_2+dy\right)^{\lfloor \frac{2n+2r}{2} \rfloor} - \widehat{\sigma}_{\xi(2n+2r)}\left(\beta_1+dy\right)^{\lfloor \frac{2n+2r}{2} \rfloor}}{\beta_1-\beta_2} \\ &= \frac{\widehat{\omega}_0\left(\beta_2+dy\right)^{n+r} - \widehat{\sigma}_0\left(\beta_1+dy\right)^{n+r}}{\beta_1-\beta_2} \end{aligned} \tag{31}$$

$$\begin{aligned} LH_{2(n-r)}(x,y) &= \frac{\widehat{\omega}_{\xi(2n-2r)}\left(\beta_2+dy\right)^{\lfloor \frac{2n-2r}{2} \rfloor} - \widehat{\sigma}_{\xi(2n-2r)}\left(\beta_1+dy\right)^{\lfloor \frac{2n-2r}{2} \rfloor}}{\beta_1-\beta_2} \\ &= \frac{\widehat{\omega}_0\left(\beta_2+dy\right)^{n-r} - \widehat{\sigma}_0\left(\beta_1+dy\right)^{n-r}}{\beta_1-\beta_2} \end{aligned} \tag{32}$$

$$\begin{aligned} LH_{2n}(x,y) &= \frac{\widehat{\omega}_{\xi(2n)}\left(\beta_2+dy\right)^{\lfloor \frac{2n}{2} \rfloor} - \widehat{\sigma}_{\xi(2n)}\left(\beta_1+dy\right)^{\lfloor \frac{2n}{2} \rfloor}}{\beta_1-\beta_2} \\ &= \frac{\widehat{\omega}_0\left(\beta_2+dy\right)^{n} - \widehat{\sigma}_0\left(\beta_1+dy\right)^{n}}{\beta_1-\beta_2} \end{aligned} \tag{33}$$

By virtue of (31), (32) and (33), we have

$$\begin{aligned} &LH_{2(n+r)}(x,y)LH_{2(n-r)}(x,y) - \left(LH_{2n}(x,y)\right)^2 \\ &\qquad = \frac{\widehat{\omega}_0\widehat{\sigma}_0\left(\beta_2+dy\right)^n\left(\beta_1+dy\right)^n\left[1-\left(\frac{\beta_2+dy}{\beta_1+dy}\right)^r\right]}{\left(\beta_1-\beta_2\right)^2} \\ &\qquad\quad + \frac{\widehat{\sigma}_0\widehat{\omega}_0\left(\beta_1+dy\right)^n\left(\beta_2+dy\right)^n\left[1-\left(\frac{\beta_1+dy}{\beta_2+dy}\right)^r\right]}{\left(\beta_1-\beta_2\right)^2}. \end{aligned}$$

Case $i=1$:

$$LH_{2(n+r)+1}(x,y) = \frac{\widehat{\omega}_1\left(\beta_2+dy\right)^{n+r} - \widehat{\sigma}_1\left(\beta_1+dy\right)^{n+r}}{\beta_1-\beta_2} \tag{34}$$

$$LH_{2(n-r)+1}(x,y) = \frac{\widehat{\omega}_1\left(\beta_2+dy\right)^{n-r} - \widehat{\sigma}_1\left(\beta_1+dy\right)^{n-r}}{\beta_1-\beta_2} \tag{35}$$

$$LH_{2n+1}(x,y) = \frac{\widehat{\omega}_1\left(\beta_2+dy\right)^{n} - \widehat{\sigma}_1\left(\beta_1+dy\right)^{n}}{\beta_1-\beta_2} \tag{36}$$

By virtue of (34), (35) and (36), we have

$$LH_{2(n+r)+1}(x,y)LH_{2(n-r)+1}(x,y)-\left(LH_{2n+1}(x,y)\right)^2$$
$$=\frac{\widehat{\omega}_1\widehat{\sigma}_1\left(\beta_2+dy\right)^n\left(\beta_1+dy\right)^n\left[1-\left(\frac{\beta_2+dy}{\beta_1+dy}\right)^r\right]}{\left(\beta_1-\beta_2\right)^2}$$
$$+\frac{\widehat{\sigma}_1\widehat{\omega}_1\left(\beta_1+dy\right)^n\left(\beta_2+dy\right)^n\left[1-\left(\frac{\beta_1+dy}{\beta_2+dy}\right)^r\right]}{\left(\beta_1-\beta_2\right)^2}.$$

Thus, the proof is completed. □

Now, we give the Cassini's identity of the bivariate conditional Lucas hybrinomial $LH_n(x,y)$.

**Corollary 2.** *For $n\geq 0$, we get*

$$LH_{2(n+1)+\xi(i)}(x,y)LH_{2(n-1)+\xi(i)}(x,y)-\left(LH_{2n+\xi(i)}(x,y)\right)^2$$
$$=\frac{\widehat{\sigma}_{\xi(i)}\widehat{\omega}_{\xi(i)}\left(\beta_1+dy\right)^n\left(\beta_2+dy\right)^n\left[1-\left(\frac{\beta_1+dy}{\beta_2+dy}\right)\right]}{\left(\beta_1-\beta_2\right)^2}$$
$$+\frac{\widehat{\sigma}_{\xi(i)}\widehat{\omega}_{\xi(i)}\left(\beta_1+dy\right)^n\left(\beta_2+dy\right)^n\left[1-\left(\frac{\beta_1+dy}{\beta_2+dy}\right)\right]}{\left(\beta_1-\beta_2\right)^2}$$

*where $\widehat{\omega}_{\xi(i)}$ and $\widehat{\sigma}_{\xi(i)}$ are defined in Theorem(7) and $i\in\{0,1\}$.*

*Proof.* Taking $r=1$ in Catalan's identity the proof is completed. □

## 4. Conclusion

The Fibonacci and Lucas numbers are well–known numbers, which have been studied by many researchers for years. These numbers arise in the applications of mathematics, computer science, physics, biology and statistics [9]. In this paper, by combining the Fibonacci and Lucas numbers with hybrid numbers, we present the generalized bivariate conditional Fibonacci and Lucas hybrinomials which are generalization of many works in the literature. Moreover, we derive many properties of generalized bivariate conditional Fibonacci and Lucas hybrinomials such as Binet's formulas, Catalan's identity, Cassini's identity of the hybrinomials.

**Author Contribution Statements** All authors contributed equally to this manuscript and they read and approved the final manuscript.

**Declaration of Competing Interests** The authors declare that they have no competing interest.

**Acknowledgment** This study is a part of the second author's Master Thesis.

## References


[1] Ait-Amrane, N. R., Belbachir, H., Bi-periodic r-Fibonacci sequence and bi-periodic r-Lucas sequence of type s, *Hacettepe Journal of Mathematics and Statistics, 51* (3) (2022), 680–699, https://dx.doi.org/10.15672/hujms.825908.

[2] Ait-Amrane, N. R., Belbachir, H., Tan, E., On generalized Fibonacci and Lucas hybrid polynomials, *Turkish Journal of Mathematics, 46* (6) (2022), 2069–2077, https://dx.doi.org/10.55730/1300-0098.3254.

[3] Bala, A., Verma, V., Some properties of bi-variate bi-periodic Lucas polynomials, *Annals of the Romanian Society for Cell Biology* (2021), 8778–8784.

[4] Belbachir, H., Bencherif, F., On some properties on bivariate Fibonacci and Lucas polynomials, *arXiv preprint arXiv:0710.1451* (2007), https://dx.doi.org/10.48550/arXiv.0710.1451.

[5] Bilgici, G., Two generalizations of Lucas sequence, *Applied Mathematics and Computation, 245* (2014), 526–538, https://dx.doi.org/10.1016/j.amc.2014.07.111.

[6] Edson, M., Yayenie, O., A new generalization of Fibonacci sequence & extended Binet's formula, *Integers, 9* (6) (2009), 639–654, https://dx.doi.org/10.1515/INTEG.2009.051.

[7] Falcón, S., Plaza, Á., The $k-$Fibonacci sequence and the Pascal 2-triangle, *Chaos, Solitons & Fractals, 33* (1) (2007), 38–49, https://dx.doi.org/10.1016/j.chaos.2006.10.022.

[8] Kızılateş, C., A new generalization of Fibonacci hybrid and Lucas hybrid numbers, *Chaos, Solitons & Fractals, 130* (2020), 109449, https://dx.doi.org/10.1016/j.chaos.2019.109449.

[9] Koshy, T., Fibonacci and Lucas Numbers with Applications, Volume 2, John Wiley & Sons, 2019.

[10] Özdemir, M., Introduction to hybrid numbers, *Advances in applied Clifford algebras, 28* (2018), 1–32, https://dx.doi.org/10.1007/s00006-018-0833-3.

[11] Panwar, Y. K., Singh, M., Generalized bivariate Fibonacci-like polynomials, *International Journal of Pure Mathematics, 1* (8) (2014), 13.

[12] Sevgi, E., The generalized Lucas hybrinomials with two variables, *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 70* (2) (2021), 622–630, https://dx.doi.org/10.31801/cfsuasmas.854761.

[13] Szynal-Liana, A., The Horadam hybrid numbers., *Discussiones Mathematicae: General Algebra & Applications, 38* (1) (2018), https://dx.doi.org/10.7151/dmgaa.1287.

[14] Szynal-Liana, A., Włoch, I., Introduction to Fibonacci and Lucas hybrinomials, *Complex Variables and Elliptic Equations, 65* (10) (2020), 1736–1747, https://dx.doi.org/10.1080/17476933.2019.1681416.

[15] Verma, A. B., Bala, A., On properties of generalized bi-variate bi-periodic Fibonacci polynomials, *International journal of Advanced science and Technology, 29* (3) (2020), 8065–8072.

[16] Yayenie, O., A note on generalized Fibonacci sequences, *Applied Mathematics and computation, 217* (12) (2011), 5603–5611, https://dx.doi.org/10.1016/j.amc.2010.12.038.

[17] Yazlik, Y., Köme, C., Madhusudanan, V., A new generalization of Fibonacci and Lucas p-numbers, *Journal of computational analysis and applications, 25* (4) (2018), 657–669.

[18] Yilmaz, N., Coskun, A., Taskara, N., On properties of bi-periodic Fibonacci and Lucas polynomials, In *AIP Conference Proceedings* (2017), vol. 1863, AIP Publishing LLC, p. 310002, https://dx.doi.org/10.1063/1.4992478.

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 64–75 (2024)
DOI:10.31801/cfsuasmas.1170867
ISSN 1303-5991 E-ISSN 2618-6470

*Research Article; Received: September 4, 2022; Accepted: October 23, 2023*

# TRAJECTORY CURVES AND SURFACES: A NEW PERSPECTIVE VIA PROJECTIVE GEOMETRIC ALGEBRA

Ferhat TAŞ

Department of Mathematics, Faculty of Science, İstanbul University, İstanbul 34134, TÜRKİYE

Abstract. The aim of this work is to define quaternion curves and surfaces and their conjugates via operators in Euclidean projective geometric algebra (EPGA). In this space, quaternions were obtained by the geometric product of vector fields. New vector fields, which we call trajectory curves and surfaces, were obtained by using this new quaternion operator. Moreover, dual quaternion curves are determined by a similar method and then their generated motion is studied. Illustrative examples are given.

## 1. Introduction

Understanding what complex numbers indicate geometrically has always been a matter of curiosity. Since the problem of finding the roots of a quadratic equation, we use a combination of a complex unit and real numbers, or their ordered binary representation, to show complex numbers. So what does this imaginary part show geometrically? Common usage is an axis orthogonal to the real axis. Thus, it shows the 2-dimensional real space in binary terms. However, complex numbers seem to contain more than that.

While working on the algebra of complex numbers of the form $a + bi$, Argand noticed that when the complex number is multiplied by the imaginary unit $i$, i.e. $i(a + bi) = -b + ai$ represents the rotation of this point, a geometric indicator of the complex number, about the origin in the plane by $90^\circ$ [1,2]. Hamilton thought that this rotational property of complex numbers might also have a counterpart in 3-dimensional space. So he predicted that an ordered 3 with two complex units could show the rotation in 3-dimensional space. However, it was not that easy to establish the algebra. An undesirable complex expression was coming from the product of two triplets. He used a combination of three imaginary numbers and

2020 *Mathematics Subject Classification.* 15A66, 16H05, 53A04, 53A05, 14J26.
*Keywords.* Quaternions, geometric algebra, rotation, projective geometric algebra.
tasf@istanbul.edu.tr; 0000-0001-5903-2881.

a real number to overcome this problem, and thus multiplication must be closed. Therefore, he invented the quaternions [3]:

A (real) quaternion is represented by $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ (general assumption) where $a, b, c, d \in \mathbb{R}$ and the conditions for units

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

held.

Thanks to quaternion algebra, one can obtain information about the rotation of any rigid body in space. Especially, in computer graphics, there are two options to rotate an object: use a rotation matrix (3x3 matrix) or a quaternion. Besides, there are some advantages to using quaternion like memory space, speed, and not showing gimbal locks as matrices do.

One can assume coefficients of the quaternion as dual numbers and then get the dual quaternion, i.e. $A, B, C, D \in \mathbb{D}$ [4–6].
Dual quaternions store both rotation and translation information about a rigid body in space. A similar operation can be done with 4x4 matrices. It is used quite often in the theory of mechanisms. This type of quaternion pair (called a dual quaternion) stores the same type of information but in two different quaternions. While the dual part of the dual quaternion represents translation information, the real one represents information about rotation. When mentioned together with these features, it is understood that it is a tool for software in robotics. Rather than its usual general definitions, it will be given here in accordance with projective geometric algebra.

Geometric algebra has very useful content to determine objects, especially complex ones geometrically, and transform one to another in space. Generally, geometric algebra over n-dimensional vector space is represented by the set $\mathcal{G}(n, 0)$ where $n$ is the dimension of vector space and the second part is the grade of the inner product. Since the vector space studied here is real, the grade of the inner product is 0. Basically, geometric algebra allows us to symbolize scalars, vectors, areas, and volumes using a simple and consistent notation. Such items are closed under the algebraic operation. It is not difficult to see that many variables of this type are closed under addition. However, the product is somewhat unusual. So these expressions can be tough to visualize. Furthermore, the orientation of an object in space is important in physics research, for instance, spinors occupy an important place in quantum mechanics. Geometric algebra tools also provide the orientation of objects.

There are two expansions of this algebra: Conformal geometric algebra (CGA) and projective geometric algebra (PGA).

EPGA is based on duality: that is, we can represent work (wedge) done in one (exterior algebra) to be equivalent (join- $\vee$) in the other (dual exterior algebra). Similar to this situation, the meet ($\wedge$) operator is also defined. The reason we work in dual space is because all of the Euclidean operations can be represented. Working in a projective dual space also prevents special cases from occurring.

In this paper, we use Euclidean curves to generate quaternion curves via geometric product. Thus, we describe what a quaternion operator looks like visually. Bearing this motivation, we wonder about the motion that will occur around a moving (dual) quaternion rather than around a fixed (dual) quaternion in the EPGA language.

## 2. Fundamentals

In this section, we try to explain the properties of GA and PGA. There are some operators. We explore how these operators lead to rotations just as complex numbers do.

One of the most remarkable works on quaternion algebra in the literature is Aslan and Yaylı, [22], where they define quaternion operators on curves and surfaces in Euclidean 3-space using geometric algebra. These operators generate motions that have orbits along the generated curve or surface and can be expressed as 1-parameter or 2-parameter homothetic motions. Besides, Shoemake presents a new kind of spline curve suitable for smoothly interpolating sequences of arbitrary rotations. The motion generated is smooth and natural, without quirks found in earlier methods, [23].

Since geometric algebra is a very broad topic from kinematics [7–9] to robot dynamics [10], from neuroscience [17] to modeling [11], our aim here is not to explain all the basic topics of geometric algebra. For more details, see [12, 15–21]. Only definitions required in the article will be given.

2.1. **Geometric algebra.** Let $\mathbb{R}^2$, 2D-real space, be spanned by two linear independent orthonormal vectors: $\{\mathbf{e}_1, \mathbf{e}_2\}$. The inner and outer product of these elements produces new bases for geometric algebra:

$$< \mathbf{e}_1, \mathbf{e}_1 > = < \mathbf{e}_2, \mathbf{e}_2 > = 1 \text{ and } \mathbf{e}_1 \wedge \mathbf{e}_2$$

Bearing with these elements we have the following bases for geometric algebra $\mathcal{G}(2,0)$:

$$\{1(\text{scalar}), \mathbf{e}_1, \mathbf{e}_2(\text{vectors}), \mathbf{e}_1 \wedge \mathbf{e}_2(\text{pseudo-scalar})\}.$$

The algebra has also general elements called multivectors like that: $a1 + b\mathbf{e}_1 + c\mathbf{e}_2 + d\mathbf{e}_1 \wedge \mathbf{e}_2$, where $a, b, c, d \in \mathbb{R}$. Partially, $x1 + y\mathbf{e}_1 \wedge \mathbf{e}_2 = x + iy$, represent a complex number.

$\mathbb{R}^3$, 3D-real space, is spanned by three independent vectors: $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. New basis for geometric algebra over $\mathbb{R}^3$ are:

$$< \mathbf{e}_1, \mathbf{e}_1 > = < \mathbf{e}_2, \mathbf{e}_2 > = < \mathbf{e}_3, \mathbf{e}_3 > = 1$$

and

$$\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_3, \mathbf{e}_2 \wedge \mathbf{e}_3.$$

So vector spaces of the geometric algebra $\mathcal{G}(3,0)$ are

$$\{\underbrace{1}_{\text{scalars}},\ \underbrace{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3}_{\text{vector space}},\underbrace{\mathbf{e}_1\wedge\mathbf{e}_2,\mathbf{e}_2\wedge\mathbf{e}_3,\mathbf{e}_3\wedge\mathbf{e}_1}_{\text{bivector space}},\ \underbrace{\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3}_{\text{trivector space.}}\}$$

All geometric algebra has $\sum_{k=0}^{3}\begin{pmatrix}3\\k\end{pmatrix}=2^3=8$ elements where the number of $k$-blades of geometric algebra over $\mathbb{R}^3$ is computed by $\begin{pmatrix}3\\k\end{pmatrix}$ combination. So any multi-vector is of the form: $a1+b\mathbf{e}_1+c\mathbf{e}_2+d\mathbf{e}_3+e\mathbf{e}_1\wedge\mathbf{e}_2+f\mathbf{e}_1\wedge\mathbf{e}_3+g\mathbf{e}_2\wedge\mathbf{e}_3+h\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3$, where $a,b,c,d,e,f,g,h\in\mathbb{R}$.

Operations are important for the derivation of elements in a mathematical structure such as vector space or algebra. These processes should always show the feature of closure. The next definition gives the fundamental operator for geometric algebra.

**Definition 1.** *Clifford defined the geometric product of two vectors,* $\mathbf{u}$ *and* $\mathbf{v}$*, as follows [9]: Let* $\mathbf{u}=\sum_{i=1}^{n}u_i\mathbf{e}_i$ *and* $\mathbf{v}=\sum_{i=1}^{n}v_i\mathbf{e}_i$ *then*

$$\mathbf{u}\mathbf{v}=<\mathbf{u},\mathbf{v}>+\mathbf{u}\wedge\mathbf{v}=\sum_{i=1}^{n}u_iv_i+\sum_{i,j=1}^{n}u_iv_j\mathbf{e}_i\mathbf{e}_j. \tag{1}$$

*where* $\mathbf{e}_i$ *are unit bases of* $\mathbb{R}^n$*. Here we use* $\mathbf{e}_{ij}=\mathbf{e}_i\mathbf{e}_j=\mathbf{e}_i\wedge\mathbf{e}_j$ *for simplicity.*

**Proposition 1.** *Let* $\mathbf{u}$*,* $\mathbf{v}$ *and* $\mathbf{w}$ *are vectors in* $\mathbb{R}^n$ *then following rules are provided*

*(i) Associativity:*

$$\mathbf{u}(\mathbf{v}\mathbf{w})=\mathbf{u}(\mathbf{v})\mathbf{w}=\mathbf{u}\mathbf{v}\mathbf{w}$$

*(ii) Distributivity:*

$$\mathbf{u}(\mathbf{v}+\mathbf{w})=\mathbf{u}\mathbf{v}+\mathbf{u}\mathbf{w}$$

*and*

$$(\mathbf{v}+\mathbf{w})\mathbf{u}=\mathbf{v}\mathbf{u}+\mathbf{w}\mathbf{u}$$

*(iii) Modulus:*

$$\|\mathbf{u}\|^2=\mathbf{u}\mathbf{u}=<\mathbf{u},\mathbf{u}>.$$

Products of fundamental elements of geometric algebra are given as follows,

$$\mathbf{e}_i\mathbf{e}_i=1,\mathbf{e}_i\mathbf{e}_j=\mathbf{e}_i\wedge\mathbf{e}_j=-\mathbf{e}_j\wedge\mathbf{e}_i=-\mathbf{e}_j\mathbf{e}_i$$
$$(\mathbf{e}_i\wedge\mathbf{e}_j)(\mathbf{e}_i\wedge\mathbf{e}_j)=\mathbf{e}_i\mathbf{e}_j\mathbf{e}_i\mathbf{e}_j=-\mathbf{e}_i\mathbf{e}_i=-1$$
$$(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)=-\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1\mathbf{e}_3=-1$$

where $i,j=1,2,3$ (for $i\neq j$).

**Definition 2.** *In geometric algebra, there is a Hodge duality for elements and defined as follows [18]: Let $I = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ be the pseudo-scalar then,*

$$1^* = -1I = -I,$$

So, scalar and pseudo-scalar are Hodge dual of each other (Hodge duality generally represented by $^*$). Similarly, vectors and blades are Hodge dual of each other:

$$\begin{aligned}
\mathbf{e}_1^* &= -\mathbf{e}_1 I = -\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = -\mathbf{e}_2\mathbf{e}_3,\\
\mathbf{e}_2^* &= -\mathbf{e}_2 I = -\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_3,\\
\mathbf{e}_3^* &= -\mathbf{e}_3 I = -\mathbf{e}_3\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = -\mathbf{e}_1\mathbf{e}_2.
\end{aligned}$$

**Definition 3.** *Let $\mathbf{w}$ be a vector and $B = \mathbf{u} \wedge \mathbf{v}$ be a 2-vector, then*

$$\mathbf{w}B = \langle \mathbf{w}, B\rangle + \mathbf{w} \wedge B = (\mathbf{w}B)_1 + (\mathbf{w}B)_3.$$

*where $(.)_k$ represent the grade of element.*

2.2. **Projective geometric algebra.** Although affine transformations of geometric objects can be achieved with vector algebra, this can cause some difficulties in 3D computer graphics, such as the algebraic separation of point and vector. For this, it is the algebraic structure that we call homogeneous coordinates and allows us to represent n-dimensional real space in (n+1)-dimensional space. Let $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ be a point in Euclidean 3-space then $X = \mathbf{x} + \mathbf{e}_0$ be a homogeneous point in 4D geometric algebra.

This space can also be defined in geometric algebra by adding an explicit extra basis: $\mathbf{e}_0$, satisfying $\mathbf{e}_0^2 = 0$, which corresponds to a null vector providing only linear terms expansion of an exponential function. So the metric structure would be $\mathbf{e}_i.\mathbf{e}_j = \delta_{ij}$ for $i, j = 1, 2$ and $\mathbf{e}_i.\mathbf{e}_0 = 0$. Since this metric structure is the same as the Euclidean metric, it also preserves isometry. There are also other bases like $\mathbf{e}^2 = -1$ or 1 and generate higher dimensional projective geometric algebra. In general, these unusual bases are called geometric numbers. Summing up, its general notation in this point-based structure is $\mathbf{P}(\mathbb{R}_{p,n,z})$ where $p, n, z$ stand for positive, negative and zero, respectively. Besides, plane-based model, for instance, the algebra $\mathbf{P}\left(\mathbb{R}^*_{2,0,1}\right)$ represents the proper 2D Euclidean space. As far as we know from linear algebra, if $V$ is a vector space then there is a dual vector space $V^*$. So each geometric object in the exterior algebras in $\mathbf{P}(V)$ and $\mathbf{P}(V^*)$ has a representation in both. This is the Poincaré duality [13, 14].

$\mathbf{P}\left(\mathbb{R}^*_{3,0,1}\right)$ provides coordinate-free, uniform representation for Euclidean elements: points, lines, and planes.

| $\wedge$ | $\mathbf{e}_0$ | $\mathbf{e}_1$ | $\mathbf{e}_2$ | $\mathbf{e}_3$ |
|---|---|---|---|---|
| $\mathbf{e}_0$ | $0$ | $\mathbf{e}_{01}$ | $\mathbf{e}_{02}$ | $\mathbf{e}_{03}$ |
| $\mathbf{e}_1$ | $\mathbf{e}_{10}$ | $0$ | $\mathbf{e}_{12}$ | $\mathbf{e}_{13}$ |
| $\mathbf{e}_2$ | $\mathbf{e}_{20}$ | $\mathbf{e}_{21}$ | $0$ | $\mathbf{e}_{23}$ |
| $\mathbf{e}_3$ | $\mathbf{e}_{30}$ | $\mathbf{e}_{31}$ | $\mathbf{e}_{32}$ | $0$ |

Although it has degenerate metric we can explain the reason why this algebra shows Euclidean isometries as follows: The basic elements in geometric algebra do not actually have Euclidean representations. Therefore, we can understand what they are geometrically by looking at their dual structure. However, dual PGA performs its operations directly with Euclidean elements. So we can call it EPGA for short. The basic linear elements of this algebra are planes (1-vector), and they are defined as follows,

$$\mathbf{a} = \sum_{i=0}^{3} a_i \mathbf{e}_i.$$

It also includes meet and join operators. These operators decrease and increase of grades of elements of algebra, respectively. Thus, the union and intersection operations of points, lines, and planes in PGA can be done with wedge and progressive product, respectively.

Quaternions are also even subalgebra (zero and two graded) of projective geometric algebra $\mathbf{P}\left(\mathbb{R}^*_{3,0,1}\right)$ and a quaternion is represented by

$$q = a + b\mathbf{e}_{12} + c\mathbf{e}_{31} + d\mathbf{e}_{23}$$

where $a, b, c, d \in \mathbb{R}$.

## 3. Quaternion Curves and Surfaces

Grounded in the elegant framework of geometric algebra, quaternion curves and surfaces stand as a cornerstone in the field of 3D geometry and computer graphics. In this chapter, we enter a world where advanced mathematics meets practical applications, exploring the profound implications of quaternions in representing and manipulating curves and surfaces.

As a hyper-complex extension of complex numbers, quaternions offer a concise and efficient way to handle 3D rotations and orientations, finding extensive applications in fields ranging from computer graphics and robotics to physics simulations. Represented by geometric algebra, these quaternions are not just abstract mathematical constructs, but powerful tools that allow us to describe the complex motion of objects in space with remarkable precision and flexibility.

**Definition 4.** *Let $\mathbf{a}(t), \mathbf{b}(t)$ be vector fields then*

$$q(t) = \mathbf{a}(t)\mathbf{b}(t) = < \mathbf{a}(t), \mathbf{b}(t) > + \mathbf{a}(t) \wedge \mathbf{b}(t)$$

*is a quaternion curve and its conjugate is given by reverse order product*

$$\widetilde{q}(t) = \mathbf{b}(t)\mathbf{a}(t) = < \mathbf{b}(t), \mathbf{a}(t) > + \mathbf{b}(t) \wedge \mathbf{a}(t).$$

**Definition 5.** *Let* $\mathbf{a}(t), \mathbf{b}(s)$ *be vector fields then*

$$q(t,s) = \mathbf{a}(t)\mathbf{b}(s) = < \mathbf{a}(t), \mathbf{b}(s) > + \mathbf{a}(t) \wedge \mathbf{b}(s)$$

*is a quaternion surface.*

To see the behavior of these quaternion curves in 3-dimensional space, it is necessary to apply them to a point. Let us now formulate here the rotation of a point through a quaternion operator generated by the meeting of two unit vectors in space:

Let $\|\mathbf{u}\| = \|\mathbf{v}\| = 1, q = \mathbf{uv} = \cos(\theta) + \sin(\theta)\mathrm{B}$ and its conjugate $\widetilde{q} = \mathbf{vu} = \cos(\theta) - \sin(\theta)\mathrm{B}$. Take a point in EPGA as $P = \mathbf{e}_{123} + P_E$ then the rotated point $P_r$ is as follows;

$$P_r = qP\widetilde{q} = \mathbf{e}_{123} + qP_E\widetilde{q} \tag{2}$$

Thus, applying real quaternion operators to EPGA elements gives the same result as in EGA.

3.1. **Trajectory curves.** The concept of trajectories generated by quaternion curves involves representing rotations in 3D space using quaternions and understanding how these rotations affect the orientation of objects over time.

In the context of trajectories, quaternions are used to smoothly interpolate between different orientations of an object, creating a continuous curve that describes the object's rotation over time. Quaternions have certain advantages over other rotation representations (such as Euler angles) because they do not suffer from gimbal lock and provide smooth interpolation without singularities.

**Corollary 1.** *Let* $q(t)$ *be a quaternion curve and* $P$ *be a point in* $\mathbf{P}\left(\mathbb{R}^*_{3,0,1}\right)$ *then*

$$\alpha(t) = q(t)P\widetilde{q}(t) \tag{3}$$

*is a trajectory curve.*

**Example 1.** *Let* $\mathbf{u}(t) = \cos(t)\mathbf{e}_1 + \sin(t)\mathbf{e}_2$*,* $\mathbf{v}(t) = \cos(t)\mathbf{e}_1 + \sin(t)\mathbf{e}_3$*,* $P = \mathbf{e}_{123} + \mathbf{e}_{032}$ *be unit vector fields and a point in* $\in \mathbf{P}\left(\mathbb{R}^*_{3,0,1}\right)$*, respectively. Then*

$$\begin{aligned}
q(t) &= \mathbf{u}(t)\mathbf{v}(t) \\
&= < \cos(t)\mathbf{e}_1 + \sin(t)\mathbf{e}_2, \cos(t)\mathbf{e}_1 + \sin(t)\mathbf{e}_3 > \\
&+ (\cos(t)\mathbf{e}_1 + \sin(t)\mathbf{e}_2) \wedge \cos(t)\mathbf{e}_1 + \sin(t)\mathbf{e}_3 \\
&= \cos(t)^2 + \cos(t)\sin(t)\mathbf{e}_1\mathbf{e}_3 - \cos(t)\sin(t)\mathbf{e}_1\mathbf{e}_2 \\
&+ \sin^2(t)\mathbf{e}_2\mathbf{e}_3.
\end{aligned}$$

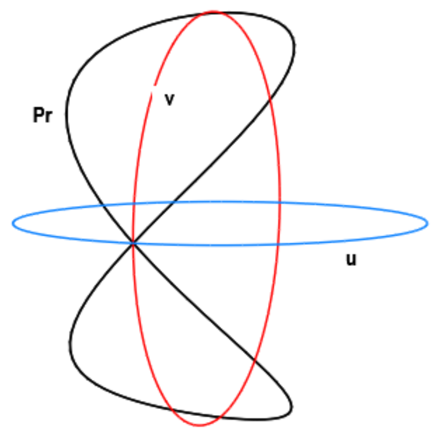



FIGURE 1. The curve that is generated by the quaternion curve.

*So the curve of quaternion rotation is (see Fig.1)*

$$\begin{aligned}\alpha(t) =& q(t)P\widetilde{q}(t)\\ =& \mathbf{e}_{123} + \cos(2t)^2\mathbf{e}_{230} + \frac{\sin(4t)}{2}\mathbf{e}_{301} - \sin(2t)\mathbf{e}_{012}.\end{aligned}$$

*This curve is also called Viviani's curve.*

3.2. **Trajectory surfaces.** Similar situations to the trajectory curves can also be done for surfaces. The only difference here is that the parameters of the vector fields forming the quaternion are different from each other.

**Corollary 2.** *Let $q(t,s)$ be a quaternion surface and $P$ be a point in $\mathbf{P}\left(\mathbb{R}^*_{3,0,1}\right)$ then*

$$X(t,s) = q(t,s)P\widetilde{q}(t,s) \tag{4}$$

*is a trajectory surface.*

**Corollary 3.** *Let $q_\theta$ be a quaternion, where $\theta$ is the angle between vectors that are constructing the quaternion, and $\alpha(t)$ be a regular curve in $\mathbf{P}\left(\mathbb{R}^*_{3,0,1}\right)$ then*

$$\beta(t,\theta) = q_\theta\alpha(t)\widetilde{q_\theta} \tag{5}$$

*is a rotational surface. This is also a special case of trajectory curves.*

**Corollary 4.** *Let $q(t)$ be a quaternion curve $\alpha(s)$ be a regular curve in $\mathbf{P}\left(\mathbb{R}^*_{3,0,1}\right)$ then*

$$\beta(t,s) = q(t)\alpha(s)\widetilde{q}(t) \tag{6}$$

*is a trajectory surface.*

Let's define some surfaces as trajectory surfaces with the tool we have generated.

3.2.1. *Sphere as a trajectory surface.* This example can be used for surfaces (one can take a different parameter for the second vector field as we defined in Def 3.2). The resulting trajectory surface is this time, a 2-sphere. This is just like the product of two curves in the topological sense: $\mathbb{S}^2 \simeq \mathbb{S}^1 \times \mathbb{S}^1$.

3.2.2. *Cone as a trajectory surface.* It is easy to obtain a cone surface using this tool. As is well known, a cone is formed by a line passing through two points in space, one of the points being fixed and the other orbiting on a circle. We can make this imaginary geometric idea as it is with the help of geometric algebra. The flexibility in the choice of the vector fields that make up the quaternion and the object to be rotated in this tool offers us different ways and offers us the opportunity to create the geometric object to be created in different ways. Everything depends on your imagination. For example, for this cone surface, we can choose one of the vector fields forming the quaternion as a unit circle and the other as a point perpendicular to the plane in which this circle is located. The geometric object to be rotated is then a line.

3.3. **Dual quaternions and rigid motions.** The projective 4D analog of a quaternion is called a dual quaternion. This is where the real difference of EPGA comes into play. So all Euclidean motions can be described in this space and we can assume the dual number unit $\epsilon$ as a pseudo-scalar $\mathbf{e}_{0123}$, i.e. provide the mystique properties of its: $\mathbf{e}_{0123}^2 = 0$. Thus, we define the dual element as an algebraic basis.

Let $q = x_0 + x_1e_2e_3 + x_2e_3e_1 + x_3e_1e_2, r = y_0 + y_1e_2e_3 + y_2e_3e_1 + y_3e_1e_2$ be two quaternions. Then we can construct the dual quaternions in geometric algebra way as follows [10]:

$$\begin{aligned} Q &= q + r\mathbf{e}_{0123} = q - r^* \\ &= x_0 + x_1e_2e_3 + x_2e_3e_1 + x_3e_1e_2 \\ &+ y_0e_0e_1e_2e_3 + y_1e_1e_0 + y_2e_2e_0 + y_3e_3e_0 \end{aligned}$$

Therefore, $P_m = QP\overline{Q}$ represents a rigid transformation of a point in the space.

**Definition 6.** *Let $q(t), r(t)$ be quaternion curves, then*

$$Q(t) = q(t) + r(t)\mathbf{e}_{0123}$$

*is a dual quaternion curve.*


FIGURE 2. Trajectory curves that are generated by the dual quaternion.

**Corollary 5.** *Let $Q(t)$ be dual quaternion curve and $P$ be a point in $\mathbf{P}\left(\mathbb{R}^*_{3,0,1}\right)$ then*

$$X(t) = Q(t)P\overline{Q}(t)$$

*is a trajectory curve.*

**Example 2.** *For the most basic situation, let $Q(t) = \cos(t)\mathbf{e}_{12} + \sin(t)\mathbf{e}_{23} + \mathbf{e}_{01} + \mathbf{e}_{0123}$ be a dual quaternion curve, then for points $P_x = \mathbf{e}_{123} + \mathbf{e}_{032}$, $P_y = \mathbf{e}_{123} + \mathbf{e}_{013}$, $P_z = \mathbf{e}_{123} + \mathbf{e}_{102}$ there are three trajectory curves, see Fig.2.*

An argument similar to the one above can be constructed with the basis (1-vectors) of EPGA. At this time, one can get a line that does not pass through the origin by meeting the two 1-vectors: Let $\mathbf{a}, \mathbf{b}$ be two 1-vectors and at least one of them is not passing through the origin, then their geometric product is

$$\mathbf{ab} = q_0 + \mathcal{L}.$$

where $q_0$ and $\mathcal{L}$ are scalar and plücker coordinates of the line, respectively. This operator generates a trajectory (orbit) of a point around a moving line in space.

It is decided whether a quaternion operator formed by the geometric multiplication of two 1-vectors is real or dual, by looking at whether its first terms (zeroth index) are zero. In other words, the line obtained from the intersection (i.e., the meet operator) of two planes passing through the origin represents the quaternion operator that represents a rotation around this line passing through the origin. On the other hand, if one of the first terms is nonzero, the resulting quaternion will represent a screw motion. Therefore, this quaternion acts like a dual quaternion operator.

## 4. Conclusion

Geometric algebra has lately garnered significant attention due to its profound applications for both imaging and performing fine operations on geometric objects. The field has seen remarkable progress, particularly in enhancing our capability to fantasize about complex geometric generalities. In this study, we embarked on a disquisition that extended the operation of geometric algebra from traditional vector-ground representations of curves and surfaces in classical figures to the realm of quaternions. Our purpose is to demonstrate the unique capabilities of quaternions as an important fine driver, shedding light on their part in generating topological structures when applied to classical 3D geometric objects.

The implications of our findings extend far beyond the realm of classical Euclidean geometry. While our study primarily focused on classical 3D geometric objects, the inherent flexibility of quaternions suggests that similar investigations can be carried out in the domain of non-Euclidean geometries. This exciting prospect hints at a wealth of fascinating results waiting to be uncovered, as the interplay between quaternions and non-Euclidean geometries promises to yield profound insights and applications in various scientific and engineering disciplines.

In summary, our research represents a pivotal contribution to the field of geometric algebra by showcasing the remarkable utility of quaternions as operators in transforming classical geometric objects and elucidating the emergence of topological structures. This work not only deepens our understanding of the relationship between algebra and geometry but also opens up a tantalizing avenue for future research, where quaternions can be harnessed to explore the rich landscapes of non-Euclidean geometries, potentially revolutionizing how we perceive and interact with the mathematical underpinnings of the physical world.

**Declaration of Competing Interests** The author has no competing interests to declare.

## References

[1] Vince, J., Imaginary Mathematics for Computer Science, Springer, ISBN-10: 3319946366, 2018. https://doi.org/10.1007/978-3-319-94637-5

[2] Argand, J. R., Essai Sur Une Manière de Représenter des Quantités Imaginaires Dans les Constructions Géométriques, 2nd edn. Gauthier-Villars, Paris, 1874.

[3] Hamilton, W. R., On quaternions: or a new system of imaginaries in algebra, *Phil. Mag.3rd ser.*, 25(163) (1844), 10-13. doi.org/10.1080/14786444408644923

[4] Clifford, W. K., Preliminary sketch of bi-quaternions, *Proceedings of the London Mathematical Society*, s1–4(1) (1873), 381–395. doi.org/10.1112/plms/s1-4.1.381

[5] Study, E., Geometrie der Dynamen, Teubner, Leipzig, 1901.

[6] Bottema, O., Roth, B., Theoretical Kinematics (Vol. 24), Courier Corporation, ISBN 10: 0486663469, ISBN 13: 9780486663463, 1990.

[7] Hestenes, D., New Foundations for Classical Mechanics (Vol. 15), Springer Science & Business Media, 2012. doi.org/10.1007/0-306-47122-1

[8] Hestenes, D., Sobczyk, G., Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics (Vol. 5), Springer Science & Business Media, 2012. doi.org/10.1007/978-94-009-6292-7

[9] Hestenes, D., Space-Time Algebra (Vol. 67), Birkhäuser, Basel, 2015. doi.org/10.1007/978-3-319-18413-5

[10] Selig, J. M., Geometric Fundamentals of Robotics, Springer Science & Business Media, 2004. doi.org/10.1007/b138859

[11] Hildenbrand, D., Geometric computing in computer graphics using conformal geometric algebra *Computers & Graphics*, 29(5) (2005), 795-803. doi.org/10.1016/j.cag.2005.08.028

[12] Dorst, L., Fontijne, D., Mann, S., Geometric Algebra for Computer Science: An Object-Oriented Approach to Geometry, Elsevier, ISBN: 9780123749420, 2010.

[13] Gunn, C. G., Doing Euclidean plane geometry using projective geometric algebra, *Adv. Appl. Clifford Algebras*, 27 (2017), 1203–1232. https://doi.org/10.1007/s00006-016-0731-5

[14] Gunn, C., Geometric algebras for Euclidean geometry, *Adv. Appl. Clifford Algebras*, 27 (2017), 185–208. https://doi.org/10.1007/s00006-016-0647-0

[15] Vince, J. A., Geometric Algebra for Computer Graphics, Springer, 2008. doi.org/10.1007/978-1-84628-997-2

[16] Perwass, C., Edelsbrunner, H., Kobbelt, L., Polthier, K., Geometric Algebra with Applications in Engineering (Vol. 4), Springer, Berlin, 2009. doi.org/10.1007/978-3-540-89068-3

[17] Bayro-Corrochano, E., Daniilidis, K., Sommer, G., Motor algebra for 3D kinematics: the case of the hand-eye calibration. *Journal of Mathematical Imaging and Vision*, 13 (2000), 79–100. https://doi.org/10.1023/A:1026567812984

[18] Kanatani, K., Understanding Geometric Algebra: Hamilton, Grassmann, and Clifford for Computer Vision and Graphics, CRC Press, 2015. https://doi.org/10.1201/b18273

[19] Vaz, J., Da Rocha, J., An Introduction to Clifford Algebras and Spinors, Oxford University Press, Oxford, 2016. https://doi.org/10.1093/acprof:oso/9780198782926.001.0001

[20] Josipović, M., Geometric Multiplication of Vectors: An Introduction to Geometric Algebra in Physics, Birkhäuser, Basel, 2019. https://doi.org/10.1007/978-3-030-01756-9

[21] Lasenby, J. (Ed.)., Guide to Geometric Algebra in Practice (pp. 371-389), Springer, New York, 2011. https://doi.org/10.1007/978-0-85729-811-9

[22] Aslan, S., Yaylı, Y., Motions on curves and surfaces using geometric algebra, *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, 71(1) (2022), 39-50. doi.org/10.31801/cfsuasmas.878766

[23] Shoemake, K., Animating rotation with quaternion curves, *In Proceedings of the 12th annual conference on Computer graphics and interactive techniques*, (1985), 245-254. https://doi.org/10.1145/325334.325242

[24] https://enkimute.github.io/ganja.js/

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 76–103 (2024)
DOI:10.31801/cfsuasmas.1147449
ISSN 1303-5991 E-ISSN 2618-6470

*Research Article; Received: July 23, 2022; Accepted: October 6, 2023*

# EXPONENTIATED GENERALIZED RAMOS-LOUZADA DISTRIBUTION WITH PROPERTIES AND APPLICATIONS

Yasin ALTINISIK[1] and Emel CANKAYA[2]

[1,2]Department of Statistics, Sinop University, 57000 Sinop, TÜRKİYE

Abstract. In this paper, we propose a new generalization of Ramos-Louzada (RL) distribution based on two additional shape parameters. Along with the genesis of its distributional form, the derivation of cumulative density function (cdf), survival and hazard rate functions, the quantile function (qf), moments, moment generating function (mgf), Shannon and Renyi entropies, order statistics and a linear representation of the proposed distribution are inspected. Several estimation methods of the model parameters are discussed throughout two comprehensive simulation studies conducted to compare its performance against some lifetime distributions. Application of a real dataset is presented to illustrate the potentiality of this distribution in line with the simulation studies.

## 1. Introduction

Lifetime modeling of complex studies has created a growing interest in the generation of flexible distributions that can provide solutions to certain problems of lifetime systems. Ramos-Louzada is such a distribution recently proposed by Ramos and Louzada ( [24]) to take instantaneous failures into account that can inevitably occur in many lifetime applications. It is announced to be a worthwhile alternative to the Exponential and Lindley ( [19]) distributions and take the forms of both with a shape parameter $\lambda \geq 2$. That is, the distribution becomes the exponential distribution for large values of $\lambda$ and it resembles to the Lindley distribution as $\lambda$ decreases towards 2.

---

2020 *Mathematics Subject Classification.* 05C38, 15A15, 05A15, 15A18.

*Keywords.* Lifetime datasets, Ramos-Louzada distribution, exponentiated generalized class, EGRL distribution, information criteria.

[1] yaltinisik@sinop.edu.tr-Corresponding author; 0000-0001-9375-2276

[2] ecankaya@sinop.edu.tr; 0000-0002-2892-2520.

Let the random variable $X$ follows the RL distribution with the rate parameter $\theta = \frac{1}{\lambda}$, $\lambda \geq 2$;

$$g(x) \quad = \quad \frac{(\theta^2 x - 2\theta + 1)\theta}{1-\theta}\ e^{-\theta x}, \tag{1}$$

where $x \geq 0$ and $0.5 \geq \theta > 0$. The cdf of $X \sim RL(\theta)$ is defined as

$$G(x) \quad = \quad 1 - \frac{(\theta^2 x - \theta + 1)}{1-\theta}\ e^{-\theta x}. \tag{2}$$

Although the RL distribution is attractive for its simplicity, it fails to provide precise evaluation of many lifetime datasets, since it contains only one parameter. Many researchers benefit from generalizing baseline (stated otherwise parent or target) distributions by adding one or more parameters into the model to increase the model fit and overcome the absence of sufficient flexibility in modeling the data. In this respect, Al-Mofleh et al. ( [3]) recently proposed a two-parameter generalization of the RL distribution by inserting a power parameter into the model. They showed that the generalized Ramos-Louzada (GRL) distribution performs better than some well-known distributions such as Marshall-Olkin ( [21]), exponentiated exponential ( [11]) and generalized Lindley ( [23]) distributions with respect to some bias and accuracy measures.

This paper proposes a new three-parameter model as a competetive extension for this generalization of the RL distribution, namely the exponentiated generalized Ramos-Louzada (EGRL) distribution. The new distribution relies on the class of distributions established by Corderio et al. ( [7]). The usual definition of the probability density function (pdf) of this family of distributions is

$$f(x) = \alpha\beta g(x)\Big[1 - G(x)\Big]^{\alpha-1}\left(1 - \Big[1 - G(x)\Big]^{\alpha}\right)^{\beta-1}, \tag{3}$$

where $\alpha > 0$ and $\beta > 0$ are two shape parameters and $g(x)$ and $G(x)$ are the pdf and cdf of the baseline distribution, respectively. The shape parameters $\alpha$ and $\beta$ in equation (3) provide better flexibility in the tails of the data and increase the entropy in the center ( [7, p. 2]). The cdf of the family of distributions is of the form

$$F(x) = \left(1 - \Big[1 - G(x)\Big]^{\alpha}\right)^{\beta}. \tag{4}$$

Our basic motivation for such generalization is to provide a better fit of RL distribution to the wider range of problems in statistics. It is also of our goal to achieve reliable estimation of model parameters considering various estimation methods. This is particularly important as it affects the model selection process. Evaluation of model fit via goodness of fit statistics is a usual practice in the literature. Al-Mofleh et al. ( [3]) consider only the minus log likelihood (-$\ell$), Cramer-von Mises (C*; [9]) and Kolmogorov-Smirnov (KS*; [16, 31]) goodness of fit statistics. The assessment of model fit via these goodness of fit statistics might produce biased results, since they do not take the model complexity into account when choosing the best distribution in a set of distributions. The distributions under consideration should also be compared to each other by means of using information criteria.

Incorporating additional adequate parameter(s) into the model improves the model fit and provides more flexibility in analyzing datasets. However, caution should be taken when generalizing baseline distributions using more parameters in the model. Achieving a good model fit requires taking into account the balance between the sample size and the number of parameters in the model (bias versus variance tradeoff as used in the literature). The information criteria such as Akaike information criterion (AIC; [1, 2]) and Bayesian information criterion (BIC; [29]) are originally developed for solving this problem as they do not only rely on log likelihood values, but also on penalty values. The log likelihood represents the fit of a model to the data at hand as the penalty value penalizes the model depending on (a function of) the number of parameters in the model. Thus, we compare the EGRL distribution against a set of alternative distributions by means of not only using model fit statistics, but also different types of information criteria.

The outline of the paper is as follows. In Section 2, we derive various statistical and reliability properties of the EGRL distribution. In Section 3, we elaborate on the methods of maximum likelihood estimation (MLE), least squares estimation (LSE), weighted least squares estimation (WLSE), and Cramer-von Mises estimation (CVME) to obtain the estimates of model parameters and their standard errors for the EGRL distribution. In Section 4, we perform two simulation studies. In the first simulation study, we evaluate the performance of the methods in estimating the parameters of the EGRL distribution with respect to bias, precision, and accuracy measures. In the second simulation study, we compare the performance of the EGRL distribution to that of a set of other lifetime distributions with respect to some goodness of fit statistics and information criteria for each estimation method. In Section 5, we exemplify the applicability of EGRL distribution for a real life problem. We illustrate that the goodness of fit statistics may not be able to detect the best distribution in a set of distributions and information criteria should be used instead when comparing the performance of distributions. The paper will be concluded with a short discussion.

## 2. The EGRL Distribution

**2.1. Probability density and cumulative density functions.** Incorporating equations (1) and (2) into the general definition in equation (3), we obtain the pdf of EGRL distribution which is given by

$$f(x) = \alpha\beta \; \frac{(\theta^2 x - 2\theta + 1)\theta}{1-\theta} \; e^{-\theta x}\left[\frac{(\theta^2 x - \theta + 1)}{1-\theta} \; e^{-\theta x}\right]^{\alpha-1}$$

$$\times\left(1-\left[\frac{(\theta^2 x-\theta+1)}{1-\theta}\ e^{-\theta x}\right]^{\alpha}\right)^{\beta-1}. \tag{5}$$

For $\alpha = \beta = 1$, the distribution reduces to the RL distribution. Similarly, by replacing $G(x)$ in equation (4) with the cdf of RL distribution in equation (2), we obtain the cdf of EGRL distribution as

$$F(x) \quad = \quad \left(1-\left[\frac{(\theta^2 x-\theta+1)e^{-\theta x}}{1-\theta}\right]^{\alpha}\right)^{\beta}. \tag{6}$$

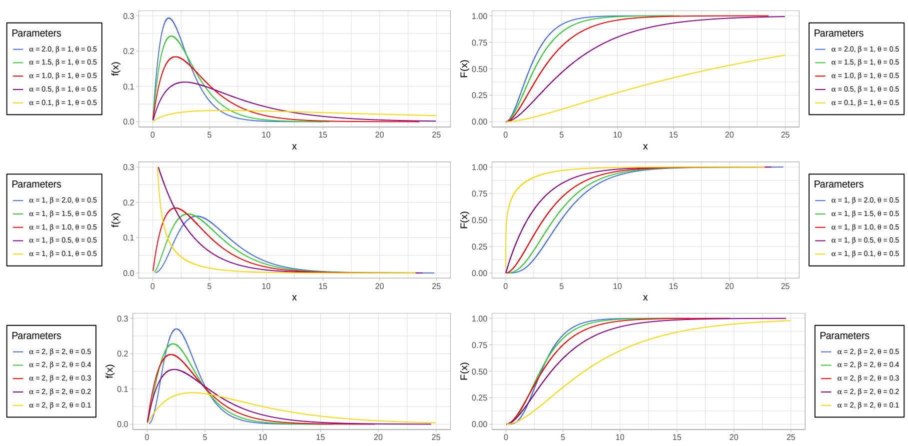



FIGURE 1. The pdf and cdf plots of EGRL distribution with varying values of $\alpha$, $\beta$ and $\theta$ parameters.

Figure 1 displays the plots for the pdf and cdf of EGRL distribution using different values of $\alpha$, $\beta$ and $\theta$ parameters. As can be seen on the left panel of the figure, the EGRL distribution is flexible in the sense that it can be positively skewed with or without reversed-J shape. The plots on the right panel of the figure show that the cdf of EGRL distribution increases towards one with increasing values of the random variable $X$ for varying values of parameters $\alpha$, $\beta$ and $\theta$.

2.2. **Survival and hazard rate functions.** The survival function (stated otherwise reliability function) is commonly used for lifetime datasets which often represents the probability of a patient's survival or an object's resistance until a predetermined time point. The survival function of EGRL distribution indicating the complement of the cdf in equation (6) is given by

$$S(x) \quad = \quad 1-\left(1-\left[\frac{(\theta^2 x-\theta+1)e^{-\theta x}}{1-\theta}\right]^{\alpha}\right)^{\beta}. \tag{7}$$

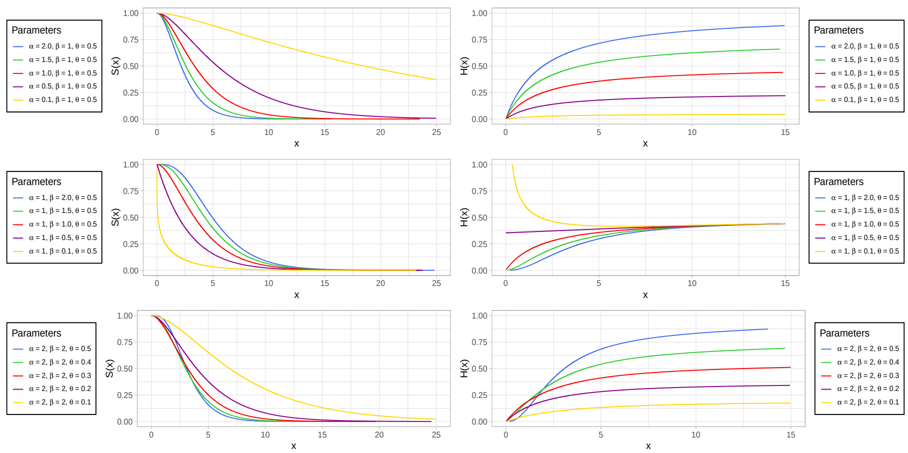


FIGURE 2. The survival and hazard rate plots of EGRL distribution with varying values of $\alpha$, $\beta$ and $\theta$ parameters.

Another widely used tool that can serve to characterize the EGRL distribution is the hazard rate function which indicates the probability of the occurence of an event. The values of hazard rate function for the EGRL distribution can easily be obtained by

$$H(x) = \frac{f(x)}{S(x)}, \tag{8}$$

where $f(x)$ is the pdf in equation (5) and $S(x)$ is the survival function in equation (7).

Figure 2 displays the plots of survival and hazard rate functions for the EGRL distribution. These plots exhibit increasing, decreasing, and reversed-J shaped hazard rate functions and decreasing survival functions with increasing values of random variable X.

2.3. **The quantile function.** The quantile function (qf) of EGRL distribution is the inverse of the cdf in equation (6). By applying $w = -\theta x - \frac{1-\theta}{\theta}$ transformation and using $w$ in Lambert form $we^w$ for $u = G(x)$ in equation (2), we obtain

$$we^w = \frac{(1-\theta)(u-1)e^{1-\frac{1}{\theta}}}{\theta}. \tag{9}$$

This means that $w$ can be defined as a Lambert function of the real argument $we^w$. The real argument $we^w \in (-\frac{1}{e}, 0)$ for $u \in (0,1)$. Thus,

$$Q_{EGRL}(u) = \frac{-\theta W_{-1}\left[\frac{(\theta-1)\left[(1-u^{\frac{1}{\beta}})^{\frac{1}{\alpha}}\right]e^{1-\frac{1}{\theta}}}{\theta}\right] + \theta - 1}{\theta^2}, \tag{10}$$

where $W_{-1}$ is the negative branch of the Lambert function, $0 < \theta \leq 0.5$, and $0 < u < 1$ (see [7, p. 2]). The values of the negative branch of Lambert function $W_{-1}$ can easily be obtained using `lambertWm1` subroutine of `lamW` package in `R` statistical software.

The Bowley skewness ( [15]) and Moorsis kurtosis ( [22]) measures for EGRL distribution are defined by

$$B = \frac{Q_{EGRL}(3/4) + Q_{EGRL}(1/4) - 2Q_{EGRL}(2/4)}{Q_{EGRL}(3/4) - Q_{EGRL}(1/4)} \tag{11}$$

and

$$M = \frac{Q_{EGRL}(3/8) - Q_{EGRL}(1/8) + Q_{EGRL}(7/8) - Q_{EGRL}(5/8)}{Q_{EGRL}(6/8) - Q_{EGRL}(2/8)}, \tag{12}$$

where, for example, $Q_{EGRL}(3/4)$ is the third quartile and $Q_{EGRL}(5/8)$ is the fifth octile of the qf for the EGRL distribution. Table 1 shows how these measures behave with varying values of parameters $\alpha$, $\beta$ and $\theta$. In line with the pdf plots in Figure 1, increasing values of Moorsis kurtosis measure are associated with the pdfs with heavier tails. Positive values of the Bowley skewness measure in this table indicate that the distributions are right skewed.

Table 1. The Bowley skewness and Moorsis kurtosis measures with varying values of $\alpha$, $\beta$, and $\theta$ parameters.

| $\alpha$ | $\beta$ | $\theta$ | Bowley skewness | Moorsis kurtosis |
|---|---|---|---|---|
| 0.1 | 1.0 | 0.5 | 0.23 | 1.25 |
| 0.5 | 1.0 | 0.5 | 0.19 | 1.30 |
| 1.0 | 1.0 | 0.5 | 0.17 | 1.32 |
| 1.5 | 1.0 | 0.5 | 0.15 | 1.33 |
| 2.0 | 1.0 | 0.5 | 0.15 | 1.34 |
| 1.0 | 0.1 | 0.5 | 0.77 | 2.49 |
| 1.0 | 0.5 | 0.5 | 0.24 | 1.16 |
| 1.0 | 1.5 | 0.5 | 0.14 | 1.46 |
| 1.0 | 2.0 | 0.5 | 0.13 | 1.56 |
| 2.0 | 2.0 | 0.1 | 0.18 | 1.31 |
| 2.0 | 2.0 | 0.2 | 0.17 | 1.32 |
| 2.0 | 2.0 | 0.3 | 0.15 | 1.33 |
| 2.0 | 2.0 | 0.4 | 0.13 | 1.41 |
| 2.0 | 2.0 | 0.5 | 0.12 | 1.61 |

2.4. **Moments.** We follow an analogous procedure to the one given in the previous subsection with a slightly different transformation

$$v = 1 - G(x) = \frac{(\theta^2 x - \theta + 1)}{1 - \theta}\, e^{-\theta x}, \tag{13}$$

where $0 < v < 1$. The $m$th moment of EGRL distribution is given by

$$E(X^m) = \int_0^\infty x^m f(x)dx = \alpha\beta \int_0^1 (-1)^{m+1} \big[z(v)\big]^m v^{\alpha-1}(1-v^\alpha)^{\beta-1}dv, \tag{14}$$

where

$$z(v) = \frac{-\theta W_{-1}\left[\frac{(1-\theta)ve^{1-\frac{1}{\theta}}}{\theta}\right] + \theta - 1}{\theta^2}.$$

The next subsection recalls some useful definitions and power series expansions that can be used to obtain the moments, mgf, Shannon and Renyi entropies, and order statistics of EGRL distribution.

2.5. **Useful definitions and power series expansions.** Let T be a random variable from the exponentiated exponential distribution which has the following pdf

$$r(t) = \alpha\beta e^{-\alpha t}(1-e^{-\alpha t})^{\beta-1}, \tag{15}$$

and cdf

$$R(t) = (1-e^{-\alpha t})^\beta, \tag{16}$$

where $\alpha$, $\beta$, $t > 0$ ( [11]). The pdf of a random variable from the exponentiated generalized family of distributions can also be defined as

$$f(x) = \frac{g(x)}{1-G(x)} r\bigg(-\log\big[1-G(x)\big]\bigg), \tag{17}$$

where $T = -\log\big[1-G(X)\big]$ follows the exponentiated exponential distribution ( [4]). By using equation (17), $u = G(x)$, and the power series expansion

$$-\log(1-u) = \sum_{i=0}^{\infty} \frac{u^{i+1}}{i+1}, \tag{18}$$

the pdf $r\bigg(-\log\big[1-G(x)\big]\bigg)$ becomes

$$r\bigg(-\log\big[1-G(x)\big]\bigg) = r\bigg(\sum_{i=0}^{\infty} \frac{u^{i+1}}{i+1}\bigg) = \alpha\beta e^{-\alpha D}(1-e^{-\alpha D})^{\beta-1}, \tag{19}$$

where $D = \sum_{i=0}^{\infty} \frac{u^{i+1}}{i+1}$. Similarly, by using equation (18), $u = G(x)$, and the power series expansion above, the corresponding cdf is defined as

$$R\bigg(-\log\big[1-G(x)\big]\bigg) = R\bigg(\sum_{i=0}^{\infty} \frac{u^{i+1}}{i+1}\bigg) = (1-e^{-\alpha D})^\beta. \tag{20}$$

By applying another useful power series expansion

$$(1-y)^a = \sum_{k=0}^{\infty} \binom{a}{k} (-1)^k |y|, |y| < 1,$$

we obtain

$$[1-F(x)]^{n-r}=\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j F^j(x), \tag{21}$$

which in turn will be used particularly to establish the order statistics for the EGRL distribution.

2.6. **Moments using power series expansions and quantile function.** By using equations (19) and (21), the $m$th moment of EGRL distribution can also be defined as follows:

$$\begin{aligned}E(X^m)=&\int_0^\infty x^m\frac{g(x)}{1-G(x)}r\Big(-\log\big[1-G(x)\big]\Big)dx=\int_0^1 Q^m(u)\frac{1}{1-u}r\Big(\sum_{i=0}^\infty\frac{u^{i+1}}{i+1}\Big)du\\ =&\alpha\beta\int_0^1 Q^m(u)\frac{1}{1-u}e^{-\alpha D}(1-e^{-\alpha D})^{\beta-1}du,\end{aligned} \tag{22}$$

where $D=\sum_{i=0}^\infty\frac{u^{i+1}}{i+1}$. Here, $Q(u)=G^{-1}(u)=x$ is the quantile function (i.e., the inverse of the cdf) in equation (9), so that $u=G(x)$ and $du=g(x)dx$.

2.7. **Moment generating function.** Following the procedure used to obtain the moments in the previous subsection, the mgf of EGRL distribution can be obtained as

$$\begin{aligned}E(e^{bX})=&\int_0^\infty e^{bx}\frac{g(x)}{1-G(x)}r\Big(-\log\big[1-G(x)\big]\Big)dx=\int_0^1 e^{bQ(u)}\frac{1}{1-u}r\Big(\sum_{i=0}^\infty\frac{u^{i+1}}{i+1}\Big)du\\ =&\alpha\beta\int_0^1 e^{bQ(u)}\frac{1}{1-u}e^{-\alpha D}(1-e^{-\alpha D})^{\beta-1}du.\end{aligned} \tag{23}$$

2.8. **Shannon entropy.** The Shannon entropy ( [30]) is a measure to ascertain the information provided by a random variable. The Shannon entropy for the random variable X from the EGRL distribution is given by

$$\eta_S=-E\left(\log\left[\frac{g(x)r(t)}{1-G(x)}\right]\right), \tag{24}$$

where $r(t)$ is used as a generator to attain the family of distributions in equation (3).

The association between the Shannon entropy for the generator variable T which has the support $[0,\infty]$ and the variable $X$ from the beta-exponential-X family of distributions can be defined by using $T=-\log\big[1-G(X)\big]$, and thus, $X=G^{-1}(1-e^{-T})$ ( [4]). This association also applies to our case for which the variable $T$ is from the exponentiated exponential distribution which has the support $[0,\infty]$ and the variable $X$ is from the EGRL distribution, since the exponentiated generalized family of distributions is a special case of the beta-exponential-X family of distributions. Thus, the Shannon entropy above is defined as

$$\begin{aligned}\eta_S =& - E\Big(\log f\big[G^{-1}(1-e^{-T})\big]\Big) + \eta_T - \mu_T,\\ =& - E\Big(\log f\big[G^{-1}(1-e^{-T})\big]\Big)\\ &+\log\big[(\alpha\beta)^{-1}\big] + \beta\Psi(\beta+1) - (\beta-1)\Psi(\beta) - \Psi(1)\\ &+\frac{\Psi(\beta+1)-\Psi(1)}{\alpha},\end{aligned} \tag{25}$$

where $\eta_T = \log\big[(\alpha\beta)^{-1}\big] + \beta\Psi(\beta+1) - (\beta-1)\Psi(\beta) - \Psi(1)$ is the Shannon entropy for random variable T, $\mu_T = \frac{\Psi(\beta+1)-\Psi(1)}{\alpha}$ is its mean and $\Psi(.)$ is the digamma function [4, p. 68].

2.9. **Renyi entropy.** The Renyi entropy ( [26]) is another widely used measure to quantify the information in random variables. The Renyi entropy is an extension of the Shannon entropy. The Renyi entropy of order $\gamma$ for the random variable $X$ from the EGRL distribution is given by

$$\begin{aligned}\eta_R =&\frac{1}{1-\gamma}\log\int_0^\infty f^\gamma(x)dx = \frac{1}{1-\gamma}\log\int_0^\infty \frac{g^\gamma(x)}{1-G^\gamma(x)} r^\gamma\Big(-\log\big[1-G(x)\big]\Big)dx\\ =&\frac{1}{1-\gamma}\log\int_0^1 \frac{g^{\gamma-1}\big[Q(u)\big]}{1-u^\gamma} r^\gamma\left(\sum_{i=0}^{\infty}\frac{u^{i+1}}{i+1}\right)du\\ =&\frac{\alpha^\gamma\beta^\gamma}{1-\gamma}\log\int_0^1 \frac{g^{\gamma-1}\big[Q(u)\big]}{1-u^\gamma} e^{-\alpha\gamma D}(1-e^{-\alpha D})^{\gamma(\beta-1)}du,\end{aligned} \tag{26}$$

where $g(x)$ is the pdf of RL distribution and $D = \sum_{i=0}^{\infty}\frac{u^{i+1}}{i+1}$. The value of order $\gamma$ influences the information obtained from random variable $X$. The Renyi entropy recovers the minimum entropy if $\gamma = \infty$, the maximum entropy if $\gamma = 0$, and Shannon's entropy if $\gamma \to 1$ ( [27]).

2.10. **Order statistics.** Let $X_{(1)} = \min\{X_1, X_2, ..., X_n\}$ and $X_{(n)} = \max\{X_1, X_2, ..., X_n\}$ are the smallest and largest values of a random sample $X_1, X_2, ..., X_n$, respectively. In line with Arnold et al. ( [5]), the pdf of the $r$th order statistic (i.e, the $r$th smallest value) is defined by

$$f_{(r)}(x) = \binom{n}{r} F^{r-1}(x)[1-F(x)]^{n-r} f(x). \tag{27}$$

By applying the power series expansion in equation (23), the pdf of the $r$th order statistic is defined by

$$f_{(r)}(x) = \binom{n}{r} F^{r-1}(x)[1-F(x)]^{n-r} f(x)$$

$$
\begin{aligned}
&= \frac{n!}{(r-1)!(n-r)!} \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j F^{j+r-1}(x) f(x) \\
&= \frac{n!}{(r-1)!(n-r)!} \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j R^{j+r-1}\Big(-\log\big[1-G(x)\big]\Big) \\
&\times \frac{g(x)}{1-G(x)} r\Big(-\log\big[1-G(x)\big]\Big) \\
&= \frac{n!}{(r-1)!(n-r)!} \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j R^{j+r-1}\left(\sum_{i=0}^{\infty} \frac{u^{i+1}}{i+1}\right) \\
&\times \frac{g(x)}{1-G(x)} r\left(\sum_{i=0}^{\infty} \frac{u^{i+1}}{i+1}\right) \\
&= \frac{n!}{(r-1)!(n-r)!} \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j (1-e^{-D})^{\beta(j+r-1)} \\
&\times \frac{g(x)}{1-G(x)} e^{-\alpha D}(1-e^{-\alpha D})^{\beta-1}, \qquad (28)
\end{aligned}
$$

where $g(x)$ and $G(x)$ are the pdf and cdf of RL distribution and $D = \sum_{i=0}^{\infty} \frac{u^{i+1}}{i+1}$.

2.11. **Linear representation.** Corderio and Lemonte ( [8]) produce the linear representations of equations (3) and (4). We summarize their procedure here by using $G(x)$ as the cdf of the baseline RL distribution. By applying the generalized binomial expansion in equation (23) twice in equation (4), the cdf of EGRL distribution is defined as

$$
\begin{aligned}
F(x) &= \left(1 - \Big[1-G(x)\Big]^{\alpha}\right)^{\beta} = \sum_{k=0}^{\infty} (-1)^k \binom{\beta}{k} \big[1-G(x)\big]^{\alpha k} \\
&= \sum_{k=0}^{\infty} (-1)^k \binom{\beta}{k} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha k}{j} G^j(x) = \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} (-1)^{j+k+1} \binom{\beta}{k} \binom{\alpha k}{j+1} G^{j+1}(x),
\end{aligned}
$$

where $G^{j+1}(x)$ is the cdf of RL distribution with a power parameter $j+1$. In other words, the cdf of EGRL distribution can be defined as a linear combination of the cdfs of RL distributions. By taking the derivative of $G^{j+1}(x)$ with respect to $x \geq 0$, we obtain the linear representation of the pdf of EGRL distribution which is given by

$$
f(x) = \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} (-1)^{j+k+1} \binom{\beta}{k} \binom{\alpha k}{j+1} (j+1) g(x) G^j(x),
$$

where $g(x)$ is the pdf and $G(x)$ is the cdf of the baseline RL distribution.

## 3. Parameter Estimation

In this section, we present the parameter estimation procedure by means of four methods: maximum likelihood estimation (MLE), least squares estimation (LSE), weighted least squares estimation (WLSE), and Cramer von Mises estimation (CVME). The LSE, WLSE, and CVME methods are included in the study as an alternative to MLE due to their ease of use.

3.1. **Maximum likelihood estimation.** The log likelihood function of a random sample $X = (X_1, X_2, ..., X_n)$ from the EGRL distribution is given by

$$\begin{aligned}\ell =& n\log(\alpha) + n\log(\beta) + n\log(\theta) - n\alpha\log(1-\theta) + \sum_{i=1}^{n}\log(\theta^2 x_i - 2\theta + 1)\\ &-\alpha\theta\sum_{i=1}^{n} x_i + (\alpha-1)\sum_{i=1}^{n}\log(\theta^2 x_i - \theta + 1)\\ &+(\beta-1)\sum_{i=1}^{n}\log\left(1-\left[\frac{(\theta^2 x_i - \theta + 1)}{1-\theta}e^{-\theta x_i}\right]^{\alpha}\right). \end{aligned} \tag{29}$$

The elements of score vector $(\frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\beta}, \frac{\partial\ell}{\partial\theta})$ containing the first derivatives (stated otherwise the gradients) of the log likelihood function with respect to parameters $\alpha$, $\beta$ and $\theta$ are given below.

$$\begin{aligned}\frac{\partial\ell}{\partial\alpha} =& \frac{n}{\alpha} - n\log(1-\theta) - \theta\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\log(\theta^2 x_i - \theta + 1)\\ &+(1-\beta)\sum_{i=1}^{n}\frac{\zeta_i^{\alpha}\left(\log(\zeta_i) - \theta x_i\right)}{e^{\alpha\theta x_i} - \zeta_i^{\alpha}},\\ \frac{\partial\ell}{\partial\beta} =& \frac{n}{\beta} + \sum_{i=1}^{n}\log\left(1 - \left(\zeta_i e^{-\theta x_i}\right)^{\alpha}\right),\\ \frac{\partial\ell}{\partial\theta} =& \frac{n}{\theta} + \frac{n\alpha}{1-\theta} + \sum_{i=1}^{n}\frac{2(\theta x_i - 1)}{\theta^2 x_i - 2\theta + 1} - \alpha\sum_{i=1}^{n} x_i + (\alpha-1)\sum_{i=1}^{n}\frac{2\theta x_i - 1}{\theta^2 x_i - \theta + 1}\\ &+(1-\beta)\sum_{i=1}^{n}\frac{\alpha x_i\zeta_i^{\alpha}\left[\theta^3 x_i - \theta^2(x_i+2) + 4\theta - 1\right]}{(1-\theta)(\theta^2 x_i - \theta + 1)(e^{\alpha\theta x_i} - \zeta_i^{\alpha})}. \end{aligned} \tag{30}$$

where $\zeta_i = \frac{(\theta^2 x_i - \theta + 1)}{1-\theta}$.

Maximum likelihood estimates (MLEs) are described analytically by setting the elements of the score vector equal to zero and solving for each parameter. The resulting equations $\frac{\partial\ell}{\partial\alpha} = 0$, $\frac{\partial\ell}{\partial\beta} = 0$ and $\frac{\partial\ell}{\partial\theta} = 0$ need to be solved simultaneously. Maximizing the log likelihood function with respect to parameters $\alpha$, $\beta$ and $\theta$ can be performed using a reliable non-linear optimization technique such as Nelder and

Mead (NM) or the Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm. Maximizing the log likelihood (or minimizing minus the log likelihood) function can be achieved by using `maxLik` and `optim` subroutines of `maxLik` and `stats` packages in `R` statistical software.

When maximizing the log likelihood function above, the initials of $\alpha$, $\beta$, and $\theta$ parameters must be specified. To easily obtain the initial for parameter $\theta$ by means of using the usual RL distribution in equation (1), the initials of parameters $\alpha$ and $\beta$ were set to 1. The initial of parameter $\theta$ was obtained by taking the inverse of the root obtained in $\mu = \frac{\lambda^2}{\lambda - 1}$ (see [24, p. 250]), where parameter $\mu$ was replaced by the sample mean $\bar{x}$. The resulting initial for this parameter is $\theta_{\text{init}} = \frac{2}{\bar{x} + \sqrt{\bar{x}^2 - 4\bar{x}}}$.

We do not provide the analytical expressions of the entries of Hessian matrix for the log likelihood function of EGRL distribution which are too complicated. The standard errors of model parameters can be obtained by an approximate Hessian matrix using the default option (i.e., the finite-difference approach) in the `maxLik` package. The square root of diagonals for the inverse of minus the Hessian matrix gives the standard errors of parameter estimates. However, the same standard errors can be obtained by using `summary` function in the `maxLik` package. Note that this approximation technique does not always converge for the standard errors of model parameters. In such a case, nonparametric bootstrapping (NB; [10]) is a reasonable alternative to estimate the standard errors of parameters. The NB can also be used to obtain an estimate of bias to compare the performance of estimation methods presented in this paper, which will be evaluated in the application section.

The estimates of model parameters and their standard errors can also be obtained by maximizing a function of the cdf of EGRL distribution or a weighted form of this function known as the method of (weighted) least squares estimation which will be presented in the next subsection.

3.2. **The method of (Weighted) Least-squares estimation.** Based on Swain et al. ( [33]), the least squares estimates of model parameters and their standard errors can be attained by maximizing

$$-\sum_{i=1}^{n}\left[F(x_{(i)}) - \frac{i}{n+1}\right]^2, \tag{31}$$

where $F(x_{(i)})$ is the cdf of the ordered random variables $x_{(1)} < x_{(2)} < ... < x_{(n)}$, see also [28, p. 181]. Thus, the least squares estimates for the EGRL distribution are obtained by maximizing

$$-\sum_{i=1}^{n}\left[\left(1 - \left[1 - G(x_{(i)})\right]^{\alpha}\right)^{\beta} - \frac{i}{n+1}\right]^2, \tag{32}$$

where $F(x_{(i)})$ in equation (33) is replaced by the cdf of the ordered random variables for the EGRL distribution. Here, $G(x_{(i)})$ represents the cdf of the ordered random variables for the baseline RL distribution in equation (2).

The weighted least squares estimation (WLSE) can be more reliable than the usual least squares estimation (LSE) when the data involve heteroscedasticity which often occurs in the presence of outlier(s). The WLSE incorporates an additional weight factor into the function above to quantify the importance of each observation in the data when estimating model parameters. The WLSE is (often) less sensitive to outliers when compared to the usual LSE[1]. The weighted least squares estimates can be obtained by maximizing

$$-\sum_{i=1}^{n} w_{(i)}\left[F(x_{(i)})-\frac{i}{n+1}\right]^2, \tag{33}$$

where $w_{(i)}=\frac{(n+1)^2(n+2)}{i(n-i+1)}$ is the value of weight factor of the $i$th observation for the data in increasing order, [28, p. 181]. Similar to the least squares estimates, the weighted least squares estimates for the EGRL distribution are obtained by maximizing

$$-\sum_{i=1}^{n} w_{(i)}\left[\left(1-\left[1-G(x_{(i)})\right]^{\alpha}\right)^{\beta}-\frac{i}{n+1}\right]^2. \tag{34}$$

Another popular estimation method that can easily be applied to estimate model parameters for the EGRL distribution is the method of Cramer-Von-Mises estimation (CVME), which will be detailed in the next subsection.

3.3. **The method of Cramer-von-Mises estimation.** The estimates of model parameters using the Cramer-von-Mises estimation (CVME; [20]) is obtained by maximizing another function of the cdf of EGRL distribution which is given by

$$-\frac{1}{12n}-\sum_{i=1}^{n}\left[F(x_{(i)})-\frac{2i-1}{2n}\right]^2. \tag{35}$$

Similar to equations (34) and (36), this function for the EGRL distribution can be defined as

$$-\frac{1}{12n}-\sum_{i=1}^{n}\left[\left(1-\left[1-G(x_{(i)})\right]^{\alpha}\right)^{\beta}-\frac{2i-1}{2n}\right]^2. \tag{36}$$

The gradients and analytical expressions of Hessian matrices with respect to the maximized functions using LSE, WLSE, and CVME methods are not presented here, but, would be made available upon request.

## 4. Simulation Studies

This section presents two simulation studies, first of which aims to investigate the performances of MLE, LSE, WLSE, and CVME methods for the EGRL distribution with respect to the bias, precision, and accuracy measures given in Walther and Moore ( [34]). The second simulation study is however set up to illustrate

---

[1]The estimates obtained by the WLSE are not always resistant to outliers. For more details on the situations in which these estimates are sensitive to outliers, see Sohn et al. ( [32]).

the potentiality of the new EGRL distribution in comparison to the some other lifetime distributions listed in Table 2. In this simulation, we show that model

TABLE 2. Some selected lifetime distributions.

| Distribution | Author(s) |
|---|---|
| Rayleigh | Rayleigh ( [25]) |
| Exponentiated generalized Normal (EGN) | Corderio et al. ( [7]) |
| Exponentiated generalized Gumbel (EGGu) | Corderio et al. ( [7]) |
| Exponentiated generalized Ramos-Louzada (EGRL) | (New) |

fit indices should be used in conjunction with information criteria to detect the best distribution in a set of distributions when analyzing the data. For the selection of best fitting models, the Cramer-von Mises (C*; [9]), Watson (W*; [35]), Kuiper (K*; [17]) and Kolmogorov-Smirnov (KS*; [16,31]) goodness of fit statistics and the log likelihood ($\ell$), Akaike information criterion (AIC; [1,2]), Consistent Akaike information criterion (CAIC; [6]), Corrected Akaike information criterion (AICc; [14]), Bayesian information criterion (BIC; [29]), and Hannan-Quinn information criterion (HQIC; [12]) are used. The goodness of fit statistics are given by

$$\begin{aligned} C^* =& \frac{1}{12n} + \sum_{k=1}^{n}\Big[\frac{2k-1}{2n} - F(X_{(k)})\Big]^2, \\ \mathrm{W}^* =& \sqrt{C^* - n\Big(\big[\frac{1}{n}\sum_{k=1}^{n} F(X_{(k)})\big] - \frac{1}{2}\Big)^2}, \\ \mathrm{K}^* =& \max\big[\frac{k}{n} - F(x_{(k)})\big] + \max\big[F(X_{(k)}) - \frac{k-1}{n}\big], \\ \mathrm{KS}^* =& \max\Big[F(X_{(k)}) - \frac{k-1}{n}, \frac{k}{n} - F(X_{(k)})\Big], \end{aligned} \tag{37}$$

where $n$ is the sample size and $F(x)$ is the cdf of the distribution under consideration for which the values of random variable X are in increasing order, namely, $x_{(1)} < x_{(2)} < ... < x_{(n)}$. The small values of these information criteria and test statistics above imply better model fits. Similarly, the information criteria are given by

$$\begin{aligned} AIC =& -2\ell + 2p, \\ \mathrm{AICc} =& -2\ell + \frac{2pn}{n-p-1}, \\ \mathrm{CAIC} =& -2\ell + p\big[\log(n) + 1\big], \end{aligned}$$

$$\mathrm{BIC} = -2\ell + p\log(n),$$

$$\mathrm{HQIC} = -2\ell + 2p\log\big[\log(n)\big], \tag{38}$$

where $n$ is the sample size and $p$ is the number of parameters in the model.

The simulation studies for two sets of population values of parameters for the EGRL distribution comprise the following steps.

(1) (a) *For the first simulation*: Set $\alpha = 1$, $\beta = 1$, and $\theta = 0.3$ as the population values of parameters for the EGRL distribution.
   (b) *For the second simulation*: Set $\alpha = 1.2$, $\beta = 1.3$, and $\theta = 0.3$ as the population values of parameters for the EGRL distribution.
(2) Set the sample size as $N = 20$, 100, or 500.
(3) Generate the values of EGRL distribution based on the population values in Step 1 and the sample size in Step 2. The data generation from the EGRL distribution is performed by using the automatic nonuniform random variate generation process presented in Hörmann et al. ( [13]). This procedure can easily be implemented using `tdr.new` and `ur` subroutines of `Runuran` package in `R` statistical software.
(4) Obtain the estimates of model parameters using MLE, LSE, WLSE, and CVME methods for the EGRL distribution in the first simulation and for the distributions in Table 2 in the second simulation.
(5) Perform Steps 3-4 for $S = 1000$ times.
(6) (a) *For the first simulation*: For the EGRL distribution and each estimation method, calculate the bias, precision, and accuracy measures given in Walther and Moore ( [34]).
   (b) *For the second simulation*: For each distribution and estimation method, calculate the log likelihood value, the values of goodness of fit statistics, and information criteria in equations (31), (39) and (40), respectively.

Notably, the measures in Step 6 (a) are obtained for each parameter of EGRL distribution in $S = 1000$ simulations. For example, the bias, precision, and accuracy measures for parameter $\alpha$ are given by

$$\mathrm{Bias}(\alpha) = \frac{1}{S}\sum_{s=1}^{S}(\hat{\alpha}_s - \alpha), \tag{39}$$

$$\mathrm{Precision}(\alpha) = \frac{1}{S}\sum_{s=1}^{S}(\hat{\alpha}_s - \bar{\alpha})^2, \tag{40}$$

$$\mathrm{Accuracy}(\alpha) = \frac{1}{S}\sum_{s=1}^{S}(\hat{\alpha}_s - \alpha)^2, \tag{41}$$

where $\alpha = 1$ is the population value of $\alpha$ in Step 1 (a), $\hat{\alpha}_s$ is the estimate of parameter $\alpha$ in the $s$th simulation, and $\bar{\alpha} = \frac{1}{S}\sum_{s=1}^{S}\hat{\alpha}_s$ for $s = 1, 2, ..., 1000$. Analogous calculations are performed for parameters $\beta$ and $\theta$. These values are displayed in

Table 3. Similarly, the values obtained in Step 6 (b) are presented in Tables 4, 5, 6, and 7 in each of which one of the estimation methods concerned are displayed in turn.

In the first and second simulation studies, parameter estimation over the data sets was performed by NM and BFGS algorithms, respectively. A data set was not accepted for inclusion in $S = 1000$ simulation trials if at least one of the following conditions occured. (1) The initial of parameter $\theta$ was not in the range of 0 and 0.5 or sample mean is smaller than 4 in line with $\theta_{\text{init}} = \frac{2}{\bar{x}+\sqrt{\bar{x}^2-4\bar{x}}}$ (see page 11). (2) The estimates of parameters were obtained outside the parameter space. For example, when the estimate of parameter $\alpha > 0$ for the EGN distribution is obtained as $\hat{\alpha} < 0$. (3) When the convergence criterion was not obtained for any of the distributions under consideration. (4) When the log likelihood value for any distribution in the set was obtained as minus infinity. Note that this last condition only applies to the second simulation study. If at least one of the conditions above occurs in the simulation, a different data was generated for the corresponding simulation trial.

Table 3 shows the values of bias, precision, and accuracy measures for each parameter of the EGRL distribution obtained from $S = 1000$ random datasets using MLE, LSE, WLSE, and CVME methods with NM algorithm. A small value in the table represents a small bias, a high precision, or a high accuracy measure. This table displays that increasing the sample size eventually reduces the bias and increases the precision and accuracy for parameters $\alpha$, $\beta$, and $\theta$ of the EGRL distribution using each estimation method. The performance of each estimation method improves as the sample size increases. It is concluded that MLE outperforms other estimation methods in terms of bias, precision, and accuracy measures.

Tables 4, 5, 6, and 7 show the average values of the (minus) log likelihood, goodness of fit statistics, and information criteria for each distribution under evaluation using MLE, LSE, WLSE, and CVME methods with BFGS algorithm, respectively. Based on these tables, one-parameter Rayleigh distribution does not provide enough flexibility in modeling the data. Because model fit statistics and (minus) log likelihood values for this distribution are larger than other distributions. It seems that the EGRL distribution often has smaller, and thus, better model fit statistics and (minus) log likelihood values when compared to other distributions. However, note that, these goodness of fit statistics are biased themselves, since they do not take the model complexity into account when choosing the best distribution in a set of distributions. The information criteria like the AIC and BIC reduce this bias by penalizing model complexity (i.e., penalizing the models containing unnecessarily more parameters). For example, when estimating model parameters using MLE for $n = 20$ in Table 4, the best distribution in the set according to the values of model fit statistics is the EGGu distribution, while it is the second best distribution after the EGRL distribution based on the values of all the information criteria under consideration. Sample size plays a crucial role for information criteria when detecting the best distribution in a set. Because small samples tend to support

TABLE 3. Bias, precision and accuracy measures for the parameters of EGRL distribution

| | MLE | | | | LSE | | | |
|---|---|---|---|---|---|---|---|---|
| | $\hat{\alpha}$ | Bias($\alpha$) | Precision($\alpha$) | Accuracy($\alpha$) | $\hat{\alpha}$ | Bias($\alpha$) | Precision($\alpha$) | Accuracy($\alpha$) |
| $n = 20$ | 0.968 | -0.032 | 0.300 | 0.301 | 0.685 | -0.315 | 0.045 | 0.144 |
| $n = 100$ | 0.991 | -0.009 | 0.267 | 0.267 | 0.838 | -0.162 | 0.048 | 0.075 |
| $n = 500$ | 1.011 | 0.011 | 0.102 | 0.103 | 0.952 | -0.048 | 0.059 | 0.061 |
| | $\hat{\beta}$ | Bias($\beta$) | Precision($\beta$) | Accuracy($\beta$) | $\hat{\beta}$ | Bias($\beta$) | Precision($\beta$) | Accuracy($\beta$) |
| $n = 20$ | 1.027 | 0.027 | 0.081 | 0.081 | 0.853 | -0.147 | 0.060 | 0.082 |
| $n = 100$ | 0.958 | -0.042 | 0.030 | 0.031 | 0.892 | -0.108 | 0.019 | 0.031 |
| $n = 500$ | 0.966 | -0.034 | 0.014 | 0.015 | 0.941 | -0.059 | 0.012 | 0.016 |
| | $\hat{\theta}$ | Bias($\theta$) | Precision($\theta$) | Accuracy($\theta$) | $\hat{\theta}$ | Bias($\theta$) | Precision($\theta$) | Accuracy($\theta$) |
| $n = 20$ | 0.336 | 0.036 | 0.006 | 0.007 | 0.368 | 0.068 | 0.005 | 0.009 |
| $n = 100$ | 0.339 | 0.039 | 0.007 | 0.009 | 0.360 | 0.060 | 0.005 | 0.009 |
| $n = 500$ | 0.317 | 0.017 | 0.006 | 0.007 | 0.326 | 0.026 | 0.006 | 0.006 |
| | WLSE | | | | CVME | | | |
| | $\hat{\alpha}$ | Bias($\alpha$) | Precision($\alpha$) | Accuracy($\alpha$) | $\hat{\alpha}$ | Bias($\alpha$) | Precision($\alpha$) | Accuracy($\alpha$) |
| $n = 20$ | 0.689 | -0.311 | 0.042 | 0.139 | 0.788 | -0.212 | 0.053 | 0.098 |
| $n = 100$ | 0.843 | -0.157 | 0.046 | 0.071 | 0.868 | -0.132 | 0.049 | 0.067 |
| $n = 500$ | 0.953 | -0.047 | 0.055 | 0.057 | 0.959 | -0.041 | 0.058 | 0.059 |
| | $\hat{\beta}$ | Bias($\beta$) | Precision($\beta$) | Accuracy($\beta$) | $\hat{\beta}$ | Bias($\beta$) | Precision($\beta$) | Accuracy($\beta$) |
| $n = 20$ | 0.849 | -0.151 | 0.054 | 0.077 | 0.977 | -0.023 | 0.088 | 0.089 |
| $n = 100$ | 0.895 | -0.105 | 0.019 | 0.030 | 0.921 | -0.079 | 0.021 | 0.027 |
| $n = 500$ | 0.946 | -0.054 | 0.011 | 0.014 | 0.948 | -0.052 | 0.012 | 0.015 |
| | $\hat{\theta}$ | Bias($\theta$) | Precision($\theta$) | Accuracy($\theta$) | $\hat{\theta}$ | Bias($\theta$) | Precision($\theta$) | Accuracy($\theta$) |
| $n = 20$ | 0.378 | 0.078 | 0.004 | 0.010 | 0.362 | 0.062 | 0.005 | 0.008 |
| $n = 100$ | 0.364 | 0.064 | 0.005 | 0.009 | 0.356 | 0.056 | 0.005 | 0.008 |
| $n = 500$ | 0.327 | 0.027 | 0.005 | 0.006 | 0.325 | 0.025 | 0.006 | 0.006 |

more parsimonious (stated otherwise simple) models, while large samples tend to support more complex models. The performance of EGRL distribution increase better than that of other distributions in the set as the sample size increases. The EGRL distribution in these tables is associated with the smallest (average) values of information criteria for $n = 500$, regardless of the type of information criterion or estimation method. Moreover, the EGRL distribution performs better than other exponentiated generalized distributions, namely, the EGN and EGGu distributions, in all cases where the sample size is $n = 100$ or $n = 500$.

TABLE 4. The values of the goodness of fit statistics, (minus) log likelihood, and information criteria for each distribution under evaluation using MLE method.

| Sample size ($n$) | Models | $\overline{\mathrm{C}^*}$ | $\overline{\mathrm{W}^*}$ | $\overline{\mathrm{K}^*}$ | $\overline{\mathrm{KS}^*}$ | $-\bar{\ell}$ | $\overline{\mathrm{AIC}}$ | $\overline{\mathrm{AICc}}$ | $\overline{\mathrm{CAIC}}$ | $\overline{\mathrm{BIC}}$ | $\overline{\mathrm{HQIC}}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 20 | Rayleigh | 0.44 | 0.43 | 0.37 | 0.28 | 57.00 | 115.99 | 116.21 | 117.99 | 116.99 | 116.18 |
| | EGRL | 0.06 | 0.23 | 0.23 | 0.14 | 51.33 | 108.67 | 111.04 | 114.66 | 111.66 | 109.25 |
| | EGN | 0.12 | 0.30 | 0.28 | 0.17 | 54.69 | 117.38 | 119.12 | 125.36 | 121.36 | 118.16 |
| | EGGu | 0.06 | 0.22 | 0.23 | 0.13 | 51.57 | 111.14 | 113.81 | 119.13 | 115.13 | 111.92 |
| 100 | Rayleigh | 1.69 | 0.79 | 0.27 | 0.22 | 276.01 | 554.02 | 554.07 | 557.63 | 556.63 | 555.08 |
| | EGRL | 0.06 | 0.22 | 0.10 | 0.06 | 251.62 | 509.23 | 509.63 | 520.05 | 517.05 | 512.40 |
| | EGN | 0.38 | 0.52 | 0.20 | 0.12 | 268.78 | 545.57 | 545.84 | 559.99 | 555.99 | 549.78 |
| | EGGu | 0.07 | 0.25 | 0.11 | 0.07 | 253.58 | 515.16 | 515.59 | 529.58 | 525.58 | 519.38 |
| 500 | Rayleigh | 7.74 | 1.66 | 0.24 | 0.20 | 1369.52 | 2741.03 | 2741.04 | 2746.25 | 2745.25 | 2742.68 |
| | EGRL | 0.05 | 0.21 | 0.05 | 0.03 | 1255.06 | 2516.11 | 2516.19 | 2531.76 | 2528.76 | 2521.08 |
| | EGN | 1.33 | 0.97 | 0.16 | 0.09 | 1328.86 | 2665.72 | 2665.77 | 2686.58 | 2682.58 | 2672.33 |
| | EGGu | 0.15 | 0.35 | 0.07 | 0.04 | 1265.08 | 2538.17 | 2538.25 | 2559.03 | 2555.03 | 2544.78 |

TABLE 5. The values of the goodness of fit statistics, (minus) log likelihood, and information criteria for each distribution under evaluation using LSE method.

| Sample size ($n$) | Models | $\overline{\mathrm{C}^*}$ | $\overline{\mathrm{W}^*}$ | $\overline{\mathrm{K}^*}$ | $\overline{\mathrm{KS}^*}$ | $-\bar{\ell}$ | $\overline{\mathrm{AIC}}$ | $\overline{\mathrm{AICc}}$ | $\overline{\mathrm{CAIC}}$ | $\overline{\mathrm{BIC}}$ | $\overline{\mathrm{HQIC}}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 20 | Rayleigh | 0.17 | 0.38 | 0.34 | 0.20 | 59.43 | 120.85 | 121.08 | 122.85 | 121.85 | 121.05 |
| | EGRL | 0.04 | 0.21 | 0.22 | 0.12 | 51.14 | 108.27 | 110.65 | 114.26 | 111.26 | 108.86 |
| | EGN | 0.06 | 0.23 | 0.24 | 0.13 | 60.90 | 129.81 | 131.54 | 137.79 | 133.79 | 130.59 |
| | EGGu | 0.04 | 0.19 | 0.20 | 0.11 | 52.29 | 112.58 | 115.25 | 120.57 | 116.57 | 113.36 |
| 100 | Rayleigh | 0.63 | 0.77 | 0.26 | 0.14 | 288.01 | 578.01 | 578.05 | 581.62 | 580.62 | 579.07 |
| | EGRL | 0.04 | 0.19 | 0.10 | 0.05 | 251.20 | 508.39 | 508.79 | 519.21 | 516.21 | 511.56 |
| | EGN | 0.14 | 0.35 | 0.16 | 0.10 | 285.02 | 578.05 | 578.32 | 592.47 | 588.47 | 582.26 |
| | EGGu | 0.04 | 0.18 | 0.09 | 0.05 | 256.61 | 521.23 | 521.65 | 535.65 | 531.65 | 525.44 |
| 500 | Rayleigh | 2.89 | 1.68 | 0.23 | 0.12 | 1437.51 | 2877.03 | 2877.03 | 2882.24 | 2881.24 | 2878.68 |
| | EGRL | 0.04 | 0.18 | 0.04 | 0.02 | 1255.46 | 2516.92 | 2516.99 | 2532.56 | 2529.56 | 2521.88 |
| | EGN | 0.54 | 0.70 | 0.13 | 0.08 | 1391.44 | 2790.87 | 2790.92 | 2811.73 | 2807.73 | 2797.49 |
| | EGGu | 0.04 | 0.20 | 0.05 | 0.03 | 1286.80 | 2581.61 | 2581.69 | 2602.46 | 2598.46 | 2588.22 |

## 5. APPLICATION

This data contain $N = 116$ observations representing a mean ozone in parts per billion at Rosevelt Island. These observations are obtained from the `airquality` dataset in `datasets` package of `R` statistical software (version 4.2.2). Table 8 shows

TABLE 6. The values of the goodness of fit statistics, (minus) log likelihood, and information criteria for each distribution under evaluation using WLSE method.

| Sample size ($n$) | Models | $\overline{\text{C}^*}$ | $\overline{\text{W}^*}$ | $\overline{\text{K}^*}$ | $\overline{\text{KS}^*}$ | $-\bar{\ell}$ | $\overline{\text{AIC}}$ | $\overline{\text{AICc}}$ | $\overline{\text{CAIC}}$ | $\overline{\text{BIC}}$ | $\overline{\text{HQIC}}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 20 | Rayleigh | 0.17 | 0.38 | 0.33 | 0.20 | 59.04 | 120.09 | 120.31 | 122.08 | 121.08 | 120.28 |
| | EGRL | 0.05 | 0.21 | 0.22 | 0.12 | 50.92 | 107.83 | 110.21 | 113.82 | 110.82 | 108.42 |
| | EGN | 0.06 | 0.24 | 0.24 | 0.13 | 60.61 | 129.22 | 130.96 | 137.21 | 133.21 | 130.00 |
| | EGGu | 0.04 | 0.19 | 0.20 | 0.11 | 52.24 | 112.48 | 115.15 | 120.47 | 116.47 | 113.26 |
| 100 | Rayleigh | 0.63 | 0.76 | 0.26 | 0.15 | 286.51 | 575.02 | 575.06 | 578.62 | 577.62 | 576.07 |
| | EGRL | 0.04 | 0.20 | 0.10 | 0.05 | 250.92 | 507.83 | 508.23 | 518.65 | 515.65 | 510.99 |
| | EGN | 0.16 | 0.38 | 0.16 | 0.09 | 372.13 | 752.26 | 752.53 | 766.68 | 762.68 | 756.48 |
| | EGGu | 0.04 | 0.18 | 0.09 | 0.05 | 256.45 | 520.91 | 521.33 | 535.33 | 531.33 | 525.12 |
| 500 | Rayleigh | 2.91 | 1.66 | 0.23 | 0.12 | 1428.25 | 2858.50 | 2858.51 | 2863.71 | 2862.71 | 2860.15 |
| | EGRL | 0.04 | 0.19 | 0.04 | 0.02 | 1255.35 | 2516.69 | 2516.77 | 2532.34 | 2529.34 | 2521.65 |
| | EGN | 0.56 | 0.72 | 0.15 | 0.10 | 3498.90 | 7005.81 | 7005.86 | 7026.67 | 7022.67 | 7012.42 |
| | EGGu | 0.04 | 0.20 | 0.05 | 0.03 | 1286.84 | 2581.68 | 2581.76 | 2602.54 | 2598.54 | 2588.29 |

TABLE 7. The values of the goodness of fit statistics, (minus) log likelihood, and information criteria for each distribution under evaluation using CVME method.

| Sample size ($n$) | Models | $\overline{\text{C}^*}$ | $\overline{\text{W}^*}$ | $\overline{\text{K}^*}$ | $\overline{\text{KS}^*}$ | $-\bar{\ell}$ | $\overline{\text{AIC}}$ | $\overline{\text{AICc}}$ | $\overline{\text{CAIC}}$ | $\overline{\text{BIC}}$ | $\overline{\text{HQIC}}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 20 | Rayleigh | 0.18 | 0.39 | 0.34 | 0.20 | 59.83 | 121.66 | 121.88 | 123.65 | 122.65 | 121.85 |
| | EGRL | 0.04 | 0.20 | 0.21 | 0.11 | 51.28 | 108.55 | 110.93 | 114.54 | 111.54 | 109.14 |
| | EGN | 0.05 | 0.22 | 0.23 | 0.13 | 62.26 | 132.52 | 134.25 | 140.50 | 136.50 | 133.30 |
| | EGGu | 0.03 | 0.18 | 0.19 | 0.10 | 52.73 | 113.45 | 116.12 | 121.44 | 117.44 | 114.23 |
| 100 | Rayleigh | 0.63 | 0.77 | 0.26 | 0.14 | 288.11 | 578.21 | 578.25 | 581.82 | 580.82 | 579.27 |
| | EGRL | 0.04 | 0.19 | 0.09 | 0.05 | 251.28 | 508.57 | 508.96 | 519.38 | 516.38 | 511.73 |
| | EGN | 0.14 | 0.35 | 0.16 | 0.09 | 283.67 | 575.33 | 575.61 | 589.75 | 585.75 | 579.55 |
| | EGGu | 0.04 | 0.18 | 0.09 | 0.05 | 257.24 | 522.47 | 522.90 | 536.90 | 532.90 | 526.69 |
| 500 | Rayleigh | 2.89 | 1.68 | 0.23 | 0.12 | 1437.64 | 2877.29 | 2877.29 | 2882.50 | 2881.50 | 2878.94 |
| | EGRL | 0.04 | 0.18 | 0.04 | 0.02 | 1255.44 | 2516.88 | 2516.95 | 2532.52 | 2529.52 | 2521.84 |
| | EGN | 0.54 | 0.70 | 0.13 | 0.08 | 1391.87 | 2791.74 | 2791.79 | 2812.60 | 2808.60 | 2798.36 |
| | EGGu | 0.04 | 0.20 | 0.05 | 0.02 | 1286.81 | 2581.61 | 2581.69 | 2602.47 | 2598.47 | 2588.23 |

the data and its descriptives. This dataset is heavily right skewed. The Q-Q plot in Figure 3 and Shapiro-Wilk normality test results ($W = 0.879$, $p < 0.001$) show that the dataset is not normally distributed. The boxplot in Figure 3 displays that the dataset contains outliers. Table 9 shows the estimates of model parameters for the Ozone data using each of the estimation methods. We provide the `R` code on how to obtain the estimates of model parameters using MLE in Appendix. The `R` code for other estimation methods and distributions are not presented in Appendix, but, would be made available upon request.

TABLE 8. The Ozone data and its descriptives.

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Data: | 41, | 36, | 12, | 18, | 28, | 23, | 19, | 8, | 7, | 16, | 11, | 14, | 18, | 14, | 34, | 6, | 30, |
| | 11, | 1, | 11, | 4, | 32, | 23, | 45, | 115, | 37, | 29, | 71, | 39, | 23, | 21, | 37, | 20, | 12, |
| | 13, | 135, | 49, | 32, | 64, | 40, | 77, | 97, | 97, | 85, | 10, | 27, | 7, | 48, | 35, | 61, | 79, |
| | 63, | 16, | 80, | 108, | 20, | 52, | 82, | 50, | 64, | 59, | 39, | 9, | 16, | 78, | 35, | 66, | 122, |
| | 89, | 110, | 44, | 28, | 65, | 22, | 59, | 23, | 31, | 44, | 21, | 9, | 45, | 168, | 73, | 76, | 118, |
| | 84, | 85, | 96, | 78, | 73, | 91, | 47, | 32, | 20, | 23, | 21, | 24, | 44, | 21, | 28, | 9, | 13, |
| | 46, | 18, | 13, | 24, | 16, | 13, | 23, | 36, | 7, | 14, | 30, | 14, | 18, | 20 | | | |

| Mean | Std. deviation | Median | Trimmed mean | Median abs. deviation | Min | Max | Range | Skewness | Kurtosis | Std. error |
|---|---|---|---|---|---|---|---|---|---|---|
| 42.13 | 32.99 | 31.50 | 37.80 | 25.95 | 1.00 | 168.00 | 167.00 | 1.21 | 1.11 | 3.06 |

TABLE 9. The estimates of model parameters using MLE, LSE, WLSE, and CVME for the Ozone data.

| | MLE | | | | LSE | | | |
|---|---|---|---|---|---|---|---|---|
| Models | $\alpha$ | $\beta$ | $\mu$ | $\theta$ | $\alpha$ | $\beta$ | $\mu$ | $\theta$ |
| Rayleigh | 37.774 | – | – | – | 28.295 | – | – | – |
| EGRL | 1.423 | 1.795 | – | 0.024 | 1.061 | 1.514 | – | 0.030 |
| EGN | 1.542 | 123.522 | -114.284 | 84.077 | 3.237 | 115.191 | -56.913 | 111.820 |
| EGGu | 0.182 | 0.602 | 16.227 | 7.177 | 0.115 | 0.648 | 12.322 | 5.188 |
| | WLSE | | | | CVM | | | |
| Models | $\alpha$ | $\beta$ | $\mu$ | $\theta$ | $\alpha$ | $\beta$ | $\mu$ | $\theta$ |
| Rayleigh | 28.735 | – | – | – | 28.288 | – | – | – |
| EGRL | 2.999 | 1.654 | – | 0.011 | 1.893 | 1.551 | – | 0.017 |
| EGN | 0.452 | 218.532 | -179.939 | 47.622 | 3.306 | 109.547 | -51.712 | 110.068 |
| EGGu | 0.131 | 0.761 | 10.899 | 5.250 | 0.114 | 0.661 | 12.198 | 5.031 |

Figure 4 shows the pdfs, cdfs and survival and hazard rate functions for each distribution using MLE. This figure shows that the EGRL and EGGu distributions fit the data better than the Rayleigh and EGN distributions. Table 10 shows that the distribution of the observed Ozone data does not deviate significantly from the EGRL and EGGu distributions, but the distribution of the data differs from the Rayleigh and EGN distributions. This can be tested by the values of Kolmogorov-Smirnov (KS*) test statistics. For doing this, the critical value for the KS test is determined for $\alpha = 0.05$, that is, $KS_t = \frac{1.36}{\sqrt{n}} = \frac{1.36}{\sqrt{116}} = 0.126$. Therefore, for example, $KS^* = 0.085 < KS_t = 0.126$ and $KS^* = 0.065 < KS_t = 0.126$ indicate that the distribution of the Ozone data is not significantly different from the EGRL

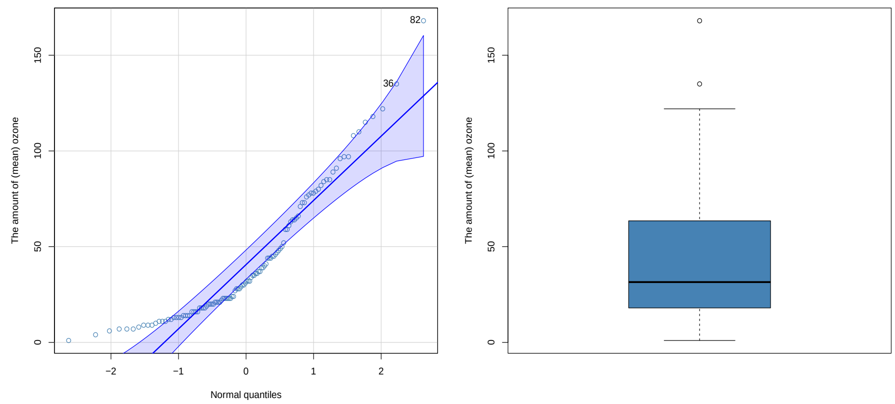



FIGURE 3. The Q-Q plot and box plot for the Ozone data.

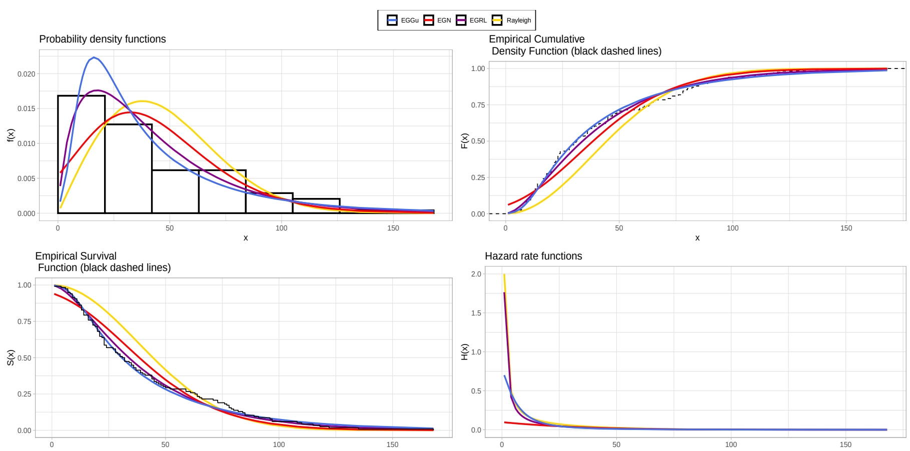



FIGURE 4. The pdf, cdf, survival and hazard rate functions for the Ozone data.

and EGGu distributions, respectively, when parameter estimation is performed by MLE. However, $KS^* = 0.248 > KS_t = 0.126$ and $KS^* = 0.214 > KS_t = 0.126$ mean that the distribution of the data significantly different from the Rayleigh and EGN distributions, respectively, when parameter estimation is performed by MLE. Table 10 also shows that the EGGu distribution provides the smallest goodness of fit statistics, regardless of the method used for parameter estimation. However, as noted in the introduction, these goodness of fit statistisc are biased as they do not take the model complexity into account. Information criteria reduce this bias by considering both the fit and complexity of the model being evaluated. In Table 10, the values of information criteria indicate that the EGRL distribution has a better balance between the model fit and complexity when compared to Rayleigh, EGN, and EGGu distributions. Thus, we conclude that the EGRL distribution can be considered as an alternative distribution in the exponential generalized family of distributions when analyzing positively skewed data using MLE, LSE, WLSE, and CVME for parameter estimation.

TABLE 10. The values of the goodness of fit statistics, (minus) log likelihood, and information criteria for each distribution under evaluation using MLE, LSE, WLSE, and CVME for the Ozone data.

| | | | | | MLE | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Models | $C^*$ | $W^*$ | $K^*$ | $KS^*$ | $-\ell$ | AIC | AICc | CAIC | BIC | HQIC |
| Rayleigh | 2.17 | 1.03 | 0.31 | 0.25 | 561.99 | 1125.97 | 1126.01 | 1129.73 | 1128.73 | 1127.09 |
| EGRL | 0.12 | 0.33 | 0.14 | 0.09 | 541.40 | 1088.79 | 1089.01 | 1100.05 | 1097.05 | 1092.15 |
| EGN | 0.51 | 0.66 | 0.21 | 0.14 | 556.53 | 1121.06 | 1121.42 | 1136.08 | 1132.08 | 1125.53 |
| EGGu | 0.06 | 0.24 | 0.12 | 0.07 | 540.46 | 1088.91 | 1089.27 | 1103.93 | 1099.93 | 1093.39 |
| | | | | | LSE | | | | | |
| Models | $C^*$ | $W^*$ | $K^*$ | $KS^*$ | $-\ell$ | AIC | AICc | CAIC | BIC | HQIC |
| Rayleigh | 0.87 | 0.93 | 0.31 | 0.18 | 585.69 | 1173.38 | 1173.41 | 1177.13 | 1176.13 | 1174.50 |
| EGRL | 0.08 | 0.28 | 0.12 | 0.06 | 542.15 | 1090.30 | 1090.52 | 1101.56 | 1098.56 | 1093.65 |
| EGN | 0.32 | 0.55 | 0.20 | 0.11 | 558.28 | 1124.56 | 1124.92 | 1139.57 | 1135.57 | 1129.03 |
| EGGu | 0.03 | 0.18 | 0.08 | 0.04 | 540.99 | 1089.98 | 1090.34 | 1105.00 | 1101.00 | 1094.45 |
| | | | | | WLSE | | | | | |
| Models | $C^*$ | $W^*$ | $K^*$ | $KS^*$ | $-\ell$ | AIC | AICc | CAIC | BIC | HQIC |
| Rayleigh | 0.87 | 0.94 | 0.31 | 0.17 | 582.99 | 1167.98 | 1168.01 | 1171.73 | 1170.73 | 1169.10 |
| EGRL | 0.09 | 0.30 | 0.13 | 0.07 | 541.58 | 1089.15 | 1089.37 | 1100.41 | 1097.41 | 1092.51 |
| EGN | 0.39 | 0.62 | 0.22 | 0.12 | 556.00 | 1120.00 | 1120.35 | 1135.01 | 1131.01 | 1124.46 |
| EGGu | 0.03 | 0.18 | 0.10 | 0.05 | 540.28 | 1088.56 | 1088.92 | 1103.57 | 1099.57 | 1093.03 |
| | | | | | CVME | | | | | |
| Models | $C^*$ | $W^*$ | $K^*$ | $KS^*$ | $-\ell$ | AIC | AICc | CAIC | BIC | HQIC |
| Rayleigh | 0.87 | 0.93 | 0.31 | 0.18 | 585.74 | 1173.48 | 1173.51 | 1177.23 | 1176.23 | 1174.59 |
| EGRL | 0.08 | 0.28 | 0.12 | 0.06 | 541.96 | 1089.92 | 1090.14 | 1101.18 | 1098.18 | 1093.28 |
| EGN | 0.32 | 0.55 | 0.20 | 0.11 | 558.29 | 1124.59 | 1124.95 | 1139.60 | 1135.60 | 1129.06 |
| EGGu | 0.03 | 0.17 | 0.08 | 0.04 | 540.98 | 1089.96 | 1090.32 | 1104.98 | 1100.98 | 1094.43 |

Goodness of fit statistics and information criteria are originally created to compare the performance of models, but not to compare the performance of estimation methods. Therefore, we do *not* recommend using the results in Table 10 to compare the performance of the methods in estimating the parameters of the EGRL distribution. For doing this, we used a bootstrap estimate of bias presented in Efron and Tibshirani ( [10]).

Let $\eta = (\alpha, \beta, \theta)$ be the vector containing the parameters of EGRL distribution. Then, the bootstrap estimate of bias for each estimation method is calculated as follows:

(1) Create $B = 1000$ bootstrap samples by resampling with replacement from the original data.
(2) Obtain the estimate of parameter vector $\eta$ for each of the bootstrap samples.
(3) Obtain the overall bootstrap estimate of parameter vector $\eta$, that is, $\eta^*$, by averaging the estimates among the bootstrap samples. The standard errors of parameter estimates (i.e., $\mathrm{SE}_\mathrm{B}^*$) are obtained by taking the square root of diagonals of the covariance matrix for the estimates in the bootstrap samples.
(4) Calculate the bootstrap estimate of bias for parameter vector $\eta$, that is, $\mathrm{Bias}_\mathrm{B} = |\eta^* - \hat{\eta}|$, where $\hat{\eta}$ is the vector of the usual estimates obtained for the original data using MLE, LSE, WLSE, or CVME.

Table 11 shows the performance evaluation of each estimation method in estimating the parameters of the EGRL distribution for the Ozone data. Efron and Tibshirani ( [10]) state that the bias can be ignored if $\frac{Bias_B}{SE_B^*} \leq 0.25$. Therefore, based on the results in Table 11, CVME outperforms other estimation methods for this particular example as it has the ratios smaller than 0.25 when estimating parameters $\alpha$, $\beta$, and $\theta$. In line with Efron and Tibshirani ( [10]), the bias-adjusted estimates (BAEs) are also provided for each estimation method using $2\hat{\eta} - \eta^*$. However, caution should be taken when using the bias-adjusted estimates in place of the usual estimates, as biases are more difficult to estimate than standard errors and correcting bias may produce higher standard errors [10, p. 138]. While the bias-adjusted estimates are reasonably close to the usual estimates for MLE and CVME, these estimates are not close to the usual estimates for LSE and WLSE. The WLSE even produces a negative bias-adjusted estimate for parameter $\theta$.

In summary, it is concluded that CVME performs better than other estimation methods in analyzing the Ozone data based on nonparametric bootstrapping bias assessment. In this sense, MLE also provides a reasonable set of parameter estimates. However, LSE and WLSE do not perform well when compared to MLE and CVME for analyzing the Ozone data using the EGRL distribution.

## 6. Discussion

In this study, we introduced a new distribution called the exponentiated generalized Ramos-Louzada distribution involving three parameters. We used four estimation methods (i.e., the MLE, LSE, WLSE, and CVME) for estimation. We assess the performance of these methods for the EGRL distribution by means of

Table 11. Performance evaluation of the estimation methods for the Ozone data in terms of the bias measure using nonparametric bootstrapping.

| Method | $\alpha$ | $\beta$ | $\theta$ |
|---|---|---|---|
| MLE ($\hat{\eta}$) | 1.423 | 1.795 | 0.024 |
| NB ($\eta^*$) | 1.016 | 1.830 | 0.046 |
| $\text{SE}^*_{\text{B}}$ | 0.411 | 0.231 | 0.035 |
| $\text{Bias}_{\text{B}} = \lvert\eta^* - \hat{\eta}\rvert$ | 0.407 | 0.035 | 0.022 |
| Ratio $= \frac{\text{Bias}_{\text{B}}}{\text{SE}^*_{\text{B}}}$ | 0.990 | 0.152 | 0.629 |
| BAE $= 2\hat{\eta} - \eta^*$ | 1.830 | 1.760 | 0.002 |
| LSE ($\hat{\eta}$) | 1.061 | 1.514 | 0.030 |
| NB ($\eta^*$) | 1.783 | 1.564 | 0.020 |
| $\text{SE}^*_{\text{B}}$ | 0.431 | 0.225 | 0.012 |
| $\text{Bias}_{\text{B}} = \lvert\eta^* - \hat{\eta}\rvert$ | 0.722 | 0.050 | 0.010 |
| Ratio $= \frac{\text{Bias}_{\text{B}}}{\text{SE}^*_{\text{B}}}$ | 1.675 | 0.222 | 0.833 |
| BAE $= 2\hat{\eta} - \eta^*$ | 0.339 | 1.464 | 0.040 |
| WLSE ($\hat{\eta}$) | 2.999 | 1.654 | 0.011 |
| NB ($\eta^*$) | 2.060 | 1.735 | 0.023 |
| $\text{SE}^*_{\text{B}}$ | 0.836 | 0.217 | 0.023 |
| $\text{Bias}_{\text{B}} = \lvert\eta^* - \hat{\eta}\rvert$ | 0.939 | 0.081 | 0.012 |
| Ratio $= \frac{\text{Bias}_{\text{B}}}{\text{SE}^*_{\text{B}}}$ | 1.123 | 0.373 | 0.522 |
| BAE $= 2\hat{\eta} - \eta^*$ | 3.938 | 1.573 | -0.001 |
| CVME ($\hat{\eta}$) | 1.893 | 1.551 | 0.017 |
| NB ($\eta^*$) | 1.792 | 1.602 | 0.020 |
| $\text{SE}^*_{\text{B}}$ | 0.439 | 0.233 | 0.013 |
| $\text{Bias}_{\text{B}} = \lvert\eta^* - \hat{\eta}\rvert$ | 0.101 | 0.051 | 0.003 |
| Ratio $= \frac{\text{Bias}_{\text{B}}}{\text{SE}^*_{\text{B}}}$ | 0.230 | 0.219 | 0.231 |
| BAE $= 2\hat{\eta} - \eta^*$ | 1.994 | 1.500 | 0.014 |

using bias, precision, and accuracy measures, the goodness of fit statistics, and information criteria. To attain this objective, we first generate the datasets from the EGRL distribution in two simulations with varying values of sample size. Then, in the first simulation, we evaluate the performance of each estimation method for the EGRL distribution by means of using bias, precision, and accuracy measures. We obtain smaller bias and better precision and accuracy measures for each parameter of EGRL distribution as the sample size increases. It is concluded that MLE outperforms other estimation methods as the sample size increases. Second simulation study is conducted to evaluate the performance of a set of distributions for each estimation method separately. It is concluded that the performance of

EGRL distribution increase better than that of other distributions as the sample size increases when the data in fact follow the EGRL distribution.

The EGRL distribution is a flexible distribution that can be used to improve the model fit when compared to other exponentiated generalized distributions such as EGN and EGGu distributions. However, caution should be taken when using this distribution to analyze datasets in some certain circumstances. We are compelled to highlight two main limitations of the EGRL distribution when using it in conjunction with the estimation methods and information criteria presented in this paper. First, the performance of the EGRL distribution on modeling the data depends on the method utilized for estimation. For example, the methods presented in this paper might produce biased parameter estimates and their standard errors in the case of the data contain many missing values and/or outliers. In such cases, a different estimation method dealing with missing values and/or outliers better should be preferred over these estimation methods. Second, the AIC is prone to overfitting, that is, falsely choosing more complicated distributions containing more parameters over the simpler (stated otherwise more parsimonious) distributions for small samples. We do not suggest the use of the EGRL distribution for small samples, since it contains relatively more parameters when compared to the usual RL distribution. In the same sense, the AIC should not be used as a decision criterion when the set of distributions contains the EGRL distribution for small samples.

**Author Contribution Statements** Writing, programming, and analysis was performed by Yasin Altınışık. All authors read, commented, and approved the final article.

**Declaration of Competing Interests** The authors declare that they have no competing interest.

## References

[1] Akaike, H., Information theory and an extension of the maximum likelihood principle, *In B.N. Petrov & F. Csaki (Eds.), Proc. 2nd Int. Symp. Information Theory, Budapest: Akademiai Kiado,* (1973), 267-281.

[2] Akaike, H., A new look at the statistical model identification, *IEEE Transaction on Automatic Control,* 19 (1974), 716-723. http://dx.doi.org/10.1109/TAC.1974.1100705

[3] Al-Mofleh, H., Afifty, A., Ibrahim, N. A., A new extended two-parameter distribution: Properties, estimation methods and applications in medicine and geology, *Mathematics,* 8 (2020), 1578. http://dx.doi.org/10.3390/math8091578

[4] Alzaatreh, A., Lee, C., Famoye, F., A new method for generating families of continuous distributions, *METRON,* 71 (2013), 63-79. http://dx.doi.org/10.1007/s40300-013-0007-y

[5] Arnold, B. C., Balakrishnan, A. N., Nagaraja, H. N., A First Course in Order Statistics, New York: Wiley-Interscience, 1992.

[6] Bozdogan, H., Model selection and Akaike's information criterion (AIC): The general theory and its analytical extensions, *Psychometrika,* 52 (1987), 345-370. http://dx.doi.org/10.1007/BF02294361

[7] Corderio, G. M., Ortega, E. M. M., Cunha, D. C. C., The exponentiated generalized class of distributions, *Journal of Data Science,* 11 (2013), 1-27. http://dx.doi.org/10.6339/JDS.2013.11(1).1086

[8] Corderio, G. M., Lemonte, A. J., The exponentiated generalized Birnbaum-Sanders distribution, *Applied Mathematics and Computation,* 247 (2014), 762-779. http://dx.doi.org/10.1016/j.amc.2014.09.054

[9] Cramer, H., On the composition of elementary errors, *Scandinavian Actuarial Journal,* 1 (1928), 13-74. http://dx.doi.org/10.1080/03461238.1928.10416862

[10] Efron, B., Tibshirani, R. J., An Introduction to the Bootstrap, New York: Chapman & Hall.

[11] Gupta, R. D., Kundu, D., Exponentiated exponential family: An alternative to gamma and Weibull distributions, *Biometrical Journal,* 43 (2001), 117-130. http://dx.doi.org/10.1002/1521-4036(200102)43:1<117::AID-BIMJ117>3.0.CO;2-R

[12] Hannan, E. J., Quinn, B. G., The determination of the order of an autoregression, *Journal of the Royal Statistical Society, Series B,* 41 (1979), 190-195. http://www.jstor.org/stable/2985032

[13] Hörmann, W., Leydold, J., Derflinger, G., Automatic Nonuniform Random Variate Generation, Springer-Verlag, Berlin Heidelberg, 2004.

[14] Hurvich, C. M., Tsai, C. L., Regression and time series model selection in small samples, *Biometrika,* 76 (1989), 297-307. http://dx.doi.org/10.1093/biomet/76.2.297

[15] Kenney, J. F., Keeping, E. S., Mathematics of Statistics, Pt. 1, 3rd ed. Princeton, NJ: Van Nostrand, 1962.

[16] Kolmogorov, A., Sulla determinazione empirica di una legge di distribuzione, *Giornale dell'Istituto Italiano degli Attuari,* 4 (1933), 83-91.

[17] Kuiper, N. H., Tests concerning random points on a circle, *Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen Series A,* 63 (1960), 38-47. http://dx.doi.org/10.1016/S1385-7258(60)50006-0

[18] Lee, E. T., Wang, J. W., Statistical Methods for Survival Data Analysis, Wiley, New York, 2003.

[19] Lindley, D. V., Fiducial distributions and Bayes' theorem, *Journal of the Royal Statistical Society Series B,* 20 (1958), 102-107. http://www.jstor.org/stable/2983909

[20] MacDonald, P. D. M., Comment on "An estimation procedure for mixtures of distributions" by Choi and Bulgren, *Journal of the Royal Statistical Society Series B,* 33 (1971), 326-329. https://www.jstor.org/stable/2985013

[21] Marshall, A. W., Olkin, I., A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families, *Biometrika,* 84 (1997), 641-652. http://dx.doi.org/10.1093/biomet/92.2.505

[22] Moors, J. J. A., A quantile alternative for kurtosis, *Journal of the Royal Statistical Society Series D (The Statistician),* 37(1988), 25-32. https://doi.org/10.2307/2348376

[23] Nadarajah, S., Bakouch, H. S., Tahmasbi, R., A generalized Lindley distribution, *Sankhya B,* 73 (2011), 331-359. https://doi.org/10.1007/s13571-011-0025-9

[24] Ramos, R. L., Louzada, F., A distribution for instantaneous failures, *Stats,* 2 (2019), 247-258. https://doi.org/10.3390/stats2020019

[25] Rayleigh, L., On the stability, or instability, of certain fluid motions, *Proceedings of the London Mathematical Society,* 11(1879), 57-72. https://doi.org/10.1112/plms/s1-11.1.57

[26] Renyi, A., On measures of entropy and information, *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability,* 1 (1961), 547-561.

[27] Rioul, O., This is IT: A Primer on Shannon's entropy and information. *In: Duplantier, B., Rivasseau, V. (eds) Information Theory. Progress in Mathematical Physics,* 78(2021). https://doi.org/10.1007/978-3-030-81480-9_2

[28] Sanku, D., Enayetur, R., Saikat, M., Statistical properties and different methods of estimation of transmuted Rayleigh distribution, *Revista Colombiana de Estadistica,* 40 (2017), 165-203. https://doi.org/10.15446/rce.v40n1.56153

[29] Schwarz, G., Estimating the dimension of a model, *Annals of Statistics,* 6 (1978), 461-464. https://doi.org/10.1214/aos/1176344136

[30] Shannon, C. E., A mathematical theory of communication, *Bell System Technical Journal,* 27 (1948), 379-432. https://doi.org/10.1002/j.1538-7305.1948.tb01338.x

[31] Smirnov, N., Table for estimating the goodness of fit of empirical distributions, *Annals of Mathematical Statistics,* 19 (1948), 279-281. https://doi.org/10.1214/aoms/1177730256

[32] Sohn, B. Y., Kim, G. B., Detection of outliers in weighted least squares regression, *Korean Journal of Computational & Applied Mathematics,* 4 (1997), 441-452. https://doi.org/10.1007/BF03014491

[33] Swain, J. J., Venkatraman, S., Wilson, J. R., Least-squares estimation of distribution functions in Johnson's translation system, *Journal of Statistical Computation and Simulation,* 29 (1988), 271-297. https://doi.org/10.1080/00949658808811068

[34] Walther, B. A., Moore, J. J., The concepts of bias, precision and accuracy, and their use in testing the performance of species richness estimators, with a literature review of estimator performance, *Ecography,* 28 (2005), 815-829. https://doi.org/10.1111/j.2005.0906-7590.04112.x

[35] Watson, G. S., Goodness-of-fit tests on a circle, *Biometrika,* 48 (1961), 109-114. https://doi.org/10.2307/2333135

## Appendix

The `R` (version 4.2.2) code used to estimate the parameters of the EGRL distribution using MLE is given below.

```
library(maxLik)
set.seed(111)
# The Ozone data
xi <- c(41, 36, 12, 18, 28, 23, 19, 8, 7, 16, 11, 14, 18, 14, 34,
6, 30, 11, 1, 11, 4, 32, 23, 45, 115, 37, 29, 71, 39, 23, 21, 37,
20, 12, 13, 135, 49, 32, 64, 40, 77, 97, 97, 85, 10, 27, 7, 48, 35,
61, 79, 63, 16, 80, 108, 20, 52, 82, 50, 64, 59, 39, 9, 16, 78, 35,
66, 122, 89, 110, 44, 28, 65, 22, 59, 23, 31, 44, 21, 9, 45, 168, 73,
76, 118, 84, 85, 96, 78, 73, 91, 47, 32, 20, 23, 21, 24, 44, 21, 28,
9, 13, 46, 18, 13, 24, 16, 13, 23, 36, 7, 14, 30, 14, 18, 20)
# Sample size
n <- length(xi)
# Determining the initials for the parameters, respectively.
alphainit <- 1
betainit <- 1
thetainit<-2/(mean(xi)+sqrt(mean(xi)^2-4*mean(xi)))
# Maximizing the log likelihood function.
logLik <- function(param) {
alpha <- param[1]
beta <- param[2]
```

```
theta <- param[3]
# The four lines below are used to ensure that the estimates updated
# in the BFGS algorithm are in line with the parameter spaces.
if (alpha < 0) {alpha <- 0.0001}
if (beta < 0) {beta <- 0.0001}
if (theta < 0) {theta <- 0.0001}
if (theta > 0.5) {theta <- 0.5}
gx <- (((1+theta^2*xi-2*theta)*(theta))/(1-theta))*(exp(-theta*xi))
Gx <- 1-((1+theta^2*xi-theta)/(1-theta))*(exp(-theta*xi))
ll <- n*log(alpha)+n*log(beta)+sum(log(gx))+(alpha-1)*(sum(log(1-Gx)))+
(beta-1)*sum(log(1-((1-Gx)^(alpha))))
}
# Obtaining the results of the BFGS algorithm. Here,
# control = list(iterlim = 100000) is used to ensure successfull
# convergence of the BFGS algorithm.
model <- maxLik(logLik, start = c(alphainit, betainit, thetainit),
method = "BFGS", control = list(iterlim = 100000))
# Displaying the results
summary(model)
--------------------------------------------
Maximum Likelihood estimation
BFGS maximization, 44 iterations
Return code 0: successful convergence
Log-Likelihood: -541.3966
3  free parameters
Estimates:
     Estimate Std. error t value  Pr(> t)
[1,]  1.42313    2.40164   0.593    0.553
[2,]  1.79495    0.24685   7.271 3.56e-13 ***
[3,]  0.02424    0.04196   0.578    0.563
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
--------------------------------------------
```

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 104–121 (2024)
DOI:10.31801/cfsuasmas.1127143
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

*Research Article; Received: June 7, 2022; Accepted: May 15, 2023*

# PPF DEPENDENT COMMON FIXED POINTS OF GENERALIZED WEAKLY CONTRACTIVE TYPE MULTI-VALUED MAPPINGS

Gutti V. R. BABU[1] and M. Vinod KUMAR[2]

[1]Department of Mathematics, Andhra University, Visakhapatnam 530003, INDIA
[2]Department of Mathematics, Anil Neerukonda Institute of Technology and Sciences (ANITS), Sangivalasa, Visakhapatnam 531162, INDIA

Abstract. In this paper, we introduce the notion of generalized weakly contractive type multi-valued mapping with respect to a single-valued mapping and prove the existence of PPF dependent coincidence points in Banach spaces. Further, we introduce the notion of generalized weakly contractive type multi-valued mappings for a pair of multi-valued mappings and prove the existence of PPF dependent common fixed points in Banach spaces. We draw some corollaries and provide nontrivial examples to illustrate our results.

## 1. Introduction

The Banach contraction principle is one of the fundemental and useful result in fixed point theory and it plays an important role in solving problems related to non-linear functional analysis. In 1969, Nadler [20] extended Banach contraction principle to the context of set valued mapping. For more works on the existence of fixed points of multi-valued maps, we refer Kaneko [16] and Mizoguchi and Takahashi [19]. In 1997, Alber and Gurre-Delabriere [1] introduced weakly contractive map which is a generalization of contraction map and obtained fixed point results in the setting of Hilbert spaces. Rhoades [22] extended this concept to metric spaces and Bae [6] considered these type of multi-valued mappings. Bose and Roychowdhury [9,10] considered some generalized versions of these mappings and proved some fixed point theorems.

2020 *Mathematics Subject Classification.* 47H10, 54H25.
*Keywords.* Multi-valued mapping, Razumikhin class, PPF dependent coincidence point, PPF dependent fixed point, PPF dependent common fixed point.

[1] gvr_babu@hotmail.com; 0000-0002-6272-2645
[2] dravinodvivek@gmail.com-Corresponding author; 0000-0001-6469-4855.

Let $(X, d)$ be a metric space and $K(X)$, the family of all non-empty compact subsets of $X$ and $H$ represents the Hausdorff distance induced by the metric $d$. i.e.,

$$H(A, B) = \max\{\sup_{a\in A} d(a, B),\ \sup_{b\in B} d(A, b)\}$$

for any $A, B \in K(X)$, where $d(a, B) = \inf_{b\in B} d(a, b)$ and $d(A, b) = \inf_{a\in A} d(a, b)$.

**Definition 1.** *[6] A point $x \in X$ is said to be a fixed point of a multi-valued mapping $T : X \to K(X)$ if $x \in Tx$.*

**Definition 2.** *A point $x \in X$ is said to be a coincidence point of two mappings $f, g : X \to X$ if $f(x) = g(x)$.*

**Definition 3.** *[9] A mapping $T : X \to X$ is said to be a generalized weakly contractive map with respect to $f : X \to X$ if*

$$\psi(d(Tx, Ty)) \le \psi(d(fx, fy)) - \phi(d(fx, fy))$$

*for all $x, y \in X$, where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous such that $\psi(t), \phi(t) > 0$ for $t \in (0, \infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, $\phi$ is non-decreasing and $\psi$ is monotonically increasing (strictly).*

If $\psi(t) = t$ for all $t \in [0, \infty)$, and $f$ is the identity map in Definition 3 then we say that $T : X \to X$ is said to be a weakly contractive map.

**Definition 4.** *[9] A multi-valued mapping $T : X \to K(X)$ is said to be a generalized weakly contractive map with respect to $f : X \to X$ if*

$$\psi(H(Tx, Ty)) \le \psi(d(fx, fy)) - \phi(d(fx, fy)),$$

*for all $x, y \in X$, where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous such that $\psi(t), \phi(t) > 0$ for $t \in (0, \infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, $\phi$ is non-decreasing and $\psi$ is monotonically increasing (strictly).*

If $f$ is the identity mapping then the multi-valued mapping $T : X \to K(X)$ is said to be generalized weakly contractive. If $\psi(t) = t$ for all $t \in [0, \infty)$, then the multi-valued mapping $T : X \to K(X)$ is said to be weakly contractive with respect to $f$.

In 1977, Bernfeld, Lakshmikantham and Reddy [8] introduced the concept of fixed point for mappings that have different domains and ranges which is called PPF (Past, Present and Future) dependent fixed point. Furthermore, they introduced the notation of Banach type contraction for a non-self mappings and proved the existence of PPF dependent fixed points of Banach type contractive mappings in the Razumikhin class. Several mathematicians proved the existence of PPF dependent fixed points of single-valued mapppings and multi-valued mappings, for more details we refer to [2–5, 7, 13, 15, 18]. In 2016, Farajzadeh, Kaewcharoen and Plubtieng [14] introduced the concept of PPF dependent fixed point of multi-valued mappings which is an extension of PPF dependent fixed point of single valued mapping and proved the existence of PPF dependent fixed point for multi-valued mappings.

Motivated by the research work of Bose and Roychowdhury [9] on weakly contractive maps, we extend the above said results for the case of PPF dependent coincidence points and PPF dependent common fixed points.

In this paper, we introduce the notion of generalized weakly contractive type multi-valued mapping with respect to a single-valued mapping and prove the existence of PPF dependent coincidence points in Banach spaces. Further, we introduce the notion of generalized weakly contractive type multi-valued mappings for a pair of multi-valued mappings and prove the existence of PPF dependent common fixed points in Banach spaces. We draw some corollaries and provide examples to illustrate our main results.

## 2. Preliminaries

In this paper, we denote the real line by $\mathbb{R}$, $\mathbb{R}^+ = [0, \infty)$, the set of all natural numbers by $\mathbb{N}$. Let $(E, ||.||_E)$ be a Banach space and we denote it by simply by $E$. Let $I = [a, b] \subseteq \mathbb{R}$ and $E_0 = C(I, E)$, the set of all continuous functions on $I$ equipped with the supremum norm $||.||_{E_0}$ and we define it by $||\phi||_{E_0} = \sup\limits_{a \leq t \leq b} ||\phi(t)||_E$ for $\phi \in E_0$.

In our discussion, let $CB(E)$ be the collection of all non-empty closed and bounded subsets of $E$. Then the Hausdorff metric $H_E$ on $CB(E)$ induced by the norm $||.||_E$ is defined by

$$H_E(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\}$$

for any $A, B \in CB(E)$, where $d(a, B) = \inf\limits_{b \in B} ||a - b||_E$ and $d(A, b) = \inf\limits_{a \in A} ||a - b||_E$.

For a fixed $c \in I$, the Razumikhin class $R_c$ of functions in $E_0$ is defined by $R_c = \left\{\phi \in E_0 \mid ||\phi||_{E_0} = ||\phi(c)||_E\right\}$ and $R_c(c) = \{\phi(c) \mid \phi \in R_c\}$. Clearly every constant function from $I$ to $E$ belongs to $R_c$ so that $R_c$ is a non-empty subset of $E_0$ .

**Definition 5.** *[8] Let $R_c$ be the Razumikhin class of continuous functions in $E_0$. Then, we say that*

(i) *the class $R_c$ is algebraically closed with respect to the difference if $\phi - \psi \in R_c$ whenever $\phi, \psi \in R_c$.*

(ii) *the class $R_c$ is topologically closed if it is closed with respect to the topology on $E_0$ by the norm $||.||_{E_0}$.*

The Razumikhin class of functions $R_c$ has the following properties.

**Theorem 1.** [2] *Let $R_c$ be the Razumikhin class of functions in $E_0$. Then*

(i) *for any $\phi \in R_c$ and $\alpha \in \mathbb{R}$, we have $\alpha\phi \in R_c$.*

(ii) *the Razumikhin class $R_c$ is topologically closed with respect to the norm defined on $E_0$.*

(iii) $\bigcap\limits_{c \in [a,b]} R_c = \{\phi \in E_0 \mid \phi : I \to E \text{ is constant}\}$.

**Definition 6.** *[8] Let $T : E_0 \to E$ be a mapping. A function $\phi \in E_0$ is said to be a PPF dependent fixed point of $T$ if $T\phi = \phi(c)$ for some $c \in I$.*

**Definition 7.** *[8] Let $T : E_0 \to E$ be a mapping. Then $T$ is called a Banach type contraction if there exists a constant $k \in [0, 1)$ such that*

$$||T\phi - T\psi||_E \leq k\, ||\phi - \psi||_{E_0}$$

*for any $\phi, \psi \in E_0$.*

**Theorem 2.** [8] *Let $T : E_0 \to E$ be a Banach type contraction. Let $R_c$ be an algebraically closed with respect to the difference and topologically closed. Then, $T$ has a unique PPF dependent fixed point in $R_c$.*

Farajzadeh, Kaewcharoen and Plubtieng [14] introduced the concept of PPF dependent fixed points of multi-valued mappings as follows.

**Definition 8.** *[14] Let $T : E_0 \to CB(E)$ be a multi-valued mapping. A point $\phi \in E_0$ is said to be a PPF dependent fixed point of $T$ if $\phi(c) \in T\phi$ for some $c \in I$.*

**Definition 9.** *[14] Let $f : E_0 \to E_0$ be a single-valued mapping and $T : E_0 \to CB(E)$ be a multi-valued mapping. A point $\phi \in E_0$ is said to be a PPF dependent coincidence point of $f$ and $T$ if $f\phi(c) \in T\phi$ for some $c \in I$.*

Here we observe that $f\phi$ is not a composition of $\phi$ and $f$.

**Definition 10.** *[14] Let $S, T : E_0 \to E$ be two single-valued mappings. A point $\phi \in E_0$ is said to be a PPF dependent common fixed point of $S$ and $T$ if $S\phi = T\phi = \phi(c)$ for some $c \in I$.*

We denote

$$\Psi = \{\psi : \mathbb{R}^+ \to \mathbb{R}^+ \mid \psi \text{ is continuous, monotonically increasing and } \psi(t) = 0 \iff t = 0\}$$

and

$$\Phi = \{\phi : \mathbb{R}^+ \to \mathbb{R}^+ \mid \phi \text{ is continuous and } \phi(t) = 0 \iff t = 0\}.$$

We use the following results in our subsequent discussions.

**Proposition 1.** *If $\{a_n\}$ and $\{b_n\}$ are two real sequences, $\{b_n\}$ is bounded, then $\liminf(a_n + b_n) \leq \liminf a_n + \limsup b_n$.*

**Lemma 1.** [20] Let $A$ and $B$ be two non-empty compact subsets of a metric space $X$. If $a \in A$ then there exists $b \in B$ such that $d(a, b) \leq H(A, B)$.

**Lemma 2.** [3] *Let $\{\phi_n\}$ be a sequence in $E_0$ such that $||\phi_n - \phi_{n+1}||_{E_0} \to 0$ as $n \to \infty$. If $\{\phi_n\}$ is not a Cauchy sequence, then there exists an $\epsilon > 0$ and two subsequences $\{\phi_{m_k}\}$ and $\{\phi_{n_k}\}$ of $\{\phi_n\}$ with $m_k > n_k > k$ such that $||\phi_{n_k} - \phi_{m_k}||_{E_0} \geq \epsilon$, $||\phi_{n_k} - \phi_{m_k-1}||_{E_0} < \epsilon$ and*

(i) $\lim\limits_{k\to\infty} ||\phi_{n_k} - \phi_{m_k+1}||_{E_0} = \epsilon,$

(ii) $\lim\limits_{k\to\infty} ||\phi_{n_k+1} - \phi_{m_k}||_{E_0} = \epsilon,$

(iii) $\lim\limits_{k\to\infty} ||\phi_{n_k} - \phi_{m_k}||_{E_0} = \epsilon,$

(iv) $\lim\limits_{k\to\infty} ||\phi_{n_k+1} - \phi_{m_k+1}||_{E_0} = \epsilon.$

## 3. Existence of PPF Dependent Coincidence Points

In this section, we introduce the concept of PPF dependent coincidence point of $f : E \to E$ and $T : E_0 \to E$.

**Definition 11.** *Let $f : E \to E$ and $T : E_0 \to E$ be two mappings. A point $\phi \in E_0$ is said to be a PPF dependent coincidence point of $f$ and $T$ if $T\phi = (f \circ \phi)(c)$ for some $c \in I$, where $f \circ \phi$ denotes the composition of $\phi$ and $f$.*

We observe that if $f$ is the identity mapping then PPF dependent coincidence point of $f$ and $T$ becomes PPF dependent fixed point of $T$.

Motivated by this idea, in the following, we now introduce the concept of PPF dependent coincidence point of $f : E \to E$ and $T : E_0 \to CB(E)$.

**Definition 12.** *Let $f : E \to E$ be a single-valued mapping and $T : E_0 \to CB(E)$ be a multi-valued mapping. A point $\phi \in E_0$ is said to be a PPF dependent coincidence point of $f$ and $T$ if $(f \circ \phi)(c) \in T\phi$ for some $c \in I$, where $f \circ \phi$ denotes the composition of $\phi$ and $f$.*

We observe that, if $f$ is an identity mapping then $\phi$ is a PPF dependent fixed point of the multi-valued mapping $T$.

**Notation:** Let $c \in I$. Let $f : E \to E$ and $\phi \in E_0$. We denote $(f \circ \phi)(c)$ by $f\phi(c)$.

In the following, we introduce the notion of generalized weakly contractive type multi-valued mappings.

**Definition 13.** *Let $T : E_0 \to CB(E)$. Let $f : E \to E$ be a continuous function. Then, $T$ is said to be a generalized weakly contractive type multi-valued mapping with respect to $f$ if there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$\psi(H_E(T\alpha, T\beta)) \le \psi(||f\alpha - f\beta||_{E_0}) - \phi(||f\alpha - f\beta||_{E_0}) \tag{1}$$

*for any $\alpha, \beta \in E_0$.*

We observe the following:

(i) if $f$ is the identity mapping in (1) then the mapping $T : E_0 \to CB(E)$ is said to be generalized weakly contractive type multi-valued mapping;

(ii) if $\psi(t) = t$ for any $t \in \mathbb{R}^+$ in (1) then the mapping $T : E_0 \to CB(E)$ is said to be weakly contractive type multi-valued mapping with respect to $f$;

(iii) if both $f$ is the identity mapping and $\psi(t) = t$ for any $t \in \mathbb{R}^+$ in (1) then the mapping $T : E_0 \to CB(E)$ is said to be weakly contractive type multi-valued mapping.

**Theorem 3.** *Let $T : E_0 \to CB(E)$ and $f : E \to E$ be functions that satisfy the following conditions:*

(i) *$T$ is a generalized weakly contractive type multi-valued mapping with respect to $f$,*

(ii) $T\phi \subseteq f(R_c)(c) = \{f\phi(c) \mid \phi \in R_c\}$ *for any* $\phi \in E_0$,
(iii) $R_c$ *is algebraically closed with respect to the difference,*
(iv) $f(R_c)$ *is complete and*
(v) $f(R_c) \subseteq R_c$.

*Then,* $T$ *and* $f$ *have a PPF dependent coincidence point in* $R_c$.

*Proof.* Let $\phi_0 \in R_c$. Then, $T\phi_0 \subseteq E$. Let $x_1 \in E$ be such that $x_1 \in T\phi_0$.
Since $T\phi_0 \subseteq f(R_c)(c)$, we choose $\phi_1$ in $R_c$ such that $x_1 = f\phi_1(c) \in T\phi_0$.
From (1), we have

$$\psi(H_E(T\phi_0, T\phi_1)) \leq \psi(||f\phi_0 - f\phi_1||_{E_0}) - \phi(||f\phi_0 - f\phi_1||_{E_0}).$$

Since $x_1 \in T\phi_0$, by Lemma 1 there exists $x_2 \in T\phi_1$ such that

$$||x_1 - x_2||_E \leq H_E(T\phi_0, T\phi_1). \tag{2}$$

Since $x_2 \in T\phi_1$ and $T\phi_1 \subseteq f(R_c)(c)$, we choose $\phi_2$ in $R_c$ such that
$x_2 = f\phi_2(c) \in T\phi_1$.
If $\phi_1 = \phi_2$ then $\phi_1$ is a PPF dependent coincidence point of $f$ and $T$.
Suppose that $\phi_1 \neq \phi_2$.
From (2), we have

$$||f\phi_1(c) - f\phi_2(c)||_E \leq H_E(T\phi_0, T\phi_1).$$

Since $R_c$ is algebraically closed with respect to the difference, we have

$$||f\phi_1 - f\phi_2||_{E_0} \leq H_E(T\phi_0, T\phi_1). \tag{3}$$

From (1), we have

$$\psi(H_E(T\phi_1, T\phi_2)) \leq \psi(||f\phi_1 - f\phi_2||_{E_0}) - \phi(||f\phi_1 - f\phi_2||_{E_0}).$$

Since $x_2 \in T\phi_1$, by Lemma 1 there exists $x_3 \in T\phi_2$ such that

$$||x_2 - x_3||_E \leq H_E(T\phi_1, T\phi_2). \tag{4}$$

Since $x_3 \in T\phi_2$ and $T\phi_2 \subseteq f(R_c)(c)$, we choose $\phi_3$ in $R_c$ such that
$x_3 = f\phi_3(c) \in T\phi_2$.
If $\phi_2 = \phi_3$ then $\phi_2$ is a PPF dependent coincident point of $f$ and $T$.
Suppose that $\phi_2 \neq \phi_3$.
From (4), we have

$$||f\phi_2(c) - f\phi_3(c)||_E \leq H_E(T\phi_1, T\phi_2).$$

Since $R_c$ is algebraically closed with respect to the difference, we have

$$||f\phi_2 - f\phi_3||_{E_0} \leq H_E(T\phi_1, T\phi_2). \tag{5}$$

On continuing this process, we get a sequence $\{f\phi_n\}$ in $R_c$ such that

$$x_n = f\phi_n(c) \in T\phi_{n-1}, ||f\phi_n - f\phi_{n+1}||_{E_0} \leq H_E(T\phi_{n-1}, T\phi_n) \text{ for all } n \in \mathbb{N}. \tag{6}$$

Clearly,

$$\begin{aligned} \psi(||f\phi_n - f\phi_{n+1}||_{E_0}) &\leq \psi(H_E(T\phi_{n-1}, T\phi_n) \\ &\leq \psi(||f\phi_{n-1} - f\phi_n)||_{E_0}) - \phi(||f\phi_{n-1} - f\phi_n||_{E_0}) \end{aligned} \tag{7}$$

$< \psi(||f\phi_{n-1} - f\phi_n||_{E_0})$.
Since $\psi$ is monotonically increasing function, we have
$||f\phi_n - f\phi_{n+1}||_{E_0} \leq ||f\phi_{n-1} - f\phi_n||_{E_0}$.
Therefore, the sequence $\{||f\phi_n - f\phi_{n+1}||_{E_0}\}$ is a decreasing sequence in $\mathbb{R}^+$ and hence it is convergent.
Let $\left|\left|f\phi_n - f\phi_{n+1}\right|\right|_{E_0} \to r$ as $n \to \infty$.
From (7), we have
$\psi(||f\phi_n - f\phi_{n+1}||_{E_0}) \leq \psi(||f\phi_{n-1} - f\phi_n||_{E_0}) - \phi(||f\phi_{n-1} - f\phi_n||_{E_0})$.
On applying limits as $n \to \infty$ on both sides, we get
$\psi(r) \leq \psi(r) - \phi(r)$ and hence $r = 0$.
Therefore,

$$\lim_{n\to\infty} \left|\left|f\phi_n - f\phi_{n+1}\right|\right|_{E_0} = 0. \tag{8}$$

We now show that $\{f\phi_n\}$ is a Cauchy sequence.
Suppose that $\{f\phi_n\}$ is not a Cauchy sequence. Then, there exists an $\epsilon > 0$ and two subsequences $\{f\phi_{m_k}\}$ and $\{f\phi_{n_k}\}$ of $\{f\phi_n\}$ such that for any $k \in \mathbb{N}, m_k > n_k > k$ such that

$$||f\phi_{n_k} - f\phi_{m_k}||_{E_0} \geq \epsilon. \tag{9}$$

Let $m_k$ be the smallest positive integer greater than $n_k$ satisfying (9).
Then, $||f\phi_{n_k} - f\phi_{m_k}||_{E_0} \geq \epsilon$ and $||f\phi_{n_k} - f\phi_{m_k-1}||_{E_0} < \epsilon$.
By Lemma 2 we have
$\lim\limits_{k\to\infty} \left|\left|f\phi_{n_k+1} - f\phi_{m_k}\right|\right|_{E_0} = \epsilon = \lim\limits_{k\to\infty} \left|\left|f\phi_{n_k} - f\phi_{m_k+1}\right|\right|_{E_0} = \lim\limits_{k\to\infty} \left|\left|f\phi_{n_k} - f\phi_{m_k}\right|\right|_{E_0}$.
Now, we show that $\lim\limits_{k\to\infty} ||f\phi_{n_k+l_1} - f\phi_{m_k+l_2}||_{E_0} = \epsilon$ for any $l_1, l_2 \in \mathbb{N}$.
Let $l_1, l_2 \in \mathbb{N}$. Now we consider

$$\begin{aligned} ||f\phi_{n_k+l_1} - f\phi_{m_k+l_2}||_{E_0} &\leq ||f\phi_{n_k+l_1} - f\phi_{n_k+l_1-1}||_{E_0} + ||f\phi_{n_k+l_1-1} - f\phi_{n_k+l_1-2}||_{E_0} \\ &+ ... + ||f\phi_{n_k+1} - f\phi_{n_k}||_{E_0} + ||f\phi_{n_k} - f\phi_{m_k+1}||_{E_0} \\ &+ ||f\phi_{m_k+1} - f\phi_{m_k+2}||_{E_0} + ... + ||f\phi_{m_k+l_2-1} - f\phi_{m_k+l_2}||_{E_0}. \end{aligned}$$

On applying limit superior as $k \to \infty$ on both sides, we get

$$\limsup_{k\to\infty} ||f\phi_{n_k+l_1} - f\phi_{m_k+l_2}||_{E_0} \leq \epsilon. \tag{10}$$

Now, we consider

$$\begin{aligned} ||f\phi_{n_k} - f\phi_{m_k+1}||_{E_0} &\leq ||f\phi_{n_k} - f\phi_{n_k+1}||_{E_0} + ||f\phi_{n_k+1} - f\phi_{n_k+2}||_{E_0} + ... \\ &+ ||f\phi_{n_k+l_1-1} - f\phi_{n_k+l_1}||_{E_0} + ||f\phi_{n_k+l_1} - f\phi_{m_k+l_2}||_{E_0} \\ &+ ||f\phi_{m_k+l_2} - f\phi_{m_k+l_2-1}||_{E_0} + ... + ||f\phi_{m_k+2} - f\phi_{m_k+1}||_{E_0}. \end{aligned}$$

Now, by applying Proposition 1 with $a_k = ||f\phi_{n_k+l_1} - f\phi_{m_k+l_2}||_{E_0}$ and
$b_k = (||f\phi_{n_k} - f\phi_{n_k+1}||_{E_0} + ||f\phi_{n_k+1} - f\phi_{n_k+2}||_{E_0} + ... + ||f\phi_{n_k+l_1-1} - f\phi_{n_k+l_1}||_{E_0} + ||f\phi_{m_k+l_2} - f\phi_{m_k+l_2-1}||_{E_0} + ... + ||f\phi_{m_k+2} - f\phi_{m_k+1}||_{E_0})$ we have

$$\begin{aligned} \epsilon \leq \liminf_{k\to\infty} ||f\phi_{n_k+l_1} - f\phi_{m_k+l_2}||_{E_0} + \limsup_{k\to\infty}(||f\phi_{n_k} - f\phi_{n_k+1}||_{E_0} \\ + ||f\phi_{n_k+1} - f\phi_{n_k+2}||_{E_0} + ... + ||f\phi_{n_k+l_1-1} - f\phi_{n_k+l_1}||_{E_0} + ||f\phi_{m_k+l_2} - f\phi_{m_k+l_2-1}||_{E_0} \\ + ... + ||f\phi_{m_k+2} - f\phi_{m_k+1}||_{E_0}). \end{aligned}$$

Hence

$$\epsilon \le \liminf_{k\to\infty} ||f\phi_{n_k+l_1} - f\phi_{m_k+l_2}||_{E_0}. \tag{11}$$

From (10) and (11), we get

$$\lim_{k\to\infty} ||f\phi_{n_k+l_1} - f\phi_{m_k+l_2}||_{E_0} = \epsilon \text{ for any } l_1, l_2 \in \mathbb{N}. \tag{12}$$

We choose $l_1, l_2 \in \mathbb{N}$ such that $(m_k + l_2) - (n_k + l_1) = 1$.
From (7), we have

$$\psi(||f\phi_{n_k+l_1} - f\phi_{m_k+l_2}||_{E_0}) \le$$
$$\psi(||f\phi_{n_k+l_1-1} - f\phi_{m_k+l_2-1}||_{E_0}) - \phi(||f\phi_{n_k+l_1-1} - f\phi_{m_k+l_2-1}||_{E_0}).$$

On applying limits as $k \to \infty$ on both sides and by using (12), we get

$$\psi(\epsilon) \le \psi(\epsilon) - \eta(\epsilon),$$

a contradiction.
Therefore, $\{f\phi_n\}$ is a Cauchy sequence in $f(R_c)$. Since $f(R_c)$ is complete, we have $f\phi_n \to \eta$ as $n \to \infty$ for some $\eta \in f(R_c)$ and hence there exists $\phi^* \in R_c$ such that $\eta = f\phi^*$ and $\lim_{n\to\infty} f\phi_n = f\phi^*$.
Now, for any $n \in \mathbb{N}$

$$d(f\phi_{n+1}(c), T\phi^*) \le H_E(T\phi_n, T\phi^*),$$

and hence

$$\psi(d(f\phi_{n+1}(c), T\phi^*)) \le \psi(H_E(T\phi_n, T\phi^*))$$
$$\le \psi(||f\phi_n - f\phi^*||_{E_0}) - \phi(||f\phi_n - f\phi^*||_{E_0}).$$

On applying limits as $n \to \infty$ on both sides, we get

$$\psi(d(f\phi^*(c), T\phi^*)) \le \psi(0) - \phi(0) \text{ and hence } \psi(d(f\phi^*(c), T\phi^*)) = 0.$$

Therefore, $f\phi^*(c) \in T\phi^*$ and hence $T$ and $f$ have a PPF dependent coincidence point in $R_c$. □

## 4. Existence of PPF Dependent Common Fixed Points

In this section, we introduce the concept of PPF dependent common fixed points for a pair of multi-valued mappings.

**Definition 14.** *Let $S, T : E_0 \to CB(E)$ be two multi-valued mappings. A point $\phi \in E_0$ is said to be a PPF dependent common fixed point of $S$ and $T$ if $\phi(c) \in S\phi$ and $\phi(c) \in T\phi$ for some $c \in I$.*

In the following we define generalized weakly contractive type mappings for a pair of multi-valued mappings.

**Definition 15.** *Let $S, T : E_0 \to CB(E)$ be two multi-valued functions. The pair $(S, T)$ is said to be a pair of generalized weakly contractive type multi-valued mappings on $E_0$ if there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$\psi(H_E(T\alpha, S\beta)) \le \psi(M(\alpha, \beta)) - \phi(M(\alpha, \beta)) \tag{13}$$

*for any* $\alpha, \beta \in E_0$, *where*
$M(\alpha,\beta) = \max\{||\alpha-\beta||_{E_0}, d(\alpha(c), T\alpha), d(\beta(c), S\beta), \frac{1}{2}[d(\beta(c), T\alpha) + d(\alpha(c), S\beta)]\}$.

**Theorem 4.** *Let* $S, T : E_0 \to CB(E)$ *be two multi-valued mappings such that:*

(i) *the pair* $(S,T)$ *is a pair of generalized weakly contractive type multi-valued mappings on* $E_0$,
(ii) $R_c$ *is algebraically closed with respect to the difference and*
(iii) $T\phi \subseteq R_c(c)$ *and* $S\phi \subseteq R_c(c)$ *for any* $\phi \in E_0$.

*Then,* $S$ *and* $T$ *have a PPF dependent common fixed point in* $R_c$.

*Proof.* Let $\phi_0 \in R_c$. Then, $T\phi_0 \subseteq E$. Let $x_1 \in E$ be such that $x_1 \in T\phi_0$.
Since $T\phi_0 \subseteq R_c(c)$, we choose $\phi_1$ in $R_c$ such that $x_1 = \phi_1(c) \in T\phi_0$.
From (13), we have
$\psi(H_E(T\phi_0, S\phi_1)) \leq \psi(M(\phi_0,\phi_1)) - \phi(M(\phi_0,\phi_1))$.
If $M(\phi_0,\phi_1) = 0$ then $\phi_0 = \phi_1$ and hence $\phi_0$ is a PPF dependent common fixed point of $S$ and $T$.
Suppose that $M(\phi_0,\phi_1) > 0$. By Lemma 1 there exists $x_2 \in S\phi_1$ such that

$$||x_1 - x_2||_E \leq H_E(T\phi_0, S\phi_1). \tag{14}$$

Since $x_2 \in S\phi_1$ and $S\phi_1 \subseteq R_c(c)$, we choose $\phi_2$ in $R_c$ such that $x_2 = \phi_2(c) \in S\phi_1$.
From (13), we have
$\psi(H_E(S\phi_1, T\phi_2)) = \psi(H_E(T\phi_2, S\phi_1)) \leq \psi(M(\phi_2,\phi_1)) - \phi(M(\phi_2,\phi_1))$.
If $M(\phi_2,\phi_1) = 0$ then $\phi_1 = \phi_2$ and hence $\phi_1$ is a PPF dependent common fixed point of $S$ and $T$.
Suppose that $M(\phi_2,\phi_1) > 0$. By Lemma 1 there exists $x_3 \in T\phi_2$ such that

$$||x_2 - x_3||_E \leq H_E(S\phi_1, T\phi_2). \tag{15}$$

Since $x_3 \in T\phi_2$ and $T\phi_2 \subseteq R_c(c)$, we choose $\phi_3$ in $R_c$ such that $x_3 = \phi_3(c) \in T\phi_2$.
Again from (13), we have
$\psi(H_E(T\phi_2, S\phi_3)) \leq \psi(M(\phi_2,\phi_3)) - \phi(M(\phi_2,\phi_3))$.
If $M(\phi_2,\phi_3) = 0$ then $\phi_2 = \phi_3$ and hence $\phi_2$ is a PPF dependent common fixed point of $S$ and $T$.
Suppose that $M(\phi_2,\phi_3) > 0$. On continuing this process, we get a sequence $\{\phi_n\}$ in $R_c$ such that

$$\phi_{2n+1}(c) \in T\phi_{2n}, \ \phi_{2n+2}(c) \in S\phi_{2n+1} \tag{16}$$

and

$$M(\phi_n, \phi_{n+1}) > 0 \tag{17}$$

with $||\phi_{2n+1}(c) - \phi_{2n+2}(c)||_E \leq H_E(T\phi_{2n}, S\phi_{2n+1})$
and $||\phi_{2n+2}(c) - \phi_{2n+3}(c)||_E \leq H_E(S\phi_{2n+1}, T\phi_{2n+2})$ for all $n \in \mathbb{N} \cup \{0\}$.
Since $R_c$ is algebraically closed with respect to the difference, for all $n \in \mathbb{N} \cup \{0\}$ we have

$$||\phi_{2n+1} - \phi_{2n+2}||_{E_0} \leq H_E(T\phi_{2n}, S\phi_{2n+1}) \tag{18}$$

and

$$||\phi_{2n+2} - \phi_{2n+3}||_{E_0} \leq H_E(S\phi_{2n+1}, T\phi_{2n+2}) = H_E(T\phi_{2n+2}, S\phi_{2n+1}). \tag{19}$$

We consider

$$\begin{aligned} M(\phi_{2n}, \phi_{2n+1}) &= \max\{||\phi_{2n} - \phi_{2n+1}||_{E_0}, d(\phi_{2n}(c), T\phi_{2n}), d(\phi_{2n+1}(c), S\phi_{2n+1}), \\ &\qquad \tfrac{1}{2}[d(\phi_{2n+1}(c), T\phi_{2n}) + d(\phi_{2n}(c), S\phi_{2n+1})]\}, \\ &\leq \max\{||\phi_{2n} - \phi_{2n+1}||_{E_0}, ||\phi_{2n}(c) - \phi_{2n+1}(c)||_E, ||\phi_{2n+1}(c) - \phi_{2n+2}(c)||_E, \\ &\qquad \tfrac{1}{2}[0 + ||\phi_{2n}(c) - \phi_{2n+2}(c)||_E\} \\ &= \max\{||\phi_{2n} - \phi_{2n+1}||_{E_0}, ||\phi_{2n+1} - \phi_{2n+2}||_{E_0}, \tfrac{1}{2}[||\phi_{2n} - \phi_{2n+2}||_{E_0}]\} \\ &\leq \max\{||\phi_{2n} - \phi_{2n+1}||_{E_0}, ||\phi_{2n+1} - \phi_{2n+2}||_{E_0}, \\ &\qquad \tfrac{1}{2}[||\phi_{2n} - \phi_{2n+1}||_{E_0} + ||\phi_{2n+1} - \phi_{2n+2}||_{E_0}]\} \\ &= \max\{||\phi_{2n} - \phi_{2n+1}||_{E_0}, ||\phi_{2n+1} - \phi_{2n+2}||_{E_0}\}, \end{aligned}$$

and hence

$$M(\phi_{2n}, \phi_{2n+1}) \leq \max\{||\phi_{2n} - \phi_{2n+1}||_{E_0}, ||\phi_{2n+1} - \phi_{2n+2}||_{E_0}\}. \tag{20}$$

Suppose that $\max\{||\phi_{2n} - \phi_{2n+1}||_{E_0}, ||\phi_{2n+1} - \phi_{2n+2}||_{E_0}\} = ||\phi_{2n+1} - \phi_{2n+2}||_{E_0}$.
Now, from (20), we have

$$M(\phi_{2n}, \phi_{2n+1}) \leq ||\phi_{2n+1} - \phi_{2n+2}||_{E_0},$$

and hence

$$\psi(M(\phi_{2n}, \phi_{2n+1})) \leq \psi(||\phi_{2n+1} - \phi_{2n+2}||_{E_0}).$$

Now, from (18), we have

$$||\phi_{2n+1} - \phi_{2n+2}||_{E_0} \leq H_E(T\phi_{2n}, S\phi_{2n+1}),$$

and hence

$$\begin{aligned} \psi(||\phi_{2n+1} - \phi_{2n+2}||_{E_0}) &\leq \psi(H_E(T\phi_{2n}, S\phi_{2n+1})) \\ &\leq \psi(M(\phi_{2n}, \phi_{2n+1})) - \phi(M(\phi_{2n}, \phi_{2n+1})) \qquad (21) \\ &\leq \psi(||\phi_{2n+1} - \phi_{2n+2}||_{E_0}) - \phi(M(\phi_{2n}, \phi_{2n+1})). \end{aligned}$$

Therefore, $f(M(\phi_{2n}, \phi_{2n+1})) = 0$ and hence $M(\phi_{2n}, \phi_{2n+1}) = 0$,
a contradiction.
Therefore,

$$\max\{||\phi_{2n} - \phi_{2n+1}||_{E_0}, ||\phi_{2n+1} - \phi_{2n+2}||_{E_0}\} = ||\phi_{2n} - \phi_{2n+1}||_{E_0}. \tag{22}$$

Now, from (20), we have

$$M(\phi_{2n}, \phi_{2n+1}) \leq ||\phi_{2n} - \phi_{2n+1}||_{E_0}. \tag{23}$$

Now, from (18), we have

$$||\phi_{2n+1} - \phi_{2n+2}||_{E_0} \leq H_E(T\phi_{2n}, S\phi_{2n+1}),$$

and hence

$$\begin{aligned} \psi(||\phi_{2n+1} - \phi_{2n+2}||_{E_0}) &\leq \psi(H_E(T\phi_{2n}, S\phi_{2n+1})) \\ &\leq \psi(M(\phi_{2n}, \phi_{2n+1})) - \phi(M(\phi_{2n}, \phi_{2n+1})) \\ &< \psi(M(\phi_{2n}, \phi_{2n+1})) \text{ (by using (17)} \\ &\leq \psi(||\phi_{2n} - \phi_{2n+1}||_{E_0}). \text{ (by using (23))} \end{aligned}$$

Since $\psi$ is monotonically increasing function, we have

$$||\phi_{2n+1} - \phi_{2n+2}||_{E_0} \leq M(\phi_{2n}, \phi_{2n+1}) \leq ||\phi_{2n} - \phi_{2n+1}||_{E_0}. \tag{24}$$

Similarly we have $||\phi_{2n+2} - \phi_{2n+3}||_{E_0} \leq M(\phi_{2n+2}, \phi_{2n+1}) \leq ||\phi_{2n+2} - \phi_{2n+1}||_{E_0}$

$$= ||\phi_{2n+1} - \phi_{2n+2}||_{E_0}. \tag{25}$$

From (24) and (25), we have $||\phi_{n+1} - \phi_n||_{E_0} \leq ||\phi_n - \phi_{n-1}||_{E_0}$ for all $n \in \mathbb{N}$. Therefore, the sequence $\{||\phi_{n+1} - \phi_n||_{E_0}\}$ is a decreasing sequence in $\mathbb{R}^+$, and hence convergent.

Let $\lim\limits_{n\to\infty} ||\phi_{n+1} - \phi_n||_{E_0} = r(\text{say})$.

From (24), we have

$$||\phi_{2n+1} - \phi_{2n+2}||_{E_0} \leq M(\phi_{2n}, \phi_{2n+1}) \leq ||\phi_{2n} - \phi_{2n+1}||_{E_0}.$$

On applying limits as $n \to \infty$, we get

$$r \leq \lim_{n\to\infty} M(\phi_{2n}, \phi_{2n+1}) \leq r \text{ and hence } \lim_{n\to\infty} M(\phi_{2n}, \phi_{2n+1}) = r.$$

From (21), we have

$$\psi(||\phi_{2n+1} - \phi_{2n+2}||_{E_0}) \leq \psi(M(\phi_{2n}, \phi_{2n+1})) - \phi(M(\phi_{2n}, \phi_{2n+1})).$$

On applying limits as $n \to \infty$, we get $\psi(r) \leq \psi(r) - \phi(r)$ and which implies that $r = 0$.

Therefore,

$$\lim_{n\to\infty} ||\phi_{n+1} - \phi_n||_{E_0} = 0. \tag{26}$$

Now, we show that $\{\phi_n\}$ is a Cauchy sequence.

From (26), to prove $\{\phi_n\}$ is a Cauchy sequence it is enough to prove that $\{\phi_{2n}\}$ is a Cauchy sequence.

Suppose that $\{\phi_{2n}\}$ is not a Cauchy sequence.

Then, there exists $\epsilon > 0$ and two subsequences $\{\phi_{2m_k}\}$ and $\{\phi_{2n_k}\}$ of $\{\phi_{2n}\}$ such that for any $k \in \mathbb{N}, m_k > n_k > k$ such that

$$||\phi_{2n_k} - \phi_{2m_k}||_{E_0} \geq \epsilon. \tag{27}$$

Let $m_k$ be the smallest positive integer greater than $n_k$ that is satisfying (27).

Then, $||\phi_{2n_k} - \phi_{2m_k}||_{E_0} \geq \epsilon$ and $||\phi_{2n_k} - \phi_{2m_k-2}||_{E_0} < \epsilon$.

We now show that $\lim\limits_{k\to\infty} ||\phi_{2n_k} - \phi_{2m_k+1}||_{E_0} = \epsilon$.

Clearly

$$\epsilon \leq ||\phi_{2n_k} - \phi_{2m_k}||_{E_0} \leq ||\phi_{2n_k} - \phi_{2m_k+1}||_{E_0} + ||\phi_{2m_k+1} - \phi_{2m_k}||_{E_0}.$$

Now, by applying Proposition 1 with $a_k = ||\phi_{2n_k} - \phi_{2m_k+1}||_{E_0}$ and $b_k = ||\phi_{2m_k+1} - \phi_{2m_k}||_{E_0}$ we have

$$\epsilon \leq \liminf_{k\to\infty} ||\phi_{2n_k} - \phi_{2m_k+1}||_{E_0} + \limsup_{k\to\infty} ||\phi_{2m_k+1} - \phi_{2m_k}||_{E_0},$$

and hence

$$\epsilon \leq \liminf_{k\to\infty} ||\phi_{2n_k} - \phi_{2m_k+1}||_{E_0}. \tag{28}$$

Clearly

$$\begin{aligned} ||\phi_{2n_k} - \phi_{2m_k+1}||_{E_0} &\leq ||\phi_{2n_k} - \phi_{2m_k-2}||_{E_0} + ||\phi_{2m_k-2} - \phi_{2m_k-1}||_{E_0} \\ &\quad + ||\phi_{2m_k-1} - \phi_{2m_k}||_{E_0} + ||\phi_{2m_k} - \phi_{2m_k+1}||_{E_0} \\ &< \epsilon + ||\phi_{2m_k-2} - \phi_{2m_k-1}||_{E_0} \end{aligned}$$

$$+ ||\phi_{2m_k-1} - \phi_{2m_k}||_{E_0} + ||\phi_{2m_k} - \phi_{2m_k+1}||_{E_0}.$$

On applying limit superior as $k \to \infty$ on both sides, we get

$$\limsup_{k\to\infty} ||\phi_{2n_k} - \phi_{2m_k+1}||_{E_0} \le \epsilon. \tag{29}$$

From (28) and (29), we get

$$\lim_{k\to\infty} ||\phi_{2n_k} - \phi_{2m_k+1}||_{E_0} = \epsilon. \tag{30}$$

We now show that $\lim\limits_{k\to\infty} ||\phi_{2n_k+l_1} - \phi_{2m_k+l_2}||_{E_0} = \epsilon$ for any $l_1, l_2 \in \mathbb{N}$.

Let $l_1, l_2 \in \mathbb{N}$.

We now consider

$$\begin{aligned} ||\phi_{2n_k+l_1} - \phi_{2m_k+l_2}||_{E_0} \le\ & ||\phi_{2n_k+l_1} - \phi_{2n_k+l_1-1}||_{E_0} + ||\phi_{2n_k+l_1-1} - \phi_{2n_k+l_2-2}||_{E_0} \\ & + ... + ||\phi_{2n_k+1} - \phi_{2n_k}||_{E_0} + ||\phi_{2n_k} - \phi_{2m_k+1}||_{E_0} \\ & + ||\phi_{2m_k+1} - \phi_{2m_k+2}||_{E_0} + ... + ||\phi_{2m_k+l_2-1} - \phi_{2m_k+l_2}||_{E_0}. \end{aligned}$$

On applying limit superior as $k \to \infty$ on both sides, we get

$$\limsup_{k\to\infty} ||\phi_{2n_k+l_1} - \phi_{2m_k+l_2}||_{E_0} \le \epsilon. \tag{31}$$

We now consider

$$\begin{aligned} ||\phi_{2n_k} - \phi_{2m_k+1}||_{E_0} \le\ & ||\phi_{2n_k} - \phi_{2n_k+1}||_{E_0} + ||\phi_{2n_k+1} - \phi_{2n_k+2}||_{E_0} + ... \\ & + ||\phi_{2n_k+l_1-1} - \phi_{2n_k+l_1}||_{E_0} + ||\phi_{2n_k+l_1} - \phi_{2m_k+l_2}||_{E_0} \\ & + ||\phi_{2m_k+l_2} - \phi_{2m_k+l_2-1}||_{E_0} + ... + ||\phi_{2m_k+2} - \phi_{2m_k+1}||_{E_0}. \end{aligned}$$

Now, by applying Proposition 1 with $a_k = ||\phi_{2n_k+l_1} - \phi_{2m_k+l_2}||_{E_0}$ and $b_k = (||\phi_{2n_k} - \phi_{2n_k+1}||_{E_0} + ||\phi_{2n_k+1} - \phi_{2n_k+2}||_{E_0} + ... + ||\phi_{2n_k+l_1-1} - \phi_{2n_k+l_1}||_{E_0} + ||\phi_{2m_k+l_2} - \phi_{2m_k+l_2-1}||_{E_0} + ... + ||\phi_{2m_k+2} - \phi_{2m_k+1}||_{E_0})$

we have

$$\begin{aligned} \epsilon \le & \liminf_{k\to\infty} ||\phi_{2n_k+l_1} - \phi_{2m_k+l_2}||_{E_0} + \limsup_{k\to\infty}(||\phi_{2n_k} - \phi_{2n_k+1}||_{E_0} \\ & + ||\phi_{2n_k+1} - \phi_{2n_k+2}||_{E_0} + ... + ||\phi_{2n_k+l_1-1} - \phi_{2n_k+l_1}||_{E_0} \\ & + ||\phi_{2m_k+l_2} - \phi_{2m_k+l_2-1}||_{E_0} + ... + ||\phi_{2m_k+2} - \phi_{2m_k+1}||_{E_0}). \end{aligned}$$

Hence

$$\epsilon \le \liminf_{k\to\infty} ||\phi_{2n_k+l_1} - \phi_{2m_k+l_2}||_{E_0}. \tag{32}$$

From (31) and (32), we get that for any $l_1, l_2 \in \mathbb{N}$

$$\lim_{k\to\infty} ||\phi_{2n_k+l_1} - \phi_{2m_k+l_2}||_{E_0} = \epsilon. \tag{33}$$

Now, we choose $l_1, l_2 \in \mathbb{N}$ such that $2n_k + l_1$ is even, $2m_k + l_2$ is odd and $(2m_k + l_2) - (2n_k + l_1) = 1$.

From (24), we have

$$||\phi_{2n_k+l_1+1} - \phi_{2m_k+l_2+1}||_{E_0} \le M(\phi_{2n_k+l_1}, \phi_{2m_k+l_2}) \le ||\phi_{2n_k+l_1} - \phi_{2m_k+l_2}||_{E_0}.$$

On applying limits as $k \to \infty$ , we get

$$\epsilon \le \lim_{k\to\infty} M(\phi_{2n_k+l_1}, \phi_{2m_k+l_2}) \le \epsilon \text{ and hence } \lim_{k\to\infty} M(\phi_{2n_k+l_1}, \phi_{2m_k+l_2}) = \epsilon.$$

From (21), we have

$$\psi(||\phi_{2n_k+l_1+1} - \phi_{2m_k+l_2+1}||_{E_0}) \le \psi(M(\phi_{2n_k+l_1}, \phi_{2m_k+l_2})) - \phi(M(\phi_{2n_k+l_1}, \phi_{2m_k+l_2})).$$

On applying limits as $k \to \infty$ we get,

$\psi(\epsilon) \le \psi(\epsilon) - \phi(\epsilon)$ and hence $\epsilon = 0$,

a contradiction.

Therefore, the sequence $\{\phi_n\}$ is a Cauchy sequence in $R_c$.

Since $E_0$ is complete, we have $\phi_n \to \phi^*$ as $n \to \infty$ for some $\phi^* \in E_0$.

Since $R_c$ is topologically closed, we have $\phi^* \in R_c$.

Now, we show that $\phi^*$ is a PPF dependent common fixed point of $S$ and $T$.

We now consider,

$$\begin{aligned}
d(\phi^*(c), S\phi^*) &\le M(\phi_{2k}, \phi^*) \\
&= \max\{||\phi_{2k} - \phi^*||_{E_0}, d(\phi_{2k}(c), T\phi_{2k}), d(\phi^*(c), S\phi^*), \\
&\qquad \tfrac{1}{2}[d(\phi^*(c), T\phi_{2k}) + d(\phi_{2k}(c), S\phi^*)]\} \\
&\le \max\{||\phi_{2k} - \phi^*||_{E_0}, ||\phi_{2k}(c) - \phi_{2k+1}(c)||_E + d(\phi_{2k+1}(c), T\phi_{2k}), d(\phi^*(c), S\phi^*), \\
&\qquad \tfrac{1}{2}[||\phi^*(c) - \phi_{2k+1}(c)||_E + d(\phi_{2k+1}(c), T\phi_{2k}) \\
&\qquad + ||\phi_{2k}(c) - \phi^*(c)||_E + d(\phi^*(c), S\phi^*)]\} \\
&= \max\{||\phi_{2k} - \phi^*||_{E_0}, ||\phi_{2k} - \phi_{2k+1}||_{E_0}, d(\phi^*(c), S\phi^*), \\
&\qquad \tfrac{1}{2}[||\phi^* - \phi_{2k+1}||_{E_0} + ||\phi_{2k} - \phi^*||_{E_0} + d(\phi^*(c), S\phi^*)]\}.
\end{aligned}$$

On applying limits as $k \to \infty$, we get

$$\begin{aligned}
d(\phi^*(c), S\phi^*) &\le \lim_{k\to\infty} M(\phi_{2k}, \phi^*) \\
&\le \max\{0, 0, d(\phi^*(c), S\phi^*), \tfrac{1}{2}[d(\phi^*(c), S\phi^*)]\} \\
&= d(\phi^*(c), S\phi^*).
\end{aligned}$$

Hence $\lim_{k\to\infty} M(\phi_{2k}, \phi^*) = d(\phi^*(c), S\phi^*)$.

Now,

$$\begin{aligned}
d(\phi^*(c), S\phi^*)) &\le ||\phi^*(c) - \phi_{2k+1}(c)||_E + d(\phi_{2k+1}(c), S\phi^*) \\
&\le ||\phi^* - \phi_{2k+1}||_{E_0} + H_E(T\phi_{2k}, S\phi^*).
\end{aligned}$$

Applying limits as $k \to \infty$ , we get

$$d(\phi^*(c), S\phi^*)) \le \lim_{k\to\infty} H_E(T\phi_{2k}, S\phi^*),$$

and hence

$$\begin{aligned}
\psi(d(\phi^*(c), S\phi^*)) &\le \lim_{k\to\infty} \psi(H_E(T\phi_{2k}, S\phi^*)) \\
&\le \lim_{k\to\infty} \psi(M(\phi_{2k}, \phi^*)) - \lim_{k\to\infty} \phi(M(\phi_{2k}, \phi^*)) \\
&= \psi(d(\phi^*(c), S\phi^*)) - \phi(d(\phi^*(c), S\phi^*)).
\end{aligned}$$

Therefore, $\phi(d(\phi^*(c), S\phi^*)) = 0$ and hence $\phi^*(c) \in S\phi^*$.

Similarly we can prove that $\phi^*(c) \in T\phi^*$.

Therefore, $\phi^*$ is a PPF dependent common fixed point of $S$ and $T$. □

## 5. Corollaries and Examples

**Corollary 1.** *Let $T : E_0 \to CB(E)$ and $f : E \to E$ be a function that satisfy the following conditions:*

(i) *$T$ is weakly contractive type multi-valued mapping with respect to $f$,*
(ii) *$T\phi \subseteq f(R_c)(c)$ for any $\phi \in E_0$,*
(iii) *$R_c$ is algebraically closed with respect to the difference,*

(iv) $f(R_c)$ *is complete and*
(v) $f(R_c) \subseteq R_c$.

*Then, $T$ and $f$ have a PPF dependent coincidence point in $R_c$.*

*Proof.* Follows from Theorem 3 by choosing $\psi(t) = t,\ t \in \mathbb{R}^+$ in the inequality (1). $\square$

By choosing $f = I,\ I$ the identity map in Theorem 3, we get the following corollary.

**Corollary 2.** *Let $T : E_0 \to CB(E)$ be a multi-valued mapping. Assume that $T$ satisfy the following conditions:*

(i) *$T$ is a generalized weakly contractive type multi-valued mapping,*
(ii) *$T\phi \subseteq R_c(c)$ for any $\phi \in E_0$,*
(iii) *$R_c$ is algebraically closed with respect to the difference.*

*Then, $T$ has a PPF dependent fixed point in $R_c$.*

The following corollary follows by choosing $\psi(t) = t,\ t \in \mathbb{R}^+$ in Corollary 2.

**Corollary 3.** *Let $T : E_0 \to CB(E)$ be a mapping satisfy the following conditions:*

(i) *$T$ is weakly contractive type multi-valued mapping,*
(ii) *$T\phi \subseteq R_c(c)$ for any $\phi \in E_0$,*
(iii) *$R_c$ is algebraically closed with respect to the difference.*

*Then, $T$ has a PPF dependent fixed point in $R_c$.*

**Corollary 4.** *Let $T : E_0 \to CB(E)$ be a mapping satisfiy the following conditions:*

(i) *suppose that there exists $k \in [0, 1)$ such that*
$$H_E(T\alpha, T\beta)) \leq k\ \ ||\alpha - \beta||_{E_0} \text{ for all } \alpha, \beta \in E_0,$$
(ii) *$T\phi \subseteq R_c(c)$ for any $\phi \in E_0$,*
(iii) *$R_c$ is algebraically closed with respect to the difference.*

*Then, $T$ has a PPF dependent fixed point in $R_c$.*

*Proof.* Follows by choosing $\phi(t) = (1 - k)t, t \in \mathbb{R}^+$ in Corollary 3. $\square$

**Corollary 5.** *Let $S, T : E_0 \to CB(E)$ be two multi-valued mappings such that*

(i) $H_E(T\alpha, S\beta)) \leq k \max\{||\alpha - \beta||_{E_0}, d(\alpha(c), T\alpha), d(\beta(c), S\beta), \frac{1}{2}[d(\beta(c), T\alpha) + d(\alpha(c), S\beta)]\}$
*for any $\alpha, \beta \in E_0$,*
(ii) *$R_c$ is algebraically closed with respect to the difference and*
(iii) *$T\alpha \subseteq R_c(c)$ and $S\alpha \subseteq R_c(c)$ for all $\alpha \in E_0$.*

*Then, $S$ and $T$ have a PPF dependent common fixed point in $R_c$.*

*Proof.* Follows by choosing $\psi(t) = t$ and $\phi(t) = (1 - k)t$ for $t \in \mathbb{R}^+$ in Theorem 4. $\square$

If $S = T$ in Theorem 4 and Corollary 5, we get the following corollaries.

**Corollary 6.** *Let $T : E_0 \to CB(E)$ be a multi-valued mapping. Assume that:*

(i) *there exist two functions $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$\psi(H_E(T\alpha, T\beta)) \leq \psi(M(\alpha, \beta)) - \phi(M(\alpha, \beta)) \tag{34}$$

*for all $\alpha, \beta \in E_0$, where*
$M(\alpha, \beta) = \max\{||\alpha - \beta||_{E_0}, d(\alpha(c), T\alpha), d(\beta(c), T\beta), \frac{1}{2}[d(\beta(c), T\alpha) + d(\alpha(c), T\beta)]\}$,

(ii) *$R_c$ is algebraically closed with respect to the difference and*

(iii) *$T\phi \subseteq R_c(c)$ for any $\phi \in E_0$.*

*Then, $T$ has a PPF dependent fixed point in $R_c$.*

**Corollary 7.** *Let $T : E_0 \to CB(E)$ be two multi-valued mappings such that*

(i) *$H_E(T\alpha, T\beta)) \leq k \max\{||\alpha - \beta||_{E_0}, d(\alpha(c), T\alpha), d(\beta(c), T\beta), \frac{1}{2}[d(\beta(c), T\alpha) + d(\alpha(c), T\beta)]\}$*
*for all $\alpha, \beta \in E_0$,*

(ii) *$R_c$ is algebraically closed with respect to the difference and*

(iii) *$T\alpha \subseteq R_c(c)$ for any $\alpha \in E_0$.*

*Then, $T$ has a PPF dependent fixed point in $R_c$.*

**Example 1.** *Let $E = \mathbb{R},\ c = 1 \in I = [\frac{1}{2}, 2] \subseteq \mathbb{R},\ E_0 = C(I, E)$.*
*Let $k \geq 1$. We define $f : E \to E$ by $f(x) = kx$ for any $x \in E$.*
*Clearly, $f$ is a continuous function.*
*By definition, $R_c(c) = \{\phi(c) \mid \phi \in R_c\}$ and*
$f(R_c)(c) = \{(f \circ \phi)(c) \mid \phi \in R_c\} = \{f(\phi(c)) \mid \phi \in R_c\} = \{k\phi(c) \mid \phi \in R_c\}$.
*First we show that $f(R_c) = R_c$.*
*Let $\alpha \in R_c$. Then $\alpha = \beta$ for some $\beta \in R_c$.*
*Clearly, $\alpha = k\frac{1}{k}\beta = k\eta$ (by Theorem 1, $\eta = \frac{1}{k}\beta \in R_c$) so that*
*$\alpha(x) = k\eta(x) = f(\eta(x)) = (f \circ \eta)(x)$ for any $x \in I$.*
*Therefore, $\alpha = f \circ \eta \in f(R_c)$ and hence*

$$R_c \subseteq f(R_c). \tag{35}$$

*Now, let $\alpha \in f(R_c)$. Then $\alpha = f \circ \beta$ for some $\beta \in R_c$.*
*Clearly, $\alpha(x) = (f \circ \beta)(x) = f(\beta(x)) = k\beta(x) = (k\beta)(x)$ for any $x \in I$.*
*Therefore, $\alpha = k\beta \in R_c$ and hence*

$$f(R_c) \subseteq R_c. \tag{36}$$

*From (35) and (36), we get $f(R_c) = R_c$.*
*Since $E_0$ is complete and $R_c$ is topologically closed we have $f(R_c) = R_c$ is complete.*
*For any $\gamma \in \mathbb{R}$, we define $\phi_\gamma : I \to E$ by*

$$\phi_\gamma(x) = \begin{cases} \gamma x^2 & \text{if } x \in [\frac{1}{2}, 1] \\ \frac{\gamma}{x^2} & \text{if } x \in [1, 2]. \end{cases}$$

*Clearly $\phi_\gamma \in E_0, ||\phi_\gamma||_{E_0} = ||\phi_\gamma(c)||_E$ and hence $\phi_\gamma \in R_c$ for any $\gamma \in \mathbb{R}$.*
*Let $\mathrm{F}_0 = \{\phi_\gamma \mid \gamma \in \mathbb{R}\}$.*
*Then, $\mathrm{F}_0$ is algebraically closed with respect to the difference and $\mathrm{F}_0 \subseteq R_c$.*

*We observe that* $\mathbb{R} = \{\phi_\gamma(c) \mid \gamma \in \mathbb{R}\} = \mathrm{F}_0(c) \subseteq R_c(c)$.
*Clearly,* $R_c(c) \subseteq \mathbb{R}$ *and hence* $f(R_c)(c) = R_c(c) = \mathbb{R}$.
*We define* $T : E_0 \to CB(E)$ *by* $T\phi = [0,\ \frac{k}{4}||\phi(c)||_E]$ *for any* $\phi \in E_0$.
*Clearly,* $T\phi \subseteq \mathbb{R} = R_c(c) = f(R_c)(c)$.
*We define* $\psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ *by* $\psi(t) = \frac{t^2}{2}$ *and*

$$\phi(t) = \begin{cases} \frac{15\ t^3}{32} & \text{if } t \in [0,1] \\ \frac{15\ t}{32} & \text{if } \ t \geq 1. \end{cases}$$

*Clearly,* $\psi \in \Psi$ *and* $\phi \in \Phi$.
*From the definition of Hausdorff distance, it follows that, for any* $\alpha, \beta \in E_0$

$$H_E(T\alpha, T\beta) = \frac{k}{4} \begin{cases} ||\alpha(c)||_E - ||\beta(c)||_E & \text{if } ||\alpha(c)||_E \geq ||\beta(c)||_E \\ ||\beta(c)||_E - ||\alpha(c)||_E & \text{if } ||\beta(c)||_E \geq ||\alpha(c)||_E \end{cases}$$

$$\begin{aligned} &= \tfrac{k}{4}|\ ||\alpha(c)||_E - ||\beta(c)||_E\ | = \tfrac{1}{4}|\ ||k\alpha(c)||_E - ||k\beta(c)||_E\ | \\ &\leq \tfrac{1}{4}|k\alpha(c) - k\beta(c)| = \tfrac{1}{4}|(f \circ \alpha)(c) - (f \circ \beta)(c)| \\ &= \tfrac{1}{4}||(f\alpha - f\beta)(c)||_E \\ &\leq \tfrac{1}{4}||f\alpha - f\beta||_{E_0}. \end{aligned}$$

*Therefore,*

$$\begin{aligned} \psi(H_E(T\alpha, T\beta)) &\leq \psi(\tfrac{1}{4}||f\alpha - f\beta||_{E_0}) = \tfrac{1}{32}[\ ||f\alpha - f\beta||_{E_0}\ ]^2 \\ &\leq \psi(||f\alpha - f\beta||_{E_0}) - \phi(||f\alpha - f\beta||_{E_0}). \end{aligned}$$

*Therefore,* $T$ *and* $f$ *satisfy all the hypotheses of Theorem 3 and* $\phi_0 \in R_c$ *is a PPF dependent coincidence point of* $T$ *and* $f$.

**Example 2.** *Let* $E = \mathbb{R},\ c = 1 \in I = [\frac{1}{2}, 2] \subseteq \mathbb{R},\ E_0 = C(I, E)$.
*On continuing the same procedure as in the Example 1, we get* $R_c(c) = \mathbb{R}$.
*We define* $T : E_0 \to CB(E)$ *by* $T\phi = [0,\ \frac{1}{5}||\phi(c)||_E]$ *for any* $\phi \in E_0$.
*Clearly* $T\phi \subseteq R_c(c)$.
*We define* $\psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ *by* $\psi(t) = 2t$ *and* $\phi(t) = \frac{6t}{5}$ *for any* $t \in \mathbb{R}^+$.
*Clearly,* $\psi \in \Psi$ *and* $\phi \in \Phi$.
*Clearly, for any* $\alpha, \beta \in E_0$, *we have*

$$\begin{aligned} H_E(T\alpha, T\beta) &\leq \tfrac{1}{5}||\alpha - \beta||_{E_0} \\ &\leq \tfrac{1}{5}\max\{||\alpha - \beta||_{E_0}, d(\alpha(c), T\alpha), d(\beta(c), T\beta), \tfrac{1}{2}[d(\beta(c), T\alpha) + d(\alpha(c), T\beta)]\} \\ &= \tfrac{1}{5}M(\alpha, \beta). \end{aligned}$$

*Therefore,*

$$\begin{aligned} \psi(H_E(T\alpha, T\beta)) &\leq \psi(\tfrac{1}{5}M(\alpha, \beta)) = \tfrac{2}{5}M(\alpha, \beta) \\ &\leq 2M(\alpha, \beta) - \tfrac{6}{5}M(\alpha, \beta) \\ &= \psi(M(\alpha, \beta)) - \phi(M(\alpha, \beta)). \end{aligned}$$

*Therefore,* $T$ *satisfies all the hypotheses of Corollary 6 and* $\phi_0 \in R_c$ *is a PPF dependent fixed point of* $T$.

**Author Contribution Statements** Each author declares substantial contributions through the following: (1) the conception and design of the study, or acquisition of data, or analysis and interpretation of data, (2) drafting the article or revising it critically for important intellectual content.

**Declaration of Competing Interests** The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## References

[1] Alber,Ya. I., Guerre-Delabriere S., Principles of weakly contractive maps in Hilbert spaces New results in Operator theory, *Adv. Appl., Birkhauser Verlag*, 98(1997), 7-22.

[2] Babu, G.V.R., Satyanarayana, G., Vinod Kumar, M., Properties of Razumikhin class of functions and PPF dependent fixed points of Weakly contractive type mappings, *Bull. Int. Math. Virtual Institute*, 9 (2019), 65-72. DOI:10.7251/BIMVI1901065B

[3] Babu, G.V.R., Vinod Kumar, M., PPF dependent fixed points of generalized Suzuki type contractions via simulation functions, *Advances in the Theory of Nonlinear Anal. and its Appl.*, 3 (2019), 121-140. https://doi.org/10.31197/atnaa.588945

[4] Babu, G.V.R., Vinod Kumar, M., PPF dependent fixed points of generalized contractions via $C_G-$simulation functions, *Communications in Nonlinear Anal.*, 7 (2019), 1-16.

[5] Babu, G.V.R., Vinod Kumar, M., PPF dependent fixed points of generalized weakly contraction maps via $C_G-$simulation functions, *Maltepe J. Math.*, 1 (2019), 66-88.

[6] Bae, J. S., Fixed point theorems for weakly contractive multivalued maps, *J. Math. Anal. Appl.*, 284 (2003), 690-697. https://doi.org/10.1016/S0022-247X(03)00387-1

[7] Bapurao C. Dhage, On some common fixed point theorems with PPF dependence in Banach space, *J. Nonlinear Sci. Appl.*, 5 (2012), 220-232. https://dx.doi.org/10.22436/jnsa.005.03.06

[8] Bernfeld, S. R., Lakshmikantham, V., Reddy, Y. M., Fixed point theorems of operators with PPF dependence in Banach spaces, *Appl. Anal.*, 6 (1977), 271-280.

[9] Bose, R. K., Roychowdhury, M. K., Fixed point theorems for generalized weakly contractive mappings, *Surveys in Math. Appl.*, 4 (2009), 215-238.

[10] Bose, R. K., Roychowdhury, M. K., Fixed point theorems for multi-valued mapping and fuzzy mappings, *Int. J. Pure and App. Math.*, 61 (2010), 53-72.

[11] Chatterjea, S. K., Fixed point theorems, *C.R.Acad. Bulgare Sci.*, 25(1972), 727-730.

[12] Choudhury, B. S., Unique fixed point theorems for weackly C-Contractive mappings, *Khatmandu University J. Sci. Tech.*, 1 (2009), 6-12. https://doi.org/10.3126/kuset.v5i1.2842

[13] Dirci, Z., McRae, F. A., Vasundharadevi, J., Fixed point theorems in partially ordered metric spaces for operators with PPF dependence, *Nonlinear Anal.*, 67 (2007), 641-647. DOI:10.1016/j.na.2006.06.022

[14] Farajzadeh, A., Kaewcharoen, A., Plubtieng, S., PPF dependent fixed point theorems for multivalued mappings in Banach spaces, *Bull. Iranian Math.Soc.*, 42 (2016), 1583-1595.

[15] Hussain, N., Khaleghizadeh, S., Salimi, P., Akbar, F., New Fixed Point Results with PPF dependence in Banach Spaces Endowed with a Graph, *Abstr. Appl. Anal.*, 2013, Article ID 827205. https://doi.org/10.1155/2013/827205

[16] Kaneko, H., Generalized contractive multi-valued mappings and their fixed points, *Math. Japon.*, 33(1988), 57-64.

[17] Kannan, R., Some results on fixed points, *Bull. Calcutta Math. Soc.*, 60 (1968), 71-76.

[18] Marwan Amin Kutbi, Wutiphol Sintunavarat, On sufficient coniditons for the existence of Past-Present-Future dependent fixed point in Razumikhin class and application, *Abstr. Appl. Anal.*, 2014, Article ID 342684. http://dx.doi.org/10.1155/2014/342687

[19] Mizoguchi, N., Takahashi, W., Fixed point theorems for multi-valued mappings on complete metric spaces, *J. Math. Anal. Appl.*, 141 (1989), 177-188.

[20] Nadler, S.B., Multivalued contraction mappings, *Pacific J. Math.*, 30 (1969), 475-488.

[21] Rhoades, B. E., A comparison of various definitions of contractive mappings, *Trans. Amer. Math. Soc.*, 226 (1977), 257-290.

[22] Rhoades, B. E., Some theorems on weakly contractive mappings, *Nonlinear Anal.*, 47 (2001), 2683-2693. https://doi.org/10.1016/S0362-546X(01)00388-1

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 122–130 (2024)
DOI:10.31801/cfsuasmas.1275521
ISSN 1303-5991 E-ISSN 2618-6470

*Research Article; Received: April 2, 2023; Accepted: October 6, 2023*

# MAJORIZATION PROPERTY FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH GENERAL OPERATOR

Pranay GOSWAMI[1], Murli Manohar GOUR[2], Som Prakash GOYAL[3] and Lokesh Kumar YADAV[4]

[1]School of Liberal Studies, Dr B.R, Ambedkar University Delhi, INDIA
[2,4]Department of Mathematics, Vivekananda Global University Jaipur, INDIA
[3]Department of Mathematics, University of Rajasthan, Jaipur, INDIA

Abstract. In this study, we introduce two new classes $S_k[E, F; \mu; \gamma]$ and $T_k(\theta, \mu, \gamma)$ of analytic functions using the general integral operator. For these two classes, we study the majorization properties. Some applications of the results are discussed in the form of corollaries.

## 1. Introduction and Definitions

The Majorization for two analytic functions $u$ and $v$ is defined as follows (see [17])

$$u(\xi) \prec\prec v(\xi); \quad (\xi \in D),$$

if there is an analytic function $\psi(\xi)$, such that

$$|\psi(\xi)| \leq 1 \text{ and } u(\xi) = \psi(\xi)\, v(\xi); \quad (\xi \in D), \tag{1}$$

where $D = \{\xi \in \mathbb{C} : |\xi| < 1\}$ is an open unit disk.
The function $u$ is subordinate to $v$ and defined as $u(\xi) \prec v(\xi)$, if there is a *schwarz function* $w$, that is analytic in $D$ with $|w(\xi)| < 1$, $w(0) = 0$, $\xi \in D$ such that $u(\xi) = v(w(\xi))$, $\xi \in D$.

Thus, by combining subordination and majorization, we may define quasi-subordination as follows:

2020 *Mathematics Subject Classification.* 30C45.
*Keywords.* Univalent functions, quasi-subordination, subordination, majorization property.
[1] pranaygoswami83@gmail.com; 0000-0003-1205-1975
[2] murlimanohar.gaur@vgu.ac.in-Corresponding author; 0000-0003-2907-8226
[3] somprg@gmail.com; 0009-0007-9016-9529
[4] lokeshmanjeetyadav8@gmail.com; 0000-0003-0896-5723.

We say that the function $u$ is quasi-subordinate relative to $\phi(z)$ to the function $v$ and defined as (See [19])

$$u(\xi) \prec_q v(\xi); \quad (\xi \in D).$$

If there are two analytic functions $\psi(\xi)$ and $w(\xi)$ in $D$ such that $\frac{u(\xi)}{\psi(\xi)}$ is analytic and subordinate to $v(\xi)$ in $D$ and

$$|\psi(\xi)| \leq 1 \text{ and } w(0) = 0, \ |w(\xi)| \leq 1; \quad (\xi \in D),$$

satisfying

$$u(\xi) = \psi(\xi)\, v\big(w(\xi)\big); \quad (\xi \in D). \tag{2}$$

**Remark 1.** *(i) We have the conventional definition of subordination if we put $\psi(\xi) = 1$ in (2).*
*(ii) We have the conventional definition of majorization if we put $w(\xi) = \xi$ in (2).*

Let $\mathcal{A}$ be the class of all functions of the form

$$f(\xi) = \xi + \sum_{\mathfrak{K}=2}^{\infty} a_{\mathfrak{K}}\, \xi^{\mathfrak{K}}\,; \quad (\xi \in D), \tag{3}$$

which are analytic in open unit disk $D$, and consider $H_s : \mathcal{A} \to \mathcal{A}$ be an operator such that $\frac{\xi H'_{s+1}(f)(\xi)}{H_{s+1}(f)(\xi)}$ is analytic in $D$ with

$$\left. \frac{\xi H'_{s+1}(f)(\xi)}{H_{s+1}(f)(\xi)} \right|_{\xi=0} = \beta + k + \gamma.$$

and satisfies

$$\xi H'_{s+1}(f)(\xi) = k\, H_{s+1}(f)(\xi) + m\, H_s(f)(\xi), \quad \forall f \in \mathcal{A}. \tag{4}$$

for some $\gamma$, $m$, $k \in \mathbb{C}$, and $\beta$ is a real number with $\beta > 0$ (See [2]).

**Remark 2.** *(i) If we take $k = -n$, $m = n + 1$, $\beta = 1 - \eta$, and $\gamma = \eta + n$ for some integers $n > -1$ and $0 \leq \eta < 1$, then the operator $H_s$ reduced into the integral operator $\mathcal{I}_n$ introduced by Liu and Noor in [16].*
*(ii) If we take $k = -b$, $m = 1+b$, $\mu = 1-\alpha$ and $\gamma = \alpha+b$, for $b \in \mathbb{C}\setminus Z_0^-$, $0 \leq \alpha < 1$, then the operator $H_s$ reduced into the Srivastava-Attiya operator $J_{s,b}$, (see [12] and [20]).*

Now, using the operator $H_s$, we express the following classes of analytic functions.

**Definition 1.** *The function $f \in \mathcal{A}$ is stated to be in the class $S_k[E, F; \mu; \gamma]$ if and only if*

$$1 + \frac{1}{\mu}\left( \frac{\xi\left(H_s f(\xi)\right)'}{H_s f(\xi)} - k - \gamma \right) \prec \frac{1 + E\,\xi}{1 + F\,\xi}, \tag{5}$$

*with $k$, $\gamma \in \mathbb{C}$, $\mu \in \mathbb{C} \setminus \{0\}$ and $-1 \leq F < E \leq 1$.*

*If we take the value of $k$, $m$, $\beta$ and $\gamma$ as defined in Remark (1.2)(i), then this class becomes $S_n[E, F; \mu; \eta]$ which is defined by Liu and Noor in [16].*
*Again if we take the value of $k$, $m$, $\mu$ and $\gamma$ as defined in Remark (1.2)(ii), then this class becomes $H_{s,\,b,\,\alpha}(E,\,F)$ which is defined by Kutbi and Attiya in [12].*

**Definition 2.** *The function $f \in \mathcal{A}$ is stated to be in the class $T_k(\theta,\,\mu,\,\gamma)$ if and only if*

$$\frac{e^{i\theta}}{\mu+k+\gamma}\left(\frac{\xi\big(H_s f(\xi)\big)'}{H_s f(\xi)}\right) \prec e^{\xi} cos\theta + isin\theta; \quad (\xi \in D), \tag{6}$$

*where $k,\,\gamma \in \mathbb{C}$, $\mu \in \mathbb{C}\setminus\{0\}$ and $-\frac{\Pi}{2} < \theta < \frac{\Pi}{2}$.*
*If we take the value of $k$, $m$, $\beta$ and $\gamma$ as defined in Remark (1.2)(i), then this class become as $T_n[\theta; \mu; \eta]$.*
*If we take the value of $k$, $m$, $\mu$ and $\gamma$ as defined in Remark (1.2)(ii), then this class becomes $T_{b,\,\alpha}$.*

Numerous mathematicians have recently investigated various majorization problems for univalent and multivalent functions as well as meromorphic and multivalent comprising distinct operators and different groups, (see [1], [6], [7], [8], [9], [10], [21], [22]).

The majorization problems of the classes $S_k[E, F; \mu; \gamma]$ and $T_k(\theta,\,\mu,\,\gamma)$ are explored in this study as follows:

## 2. Main Results

**Theorem 1.** *Assume the function $f \in \mathcal{A}$ and that $g \in S_k[E, F; \mu; \gamma]$. If $H_s f(\xi)$ is majorized by $H_s g(\xi)$ in $D$, then*

$$|H_{s-1}\, f(\xi)| \le |H_{s-1}\, g(\xi)|, \quad for \quad |\xi| \le \epsilon_0, \tag{7}$$

*where the least positive root of following equation is $\epsilon_0$.*

$$\begin{aligned} &|\mu(E-F)+\gamma\,F|\epsilon^3 - (2|F|+|\gamma|)\epsilon^2 - \big[2+|\mu(E-F)+\gamma\,F|\big]\epsilon \\ &+|\gamma| = 0, \end{aligned} \tag{8}$$

*and $-1 \le F < E \le 1$, $k,\,\gamma,\,m \in \mathbb{C}$, $\mu \in \mathbb{C}\setminus\{0\}$.*

*Proof.* Since $g \in S_k[E, F; \mu; \gamma]$ then, from (5) and definition of majorization

$$1+\frac{1}{\mu}\left(\frac{\xi\big(H_s' g(\xi)\big)}{H_s g(\xi)} - k - \gamma\right) = \frac{1+E\,w(\xi)}{1+F\,w(\xi)},$$

with $w(0)=0$ and $|w(\xi)| \le |\xi| < 1, \;\; \forall \xi \in D$.
Now, from the above equality

$$\frac{\xi\big(H_s' g(\xi)\big)}{H_s g(\xi)} = \frac{(k+\gamma)+(\mu\,(E-F)+(k+\gamma)F)\,w(\xi)}{1+F\,w(\xi)}. \tag{9}$$

Using the relation (4), that is,

$$\xi\Big(H_s' g(\xi)\Big) = kH_S\, g(\xi) + m\, H_{S-1}\, g(\xi),$$

for $k,\, m \in \mathbb{C}$, we have from (9) as

$$\frac{H_{S-1}\, g(\xi)}{H_s\, g(\xi)} = \frac{\gamma + \big(\mu(E-F) + \gamma\, F\big) w(\xi)}{m\, \big(1 + F\, w(\xi)\big)},$$

which implies that

$$|H_s\, g(\xi)| \le \frac{|m|\, (1 + |F|\, |\xi|)\, |H_{S-1}\, g(\xi)|}{|\gamma| - \big|\mu(E-F) + \gamma\, F\big|\, |\xi|}. \tag{10}$$

As $H_s\, f(\xi)$ is majorized by $H_S\, g(\xi)$ in open unit disk $D$, then

$$H_s f(\xi) = \psi(\xi)\, H_s g(\xi). \tag{11}$$

Multiplying (11) by $\xi$ after differentiating with respect to $\xi$, we get

$$\xi\big(H_s'\, f(\xi)\big) = \xi\, \psi(\xi)\, \big(H_s'\, g(\xi)\big) + \xi\, \psi^{'}(\xi)\, H_s\, g(\xi),$$

on using relation (4), we have

$$m\;\, H_{s-1}\, f(\xi) = \xi\, \psi^{'}(\xi)\, H_s\, g(\xi) + m\, \psi(\xi)\, H_{s-1}\, g(\xi)$$

that implies

$$|m|\, |H_{s-1}\, f(\xi)| \le |\xi|\, |\psi^{'}(\xi)|\, |H_s\, g(\xi)| + |m|\, |\psi(\xi)|\, |H_{s-1}\, g(\xi)|. \tag{12}$$

As a consequence, considering that the $\psi$ (*Schwarz function*) meets the inequality, (see [18])

$$|\psi^{'}(\xi)| \le \frac{1 - |\psi(\xi)|^2}{1 - |\xi|^2}; \qquad (\xi \in D), \tag{13}$$

on using (10) and (13) in (12), we have

$$|H_{s-1}\, f(\xi)| \le \left[\frac{|\xi|(1 - |\psi(\xi)|^2)(1 + |F|\, |\xi|)}{(1 - |\xi|^2)\Big(|\gamma| - |\mu(E-F) + \gamma\, F|\, |\xi|\Big)} + |\psi(\xi)|\right] |H_{s-1}\, g(\xi)|. \tag{14}$$

Setting $|\xi| = \epsilon$, $|\psi(\xi)| = \kappa$, then inequality (14) leads to

$$|H_{s-1}\, f(\xi)| \le \frac{\zeta(\epsilon,\kappa)\, |H_{s-1}\, g(\xi)|}{(1 - \epsilon^2)\Big(|\gamma| - |\mu(E-F) + \gamma\, F|\;\, \epsilon\Big)}, \tag{15}$$

where

$$\zeta(\epsilon,\kappa) = \epsilon(1-\kappa^2)(1 + |F|\epsilon) + \kappa(1-\epsilon^2)\big[|\gamma| - |\mu(E-F) + \gamma\, F|\epsilon\big].$$

Then, from (15)

$$|H_{s-1}\, f(\xi)| \le \mathfrak{T}(\epsilon,\kappa)\, |H_{s-1}\, g(\xi)|, \tag{16}$$

where

$$\mathfrak{T}(\epsilon,\kappa)=\frac{\zeta(\epsilon,\kappa)}{(1-\epsilon^2)\Big(|\gamma|-|\mu(E-F)+\gamma\,F|\,\epsilon\Big)},\tag{17}$$

from relation (16), in an attempt to prove our result, we have to specify

$$\begin{aligned}\epsilon_0&=max\big\{\epsilon\in[0,1);\ \mathfrak{T}(\epsilon,\kappa)\le 1;\quad \forall\kappa\in[0,1]\big\}\\&=max\big\{\epsilon\in[0,1);\ G(\epsilon,\kappa)\ge 0;\quad \forall\kappa\in[0,1]\big\},\end{aligned}$$

where

$$\begin{aligned}G(\epsilon,\kappa)=&(1-\epsilon^2)(1-\kappa)\Big[|\gamma|-|\mu(E-F)+\gamma\,F|\,\epsilon\Big]\\&-\epsilon(1-\kappa^2)(1+|F|\epsilon).\end{aligned}$$

A simple calculation shows that the $G(\epsilon,\kappa)\ge 0$ inequality is equivalent to

$$\begin{aligned}u(\epsilon,\kappa)=&\Big[|\gamma|-|\mu(E-F)+\gamma\,F|\epsilon\Big]\ (1-\epsilon^2)\\&-\epsilon(1+\kappa)(1+|F|\epsilon)\ge 0,\end{aligned}$$

while the function $u(\epsilon,\kappa)$ has a least value at $\kappa=1$, i.e.
$min\{u(\epsilon,\kappa):\kappa\in[0,1]\}=u(\epsilon,1)=v(\epsilon)$,
where

$$\begin{aligned}v(\epsilon)=&|\mu(E-F)+\gamma\,F|\epsilon^3-(2|F|+|\gamma|)\epsilon^2\\&-\big[2+|\mu(E-F)+\gamma\,F|\big]\epsilon+|\gamma|=0,\end{aligned}$$

it follows that $v(\epsilon)\ge 0;\quad \forall\epsilon\in[0,\epsilon_0]$, where $\epsilon_0=\epsilon_0(\mu,\gamma,E,F)$ is the least positive root of equation (8), which proves the conclusion of (7). □

**Theorem 2.** *Assume the function $f\in\mathcal{A}$ and that $g\in T_k(\theta,\,\mu,\,\gamma)$. If $H_sf(\xi)$ is majorized by $H_sg(\xi)$ in $D$, therefore*

$$|H_{s-1}f(\xi)|\le|H_{s-1}g(\xi)|\qquad for\quad |\xi|\le\epsilon_1,\tag{18}$$

*where the least positive root of following equation is $\epsilon_1$.*

$$\epsilon^2\big(|\mu+k+\gamma|\,e^{\epsilon}-|k|-|\mu+\gamma||tan\theta|\big)+2\epsilon|sec\theta|-\big(|\mu+k+\gamma|\,e^{\epsilon}-|k|-|\mu+\gamma||tan\theta|\big)=0,\tag{19}$$

*and $\gamma,\,k\in\mathbb{C}$, $-\frac{\Pi}{2}<\theta<\frac{\Pi}{2}$, $\mu\in\mathbb{C}\setminus\{0\}$.*

*Proof.* Since, $g\in T_k(\theta,\,\mu,\,\gamma)$ then, from (1) and the subordination relation

$$\frac{e^{i\theta}}{\mu+k+\gamma}\left(\frac{\xi\big(H_s'\,g(\xi)\big)}{H_sg(\xi)}\right)=e^{w(\xi)}cos\theta\;+isin\theta,\tag{20}$$

with $w(0)=0$ and $|w(\xi)|\le 1\quad\forall\xi\in D$.
From (20), we have

$$\frac{\xi\,H_s'\,g(\xi)}{H_sg(\xi)}=(\mu+k+\gamma)\left(\frac{e^{w(\xi)}+itan\theta}{1+itan\theta}\right).\tag{21}$$

Now, using (4) in (21), for $\gamma$, $m$, $k \in \mathbb{C}$ and $\mu \in \mathbb{C} \setminus \{0\}$, we have the following.

$$\frac{H_{s-1}\, g(\xi)}{H_s g(\xi)} = \frac{(\mu + k + \gamma)\, e^{w(\xi)} - k + (\gamma + \mu) itan\theta}{m(1 + itan\theta)}$$

which implies that

$$|H_s g(\xi)| \leq \frac{|m|\, |sec\theta|}{\big(|\mu + k + \gamma|\, e^{|\xi|} - |k| - |\mu + \gamma||tan\theta|\big)} |H_{s-1}\, g(\xi)|. \tag{22}$$

Now, since $H_s\, f(\xi)$ is majorized by $H_s\, g(\xi)$ in $D$, we have

$$H_s f(\xi) = \psi(\xi) H_s g(\xi). \tag{23}$$

Multiplying (23) by $\xi$ after differentiating with respect to $\xi$, we get

$$\xi\big(H_s'\, f(\xi)\big) = \xi\, \psi(\xi)\, \big(H_s'\, g(\xi)\big) + \xi\, \psi^{'}(\xi)\, H_s\, g(\xi),$$

on using relation (4), we have

$$m\ H_{s-1}\, f(\xi) = \xi\, \psi^{'}(\xi)\, H_s\, g(\xi) + m\, \psi(\xi)\, H_{s-1}\, g(\xi)$$

that implies

$$|m|\, |H_{s-1}\, f(\xi)| \leq |\xi|\, |\psi^{'}(\xi)|\, |H_s\, g(\xi)| + |m|\, |\psi(\xi)|\, |H_{s-1}\, g(\xi)|. \tag{24}$$

As a consequence, considering that the $\psi$ (*Schwarz function*) meets the inequality, (see [18])

$$|\psi^{'}(\xi)| \leq \frac{1 - |\psi(\xi)|^2}{1 - |\xi|^2}; \qquad (\xi \in D), \tag{25}$$

using (22) and (25) in (24), we have

$$|H_{s-1}\, f(\xi)| \leq \left( \frac{|\xi|(1 - |\psi(\xi)|^2)|sec\theta|}{(1 - |\xi|^2)\big(|\mu + k + \gamma|\, e^{|\xi|} - |k| - |\mu + \gamma||tan\theta|\big)} + |\psi(\xi)| \right) |H_{s-1}\, g(\xi)|. \tag{26}$$

Setting $|\xi| = \epsilon$, $|\psi(\xi)| = \kappa \quad (0 \leq \kappa \leq 1)$, then inequality (26) leads to

$$|H_{s-1}\, f(\xi)| \leq \frac{\zeta_1(\epsilon, \kappa)}{(1 - \epsilon^2)\, \big(|\mu + k + \gamma|\, e^{\epsilon} - |k| - |\mu + \gamma||tan\theta|\big)} |H_{s-1}\, g(\xi)|, \tag{27}$$

where

$$\zeta_1(\epsilon, \kappa) = \epsilon(1 - \kappa^2)|sec\theta| + \kappa(1 - \epsilon^2)\big(|\mu + k + \gamma|\, e^{\epsilon} - |k| - |\mu + \gamma||tan\theta|\big).$$

Then, from (27)

$$|H_{s-1}\, f(\xi)| \leq \mathfrak{T}_1(\epsilon, \kappa)\ |H_{s-1}\, g(\xi)|, \tag{28}$$

where

$$\mathfrak{T}_1(\epsilon, \kappa) = \frac{\zeta_1(\epsilon, \kappa)}{(1 - \epsilon^2)\, \big(|\mu + k + \gamma|\, e^{\epsilon} - |k| - |\mu + \gamma||tan\theta|\big)}, \tag{29}$$

From relation (28), in order to prove our result, we have to specify

$$\epsilon_1 = max\big\{\epsilon \in [0, 1);\ \mathfrak{T}_1(\epsilon, \kappa) \leq 1 \quad \forall \kappa \in [0, 1]\big\}$$

$$= max\{\epsilon \in [0,1);\ G_1(\epsilon,\kappa) \geq 0 \quad \forall \kappa \in [0,1]\},$$

where

$$G_1(\epsilon,\kappa) = (1-\epsilon^2)(1-\kappa)\big(|\mu+k+\gamma|\,e^{\epsilon} - |k| - |\mu+\gamma||tan\theta|\big) - \epsilon(1-\kappa^2)|sec\theta|.$$

A quick calculation illustrates that the inequality $G_1(\epsilon,\kappa) \geq 0$ is equivalent to

$$u_1(\epsilon,\kappa) = (1-\epsilon^2)\big(|\mu+k+\gamma|\,e^{\epsilon} - |k| - |\mu+\gamma||tan\theta|\big) - \epsilon(1+\kappa)|sec\theta| \geq 0,$$

while the function $u_1(\epsilon,\kappa)$ takes its lowest value at $\kappa = 1$, that is,

$$min\{u_1(\epsilon,\kappa) : \kappa \in [0,1]\} = u_1(\epsilon,1) = v_1(\epsilon),$$

where

$$v_1(\epsilon) = (1-\epsilon^2)\big(|\mu+k+\gamma|\,e^{\epsilon} - |k| - |\mu+\gamma||tan\theta|\big) - 2\epsilon|sec\theta| = 0,$$

It follows that $v_2(\epsilon) \geq 0 \quad \forall \epsilon \in [0,\epsilon_1]$, where $\epsilon_1 = \epsilon_1(\theta,\,\gamma,\,\mu,\,k)$ is the least positive root of equation (19), which proves the conclusion of (18). □

## 3. Corollaries and Consequences

**Corollary 1.** *Assume the function $f \in \mathcal{A}$ and that $g \in S_n[E,F;\mu;\eta]$. If $\mathcal{I}_n\,f(\xi)$ is majorized by $\mathcal{I}_n\,g(\xi)$ in $D$, then*

$$|\mathcal{I}_{n-1}\,f(\xi)| \leq |\mathcal{I}_{n-1}\,g(\xi)| \quad for \quad |\xi| \leq \epsilon_2, \tag{30}$$

*where the least positive root of following equation is $\epsilon_2$.*

$$\big|\mu\,E+(n+\eta-\mu)F\big|\epsilon^3-\big(2|F|+|n+\eta|\big)\epsilon^2-\big(2+|\mu\,E+(\eta+n-\mu)F|\big)\epsilon+|\eta+n| = 0, \tag{31}$$

*and $-1 \leq F < E \leq 1$, $\mu \in \mathbb{C}\setminus\{0\}$, $n > -1$, $0 \leq \eta < 1$,.*

**Corollary 2.** *Assume the function $f \in \mathcal{A}$ and that $g \in T_n[\theta;\mu;\eta]$. If $\mathcal{I}_n\,f(\xi)$ is majorized by $\mathcal{I}_n\,g(\xi)$ in $D$, then*

$$|\mathcal{I}_{n-1}\,f(\xi)| \leq |\mathcal{I}_{n-1}\,g(\xi)| \quad for \quad |\xi| \leq \epsilon_3, \tag{32}$$

*where the least positive root of following equation is $\epsilon_3$.*

$$(|\mu+\eta|e^{\epsilon}-|n|-|\mu+\eta+n||tan\theta|)\epsilon^2-2|sec\theta|\epsilon-(-|n|-|\mu+\eta+n||tan\theta|+|\mu+\eta|e^{\epsilon}) = 0, \tag{33}$$

*and $n > -1$, $0 \leq \eta < 1$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$..*

**Corollary 3.** *Assume the function $f \in \mathcal{A}$ and that $g \in H_{s,\,b,\,\alpha}(E,\,F)$. If $J_{s,b}\,f(\xi)$ is majorized by $J_{s,b}\,g(\xi)$ in $D$, then*

$$|J_{s-1,b}\,f(\xi)| \leq |J_{s-1,b}\,g(\xi)| \quad for \quad |\xi| \leq \epsilon_4, \tag{34}$$

*where the least positive root of following equation is $\epsilon_4$.*

$$\begin{aligned}&\big|(1-\alpha)E+(2\alpha+b-1)\big|\epsilon^3 - \big(2|F|+|\alpha+b|\big)\epsilon^2 - \big(2+|(1-\alpha)E+(2\alpha+b-1)F|\big)\epsilon\\ &+|\alpha+b| = 0,\end{aligned} \tag{35}$$

*and $-1 \leq F < E \leq 1$, $b \in \mathbb{C}\setminus Z_0^-$, $0 \leq \alpha < 1$.*

**Corollary 4.** *Assume the function $f \in \mathcal{A}$ and that $g \in T_{b,\alpha}$. If $J_{s,b}\,f(\xi)$ is majorized by $J_{s,b}\,g(\xi)$ in $D$, then*

$$|J_{s-1,b}\,f(\xi)| \leq |J_{s-1,b}\,g(\xi)| \quad for \quad |\xi| \leq \epsilon_5, \tag{36}$$

*where the least positive root of following equation is $\epsilon_5$.*

$$\left(e^{\epsilon} - |1+b||tan\theta| - |b|\right)\epsilon^2 + 2|sec\theta|\,\epsilon - \left(e^{\epsilon} - |b| - |1+b||tan\theta|\right) = 0, \tag{37}$$

*and $b \in \mathbb{C} \setminus Z_0^-$, $0 \leq \alpha < 1$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.*

**Author Contribution Statements** Each author contributes equally to the preparation of the manuscript.

**Declaration of Competing Interests** The author declares that he has no competing interests.

**Acknowledgements** The authors thank the reviewer and the editor for their constructive comments to improve the article.

## References

[1] Altintas, O., Özkan, Ö., Srivastava, H. M., Majorization by starlike functions of complex order, *Complex Var. Elliptic Equ.*, (46)(3) (2001), 207–218. https://doi.org/10.1080/17476930108815409

[2] Attiya, A. A., Bulboacă, T., A general theorem associated with the Briot-Bouquet differential subordination, *J. Comput. Anal. Appl.*, 16(4) (2014), 722–730.

[3] Bulboacă, T., Classes of first-order differential subordinations, *Demonstr. Math.*, 35 (2) (2002), 287–292. https://doi.org/10.4236/am.2011.27114.

[4] Flett, T. M., The dual of an inequality of Hardy and Littlewood and some related inequalities, *J. Math. Anal. Appl.*, 38(3) (1972), 746–765. https://doi.org/10.1016/0022-247X(72)90081-9

[5] Fournier, R., Ruscheweyh, S., On two extremal problems related to univalent functions, *Rocky Mt. J. Math.*, 24 (1994), 529–538. https://doi.org/10.1216/rmjm/1181072416

[6] Goswami, P., Sharma B., Bulboacă, T., Majorization for certain classes of analytic functions using multiplier transformation, *Appl. Math. Lett.*, 23(5) (2010), 633–637. https://doi.org/10.1016/j.aml.2010.01.029

[7] Goswami, P., Aouf, M. K., Majorization properties for certain classes of analytic functions using the Salagean operator, *Appl. Math. Lett.*, 23(11) (2010), 1351–1354. https://doi.org/10.1016/j.aml.2010.06.030

[8] Gour, M. M., Goswami, P., Frasin B A., Althobaithi S., Majorization for certain classes of analytic functions defined by Fournier-Ruscheweyh integral operator, *Journal of Mathematics*, 2022 (2022), Article ID: 6580700. https://doi.org/10.1155/2022/6580700

[9] Goyal, S. P., Goswami, P., Majorization for certain classes of meromorphic functions defined by integral operator, *Ann. Univ. Mariae Curie-Skłodowska, Sect. A*, 66(2) (2012), 57-62. https://doi.org/10.2478/v10062-012-0013-1

[10] Goyal, S. P., Goswami, P., Majorization for certain classes of analytic functions defined by fractional derivatives, *Appl. Math. Lett.*, 22(12) (2009), 1855-1858. https://doi.org/10.1016/j.aml.2009.07.009

[11] Jung, I. B., Kim, Y. C., Srivastava, H. M., The Hardy space of analytic functions associated with certain one-parameter families of integral operators, *J. Math. Anal. Appl.*, 176(1) (1993), 138-147. https://doi.org/10.1006/jmaa.1993.1204

[12] Kutbi, M. A., Attiya, A. A., Differential subordination result with the Srivastava-Attiya integral operator, *J. Inequal. Appl.*, 2010 (2010), 1-10. https://doi.org/10.1155/2010/618523

[13] Li, J. L., Srivastava, H. M., Starlikeness of functions in the range of a class of integral operators, *Integral Transforms Spec. Funct.*, 15(2) (2004), 129-136. https://doi.org/10.1080/10652460310001600708

[14] Li, J. L., Srivastava, H. M., Some questions and conjectures in the theory of univalent functions, *Rocky Mt. J. Math.*, 28 (1998), 1035-1041. http://hdl.handle.net/1828/1649

[15] Li, J. L., Notes on Jung-Kim-Srivastava integral operator, *J. Math. Anal. Appl.*, 294(1) (2004), 96-103. https://doi.org/10.1016/j.jmaa.2004.01.040

[16] Liu, J. L., Noor, K. I., Some properties of Noor integral operator, *J. Nat. Geom.*, 21(1-2) (2002), 81–90. https://doi.org/10.1016/S0022-247X(03)00270-1

[17] MacGreogor, T. H., Majorization by univalent functions, *Duke Math. J.*, 34 (1967), 95-102. https://doi.org/10.1215/S0012-7094-67-03411-4

[18] Nehari Z., Conformal Mapping, MacGraw-Hill Book Company, New York, Toronto, London, 1955.

[19] Roberston M.S., Quasi-subordination and coefficient conjectures, *Bull. Am. Math. Soc.*, 76 (1970), 1-9.

[20] Srivastava, H. M., Attiya, A. A., An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination, *Integral Transforms Spec. Funct.*, 18(3) (2007), 207-216. https://doi.org/10.1080/10652460701208577

[21] Tang, H., Aouf, M. K., Deng, G. T., Majorization problems for certain subclasses of meromorphic multivalent functions associated with the Liu-Srivastava operator, *Filomat*, 29(4) (2015), 763-772. https://doi.org/10.2298/FIL1504763T

[22] Tang, H., Deng, G., Majorization problems for two subclasses of analytic functions connected with the Liu-Owa integral operator and exponential function, *J. Inequal. Appl.*, 277 (2018), 1-11. https://doi.org/10.1186/s13660-018-1865-x

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 131–146 (2024)
DOI:10.31801/cfsuasmas.1340330
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

*Research Article; Received: August 9, 2023; Accepted: September 8, 2023*

# CHOLESKY ALGORITHM OF A LUCAS TYPE MATRIX

Semih YILMAZ[1] and Betül ERDOĞAN[2]

[1]Department of Actuarial Sciences, Kırıkkale University, Kırıkkale, TÜRKİYE
[2]Department of Mathematics, Kırıkkale University, Kırıkkale, TÜRKİYE

Abstract. Many generalizations have been made for Fibonacci and Lucas number sequences and many properties have been found about these sequences. In the article [13], the authors obtained many features of these sequences with the Cholesky decomposition algorithm, using the 2 x 2 matrix belonging to a generalization of the Fibonacci sequence. In this study, it is shown that many different features can be found by using a 2 x 2 matrix belonging to the Lucas number sequence with the same method.

## 1. Introduction

Most identities for the Fibonacci number sequence $F_n$ and the Lucas number sequence $L_n$ are obtained by changing the recursion relations and/or initial conditions of the sequences and making sequence generalizations ( [2]- [5], [9]- [15], [17], [19]- [27]).

The Fibonacci numbers $F_n$ are defined by a quadratic recurrence relation:

$$F_{n+2} = F_{n+1} + F_n, \; n \geq 0 \tag{1}$$

with initial conditions $F_0 = 0$ and $F_1 = 1$, see [15]. Binet formula for the numbers $F_n$ is

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{2}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ . From here, it can be noted that and

$$\begin{aligned} \alpha\beta &= -1, \\ \alpha + \beta &= 1, \\ \alpha - \beta &= \sqrt{5}. \end{aligned}$$

2020 *Mathematics Subject Classification.* 11B39, 15B36, 15A23, 15A18, 15A16.
*Keywords.* Fibonacci numbers, Lucas numbers, Cholesky decomposition.

[1] syilmaz@kku.edu.tr; 0000-0003-3171-8763
[2] btl.erdgn46@gmail.com-Corresponding author; 0000-0002-6789-1528.

We also recall [15] that

$$F_n = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-j-1}{j},$$

where $\lfloor \cdot \rfloor$ denotes the greatest integer function. Using the Binet formula, we can write the following equation for negative indices:

$$F_{-n} := (-1)^{n+1} F_n.$$

Analogously, the numbers $L_n$ are defined by a quadratic recurrence relation:

$$L_{n+2} = L_{n+1} + L_n, \ n \geq 0$$

with initial conditions $L_0 = 2, L_1 = 1$, see [17]. Binet formula for the numbers $L_n$ is

$$L_n = \alpha^n + \beta^n. \tag{3}$$

Also the $F_n$ and $L_n$ numbers satisfied following identity

$$L_n = F_{n-1} + F_{n+1}. \tag{4}$$

Morever, from above equalities we have that

$$L_{-n} = (-1)^n L_n.$$

In [8] and [28], the *Cholesky decomposition* (*Cholesky factorization*) is defined as: If $A \in \mathbb{R}_n^n$ is symmetric positive definite matrix, then there exists a unique lower triangular matrix $G \in \mathbb{R}_n^n$ with positive diagonal entries such that $A = GG^T$. Here $G^T$ is the transpose matrix of the $G$. The calculation of $G$ and $G^T$ matrices is called the *Cholesky algorithm*.

Matrix method is also very useful method to obtain the properties of Fibonacci and Lucas sequences, see [6], [13], [16], [18], [22], [24], [26]. In particular, Horadam and Flipponi obtained some new features for Fibonacci and Lucas sequences by using the matrix $M_k$ which is created by the Cholesky matrix decomposition algorithm [13]. While doing this work they used the k-Fibonacci generalized sequence and the $M$ matrix belonging to this sequence.

We observed that the application of the same method for the $M$ matrix constituting the Lucas sequence creates different sequence properties. In this study, the matrix functions of the $xM_k^n$ matrix sequence, which was created by using the $M = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ matrix that produced the Lucas sequence, were examined and new results were obtained.

## 2. Main Results

From [16] let's consider the $2 \times 2$ symmetric matrix

$$M = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

which has eigenvalues $\alpha + 2$ and $\beta + 2$. For a positive integer $n$,

$$M^n = \begin{cases} 5^{\frac{n-1}{2}} \begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix}, & \text{if } n \text{ is odd,} \\ 5^{\frac{n}{2}} \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}, & \text{if } n \text{ is even,} \end{cases} \tag{5}$$

see [16]. Now let us define the matrix sequence $M_k$ in the following steps.

Let $M_1 := M$, therefore

$$M_1 = M = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

and Cholesky decomposition of $M_1$ is obtained as

$$M_1 = T_1 T_1^T = \begin{bmatrix} a_1 & 0 \\ c_1 & b_1 \end{bmatrix} \begin{bmatrix} a_1 & c_1 \\ 0 & b_1 \end{bmatrix},$$

where $T_1$ is a lower triangular matrix and $T_1^T$ is the transpose matrix of $T_1$. So $T_1^T$ is an upper triangular matrix. The $a_1$, $b_1$ and $c_1$ components of $T_1$ easily obtained with the matrix equation above. In fact, the system

$$\begin{aligned} a_1^2 &= 3, \\ a_1 c_1 &= 1, \\ b_1^2 + c_1^2 &= 2 \end{aligned}$$

can be written, whose solution is

$$\begin{aligned} a_1 &= \pm\sqrt{3} \\ c_1 &= \frac{1}{a_1} \\ b_1 &= \pm\sqrt{2 - c_1^2} \end{aligned}$$

Any of the four solutions obtained creates a Cholesky decomposition of the symmetric matrix $M_1$.

We also know that the product of a lower triangular matrix and an upper triangular matrix is generally not commutative, so it is known that the inverse product $T_1^T T_1$ gives a symmetric matrix $M_2$ similar to but different from $M_1$ [7]. If we consider the $b_1 = \sqrt{\frac{5}{3}}$ solution, we get

$$M_2 = \frac{1}{3} \begin{bmatrix} 10 & \sqrt{5} \\ \sqrt{5} & 5 \end{bmatrix},$$

when $b_1 = -\sqrt{\frac{5}{3}}$ the off-diagonal components of $M_2$ are negative.

In contrast, $M_2$ can be decomposed similarly so that

$$M_2 = T_2 T_2^{T} = \begin{bmatrix} a_2 & 0 \\ c_2 & b_2 \end{bmatrix} \begin{bmatrix} a_2 & c_2 \\ 0 & b_2 \end{bmatrix},$$

where

$$\begin{aligned} a_2 &= \pm\sqrt{\frac{10}{3}}, \\ c_2 &= \frac{\sqrt{5}}{a_2} = \pm\frac{\sqrt{6}}{6}, \\ b_2 &= \pm\frac{\sqrt{6}}{2}. \end{aligned}$$

The inverse product $T_2^{T} T_2$ gave rise to a matrix $M_3$ with the sign of the off-diagonal entries based on $b_2$.

If we repeat such a procedure indefinitely, we get the sequence $(M_k)_1^{\infty}$ of the $2\times 2$ symmetric matrices. Henceforth $M_k$ be called the *$k$-order Lucas-type Cholesky algorithm matrix*.

Due to the unclear sign of Cholesky decomposition, the above matrix sequence is not the only possible result of applications of the Cholesky algorithm to $M$. However, other possible outcomes may differ only in the sign of the off-diagonal components of the above matrix sequence, in any term of the sequence except the first term. However, from now on we will only consider the positive definite $(M_k)$ matrix sequence.

Since the matrices $M_k$ are similar, they have the same eigenvalues. $M_k$ tends to a diagonal matrix containing these eigenvalues as $k$ tends to infinity.

The following Lemma can be easily obtained from [15] and [27]

**Lemma 1.** *Let $k$ be a positive integer, then*

*i) If $k$ is odd, then $L_{k-1}L_{k+1} = 5F_k^2 + 1$.*

*ii) If $k$ is even, then $5F_{k+1} = L^2_{\frac{k}{2}+1} + L^2_{\frac{k}{2}}$.*

*iii) If $k$ is even, then $L_k^2 + 1 = F_{k+1}\left(L^2_{\frac{k}{2}-1} + L^2_{\frac{k}{2}}\right)$.*

**Theorem 1.** *Let $k$ be positive integer, then*

$$M_k = \begin{cases} \frac{1}{F_k}\begin{bmatrix} L_{k+1} & 1 \\ 1 & L_{k-1} \end{bmatrix}, & \text{if } k \text{ is odd,} \\ \frac{1}{L_k}\begin{bmatrix} L^2_{\frac{k}{2}+1} + L^2_{\frac{k}{2}} & \sqrt{5} \\ \sqrt{5} & L^2_{\frac{k}{2}} + L^2_{\frac{k}{2}-1} \end{bmatrix}, & \text{if } k \text{ is even.} \end{cases}$$

*Proof.* From the $M_1$ and $M_2$ matrices we found earlier, it can be seen that the equality is achieved in the case of $k = 1$ and $k = 2$.

If $k$ is odd:

$$M_k = \frac{1}{F_k}\begin{bmatrix} L_{k+1} & 1 \\ 1 & L_{k-1} \end{bmatrix} = T_k T_k^T$$

hence, using Lemma 1, we obtain

$$T_k = \begin{bmatrix} \frac{\sqrt{L_{k+1}}}{\sqrt{F_k}} & 0 \\ \frac{1}{\sqrt{F_k L_{k+1}}} & \sqrt{5}\frac{\sqrt{F_k}}{\sqrt{L_{k+1}}} \end{bmatrix}.$$

Therefore

$$\begin{aligned} M_{k+1} &= T_k^T T_k = \begin{bmatrix} \frac{\sqrt{L_{k+1}}}{\sqrt{F_k}} & \frac{1}{\sqrt{F_k L_{k+1}}} \\ 0 & \sqrt{5}\frac{\sqrt{F_k}}{\sqrt{L_{k+1}}} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{L_{k+1}}}{\sqrt{F_k}} & 0 \\ \frac{1}{\sqrt{F_k L_{k+1}}} & \sqrt{5}\frac{\sqrt{F_k}}{\sqrt{L_{k+1}}} \end{bmatrix} \\ &= \frac{1}{L_{k+1}}\begin{bmatrix} \frac{L_{k+1}^2+1}{F_k} & \sqrt{5} \\ \sqrt{5} & 5F_k \end{bmatrix}. \end{aligned}$$

Here, using the Lemma 1

$$M_{k+1} = \frac{1}{L_{k+1}}\begin{bmatrix} L^2_{\frac{k+1}{2}+1} + L^2_{\frac{k+1}{2}} & \sqrt{5} \\ \sqrt{5} & L^2_{\frac{k+1}{2}} + L^2_{\frac{k-1}{2}} \end{bmatrix}$$

is obtained.

If $k$ is even:

$$M_k = \frac{1}{L_k}\begin{bmatrix} L^2_{\frac{k}{2}+1} + L^2_{\frac{k}{2}} & \sqrt{5} \\ \sqrt{5} & L^2_{\frac{k}{2}} + L^2_{\frac{k}{2}-1} \end{bmatrix}$$

hence, using Lemma 1, we obtain

$$T_k = \begin{bmatrix} \sqrt{5}\frac{\sqrt{F_{k+1}}}{\sqrt{L_k}} & 0 \\ \frac{1}{\sqrt{L_k}\sqrt{F_{k+1}}} & \frac{\sqrt{L_k}}{\sqrt{F_{k+1}}} \end{bmatrix}.$$

Therefore

$$\begin{aligned} M_{k+1} &= T_k^T T_k = \begin{bmatrix} \sqrt{5}\frac{\sqrt{F_{k+1}}}{\sqrt{L_k}} & \frac{1}{\sqrt{L_k}\sqrt{F_{k+1}}} \\ 0 & \frac{\sqrt{L_k}}{\sqrt{F_{k+1}}} \end{bmatrix} \begin{bmatrix} \sqrt{5}\frac{\sqrt{F_{k+1}}}{\sqrt{L_k}} & 0 \\ \frac{1}{\sqrt{L_k}\sqrt{F_{k+1}}} & \frac{\sqrt{L_k}}{\sqrt{F_{k+1}}} \end{bmatrix} \\ &= \frac{1}{F_{k+1}}\begin{bmatrix} \frac{5F_{k+1}^2+1}{L_k} & 1 \\ 1 & L_k \end{bmatrix} \\ &= \frac{1}{F_{k+1}}\begin{bmatrix} L_{k+2} & 1 \\ 1 & L_k \end{bmatrix}. \end{aligned}$$

Here, the equation $L_{k+2}\,L_k = 5F_{k+1}^2 + 1$ obtained from $L_{2m}\,L_{2n} = 5F_{m+n}^2 + L_{m-n}^2$ in [15, p.109] is used. $\square$

**Theorem 2.** *If we apply the Cholesky algorithm to $M^n$, we obtain the followings:*

$$(M^n)_k = \begin{cases} \frac{5^{\frac{n-1}{2}}}{F_k}\begin{bmatrix} L_{n+k} & L_n \\ L_n & L_{n-k} \end{bmatrix}, & \text{if } k \text{ is odd and } n \text{ is odd,} \\ \frac{5^{\frac{n}{2}}}{F_k}\begin{bmatrix} F_{n+k} & F_n \\ F_n & F_{n-k} \end{bmatrix}, & \text{if } k \text{ is odd and } n \text{ is even,} \\ \frac{5^{\frac{n}{2}}}{L_k}\begin{bmatrix} L_{n+k} & F_n\sqrt{5} \\ F_n\sqrt{5} & L_{n-k} \end{bmatrix}, & \text{if } k \text{ is even and } n \text{ is even,} \\ \frac{5^{\frac{n-1}{2}}}{L_k}\begin{bmatrix} 5F_{n+k} & L_n\sqrt{5} \\ L_n\sqrt{5} & 5F_{n-k} \end{bmatrix}, & \text{if } k \text{ is even and } n \text{ is odd.} \end{cases}$$

*We can also see that the equation $(M_k)^n = (M^n)_k$ and for simplicity we will use the notation $M_k^n := (M_k)^n = (M^n)_k$.*

*Proof.* It can be easily seen by induction using Theorem 1 and equation (5) . □

Here, suppose the above power equation is true for some value of $n$, say $N$. In this case, $(M_k)^N = (M^N)_k$. From this, it can be easily seen that $(M_k)^{N+1} = M_k(M_k)^N = M_k(M^N)_k = (M^{N+1})_k$ so if the above power equation is true for $N$, it is also true for $N+1$.

2.1. **Functions of the Matrix $xM_k^n$.** From the theory of functions of matrices [7], if the function $f$ is a function defined on the spectrum of a $2 \times 2$ matrix $A = [a_{ij}]$ with distinct eigenvalues $\lambda_1$ and $\lambda_2$, then

$$f(A) = X = [x_{ij}] = c_0 I + c_1 A, \tag{6}$$

where $I$ is the $2 \times 2$ identity matrix and the coefficients $c_0$ and $c_1$ are given by the solution of the system

$$\begin{aligned} c_0 + c_1\lambda_1 &= f(\lambda_1), \\ c_0 + c_1\lambda_2 &= f(\lambda_2). \end{aligned}$$

Therefore

$$\begin{aligned} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} &= c_0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_1\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \\ &= \begin{bmatrix} c_0 + c_1 a_{11} & c_1 a_{12} \\ c_1 a_{21} & c_0 + c_1 a_{22} \end{bmatrix}. \end{aligned}$$

From the last equation, we get

$$\begin{aligned} x_{11} &= c_0 + c_1 a_{11}, \\ x_{12} &= c_1 a_{12}, \end{aligned}$$

$$\begin{aligned} x_{21} &= c_1 a_{21}, \\ x_{22} &= c_0 + c_1 a_{22}. \end{aligned}$$

In equation (6), let us write $\lambda_1$ and $\lambda_2$ instead of $A$ and find $c_0$ and $c_1$ values

$$\begin{aligned} c_0 &= \frac{(\beta+2)^n f(x(\alpha+2)^n) - (\alpha+2)^n f(x(\beta+2)^n)}{(\beta+2)^n - (\alpha+2)^n}, \\ c_1 &= \frac{f(x(\beta+2)^n) - f(x(\alpha+2)^n)}{(\beta+2)^n - (\alpha+2)^n} \end{aligned}$$

and then

$$\begin{aligned} x_{11} &= [(a_{11}-\lambda_1)f(\lambda_2) - (a_{11}-\lambda_2)f(\lambda_1)]/(\lambda_2-\lambda_1), \\ x_{12} &= a_{12}[f(\lambda_2) - f(\lambda_1)]/(\lambda_2-\lambda_1), \\ x_{21} &= a_{21}[f(\lambda_2) - f(\lambda_1)]/(\lambda_2-\lambda_1), \\ x_{22} &= [(a_{22}-\lambda_1)f(\lambda_2) - (a_{22}-\lambda_2)f(\lambda_1)]/(\lambda_2-\lambda_1). \end{aligned}$$

**Lemma 2.** *Let $k$ and $n$ be arbitrary positive integers. For $x$ an arbitrary quantity, let us consider the matrix $xM_k^n$ having eigenvalues*

$$\begin{aligned} \lambda_1 &= x(\alpha+2)^n, \\ \lambda_2 &= x(\beta+2)^n. \end{aligned}$$

*Proof.* It is easily seen by induction. □

To express the $y_{ij}$ components of $Y = [y_{ij}] = f(xM_k^n)$ in separate formulas, we can give the following theorem with

$$\lambda := \frac{(\beta+2)^n f(x(\alpha+2)^n) - (\alpha+2)^n f(x(\beta+2)^n)}{(\beta+2)^n - (\alpha+2)^n}$$

and

$$\phi := \frac{f(x(\beta+2)^n) - f(x(\alpha+2)^n)}{(\beta+2)^n - (\alpha+2)^n}.$$

**Theorem 3.** *Let $k$ and $n$ be arbitrary positive integers.*

*i) If $n$ is even and $k$ is odd, then*

$$Y = \frac{5^{\frac{n}{2}}}{F_k} \begin{bmatrix} \lambda F_k + \phi F_{n+k} & \phi F_n \\ \phi F_n & \lambda F_k + \phi F_{n-k} \end{bmatrix}.$$

*ii) If $n$ is odd and $k$ is odd, then*

$$Y = \frac{5^{\frac{n-1}{2}}}{F_k} \begin{bmatrix} \lambda F_k + \phi L_{n+k} & \phi L_n \\ \phi L_n & \lambda F_k + \phi L_{n-k} \end{bmatrix}.$$

*iii) If $n$ is odd and $k$ is even, then*

$$Y = \frac{5^{\frac{n-1}{2}}}{L_k} \begin{bmatrix} \lambda L_k + 5\phi F_{n+k} & \sqrt{5}\phi L_n \\ \sqrt{5}\phi L_n & \lambda L_k + 5\phi F_{n-k} \end{bmatrix}.$$

*iv) If $n$ is even and $k$ is even, then*

$$Y = \frac{5^{\frac{n}{2}}}{L_k} \begin{bmatrix} \lambda L_k + \phi L_{n+k} & \sqrt{5}\phi F_n \\ \sqrt{5}\phi F_n & \lambda L_k + \phi L_{n-k} \end{bmatrix}.$$

*Proof.* Taking $xM_k^n$ as matrix $A$ in equation (6) and applying the above steps using Lemma 2 the desired result is obtained. □

**Theorem 4.** *If $f$ is the matrix inversion function then*

$$(xM_k^n)^{-1} = \begin{cases} \frac{5^{\frac{-n-1}{2}}}{xF_k} \begin{bmatrix} L_{n-k} & -L_n \\ -L_n & L_{n+k} \end{bmatrix}, & \text{if } k \text{ is odd and } n \text{ is odd,} \\ \frac{5^{\frac{-n}{2}}}{xF_k} \begin{bmatrix} F_{n-k} & -F_n \\ -F_n & F_{n+k} \end{bmatrix}, & \text{if } k \text{ is odd and } n \text{ is even,} \\ \frac{5^{\frac{-n}{2}}}{xL_k} \begin{bmatrix} L_{n-k} & -F_n\sqrt{5} \\ -F_n\sqrt{5} & L_{n+k} \end{bmatrix}, & \text{if } k \text{ is even and } n \text{ is even,} \\ \frac{5^{\frac{-n-1}{2}}}{xL_k} \begin{bmatrix} 5F_{n-k} & -L_n\sqrt{5} \\ -L_n\sqrt{5} & 5F_{n+k} \end{bmatrix}, & \text{if } k \text{ is even and } n \text{ is odd.} \end{cases}$$

*Proof.* It can be easily seen using the identity $(xM_k^n)^{-1} = \frac{1}{x}M_k^{-n}$, $(x \neq 0)$. □

## 3. Relations with Some Finite Series

In this section, sums of some finite series containing $F_n$ and $L_n$ are found using some properties of the Lucas-type Cholesky algorithm matrix $M_k$.

**Lemma 3.** *If $k$ is a positive integer, then*

$$M_k^2 = 5M_k - 5I, \tag{7}$$

*and*

$$M_k^{-1} = I - \frac{1}{5}M_k. \tag{8}$$

*Proof.* Using equation (1), it easily be obtained from equations Theorem 1 and Theorem 2. □

**Lemma 4.** *If $x$ is an arbitrary quantity with the constraints $x \neq \dfrac{1}{\alpha^n}$ and $x \neq \dfrac{1}{\beta^n}$ then*

$$(xM_k^n - I)^{-1} = \begin{cases} \dfrac{\left(5^{\frac{n+1}{2}} F_n x - 1\right) I - xM_k^n}{5^n x^2 - 5^{\frac{n+1}{2}} F_n x + 1}, & \text{if } n \text{ is odd,} \\ \dfrac{\left(5^{\frac{n}{2}} L_n x - 1\right) I - xM_k^n}{5^n x^2 - 5^{\frac{n}{2}} L_n x + 1}, & \text{if } n \text{ is even.} \end{cases}$$

*Proof.* It can be easily seen using equations (2), (3), (4) and Lemma 3 and the following equations

$$\begin{aligned} L_{k+n} - 5F_nF_k &= -L_{n-k} && \text{if } k \text{ is odd and } n \text{ is odd [15, p. 111, 83.]}, \\ F_{k+n} - F_kL_n &= -F_{n-k} && \text{if } k \text{ is odd and } n \text{ is even [15, p. 118, 58.]}, \\ L_{k+n} - L_nL_k &= -L_{n-k} && \text{if } k \text{ is even and } n \text{ is even [15, p. 111, 83.]}, \\ F_{k+n} - L_kF_n &= -F_{n-k} && \text{if } k \text{ is even and } n \text{ is odd [15, p. 118, 58.]}. \end{aligned}$$

□

**Lemma 5.** *For positive numbers $k$ and $n$ the following equality holds*

$$M_k^n = \sum_{j=0}^{n} 5^{-j} \binom{n}{j} M_k^{2j}.$$

*Proof.* From equation (7) we can write $(M_k^2 + 5I)^n = (5M_k)^n$, from which the proof can be obtained by using the binomial expansion. □

**Theorem 5.** *i) Let $n$ be a nonnegative even integer and $k$ be an arbitrary positive integer. Then we have*

$$F_{n\mp k} = 5^{-\frac{n}{2}} \sum_{j=0}^{n} \binom{n}{j} F_{2j\mp k},$$

$$L_{n\mp k} = 5^{-\frac{n}{2}} \sum_{j=0}^{n} \binom{n}{j} L_{2j\mp k}.$$

*ii) Let $n$ be a nonnegative odd integer and $k$ be an arbitrary positive integer. Then we have*

$$F_{n\mp k} = 5^{\frac{-n-1}{2}} \sum_{j=0}^{n} \binom{n}{j} F_{2j\mp k},$$

$$L_{n\mp k} = 5^{\frac{-n+1}{2}} \sum_{j=0}^{n} \binom{n}{j} L_{2j\mp k}.$$

*Proof.* If $n$ is even positive integer and $k$ is odd positive integer, then from Theorem 2 and Lemma 5,

$$M_k^n = \frac{5^{\frac{n}{2}}}{F_k} \begin{bmatrix} F_{n+k} & F_n \\ F_n & F_{n-k} \end{bmatrix} = \sum_{j=0}^{n} 5^{-j} \binom{n}{j} \frac{5^j}{F_k} \begin{bmatrix} F_{2j+k} & F_{2j} \\ F_{2j} & F_{2j-k} \end{bmatrix},$$

hence,

$$5^{\frac{n}{2}} \begin{bmatrix} F_{n+k} & F_n \\ F_n & F_{n-k} \end{bmatrix} = \begin{bmatrix} \sum\limits_{j=0}^{n} \binom{n}{j} F_{2j+k} & \sum\limits_{j=0}^{n} \binom{n}{j} F_{2j} \\ \sum\limits_{j=0}^{n} \binom{n}{j} F_{2j} & \sum\limits_{j=0}^{n} \binom{n}{j} F_{2j-k} \end{bmatrix},$$

therefore,

$$F_{n \mp k} = 5^{-\frac{n}{2}} \sum_{j=0}^{n} \binom{n}{j} F_{2j \mp k}.$$

Other equations are obtained in a similar way. □

**Lemma 6.** *For positive integers $k, n, s$ the following equality holds*

$$M_k^{2n+s} = 5^n \sum_{j=0}^{n} (-1)^{n+j} \binom{n}{j} M_k^{s+j}.$$

*Proof.* From equation (7), we can write

$$(5M_k - 5I)^n M_k^s = M_k^{2n+s} \tag{9}$$

from which the proof can be obtained by using the binomial expansion. □

**Theorem 6.** *For positive integers $n$ and $s$ the following equality holds*

$$L_{2n+s} = \sum_{j=0}^{n} \binom{n}{j} \begin{cases} (-1)^{n+1} 5^{\frac{j+1}{2}} F_{s+j}, & \text{if } j \text{ is odd,} \\ (-1)^n 5^{\frac{j}{2}} L_{s+j}, & \text{if } j \text{ is even,} \end{cases},$$

$$F_{2n+s} = \sum_{j=0}^{n} \binom{n}{j} \begin{cases} (-1)^{n+1} 5^{\frac{j-1}{2}} L_{s+j}, & \text{if } j \text{ is odd,} \\ (-1)^n 5^{\frac{j}{2}} F_{s+j}, & \text{if } j \text{ is even.} \end{cases}.$$

*Proof.* It can be easily seen with Lemma 6 and Theorem 2. □

**Theorem 7.** *For positive integers $k$ and $n$ the followings holds*

$$L_{n\pm k}=\begin{cases}\sum_{j=0}^{n}\binom{n}{j}\begin{cases}-5^{\frac{n-j}{2}}L_{j\mp k}, & \text{if } j \text{ is odd,}\\ 5^{\frac{n-j+1}{2}}F_{j\mp k}, & \text{if } j \text{ is even,}\end{cases} & \text{if } k \text{ is odd and } n \text{ is odd,}\\ \sum_{j=0}^{n}\binom{n}{j}\begin{cases}5^{\frac{n-j}{2}}L_{j\mp k}, & \text{if } j \text{ is odd,}\\ -5^{\frac{n-j+1}{2}}F_{j\mp k}, & \text{if } j \text{ is even,}\end{cases} & \text{if } k \text{ is even and } n \text{ is odd,}\\ \sum_{j=0}^{n}\binom{n}{j}\begin{cases}5^{\frac{n-j+1}{2}}F_{j\mp k}, & \text{if } j \text{ is odd,}\\ -5^{\frac{n-j}{2}}L_{j\mp k}, & \text{if } j \text{ is even,}\end{cases} & \text{if } k \text{ is odd and } n \text{ is even,}\\ \sum_{j=0}^{n}\binom{n}{j}\begin{cases}-5^{\frac{n-j+1}{2}}F_{j\mp k}, & \text{if } j \text{ is odd,}\\ 5^{\frac{n-j}{2}}L_{j\mp k}, & \text{if } j \text{ is even,}\end{cases} & \text{if } k \text{ is even and } n \text{ is even,}\end{cases}$$

$$F_{n\pm k}=\begin{cases}\sum_{j=0}^{n}\binom{n}{j}\begin{cases}5^{\frac{n-j}{2}}F_{j\mp k}, & \text{if } j \text{ is odd,}\\ -5^{\frac{n-j-1}{2}}L_{j\mp k}, & \text{if } j \text{ is even,}\end{cases} & \text{if } k \text{ is odd and } n \text{ is odd,}\\ \sum_{j=0}^{n}\binom{n}{j}\begin{cases}-5^{\frac{n-j}{2}}F_{j\mp k}, & \text{if } j \text{ is odd,}\\ 5^{\frac{n-j-1}{2}}L_{j\mp k}, & \text{if } j \text{ is even,}\end{cases} & \text{if } k \text{ is even and } n \text{ is odd,}\\ \sum_{j=0}^{n}\binom{n}{j}\begin{cases}-5^{\frac{n-j-1}{2}}L_{j\mp k}, & \text{if } j \text{ is odd,}\\ 5^{\frac{n-j}{2}}F_{j\mp k}, & \text{if } j \text{ is even,}\end{cases} & \text{if } k \text{ is odd and } n \text{ is even,}\\ \sum_{j=0}^{n}\binom{n}{j}\begin{cases}5^{\frac{n-j-1}{2}}L_{j\mp k}, & \text{if } j \text{ is odd,}\\ -5^{\frac{n-j}{2}}F_{j\mp k}, & \text{if } j \text{ is even,}\end{cases} & \text{if } k \text{ is even and } n \text{ is even.}\end{cases}$$

*Proof.* Using equation (8) we can write $(I-\frac{1}{5}M_k)^n=(M_k^n)^{-1}$. Here,

$$(I-\frac{1}{5}M_k)^n=\sum_{j=0}^{n}\binom{n}{j}(-1)^j\,\frac{1}{5^j}M_k^j=(M_k^n)^{-1}.$$

Let $n,k$ be odd positive integers.

$$\sum_{\substack{j=0\\ j\text{ odd}}}^{n}\binom{n}{j}(-1)^j\,\frac{1}{5^j}\frac{5^{\frac{j-1}{2}}}{F_k}\begin{bmatrix}L_{j+k} & L_j\\ L_j & L_{j-k}\end{bmatrix}+\sum_{\substack{j=0\\ j\text{ even}}}^{n}\binom{n}{j}(-1)^j\,\frac{1}{5^j}\frac{5^{\frac{j}{2}}}{F_k}\begin{bmatrix}F_{j+k} & F_j\\ F_j & F_{j-k}\end{bmatrix}$$

$$=\frac{5^{\frac{-(n+1)}{2}}}{F_k}\begin{bmatrix}L_{n-k} & -L_n\\ -L_n & L_{n+k}\end{bmatrix},$$

hence,

$$\begin{bmatrix}L_{n-k} & -L_n\\ -L_n & L_{n+k}\end{bmatrix}=\sum_{j=0}^{n}\binom{n}{j}(-1)^j\begin{cases}5^{\frac{n-j}{2}}\begin{bmatrix}L_{j+k} & L_j\\ L_j & L_{j-k}\end{bmatrix}, & \text{if } j \text{ is odd,}\\ 5^{\frac{n-j+1}{2}}\begin{bmatrix}F_{j+k} & F_j\\ F_j & F_{j-k}\end{bmatrix}, & \text{if } j \text{ is even,}\end{cases},$$

from which the following result is obtained

$$\begin{aligned} L_{n-k} &= \sum_{j=0}^{n}\binom{n}{j}\begin{cases} -5^{\frac{n-j}{2}}L_{j+k}, & \text{if } j \text{ is odd,}\\ 5^{\frac{n-j+1}{2}}F_{j+k}, & \text{if } j \text{ is even,}\end{cases}\\ L_{n+k} &= \sum_{j=0}^{n}\binom{n}{j}\begin{cases} -5^{\frac{n-j}{2}}L_{j-k}, & \text{if } j \text{ is odd,}\\ 5^{\frac{n-j+1}{2}}F_{j-k}, & \text{if } j \text{ is even.}\end{cases}\end{aligned}$$

Other equations are obtained in a similar way. □

**Theorem 8.** *Let $h$, $k$ and $n$ be positive integers and*

$$\theta(n) := 5^{\frac{n+1}{2}}F_n x - 1, \quad \vartheta(n) := 5^{\frac{n}{2}}L_n x - 1.$$

*i) If $n$ is odd, then*

$$\begin{aligned}\sum_{j=0}^{h} x^j M_k^{nj} &= \frac{\theta(n)I - xM_k^n}{5^n x^2 - \theta(n)}\left(x^{h+1}M_k^{n(h+1)} - I\right)\\ &= -\frac{x^{h+2}M_k^{n(h+2)} - xM_k^n - \theta(n)\left(xM_k^n\right)^{h+1} + \theta(n)I}{5^n x^2 - \theta(n)}.\end{aligned}$$

*ii) If $n$ is even, then*

$$\begin{aligned}\sum_{j=0}^{h} x^j M_k^{nj} &= \frac{\vartheta(n)I - xM_k^n}{5^n x^2 - \vartheta(n)}\left(x^{h+1}M_k^{n(h+1)} - I\right)\\ &= -\frac{x^{h+2}M_k^{n(h+2)} - xM_k^n - \vartheta(n)\left(xM_k^n\right)^{h+1} + \vartheta(n)I}{5^n x^2 - \vartheta(n)}.\end{aligned}$$

*Proof.*

$$(xA^n - I)\sum_{j=0}^{h} x^j A^{nj} = x^{h+1}A^{n(h+1)} - I, \tag{10}$$

is valid for every square matrix A. Using equation (10) and Lemma 4, i) and ii) can easily be shown. □

**Theorem 9.** *Let $n$ and $s$ be arbitrary integers where $x \neq \dfrac{1}{\alpha^n}$ and $x \neq \dfrac{1}{\beta^n}$, the following equations are satisfied:*

*i)*

$$\sum_{j=0}^{h} x^j F_{nj+s} = \frac{(-1)^{n-1}x^{h+2}F_{nh+s} + x^{h+1}F_{n(h+1)+s} - (-1)^s xF_{n-s} - F_s}{(-1)^{n-1}x^2 + L_n x - 1},$$

*ii)*

$$\sum_{j=0}^{h} x^j L_{nj+s} = \frac{(-1)^{n-1}x^{h+2}L_{nh+s} + x^{h+1}L_{n(h+1)+s} + (-1)^s x L_{n-s} - L_s}{(-1)^{n-1}x^2 + L_n x - 1}.$$

*Proof.* The equation i) can be obtained by using the Lemma 4 and Theorem 2. By substitute $s \pm 1$ for $s$ in equation i) we obtained ii). □

## 4. Relationships with Some Infinite Series

In this section, we consider a method using functions of the matrix $xM_k^n$ to find sums of infinite series containing $F_n$ and $L_n$. Under certain restrictions, some sum formulas can be computed using the results given in Section 3.

**Theorem 10.** *If*

$$-\frac{1}{\alpha^n} < x < \frac{1}{\alpha^n}$$

*then,*

$$\begin{aligned} \sum_{j=0}^{\infty} x^j F_{nj+s} &= \frac{(-1)^{s-1}xF_{n-s} - F_s}{(-1)^{n-1}x^2 + L_n x - 1}, \\ \sum_{j=0}^{\infty} x^j L_{nj+s} &= \frac{(-1)^{s}xL_{n-s} - L_s}{(-1)^{n-1}x^2 + L_n x - 1}. \end{aligned}$$

*Proof.* If the limits of i) and ii) in Theorem 9 are taken on both sides as $h$ goes to infinity, we get the equations. □

### 4.1. Calculation of Certain Functions of $xM_k^n$.

In [7] and [13] we see that the authors obtain some identity with the matrix functions. Similarly, we can examine some series of Fibonacci and Lucas sequences using the $xM_k^n$ matrices.

**Theorem 11.** *For positive numbers $k$, $n$ the following equality holds*

$$Y = \exp(xM_k^n) = \sum_{j=0}^{\infty} \frac{x^j M_k^{jn}}{j!}.$$

*Proof.* If we take $A = xM_k^n$ in the equation given in [7, p. 113] for the exponential function of a matrix $A$, we get the result. □

**Theorem 12.** *For positive integers $k$ and $n$ the following identities holds*

$$\sum_{j=0}^{\infty} \frac{x^j L_{jn+k}}{j!} = \alpha^k \exp(x\alpha^n) + \beta^k \exp(x\beta^n),$$

$$\sum_{j=0}^{\infty} \frac{x^j L_{jn}}{j!} = \exp(x\alpha^n) + \exp(x\beta^n),$$
$$\sum_{j=0}^{\infty} \frac{x^j L_{jn-k}}{j!} = (-1)^k[\alpha^k \exp(x\alpha^n) + \beta^k \exp(x\beta^n)],$$
$$\sum_{j=0}^{\infty} \frac{x^j F_{jn+k}}{j!} = \frac{\alpha^k \exp(x\alpha^n) - \beta^k \exp(x\beta^n)}{\alpha - \beta},$$
$$\sum_{j=0}^{\infty} \frac{x^j F_{jn}}{j!} = \frac{\exp(x\alpha^n) - \exp(x\beta^n)}{\alpha - \beta},$$
$$\sum_{j=0}^{\infty} \frac{x^j F_{jn-k}}{j!} = (-1)^{k-1}\left[\frac{\alpha^k \exp(x\beta^n) - \beta^k \exp(x\alpha^n)}{\alpha - \beta}\right].$$

*Proof.* When $f$ is an exponential function, if we replace $Y$ in Theorem 3 by its equivalent given in Theorem 11, we obtain these identities from the matrix equation. □

The technique presented above allows us to consider a very large number of infinite series involving $F_n$ and $L_n$ by considering power series expansions ( [1], [7], [21]) of other functions of the matrix $xM_n^k$. Finally, let us examine the expansion of $\tan^{-1} y$.

**Theorem 13.** *Under the constraint*
$$-\frac{1}{\alpha^n} \leq x \leq \frac{1}{\alpha^n}$$
*we have*
$$\sum_{j=1}^{\infty} \frac{(-1)^{j+1} x^{2j-1} L_{n(2j-1)+s}}{2j-1} = \alpha^s \tan^{-1}(x\alpha^n) + \beta^s \tan^{-1}(x\beta^n).$$

## 5. Conclusion

In this work, many identities for Fibonacci and Lucas sequences have been obtained. Although some of these are identities that can be obtained more simply in different ways, they are not found in the literature. What we really want to do here is to show how productive the Cholesky decomposition method is.

**Author Contribution Statements** The authors contributed equally to this article.

**Declaration of Competing Interests** The authors declare that they have no competing interest.

**Acknowledgements** This paper is dedicated to Prof. Dr. İlker AKKUŞ.

## References

[1] Abramowitz, M., Stegun, I., Handbook of Mathematical Functions, Dover, New York, 1972.
[2] Basin, S. L., Hoggatts V. E. Jr., A primer on the Fibonacci sequence: Part II., *Fibonacci Quarterly*, 1(2) (1963), 47-52.
[3] Bergum, G. E., Hoggatt, V. E. Jr., An application of the characteristic of the generalized Fibonacci sequence, *Fibonacci Quarterly*, 15(3) (1977), 215-220.
[4] Bicknell, M., A primer on the Pell Sequence and related sequences, *Fibonacci Quarterly*, 13(4) (1975), 345-49.
[5] Clarke, J. H., Shannon, A. G., Some generalized Lucas sequences, *Fibonacci Quarterly*, 23(2) (1985), 120-25.
[6] Filipponi, P., Horadam A. F., A matrix approach to certain identities, *Fibonacci Quarterly*, 26(2) (1988), 115-26.
[7] Gantmacher, F. R., The Theory of Matrices, Ams Chelsea, 1960.
[8] Golub, G. H., Van Loan, C. F., Matrix Computations (3rd ed.), Johns Hopkins, 1996.
[9] Hoggatt, V. E. Jr., Bicknell, M. J., Generalized Lucas sequences, *Fibonacci Quarterly*, 15(2) (1977), 131-39.
[10] Horadam, A. F., A generalized Fibonacci sequence, *Amer. Math. Monthly*, 68(5) (1961), 455-59. https://doi.org/10.2307/2311099
[11] Horadam, A. F., Generating functions for powers of certain generalized sequences of numbers, *Duke Math. J.*, 32 (1965), 437-59. https://doi.org/10.1215/S0012-7094-65-03244-8
[12] Horadam, A. F., Mahon, J. M., Pell and Pell-Lucas polynomials, *Fibonacci Quarterly*, 23(1) (1985), 7-20.
[13] Horadam, A. F., Filipponi, P., Cholesky algoritm matrices of Fibonacci type and properties of generalized sequences, *Fibonacci Quarterly*, 29(2) (1991), 164-173.
[14] Kalman, D., Mena, R., The Fibonacci numbers-exposed, *Math. Mag.*, 76 (2003), 167-181. https://doi.org/10.2307/3219318
[15] Koshy, T., Fibonacci and Lucas Numbers with Applications, John Wiley & Sons, 2017.
[16] Köken, F., Bozkurt, D., On Lucas numbers by the matrix method, *Hacettepe Journal of Mathematics and Statistics*, 39(4) (2010), 471-475. https://doi.org/10.1080/03610910903480834
[17] Lucas, E., Theorie des Nombres, Blanchard, 1961.
[18] Mahon, J. M., Horadam, A. F., Matrix and other summation techniques for Pell polynomials, *Fibonacci Quarterly*, 24(4) (1986), 290-308.
[19] Pethe, S., Horadam, A. F., Generalized Gaussian Fibonacci numbers, *Bull. of the Australian Math. Soc.*, 33(1) (1986), 37-48. https://doi.org/10.1017/S0004972700002847
[20] Shannon, A. G., Fibonacci annalogs of the classical polynomials, *Math. Magazine*, 48 (1975), 123-130. https://doi.org/10.1080/0025570X.1975.11976463
[21] Siegel, M. R., Manualedi Matematica. Milan: ETAS Libris S.p.A., 1974.
[22] Tan, E., On bi-periodic Fibonacci and Lucas numbers by matrix method, *Ars Combinatoria*, 133 (2017), 107–113.
[23] Tan, E., Some properties of the bi-periodic Horadam sequences, *Notes On Number Theory and Discrete Mathematics*, 23(4) (2017), 56-65.

[24] Tan, E., Ekin, A. B., Some identities on conditional sequences by using matrix method, *Miskolc Math. Notes*, 18(1) (2017), 469-477.
[25] Tan, E., Dağlı, M., Belkhir, A., Biperiodic incomplete Horadam numbers, *Turkish Journal of Mathematics*, 47 (2023), 554-564. https://doi.org/10.55730/1300-0098.3378
[26] Tan, E., Leung, H. H., Some basic properties of the generalized bi-periodic Fibonacci and Lucas sequences, *Advances in Difference Equations*, 26 (2020), 1-11.
[27] Vajda, S., Fibonacci & Lucas Numbers and the Golden Section, John Wiley & Sons, 1989.
[28] Wilkinson, J. H., The Algebraic Eigenvalue Problem, Oxford: Clarendon Press, 1965.

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 147–152 (2024)
DOI:10.31801/cfsuasmas.1265768
ISSN 1303-5991 E-ISSN 2618-6470

*Research Article; Received: March 15, 2023; Accepted: October 9, 2023*

# NORMAL AUTOMORPHISMS OF FREE METABELIAN LEIBNIZ ALGEBRAS

Zeynep ÖZKURT

Department of Mathematics, Çukurova University, Adana, TÜRKİYE

Abstract. Let $\mathfrak{M}$ be a free metabelian Leibniz algebra with generating set $X = \{x_1, ..., x_n\}$ over the field $\mathfrak{K}$ of characteristic 0. An automorphism $\phi$ of $\mathfrak{M}$ is said to be normal automorphism if each ideal of $\mathfrak{M}$ is invariant under $\phi$. In this work, it is proven that every normal automorphism of $\mathfrak{M}$ is an IA-automorphism and the group of normal automorphisms coincides with the group of inner automorphisms.

## 1. Introduction

Leibniz algebras were discovered in 1965 by A. Bloh [2] and forgotten for nearly thirty years. In the early 1990s Leibniz algebras were rediscovered by Loday as a generalization of Lie algebras [8]. In 1993, Loday and Pirashvili studied these algebras and they described the free Leibniz algebras [9]. In 2001, Mikhalev and Umirbaev obtained some important results on subalgebras of free Leibniz algebras [11]. Then automorphisms of free Leibniz algebras of rank two were described by Abdykhalykov at.al. [1]. In [13], the author studied on automorphic orbits of free Leibniz algebras of rank two. In [16], Hall bases of free Leibniz algebras were defined by Shahryari. In 2002, it was given a description of free metabelian Leibniz algebras by Drensky and Cattaneo [3]. Let $\mathfrak{M}$ be a free metabelian Leibniz algebra of rank $n$. Denote by $\mathfrak{M}'$, the commutator ideal of $\mathfrak{M}$. We write $\mathrm{Aut}(\mathfrak{M})$ for the automorphism group of $\mathfrak{M}$. Let

$$\pi : \mathrm{Aut}(\mathfrak{M}) \to \mathrm{Aut}(\mathfrak{M}/\mathfrak{M}')$$

be the canonical homomorphism with kernel consisting of automorphisms that induce the identity mapping on $\mathfrak{M}/\mathfrak{M}'$. The kernel of $\pi$ is called the IA-automorphism group and denoted by $\mathrm{IAut}(\mathfrak{M})$. In [17,18], the author and Taş Adıyaman described a generating set for $\mathrm{IAut}(\mathfrak{M})$ of rank three and $n$, respectively. Recently, symmetric

2020 *Mathematics Subject Classification.* 17A32; 17A36; 17A50.
*Keywords.* Leibniz algebra, normal automorphism, inner automorphism.
zyapti@cu.edu.tr; 0000-0001-9703-3463.

polynomials of $\mathfrak{M}$ were considered in [7]. An automorphism $\theta$ of $\mathfrak{M}$ is said to be a normal automorphism if $\theta(I) = I$ for each ideal $I$ of $\mathfrak{M}$. Normal automorphism group $\mathrm{Aut}(\mathfrak{N})$ is a normal subgroup of $\mathrm{Aut}(\mathfrak{M})$. For an element $u$ of $\mathfrak{M}'$ the adjoint operator

$$\mathrm{ad}u : \mathfrak{M} \longrightarrow \mathfrak{M}$$

defined by $\mathrm{ad}u(v) = [v, u]$, for every $v \in \mathfrak{M}$ is nilpotent since $\mathrm{ad}^2 u = 0$. Hence $\exp(\mathrm{ad}u) = 1 + \mathrm{ad}u$ is an automorphism of $\mathfrak{M}$ called an inner automorphism. Denote by $\mathrm{Inn}(\mathfrak{M})$, the inner automorphism group of $\mathfrak{M}$. It is known that $\mathrm{Aut}(\mathfrak{N})$ contains $\mathrm{Inn}(\mathfrak{M})$. There exist many groups whose normal automorphisms are inner. See the papers [5, 10, 14, 15, 19]. In [4], Endimioni studied normal automorphisms of a free metabelian nilpotent group. Normal automorphisms are important for algebras. In [6], normal automorphisms of free metabelian nilpotent Lie algebras were considered. In [12], Öğüşlü proved that each normal automorphism of the metabelian product of abelian Lie algebras is an IA-automorphism and acts identically on the commutator algebra. It is natural to generalize results of Lie algebras to Leibniz algebras.

In this work, an analogue of the result in [12] is established for Leibniz algebras over a field of characteristic 0 and it is proven that each normal automorphism of $\mathfrak{M}$ is an IA-automorphism. Then it is proven that $\mathrm{Aut}(\mathfrak{N}) = \mathrm{Inn}(\mathfrak{M})$.

## 2. Preliminaries

Let $\mathfrak{K}$ be a field of characteristic 0. The vector space $\mathfrak{L}$ over $\mathfrak{K}$ equipped with a bilinear map $[,] : \mathfrak{L} \times \mathfrak{L} \longrightarrow \mathfrak{L}$ is called a Leibniz algebra if it satisfies the Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

for all $x, y, z \in \mathfrak{L}$. In the general case a Leibniz algebra $\mathfrak{L}$ is a non-associtive and non-commutative algebra. If the condition $[x, x] = 0$ for all $x \in \mathfrak{L}$ is satisfied, then $\mathfrak{L}$ is a Lie algebra. Every commutator is reduced to a linear combination of left normed commutators by the Leibniz identity. Denote by $\mathrm{Ann}(\mathfrak{L})$, the ideal of $\mathfrak{L}$ generated by elements $\{[a, a] : a \in \mathfrak{L}\}$. It is known (see [9]) that $r_z = 0 \Leftrightarrow z \in \mathrm{Ann}(\mathfrak{L})$, where $r_z = \mathrm{ad}z$.

Let $\mathfrak{F}$ be the free Leibniz algebra with a generating set $\{x_1, \ldots, x_n\}$ over the field $\mathfrak{K}$ of characteristic 0 (see [9]) and let $\mathfrak{F}'$ and $\mathfrak{F}''$ be the commutator subalgebras of $\mathfrak{F}$ and $\mathfrak{F}'$, respectively. Then $\mathfrak{F}/\mathfrak{F}'$ and $\mathfrak{F}'/\mathfrak{F}''$ are abelian Leibniz algebras over $\mathfrak{K}$. We fix the notation $\mathfrak{M} = \mathfrak{F}/\mathfrak{F}''$ for the free metabelian Leibniz algebra over the field $\mathfrak{K}$. Then $\mathfrak{M}' = \mathfrak{F}'/\mathfrak{F}''$. Denote by $\langle \mathfrak{S} \rangle$, the ideal of $\mathfrak{M}$ generated by a set $\mathfrak{S}$.

The generators of $\mathrm{Aut}(\mathfrak{M})$ are given in the following theorem from [18].

**Theorem 1.** *Let $\mathfrak{M}$ be the free metabelian Leibniz algebra with a generating set $\{x_1, \ldots, x_n\}$. Then $\mathrm{Aut}(\mathfrak{M})$ is generated by the general linear group together with the inner automorphisms and the following IA-automorphisms*

$$\phi \quad : \quad x_1 \to x_1 + [z, x_1]$$

$$x_j \to x_j - [x_j, z]$$

*where* $z \in \mathfrak{M}'$ *and* $z \in \langle x_2 \rangle \oplus \ldots \oplus \langle x_n \rangle$,

$$\sigma \quad : \quad x_j \to x_j + [z, x_j]$$

*where* $z$ *is generated by the elements of the form* $[x, y] - [y, x]$ *where* $x, y \in \{x_1, \ldots, x_n\}$,

$$\begin{aligned} \tau \quad : \quad & x_1 \to x_1 + u \\ & x_i \to x_i \end{aligned}$$

*where* $i \neq 1$, $u \in \mathrm{Ann}(\mathfrak{M})$ *depends on* $x_t{}'\mathrm{s}$, $t \in \{2, \ldots, n\}$,

$$\begin{aligned} \psi \quad : \quad & x_1 \to x_1 + v \\ & x_i \to x_i \end{aligned}$$

*where* $v \in \langle [x_j, x_k] \rangle$, $j \neq k \neq 1$,$i \neq 1$.

## 3. Normal Automorphisms

**Theorem 2.** *Let* $\theta \in \mathrm{Aut}(\mathfrak{N})$. *Then* $\theta \in \mathrm{IAut}(\mathfrak{M})$.

*Proof.* Let $\mathfrak{M}$ be a free metabelian Leibniz algebra with the generating set $\{x_1, \ldots, x_n\}$. Every automorphism $\theta$ of $\mathfrak{M}$ is defined by

$$\theta \quad : \quad x_i \to k_{i1}x_1 + k_{i2}x_2 + ... + k_{in}x_n + u_i,$$

where the linear part is invertible, $u_i \in \mathfrak{M}', i = 1, \ldots, n$, $k_{ij} \in \mathfrak{K}$ [18]. Let $\theta \in \mathrm{Aut}(\mathfrak{N})$. Consider the ideal $\langle x_i \rangle$ of $\mathfrak{M}$. We have $\theta(x_i) \in \langle x_i \rangle$. Then $k_{i1}x_1 + \ldots + k_{ii}x_i + \ldots + k_{in}x_n + u_i \in \langle x_i \rangle$. By grading $k_{i1}x_1 + \ldots + k_{ii}x_i + \ldots + k_{in}x_n \in \langle x_i \rangle$ and $u_i \in \langle x_i \rangle$ are obtained. Since $x_1, x_2, \ldots, x_n$ are free generators, we obtain $k_{ij} = 0$ for $i \neq j$. Hence we have

$$\theta \quad : \quad x_i \to k_{ii}x_i + u_i,$$

where $k_{ii} \in \mathfrak{K}$. Consider the ideal $\langle \sum_{i=1}^n x_i \rangle$ of $\mathfrak{M}$. We obtain $\theta(\sum_{i=1}^n x_i) \in \langle \sum_{i=1}^n x_i \rangle$. Clearly

$$\theta(x_1 + x_2 + ... + x_n) = k_{11}x_1 + k_{22}x_2 + \ldots + k_{nn}x_n + u_1 + u_2 + \ldots + u_n$$

and

$$k_{11}x_1 + k_{22}x_2 + \ldots + k_{nn}x_n + u_1 + u_2 + \ldots + u_n \in \langle x_1 + x_2 + \ldots + x_n \rangle.$$

By grading we have

$$k_{11}x_1 + k_{22}x_2 + \ldots + k_{nn}x_n = k(x_1 + x_2 + \ldots + x_n)$$

for a coefficient $k \in \mathfrak{K}$. It implies

$$(k_{11} - k)x_1 + (k_{22} - k)x_2 + \ldots + (k_{nn} - k)x_n = 0,$$

by the linearly independence $k_{ii} - k = 0$, and $k_{ii} = k$ for $i = 1, 2, \ldots, n$. Therefore,

$$\theta \quad : \quad x_i \to kx_i + u_i.$$

Consider the ideal $\langle x_i + [x_i, x_i] \rangle$ of $\mathfrak{M}$,

$$\theta(x_i + [x_i, x_i]) = kx_i + k^2[x_i, x_i] + u_i + k[u_i, x_i] + k[x_i, u_i] \in \langle x_i + [x_i, x_i] \rangle.$$

By Theorem 1, $u_i \neq [x_i, x_i]$. Clearly it yields

$$kx_i + k^2[x_i, x_i] + u_i + k[u_i, x_i + k[x_i, u_i] = c(x_i + [x_i, x_i]) + z$$

where $c \in \mathfrak{K}, z \in \langle x_i + [x_i, x_i] \rangle$. By this equality, we obtain $k = c$, $k^2 = c$. Then we see that $k = k^2$ and $0 = k - k^2 = k(1 - k)$. Hence $k = 1$. $\square$

**Theorem 3.** $\mathrm{Aut}(\mathfrak{N}) = \mathrm{Inn}(\mathfrak{M})$.

*Proof.* Let $\theta \in \mathrm{Aut}(\mathfrak{N})$. Then $\theta$ is an IA-automorphism by Theorem 2. Hence, it can be defined by

$$\theta \quad : \quad x_i \to x_i + u_i$$

where $u_i \in \mathfrak{M}'$. Using the generating set of IA-automorphisms by Theorem 1, we can write the elements $u_i, i = 1, 2, \ldots, n$ as in the following forms;
**Case 1.** $u_i = [x_i, w]$ for $i = 1, 2, \ldots, n$ and $w \in \mathfrak{M}'$. In this form, $\theta$ is an inner automorphism.
**Case 2.** $u_1 = [w, x_1]$, $u_j = -[x_j, w]$, for $j = 2, \ldots, n$, where $w \in \mathfrak{M}'$ and $w \in \langle x_2 \rangle \oplus \ldots \oplus \langle x_i \rangle \oplus \ldots \oplus \langle x_n \rangle, i \neq 1$. Now take $[x_1, x_2] \in \mathfrak{M}'$. Consider the ideal $\langle [x_1, x_2] \rangle$ of $\mathfrak{M}$. Then

$$\theta([x_1, x_2]) = [x_1, x_2] + [u_1, x_2] + [x_1, u_2] = [x_1, x_2] + [[w, x_1], x_2] - [x_1, [x_2, w]].$$

Since $[[w, x_1], x_2] - [x_1, [x_2, w]] \notin \langle [x_1, x_2] \rangle$, then $\theta([x_1, x_2]) \notin \langle [x_1, x_2] \rangle$. This is a contradiction.
**Case 3.** $u_i = [w, x_i]$, for $i = 1, 2, \ldots, n$, where $w$ is generated by the elements of the form $[x, y] - [y, x]$, for $x, y \in \{x_1, ..., x_n\}$. Consider the ideal $\langle [x_1, x_2] \rangle$ of $\mathfrak{M}$. Then

$$\theta([x_1, x_2]) = [x_1, x_2] + [u_1, x_2] + [x_1, u_2] = [x_1, x_2] + [[w, x_1], x_2] + [x_1, [w, x_2]].$$

Since $[[w, x_1], x_2] + [x_1, [w, x_2]] \notin \langle [x_1, x_2] \rangle$, then $\theta([x_1, x_2]) \notin \langle [x_1, x_2] \rangle$. This is a contradiction.
**Case 4.** $u_1 \in \mathrm{Ann}(\mathfrak{M})$ depends on $x_t$'s, $t \in \{2, \ldots, n\}$, and $u_j = 0$ for $j = 2, \ldots, n$. We have

$$\theta([x_1, x_2]) = [x_1, x_2] + [u_1, x_2].$$

Since $[u_1, x_2] \notin \langle [x_1, x_2] \rangle$, this automorphism is not a normal automorphism.
**Case 5.** $u_1 = \langle [x_j, x_k] \rangle$, $j \neq k \neq 1$, and $u_j = 0$, for $j = 2, \ldots, n$. We obtain

$$\theta([x_1, x_2]) = [x_1, x_2] + [u_1, x_2].$$

Since the element $[u_1, x_2] \notin \langle [x_1, x_2] \rangle$, then $\theta([x_1, x_2]) \notin \langle [x_1, x_2] \rangle$. This is a contradiction.

Therefore, the elements $u_i$ are only as in Case 1. Hence $\theta$ is an inner automorphism. $\square$

**Declaration of Competing Interests** The author declares no competing interests.

**Acknowledgements** The author is very grateful to the anonymous referees for the careful reading of the manuscript and the valuable suggestions for the improvement of the exposition.

## References

[1] Abdykhalykov, A. T., Mikhalev, A. A., Umirbaev, U. U., Automorphisms of two-generated free Leibniz algebras, *Comm. Algebra*, 29(7) (2001), 2953-2960. https://doi.org/10.1081/AGB-4998

[2] Bloh, A., A generalization of the concept of a Lie algebra, *Dokl. Akad. Nauk SSSR*, 165(3) (1965), 471-473. English translation:*Sov. Math., Dokl.*, 6 (1965), 1450-1452.

[3] Drensky, V., Cattaneo, G. M. P., Varieties of metabelian Leibniz algebras, *Journal of Algebra and Its Appl.*, 1(1) (2002), 31-50. https://doi.org/10.1142/S0219498802000033

[4] Endimioni, G., Normal automorphisms of a free metabelian nilpotent group, *Glasgow Math. J.*, 52 (2010), 169-177. https://doi.org/10.1017/S0017089509990267

[5] Jarden, M., Ritter, J., Normal automorphisms of absolute Galois groups of p-adic fields, *Duke Math. J.*, 47 (1980). https://doi.org/10.1215/S0012-7094-80-04705-5

[6] Fındık, Ş., Normal and normally outer automorphisms of free metabelian nilpotent Lie algebras, *Serdica Math. J.*, 36 (2010), 171-210.

[7] Fındık, Ş., Özkurt, Z., Symmetric polynomials in Leibniz algebras and their inner automorphisms, *Turkish Journal of Mathematics*, 44 (2020), 2306-2311. https://doi.org/10.3906/mat-2006-44

[8] Loday, J. L., Une version non commutative des algèbres de Lie: les algèbres de Leibniz, *Enseign. Math.*, 39 (1993), 269-293.

[9] Loday, J. L., Pirashvili, T., Universal enveloping algebras of Leibniz algebras and (co)homology, *Math. Ann.*, 296 (1993), 139-158. https://doi.org/10.1007/BF01445099

[10] Lubotzky, A., Normal automorphisms of free groups, *J. Algebra*, 63 (1980), 494. https://doi.org/10.1016/0021-8693(80)90086-1

[11] Mikhalev, A. A., Umirbaev, U. U., Subalgebras of free Leibniz algebras, *Communications in Algebra*, 26 (1998), 435-446. https://doi.org/10.1080/00927879808826139

[12] Öğüşlü, N. Ş., Normal automorphisms of the metabelian product of free abelian Lie algebra, *Algebra and Discrete Mathematics*, 30(2) (2020), 230-234. http://dx.doi.org/10.12958/adm1258

[13] Özkurt, Z., Orbits and test elements in free Leibniz algebras of rank two, *Communications in Algebra*, 43(8) (2015), 3534-3544. https://doi.org/10.1080/00927872.2014.982806

[14] Romankov, V. A., Normal automorphisms of discrete groups, *Sib. Mat. Zh.*, 24(4) (1983), 138-149. English translation: *Siberian Math.J.*, 24 (1983), 604-614.

[15] Romanovskii, N. S., Normal automorphisms of free solvable pro-p-groups, *Algebra Logika*, 36(4) (1997), 441-453, English translation: *Algebra Log.*, 36(4) (1997), 257-263. https://doi.org/10.1007/BF02261748

[16] Shahryari, M., Hall bases for free Leibniz algebras, *Bull. Iranian Math. Soc.*, 45(2) (2019), 617-625. https://doi.org/10.1007/s41980-018-0153-3

[17] Taş Adiyaman, T., Özkurt, Z., Automorphisms of free metabelian Leibniz algebras of rank three, *Turk. J. Math.*, 43(5) (2019), 2262-2274. https://doi.org/10.3906/mat-1903-104

[18] Taş Adiyaman, T., Özkurt, Z., Automorphisms of free metabelian Leibniz algebras. *Comm. Algebra*, 49(10) (2021), 4348-4359. https://doi.org/10.1080/00927872.2021.1919690

[19] Timoshenko, E. I., Normal automorphisms of a soluble product of abelian groups, *Sib. Mat. Zh.*, 56(1) (2015), 227-236. English translation: *Siberian Math. J.*, 56(1) (2015), 191-198. https://doi.org/10.1134/S0037446615010188

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 153–164 (2024)
DOI:10.31801/cfsuasmas.1282587
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

*Research Article; Received: March 15, 2023; Accepted: October 9, 2023*

# AFFINE MAPPINGS AND MULTIPLIERS FOR WEIGHTED ORLICZ SPACES OVER THE AFFINE GROUP $\mathbb{R}_+ \times \mathbb{R}$

Rüya ÜSTER

Department of Mathematics, Faculty of Science, İstanbul University, İstanbul, TÜRKİYE

ABSTRACT. Let $\mathbb{A} = \mathbb{R}_+ \times \mathbb{R}$ be the affine group with a right Haar measure $\mu$, $\omega$ be a weight function on $\mathbb{A}$ and $\Phi$ be a Young function. We characterize the affine continuous mappings on the subsets of $L^\Phi(\mathbb{A}, \omega)$. Moreover we show that there exists an isometric isomorphism between the multiplier of the pair $(L^1(\mathbb{A}) \cap L^\Phi(\mathbb{A}), L^1(\mathbb{A}))$ and the space of bounded measures $M(\mathbb{A})$.

## 1. Introduction

Orlicz spaces are an important concept in analysis and applications (see [19, 23, 24]). This concept extends the classical concept of $L^p$ Lebesgue spaces for $p \geq 1$. A convex function $\Phi(x)$ is used in place of the function $x^p$ appearing in the definition of $L^p$ spaces. This function $\Phi$ is called a Young function. In addition to $L^p$ spaces, several function spaces can be considered as Orlicz spaces; for example $L \log^+ L$ Zygmund spaces, which are Banach spaces related to Hardy-Littlewood maximal functions. Moreover, Sobolev spaces can be also considered as subspaces of Orlicz spaces (see [5]). Most of the features of Orlicz spaces have been investigated thoroughly (see [23], for example), especially, Orlicz spaces determined on measure spaces (see for example [12, 14, 17, 23]). In recent years, Orlicz spaces and their weighted cases are examined as Banach algebras over locally compact groups (lcg). Moreover their several properties are also studied (see [1, 20–22, 27, 28]).

On the other hand one of the basic problems in harmonic analysis is the description of multipliers. Multipliers have been considered in several contexts, for example Banach algebras and Banach modules theories, partial differential equations, the existence of invariant means, etc. Our aim in this paper is to investigate the affine continuous mappings for the weighted Orlicz space $L^\Phi(\mathbb{A}, \omega)$ over the affine group $\mathbb{A}$ and study the multiplier problem for $L^\Phi(\mathbb{A}) \cap L^1(\mathbb{A})$. The affine

---

2020 *Mathematics Subject Classification.* 46H25, 43A15, 46E30, 43A22, 43A20.

*Keywords.* Affine group, affine mapping, multiplier, weighted Orlicz space.

ruya.uster@istanbul.edu.tr; 0000-0003-4063-5118.

group chosen is a prime example of a nonabelian group on which harmonic analysis and even more applied time-frequency analysis questions are studied (see [8,9]).

For $L^p$ spaces, in [16], Lau studied the affine mappings $T$ between the subsets of Lebesgue spaces. In [27], Üster and Öztop studied continuous affine mappings on the subsets of Orlicz spaces. On the other hand the characterization of multipliers for weighted Lebesgue spaces has been given by Gaudry [10]. (See also [7].) In [10], Gaudry showed that the multiplier space of $L^1(G,\omega)$ can be characterized by $M(G,\omega)$. Moreover in [28], Üster characterized the compact mulipliers of $L^\Phi(G,\omega)$. Here $G$ denotes a lcg. (See Section 2 for notation.)

The paper is organized as follows. In Section 2, we recall some basic definitions and notions on Orlicz and weighted Orlicz spaces. In Section 3, we study continuous affine mappings on subsets of weighted Orlicz space $L^\Phi(\mathbb{A},\omega)$ and we give a characterization for the multipliers of $L^\Phi(\mathbb{A})\cap L^1(\mathbb{A})$.

## 2. Preliminaries

We start this section by introducing some basic facts for an affine group and essential constructions on it.

Let $\mathbb{A} := (\mathbb{R}_+ \times \mathbb{R}, \cdot_{\mathbb{A}})$ be the affine group equipped with the multiplication

$$(s,t)\cdot_{\mathbb{A}}(x,y) = (sx, sy+t), \tag{1}$$

for $(s,t),\ (x,y) \in \mathbb{A}$. Note that $(1,0)\cdot_{\mathbb{A}}(s,t) = (s,t)\cdot_{\mathbb{A}}(1,0) = (s,t)$ and $(s,t)\cdot_{\mathbb{A}}(s^{-1},-s^{-1}t) = (s^{-1},-s^{-1}t)\cdot_{\mathbb{A}}(s,t) = (1,0)$. Thus $\mathbb{A}$, endowed with the multiplication (1), becomes a group and this group is called the affine group.

Since a mapping of the real line can be defined by $F_{s,t}:\mathbb{R}\to\mathbb{R}$ such that

$$F_{s,t}(x) = (s,t)\cdot x = sx+t,\ x\in\mathbb{R}$$

for any $(s,t)\in\mathbb{A}$, the affine group is also called the $sx+t$ group. $F_{s,t}$ is the affine mapping of the real line $\mathbb{R}$ and this operation is coherent with (1).

We can represent the affine group $\mathbb{A}$ in matrix form as

$$\mathbb{A} := \left\{ \begin{pmatrix} s & t \\ 0 & 1 \end{pmatrix} : s>0,\ t\in\mathbb{R} \right\}.$$

The inverse and the identity elements are given by

$$\begin{pmatrix} s^{-1} & -s^{-1}t \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The operations of the inversion and multiplication are continuous in the product topology. Thus the affine group $\mathbb{A}$ is a locally compact group and

$$d\nu(x,y) = \frac{dx}{x^2}\,dy$$

$$d\mu(x,y) = \frac{dx}{x}\,dy$$

are the left and right Haar measures, respectively (for more details see [13]). Now since

$$d\nu(x,y) = \frac{dx}{x^2}\,dy = \frac{1}{x}d\mu(x,y),$$

the affine group is not unimodular. The modular function on the affine group is $\Delta(x,y) = x^{-1}$.

Throughout this work we use the right Haar measure $d\mu$ on $\mathbb{A}$.

Let $f : \mathbb{A} \to \mathbb{C}$ and $(s,t) \in \mathbb{A}$. We use $L_{(s,t)}$ for the left translation and $R_{(s,t)}$ for the right translation given by

$$(L_{(s,t)}f)(x,y) := f((s,t)^{-1} \cdot_{\mathbb{A}} (x,y)) \quad \text{and} \quad (R_{(s,t)}f)(x,y) := f((x,y) \cdot_{\mathbb{A}} (s,t)^{-1}).$$

Next we give some notions regarding Orlicz spaces, weighted Orlicz spaces and Young functions. Our main references are [12] and [23].

**Definition 1.** *A function* $\Phi : [0,\infty) \to [0,\infty]$ *is called a Young function if* $\Phi$ *is convex,* $\Phi(0) = 0$ *and* $\lim\limits_{t\to\infty} \Phi(t) = +\infty$.

For a Young function $\Phi$, its conjugate function $\Psi$ is given by

$$\Psi(t) = \sup\{st - \Phi(s) : s \geq 0\} \quad (t \geq 0).$$

The pair $(\Phi, \Psi)$ of Young functions $\Phi, \Psi$ is said to be (Young) conjugate and we have

$$st \leq \Phi(s) + \Psi(t) \quad (\forall s,t \geq 0). \tag{2}$$

In this paper we only consider the real-valued Young functions. Clearly $\Phi$ is continuous and $\lim_{t\to\infty} \Phi(t) = \infty$. Note that the continuity of $\Phi$ may not imply the continuity of $\Psi$.

Let us recall the following facts about Orlicz spaces. Let $(\Phi, \Psi)$ be conjugate Young functions. Then the Orlicz space $L^{\Phi}(\mathbb{A})$ is defined to be

$$L^{\Phi}(\mathbb{A}) = \{f : \mathbb{A} \to \mathbb{C} : \int\limits_{\mathbb{A}} \Phi(\alpha|f(x,y)|)\,\frac{dx}{x}dy < \infty \text{ for some } \alpha > 0\}.$$

Here $f$ and $g$ in $L^{\Phi}(\mathbb{A})$ are equivalent if $f = g$ a.e. Recall that an Orlicz space is a Banach space with respect to (Orlicz) norm which is defined by

$$\|f\|_{\Phi} = \sup\left\{ \int\limits_{\mathbb{A}} |f(x,y)\nu(x,y)|\,\frac{dx}{x}dy : \int\limits_{\mathbb{A}} \Psi(|\nu(x,y)|)\,\frac{dx}{x}dy \leq 1 \right\}$$

for $f \in L^{\Phi}(\mathbb{A})$. Here $(\Phi, \Psi)$ are conjugate Young functions.

Another norm on an Orlicz space is the Luxemburg norm $N_{\Phi}(f)$ defined by

$$N_{\Phi}(f) = \inf\left\{\lambda > 0 : \int\limits_{\mathbb{A}} \Phi\left(\frac{|f(x,y)|}{\lambda}\right)\frac{dx}{x}dy \leq 1\right\}.$$

Note that the Orlicz and Luxemburg norms are equivalent; that is,

$$N_{\Phi}(\cdot) \leq \|\cdot\|_{\Phi} \leq 2N_{\Phi}(\cdot).$$

We shall use the following definition in the last section. In [4] and [29], the main motivation to use this definition is to estimate the norm of the dilation operator. Here we use a result of Lemma 3.3 given in [29].

Given $\gamma > 0$ one can define

$$N_{\Phi,\gamma}(f) := \inf\{\lambda > 0 : \int\limits_{\mathbb{A}} \Phi\left(\frac{|f(x,y)|}{\lambda}\right) \frac{dx}{x} dy \leq \ \gamma\}.$$

Here $N_{\Phi,1} = N_\Phi$ and these norms are equivalent on $L^\Phi(\mathbb{A})$:

$$\frac{\gamma_1}{\gamma_2} N_{\Phi,\gamma_1}(f) \leq N_{\Phi,\gamma_2}(f) \leq N_{\Phi,\gamma_1}(f)$$

for $0 < \gamma_1 \leq \gamma_2$.

For Orlicz spaces an important notion is the $\Delta_2$-condition. Let us recall the following definition.

**Definition 2.** *Let $\Phi : [0,\infty) \to [0,\infty]$ be a Young function. Then $\Phi$ is said to satisfy $\Delta_2$-condition (globally), if*

$$\Phi(2x) \leq M\Phi(x) \quad (x \geq 0)$$

*for some absolute constant $M > 0$.*

Note that if $\Phi \in \Delta_2$, then $L^\Phi(\mathbb{A})^* \cong L^\Psi(\mathbb{A})$, here $^*$ denotes the dual [23, Corollary 3.4.5]. Moreover if $\Psi \in \Delta_2$, then $L^\Phi(\mathbb{A})$ is a reflexive Banach space (see [14,23] for more general cases.)

On the other hand, the weighted Orlicz space $L^\Phi(G,\omega)$ is defined by Osançlıol and Öztop in [20] over a lcg $G$ and they consider the Banach algebra structure for $L^\Phi(G,\omega)$.

A weight function $\omega$ is a positive, locally integrable function on $\mathbb{A}$. In this paper we assume that $\omega$ is continuous (see [25, Section 3.7]). The space $L^\Phi(\mathbb{A},\omega)$ is defined by $\{f : f\omega \in L^\Phi(\mathbb{A})\}$. We also set

$$N_\Phi^\omega(f) = N_\Phi(f\omega) \tag{3}$$

for $f \in L^\Phi(\mathbb{A},\omega)$. Then $N_\Phi^\omega(\cdot)$ defines a norm on $L^\Phi(\mathbb{A},\omega)$ and $L^\Phi(\mathbb{A},\omega)$ is a Banach space with respect to this norm. Moreover, $L^\Psi(\mathbb{A},\omega^{-1})$ is the dual space of $(L^\Phi(\mathbb{A},\omega), N_\Phi^\omega(\cdot))$ if $\Phi$ fulfills the $\Delta_2$-condition. Here the duality is given by

$$\langle f,h\rangle = \int\limits_{\mathbb{A}} f(x,y)h(x,y)\frac{dx}{x}dy \quad (f \in L^\Phi(\mathbb{A},\omega),\ h \in L^\Psi(\mathbb{A},\omega^{-1})),$$

where $(\Phi,\Psi)$ are conjugate Young functions and the space $L^\Psi(\mathbb{A},\omega^{-1})$ is endowed with the norm $N_\Psi^{\omega^{-1}}(f) = N_\Psi(\frac{f}{\omega})$. So if $\Phi,\Psi$ fulfill the $\Delta_2$-condition then $L^\Phi(\mathbb{A},\omega)$ is a reflexive Banach space (for the general case see [20]).

For $\Phi(x) = \frac{x^p}{p}$, $1 < p < \infty$, the conjugate Young function is $\Psi(y) = \frac{y^q}{q}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then $L^\Phi(\mathbb{A},\omega)$ and its norm are equal to the Lebesgue space $L^p(\mathbb{A},\omega)$

and its norm. For $p = 1$ and $\Phi(x) = x$ the conjugate Young function is

$$\Psi(y) = \begin{cases} 0, & 0 \leq y \leq 1 \\ \infty, & \text{otherwise} \end{cases}$$

and we have $L^{\Phi}(\mathbb{A}, \omega) = L^1(\mathbb{A}, \omega)$. Note that for $p = 1$, the Banach algebra $L^1(\mathbb{A}, \omega)$ always has a bounded approximate identity.

As usual, $M(\mathbb{A}, \omega)$ is the set of all complex bounded regular Borel measures $\lambda$ on $\mathbb{A}$ with

$$\|\lambda\|_{\omega} = \int_{\mathbb{A}} \omega(s,t) d\lambda(s,t) < \infty.$$

We denote the space of all continuous functions $f$ on $\mathbb{A}$ vanishing at infinity by $C^0(\mathbb{A}, \omega^{-1})$ with the norm $\|f\|_{\infty,\omega^{-1}} = \|\frac{f}{\omega}\|_{\infty}$. Then $M(\mathbb{A}, \omega)$ is realized as $(C^0(\mathbb{A}, \omega^{-1}))^*$ by

$$\langle \lambda, f \rangle = \int_{\mathbb{A}} f(x,y) d\lambda(x,y)$$

(for the general case see [11]). If $\lambda \in M(\mathbb{A}, \omega)$ and $f \in L^{\Phi}(\mathbb{A}, \omega)$ the convolution of $\lambda$ and $f$ is defined by

$$(\lambda * f)(x,y) = \int_{\mathbb{A}} f((s,t)^{-1} \cdot_{\mathbb{A}} (x,y)) d\lambda(s,t).$$

Moreover if $f, g$ are measurable functions on $\mathbb{A}$ the convolution of $f$ and $g$ is defined by

$$(f * g)(x,y) = \int_{\mathbb{A}} f(s,t) g((s,t)^{-1} \cdot_{\mathbb{A}} (x,y)) \frac{ds}{s} dt \quad ((x,y) \in \mathbb{A}).$$

For each $(s,t) \in \mathbb{A}$, let $\delta_{(s,t)}(E) = 1_E(s,t)$, where $1_E$ is the characteristic function of $E \subseteq \mathbb{A}$. Then

$$(\delta_{(s,t)} * f)(x,y) = f((s,t)^{-1} \cdot_{\mathbb{A}} (x,y)) = L_{(s,t)} f(x,y) \quad ((s,t) \in \mathbb{A})$$

where $L_{(s,t)^{-1}}$ is the left translation operator. For a function $f$ on $\mathbb{A}$, we use $\widetilde{f}$ defined by $\widetilde{f}(x,y) = f((x,y)^{-1})$ for each $(x,y) \in \mathbb{A}$.

Throughout the paper we study $L^{\Phi}(\mathbb{A}, \omega)$ with the weight $\omega$ and the $\Delta_2$-condition on a Young function $\Phi$.

## 3. Main Results

In this section we characterize the affine continuous mappings for $L^{\Phi}(\mathbb{A}, \omega)$ over the affine gorup $\mathbb{A}$ and we study the multiplier problem for the space $L^{\Phi}(\mathbb{A}, \omega) \cap L^1(\mathbb{A}, \omega)$. Let us first give the following definitions.

**Definition 3.** *Let $C \subseteq L^{\Phi}(\mathbb{A}, \omega)$. Then $C$ is called left invariant if $L_{(x,y)} f \in C$ for each $f \in C$ and $(x,y) \in \mathbb{A}$.*

Notice that for $f \in L^\Phi(\mathbb{A},\omega)$ and $(x,y) \in \mathbb{A}$ we have $L_{(x,y)}f \in L^\Phi(\mathbb{A},\omega)$ and $N_\Phi^\omega(L_{(x,y)}f) \leq \omega(x,y)N_\Phi^\omega(f)$ (for the general lcgs see [20, Lemma 2.3]).

**Definition 4.** *Let $X$ and $Y$ be normed spaces and $C$, $D$ be convex subsets of $X$ and $Y$ respectively. Then a mapping $f : C \to D$ is called affine if*

$$f(\alpha x + (1-\alpha)y) = \alpha f(x) + (1-\alpha)f(y)$$

*for each $x, y \in C$ and $\alpha \in [0,1]$.*

For the subset $K$ of $L^\Phi(\mathbb{A},\omega)$, we use $\operatorname{co} K$ for the convex hull of $K$. In addition to the norm topology on $L^\Phi(\mathbb{A},\omega)$, we will take the weak topology $w$ and the weak$^*$ topology $w^*$ for the pair $(L^\Phi(\mathbb{A},\omega), L^\Phi(\mathbb{A},\omega)^*)$ where $(\Phi,\Psi)$ is a conjugate pair.

Moreover, we make use of the following subsets of $M(\mathbb{A},\omega)$:

(i) $P(\mathbb{A},\omega) = \{\mu \in M(\mathbb{A},\omega) : \|\mu\|_\omega = 1 \text{ and } \mu \geq 0\}$,

(ii) $P_1(\mathbb{A},\omega) = \{h \in L^1(\mathbb{A},\omega) : \|h\|_{1,\omega} = 1 \text{ and } h \geq 0\}$,

(iii) $E(\mathbb{A},\omega) = \{\frac{\delta_{(x,y)}}{\omega(x,y)} : (x,y) \in \mathbb{A}\}$.

We omit the proof of the following Lemma which appears in [28] for general locally compact abelian groups. One can get the same result for nonabelian groups in a similar way.

**Lemma 1.** *We have $P(\mathbb{A},\omega) = \overline{P_1(\mathbb{A},\omega)}^{w^*} = \overline{\operatorname{co} E(\mathbb{A},\omega)}^{w^*}$. Here $\overline{\cdot}^{w^*}$ indicates weak$^*$ closure.*

**Lemma 2.** *The following are true.*

(i) *Let $f \in L^\Phi(\mathbb{A},\omega)$. Then the mapping $\mu \mapsto \mu * f$ is continuous from $(M(\mathbb{A},\omega), w^*)$ to $(L^\Phi(\mathbb{A},\omega), w)$.*

(ii) *Let $f \in L^1(\mathbb{A},\omega)$. Then the mapping $h \mapsto f * h$ is continuous from $(L^\Phi(\mathbb{A},\omega), w)$ to $(L^\Phi(\mathbb{A},\omega), w)$.*

*Proof.* (i) Let $\{\mu_\alpha\}_\alpha \subseteq M(\mathbb{A},\omega)$ be a net that is weak$^*$ convergent to $\mu$, $(\Phi,\Psi)$ a conjugate Young pair and $f \in L^\Phi(\mathbb{A},\omega)$. Since $L^\Phi(\mathbb{A},\omega)$ is $M(\mathbb{A},\omega)$-module the mapping $\mu * f$ is well defined (see [20]). Let $T \in (L^\Phi(\mathbb{A},\omega))^*$, so there exists $g \in L^\Psi(\mathbb{A},\omega^{-1})$ such that

$$T(f) = \int_{\mathbb{A}} f(x,y)g(x,y)\frac{dx}{x}dy = \langle f, g\rangle.$$

Thus we obtain that

$$\begin{aligned} T(\mu_\alpha * f) &= \langle \mu_\alpha * f, g\rangle \\ &= \int_{\mathbb{A}} (\mu_\alpha * f)(x,y)g(x,y)\,\frac{dx}{x}dy \end{aligned}$$

$$= \int_{\mathbb{A}} \int_{\mathbb{A}} f((s,t)^{-1} \cdot_{\mathbb{A}} (x,y))\, d\mu_\alpha(s,t) g(x,y)\, \frac{dx}{x} dy$$

$$= \int_{\mathbb{A}} \int_{\mathbb{A}} \tilde{f}((x,y)^{-1} \cdot_{\mathbb{A}} (s,t)) g(x,y)\, \frac{dx}{x} dy\, d\mu_\alpha(s,t)$$

$$= \int_{\mathbb{A}} (g * \tilde{f})(s,t)\, d\mu_\alpha(s,t)$$

$$= \langle g * \tilde{f}, \mu_\alpha \rangle.$$

Since $f \in L^\Phi(\mathbb{A},\omega)$ and $g \in L^\Psi(\mathbb{A},\omega^{-1})$, we have $g*\tilde{f} \in C_0(\mathbb{A},\omega^{-1})$ (for the general case see [20].) This implies that $T(\mu_\alpha * f) = \langle g * \tilde{f}, \mu_\alpha\rangle \to \langle g * \tilde{f}, \mu\rangle = \langle \mu * f, g\rangle = T(\mu * f)$, i.e., $\mu_\alpha * f$ weakly converges to $\mu * f$ in $L^\Phi(\mathbb{A},\omega)$.

(ii) Let $\{h_\alpha\}_\alpha \subseteq L^\Phi(\mathbb{A},\omega)$ be a net that is weakly convergent to $h$ and $f \in L^1(\mathbb{A},\omega)$. We have $\lim_\alpha \langle h_\alpha, g\rangle = \langle h, g\rangle$ for all $g \in (L^\Phi(\mathbb{A},\omega))^*$. Thus we obtain that

$$\begin{aligned}
\langle h_\alpha * f, g\rangle &= \int_{\mathbb{A}} (h_\alpha * f)(x,y) g(x,y) \frac{dx}{x} dy \\
&= \int_{\mathbb{A}} \int_{\mathbb{A}} h_\alpha(s,t) f((s,t)^{-1} \cdot_{\mathbb{A}} (x,y)) g(x,y) \frac{ds}{s} dt \frac{dx}{x} dy \\
&= \int_{\mathbb{A}} \int_{\mathbb{A}} h_\alpha(s,t) \tilde{f}((x,y)^{-1} \cdot_{\mathbb{A}} (s,t)) g(x,y) \frac{ds}{s} dt \frac{dx}{x} dy \\
&= \int_{\mathbb{A}} h_\alpha(s,t) (g * \tilde{f})(s,t) \frac{ds}{s} dt \\
&= \langle h_\alpha, g * \tilde{f}\rangle.
\end{aligned}$$

This gives that $\langle h_\alpha * f, g\rangle = \langle h_\alpha, g * \tilde{f}\rangle \to \langle h, g * \tilde{f}\rangle = \langle h * f, g\rangle$. □

**Theorem 1.** *Let $C, D$ be convex, closed, left invariant subsets of $L^{\Phi_2}(\mathbb{A},\omega)$ and $L^{\Phi_1}(\mathbb{A},\omega)$ respectively. If $T : C \to D$ is a continuous and affine mapping then the following are equivalent.*

(i) *$T(L_{(x,y)} f) = L_{(x,y)}(Tf)$ for each $(x,y) \in \mathbb{A}$ and $f \in C$.*
(ii) *$T(\nu * f) = \nu * T(f)$ for each $\nu \in P_1(\mathbb{A},\omega)$ and $f \in C$.*

*Proof.* (i $\Rightarrow$ ii) Let $f \in C$ and assume that $T(L_{(x,y)} f) = L_{(x,y)}(Tf)$ for each $(x,y) \in \mathbb{A}$ and $\nu \in P_1(\mathbb{A},\omega)$. Using Lemma 1, there exists a net $\{\nu_\alpha\}_\alpha$ in $\operatorname{co} E(\mathbb{A},\omega)$, $\nu_\alpha = \sum_{i=1}^{n_\alpha} {\lambda_i}^\alpha \frac{\delta_{(s_i^\alpha, t_i^\alpha)}}{\omega(s_i^\alpha, t_i^\alpha)}$ and $\nu_\alpha$ weak* converges to $\nu$. Then by Lemma 2, $\{\nu_\alpha * f\}_\alpha$

weakly converges to $\nu * f$ for each $f \in C$. Thus we have

$$\nu_\alpha * f = \left( \sum_{i=1}^{n_\alpha} \lambda_i{}^\alpha \frac{\delta_{(s_i^\alpha, t_i^\alpha)}}{\omega(s_i^\alpha, t_i^\alpha)} \right) * f = \sum_{i=1}^{n_\alpha} \frac{\lambda_i^\alpha}{\omega(s_i^\alpha, t_i^\alpha)} L_{(s_i^\alpha, t_i^\alpha)^{-1}} f.$$

As $C$ is convex and left invariant, the net $\{\nu_\alpha * f\}_\alpha$ is contained in $C$. Now using Lemma 2 it follows that $\nu * f \in C$.

On the other hand since $C$ and $D$ are convex and closed they are weakly closed. Moreover since $T$ is continuous and affine $T$ is weakly continuous when $C$ and $D$ have their respective weak topologies (see [6,26]). Then we get that

$$\begin{aligned}
T(\nu * f) &= \lim_\alpha T(\nu_\alpha * f) \\
&= \lim_\alpha T\left( \left( \sum_{i=1}^{n_\alpha} \lambda_i^\alpha \frac{\delta_{(s_i^\alpha, t_i^\alpha)}}{\omega(s_i^\alpha, t_i^\alpha)} \right) * f \right) \\
&= \lim_\alpha T\left( \sum_{i=1}^{n_\alpha} \frac{\lambda_i^\alpha}{\omega(s_i^\alpha, t_i^\alpha)} \left( \delta_{(s_i^\alpha, t_i^\alpha)} * f \right) \right) \\
&= \lim_\alpha T\left( \sum_{i=1}^{n_\alpha} \frac{\lambda_i^\alpha}{\omega(s_i^\alpha, t_i^\alpha)} (L_{(s_i^\alpha, t_i^\alpha)^{-1}} f) \right) \\
&= \lim_\alpha \sum_{i=1}^{n_\alpha} \frac{\lambda_i^\alpha}{\omega(s_i^\alpha, t_i^\alpha)} T(L_{(s_i^\alpha, t_i^\alpha)^{-1}} f) \\
&= \lim_\alpha \sum_{i=1}^{n_\alpha} \frac{\lambda_i^\alpha}{\omega(s_i^\alpha, t_i^\alpha)} L_{(s_i^\alpha, t_i^\alpha)^{-1}} T(f) \\
&= \lim_\alpha \left( \sum_{i=1}^{n_\alpha} \lambda_i^\alpha \frac{\delta_{(s_i^\alpha, t_i^\alpha)}}{\omega(s_i^\alpha, t_i^\alpha)} \right) * T(f) \\
&= \lim_\alpha \nu_\alpha * T(f) \\
&= \nu * T(f).
\end{aligned}$$

(ii $\Rightarrow$ i) Conversely let $(x, y) \in \mathbb{A}$. Using Lemma 1 there exists a net $\{\nu_\alpha\}_\alpha \subseteq P_1(\mathbb{A}, \omega)$ such that $\{\nu_\alpha\}_\alpha$ converges to $\delta_{(x,y)^{-1}}$ in the weak* topology. If $T(\nu * f) = \nu * T(f)$ for each $\nu \in P_1(\mathbb{A}, \omega)$ and $f \in C$ we have that

$$\begin{aligned}
T(L_{(x,y)} f) &= T(\delta_{(x,y)^{-1}} * f) \\
&= \lim_\alpha T(\nu_\alpha * f) \\
&= \lim_\alpha \nu_\alpha * T(f) \\
&= \delta_{(x,y)^{-1}} * T(f) \\
&= L_{(x,y)} T(f).
\end{aligned}$$

This completes the proof. □

**Theorem 2.** *Let $B$ be a weakly compact, bounded, left invariant, closed subset of $L^{\Phi}(\mathbb{A},\omega)$ and $T$ be a continuous affine mapping from $P_1(\mathbb{A},\omega)$ to $B$. Then $T$ commutes with all left translations if and only if there exists an $f \in B$ such that $T(g) = g * f$ for each $g \in P_1(\mathbb{A},\omega)$.*

*Proof.* Let $(x,y) \in \mathbb{A}$ and assume that $T(L_{(x,y)}g) = L_{(x,y)}(Tg)$ for each $g \in P_1(\mathbb{A},\omega)$. Using Theorem 1 we have $T(k * g) = k * T(g)$ for $k, g \in P_1(\mathbb{A},\omega)$. Let $\{u_\alpha\}_\alpha \subseteq P_1(\mathbb{A},\omega)$ be a bounded approximate identity for $L^1(\mathbb{A},\omega)$. Since $B$ is weakly compact and $T(u_\alpha) \in B$ is bounded, there exists $f \in B$ such that $\{T(u_\alpha)\}_\alpha$ converges to $f$ weakly. Thus

$$\begin{aligned} T(g) &= \lim_\alpha T(g * u_\alpha) \\ &= \lim_\alpha g * T(u_\alpha) \\ &= g * f \end{aligned}$$

and the result follows.

For the converse let $(x,y) \in \mathbb{A}$ and assume that $f \in B$ such that $T(g) = g * f$ for all $g \in P_1(\mathbb{A},\omega)$. Then

$$\begin{aligned} L_{(x,y)}T(g) &= L_{(x,y)}(g * f) \\ &= \delta_{(x,y)^{-1}} * (g * f) \\ &= (\delta_{(x,y)^{-1}} * g) * f \\ &= L_{(x,y)}g * f \\ &= T(L_{(x,y)}g) \end{aligned}$$

which gives the required result. □

Now our purpose is to obtain a characterization for the multipliers of $L^{\Phi}(\mathbb{A}) \cap L^1(\mathbb{A})$. We observe that the following result does not work for the weighted case and we give the result for the unweighted case.

We start with the definition of the left multiplier of $L^{\Phi}(\mathbb{A})$.

**Definition 5.** *Let $T$ be a bounded linear operator from $L^{\Phi_1}(\mathbb{A})$ to $L^{\Phi_2}(\mathbb{A})$. Then $T$ is said to be a left multiplier for $(L^{\Phi_2}(\mathbb{A}), L^{\Phi_1}(\mathbb{A}))$ if $T(L_{(x,y)}f) = L_{(x,y)}(Tf)$ for all $f \in L^{\Phi_2}(\mathbb{A})$ and $(x,y) \in \mathbb{A}$. We write $\mathcal{M}(L^{\Phi_2}(\mathbb{A}), L^{\Phi_1}(\mathbb{A}))$ for the set of left multipliers of $(L^{\Phi_2}(\mathbb{A}), L^{\Phi_1}(\mathbb{A}))$.*

**Remark 1.** *Observe that the normed space $L^1(\mathbb{A}) \cap L^{\Phi}(\mathbb{A})$ is a Banach space with the norm*

$$|||f||| = \|f\|_1 + N_\Phi(f)$$

*and dense in $L^1(\mathbb{A})$.*

The following lemma is important to us for our last result (for the proof see [29, Lemma 3.3].)

**Lemma 3.** *Let $\Phi$ be a Young function satisfying the $\Delta_2$ condition. If $f \in L^\Phi(\mathbb{A})$ then $\lim_{(a,b)\to(+\infty,+\infty)} N_\Phi(f + L_{(a,b)}f) = N_{\Phi,\frac{1}{2}}(f)$.*

Now we have the tools to give a characterization of the multipliers of $L^\Phi(\mathbb{A}) \cap L^1(\mathbb{A})$.

**Theorem 3.** *Let $T : L^1(\mathbb{A}) \cap L^\Phi(\mathbb{A}) \to L^1(\mathbb{A})$ be a linear mapping. Then the following are equivalent.*

(i) $T \in \mathcal{M}(L^1(\mathbb{A}) \cap L^\Phi(\mathbb{A}), L^1(\mathbb{A}))$.

(ii) *There exists a unique measure $\mu \in M(\mathbb{A})$ such that $Tf = \mu * f$ for each $f \in L^1(\mathbb{A}) \cap L^\Phi(\mathbb{A})$.*
*Furthermore the correspondence between $T$ and $\mu$ defines an isometric isomorphism of $\mathcal{M}(L^1(\mathbb{A}) \cap L^\Phi(\mathbb{A}), L^1(\mathbb{A}))$ onto $M(\mathbb{A})$.*

*Proof.* Assume that $T \in \mathcal{M}(L^1(\mathbb{A}) \cap L^\Phi(\mathbb{A}), L^1(\mathbb{A}))$. Then for each $f \in L^1(\mathbb{A}) \cap L^\Phi(\mathbb{A})$ we obtain that

$$\|Tf\|_1 \le \|T\|(\|f\|_1 + N_\Phi(f)). \tag{4}$$

By Lemma 3 we have $\lim_{(s,t)\to(\infty,\infty)} N_\Phi(f + L_{(s,t)}f) = N_{\Phi,\frac{1}{2}}(f)$. Using this fact together with (4) we have that

$$\begin{aligned} 2\|Tf\|_1 &= \lim_{(s,t)\to(\infty,\infty)} \|Tf + L_{(s,t)}Tf\|_1 \\ &= \lim_{(s,t)\to(\infty,\infty)} \|T(f + L_{(s,t)}f)\|_1 \\ &\le \lim_{(s,t)\to(\infty,\infty)} \|T\|(\|f + L_{(s,t)}f\|_1 + N_\Phi(f + L_{(s,t)}f)) \\ &= \|T\|(2\|f\|_1 + N_{\Phi,\frac{1}{2}}(f)) \end{aligned}$$

for each $f \in L^1(\mathbb{A}) \cap L^\Phi(\mathbb{A})$. Therefore we obtain

$$\|Tf\|_1 \le \|T\|(\|f\|_1 + 2^{-1}N_{\Phi,\frac{1}{2}}(f)).$$

Applying this step $n$ times we obtain

$$\|Tf\|_1 \le \|T\|(\|f\|_1 + 2^{-n}N_{\Phi,\frac{1}{2}}(f))$$

for $f \in L^1(\mathbb{A}) \cap L^\Phi(\mathbb{A})$. Since $\lim_{n\to\infty} 2^{-n} = 0$ we deduce that $\|Tf\|_1 \le \|T\|\|f\|_1$.

Thus $T$ defines a linear continuous mapping from $L^1(\mathbb{A}) \cap L^\Phi(\mathbb{A})$ to $L^1(\mathbb{A})$ commuting with left translations. Moreover since $L^1(\mathbb{A}) \cap L^\Phi(\mathbb{A})$ is dense in $L^1(\mathbb{A})$, $T$ determines a unique map $S \in \mathcal{M}(L^1(\mathbb{A}))$ and $\|S\| \le \|T\|$. Moreover there exists a unique $\mu \in M(\mathbb{A})$ such that $Sf = \mu * f$ for each $f \in L^1(\mathbb{A})$ and $\|\mu\| = \|S\|$ (see [30]). Therefore $Tf = \mu * f$ for each $f \in L^1(\mathbb{A}) \cap L^\Phi(\mathbb{A})$ and $\|\mu\| \le \|T\|$.

Conversely, if $\mu \in M(\mathbb{A})$ and $Tf = \mu * f$ for each $f \in L^1(\mathbb{A}) \cap L^\Phi(\mathbb{A})$ we obtain

$$\|Tf\|_1 = \|\mu * f\|_1 \le \|\mu\|\|f\|_1 \le \|\mu\|\, |||f|||.$$

Therefore $T \in \mathcal{M}(L^1(\mathbb{A}) \cap L^\Phi(\mathbb{A}), L^1(\mathbb{A}))$ and $\|T\| \leq \|\mu\|$.

This gives to equivalence of (i) and (ii).

It is clear that the correspondence between $T$ and $\mu$ defines an isometric isomorphism from $\mathcal{M}(L^1(\mathbb{A}) \cap L^\Phi(\mathbb{A}), L^1(\mathbb{A}))$ onto $M(\mathbb{A})$. □

**Declaration of Competing Interests** The author declares that there are no conflicts of interest regarding the publication of this paper.
**Acknowledgements** The author is grateful to an anonymous referee for a careful reading of the manuscript and for helpful comments.

## References

[1] Akbarbaglu, I., Maghsoudi, S., Banach-Orlicz algebras on a locally compact group, *Mediterranean Journal of Mathematics*, 10 (2013), 1937-1974. https://doi.org/10.1007/s00009-013-0267-z
[2] Bennet, G., Sharpley, R., Interpolation of Operators, Academic Press London, 1988.
[3] Birnbaum, Z. W., Orlicz, W., Über die Verallgemeinerung des Begriffes der zueinander konjugerten Potenzen, *Studia Math.*, 3 (1931), 1-67.
[4] Blasco, O., Osançlıol, A., Notes on bilinear multipliers on Orlicz spaces, *Mathematische Nachrichten*, 292(12) (2019), 2522-2536. https://doi.org/10.1002/mana.201800551
[5] Cianchi, A., Pick, L., Slavíkova, L., Sobolev embeddings in Orlicz and Lorentz spaces with measures, *Journal of Mathematical Analysis and Applications*, 485 (2020), Paper no. 123827. https://doi.org/10.1016/j.jmaa.2019.123827
[6] Conway, J. B., A Course in Functional Analysis, 2nd Edition, Graduate Text in Mathematics, Springer-Verlag, New York, 1990. https://doi.org/10.1007/978-1-4757-4383-8
[7] Edwards, R. E., The stability of weighted Lebesgue spaces, *Trans. Amer. Math. Soc.*, 93 (1959), 369-394.
[8] Berge, E., Berge, S. M., Luef, F., Skrettingland, E., Affine quantum harmonic analysis, *Journal of Functional Analysis*, 282 (2022), 109327. https://doi.org/10.1016/j.jfa.2021.109327
[9] Berge, E., Berge, S. M., Luef, F., The affine Wigner distribution, *Applied and Computational Harmonic Analysis*, 56 (2022), 150-175. https://doi.org/10.1016/j.acha.2021.08.006
[10] Gaudry, G. I., Multipliers of weighted Lebesgue and measure spaces, *Proc. London Math. Soc.*, 19 (1969), 327-340. https://doi.org/10.1112/plms/s3-19.2.327
[11] Ghahramani, F., Automorphism of weighted measure algebras, Conference on Automatic Continuity and Banach Algebras, Canberra, *Proc. Centre Math. Anal. Austral. Nat. Univ.*, 21 (1989), 144-154.
[12] Harjulehto, P., Hästö, P., Orlicz Spaces and Generalized Orlicz Spaces, Lecture notes in mathematics, 2236, Springer, 2019. https://doi.org/10.1007/978-3-030-15100-3
[13] Kaniuth, E., Taylor, K. F., Induced Representations of Locally Compact Groups, Cambridge University Press, 197, 2013. https://doi.org/10.1017/CBO9781139045391
[14] Krasnosel'skii, M. A., Rutickii, Ja. B., Convex Functions and Orlicz Spaces, Noordhoff, Graningen, 1961.
[15] Larsen, R., An Introduction to the Theory of Multipliers, Die Grundlehren der mathematischen Wissenschaften, 175, Springer-Verlag, Berlin, Heidelberg and New York, 1971. https://doi.org/10.1007/978-3-642-65030-7
[16] Lau, A. T., Closed convex invariant subsets of $L^p(G)$, *Transactions of the American Mathematical Society*, 232 (1977), 131-142. https://doi.org/10.2307/1998929
[17] Luxemburg, W. A. J., Banach function spaces, PhD Dissertation, 1955.

[18] Majewski, W. A., Labuschagne, L. E., On applications of Orlicz spaces to statistical physics, *Annales Henri Poincaré*, 15 (2014), 1197-1221. https://doi.org/10.1007/s00023-013-0267-3

[19] Orlicz, W., Über eine gewisse klasse von Räumen vom Typus B, *Bulletin International de l'Academie Polonaise des Sciences et des Lettres Série A*, 8 (1932), 207-220.

[20] Osançlıol, A., Öztop, S., Weighted Orlicz algebras on locally compact groups, *Journal of Australian Mathematical Society*, 99 (2015), 399-414. https://doi.org/10.1017/S1446788715000257

[21] Öztop, S., Samei, E., Twisted Orlicz algebras I, *Studia Mathematica*, 236 (2017), 271-296. 10.4064/sm8562-9-2016

[22] Öztop, S., Samei, E., Twisted Orlicz algebras II, *Mathematische Nachrichten*, 292 (2019), 1122-1136. https://doi.org/10.1002/mana.201700362

[23] Rao, M. M., Ren, Z. D., Theory of Orlicz Spaces, Marcel Dekker, New York, 1991.

[24] Rao, M. M., Ren, Z. D., Applications of Orlicz Spaces, Marcel Dekker, New York, 2002. https://doi.org/10.1201/9780203910863

[25] Reiter H., Stegeman J.D., Classical Harmonic Analysis and Locally Compact Groups, Clarendon Press, Oxford, 2000.

[26] Rudin, W., Real and Complex Analysis, Third edition, McGraw-Hill Book Co, 1987.

[27] Üster, R., Öztop, S., Invariant subsets and homological properties of Orlicz modules over group algebras, *Taiwanese Journal of*, 24 (2020), 959-973. 10.11650/tjm/190903

[28] Üster, R., Multipliers for the weighted Orlicz spaces of a locally compact abelian group, *Results in Mathematics*, 76(4) (2021), Paper No. 183. https://doi.org/10.1007/s00025-021-01493-4

[29] Üster, R., A criterion for nonzero multiplier for Orlicz spaces of an affine group $\mathbb{R}_+ \times \mathbb{R}$, *Hacettepe Journal of Mathematics and Statistics*, 52(5) (2023), 1198-1205. https://doi.org/10.15672/hujms.1175682

[30] Wendel, J. G., Left centralizers and isomorphisms of group algebras, *Pacific J. Math.*, 2 (1952), 251–261. 10.2140/pjm.1952.2.251

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 165–177 (2024)
DOI:10.31801/cfsuasmas.1299533
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

*Research Article; Received:May 19, 2023; Accepted: October 23, 2023*

# TRANSLATION SURFACES GENERATING WITH SOME PARTNER CURVE

Aykut HAS,[1] Beyhan YILMAZ[2]

[1,2]Department of Mathematics, Faculty of Science, Kahramanmaras Sutcu Imam University, Kahramanmaras, TÜRKİYE

Abstract. In this article, generating curves of translation surfaces are paired with some special curve pairs. With the results obtained from these pairings, the developable and minimal translation surfaces are characterized. In addition, the surface curvatures of the translation surface are obtained. For a better understanding of the results, examples are given and their drawings are made with the help of Mathematica.

## 1. Introduction

The main purpose of differential geometry is to understand and characterize the mathematical properties of any geometric object defined in space. The most important of these objects are curves and surfaces. Researchers working on this subject often have to characterize the curve and the surface in a certain way in order to understand it. One of the most important ways to characterize the curve is to use Frenet vectors. For example, Bertrand pairs of curves were characterized by J. Bertrand in 1850 as curves whose reciprocal normal vectors are linearly dependent [1]. Similarly, the Mannheim curve pairs were characterized by the normal vector of one of the curves and the binormal vector of the other as linearly dependent by A. Mannheim in 1878 [2]. In addition, the involute-evolute curve pairs are characterized as curve pairs whose mutual tangent vectors are perpendicular [3].

The study of surfaces is one of the most captivating subjects in the field of differential geometry. Consequently, researchers have extensively investigated various types of surfaces [4–6]. Much like curves, researchers endeavor to characterize

2020 *Mathematics Subject Classification.* 53A04, 53A05.

*Keywords.* Translation surface, Bertrand partner curve, Mannheim partner curve, involute-evolute curves.

[1] ahas@ksu.edu.tr-Corresponding author; 0000-0003-0658-9365;

[2] beyhanyilmaz@ksu.edu.tr; 0000-0002-5091-3487

.

surfaces. Moreover, another significant aspect that piques researchers' interest is whether a surface is developable or minimal [7,8]. One of the interesting surfaces in Euclidean space is the translation surface produced by the two curves. The general form of translation surface is the surface that can be generated from two arbitrary space curves by translating either of them parallel to itself. In such a way that each of its points describes a curve that is a translation of the other curve. A generalized type of translation surface parameterized by

$$\chi(u,v) = \mathbf{x}(u) + \mathbf{y}(v) \tag{1}$$

where $\mathbf{x} : I \subset \mathbb{R} \to \mathbb{E}^3$ and $\mathbf{y} : J \subset \mathbb{R} \to \mathbb{E}^3$ are arbitrary generating curves of $\chi$ according to the parameters $u$ and $v$ (may be the arc-length parameters), respectively. Let $\{\mathbf{t}_{\mathbf{x}}, \mathbf{n}_{\mathbf{x}}, \mathbf{b}_{\mathbf{x}}\}$ be the Frenet frame field of $\mathbf{x}$ with curvature $\kappa_{\mathbf{x}}$ and torsion $\tau_{\mathbf{x}}$. Also, let $\{\mathbf{t}_{\mathbf{y}}, \mathbf{n}_{\mathbf{y}}, \mathbf{b}_{\mathbf{y}}\}$ be the Frenet frame field of $\mathbf{y}$ with curvature $\kappa_{\mathbf{y}}$ and torsion $\tau_{\mathbf{y}}$. A translation surface has the property that the translations of a parametric curve $u = c$ by $\mathbf{y}(v)$ remain in $\chi$ (similarly for the parametric curves $v = c$) [9–11]. Translation surfaces are the basic modeling surfaces commonly used in computer aided geometric design and geometric modeling [12]. Also, translation surfaces are common in descriptive geometry and architecture because they can be easily modeled [13,14]. Many studies are carried out on translation surfaces so far: L. Verstraelen et al. have studied minimal translation surfaces in n-dimensional Euclidean spaces [15]. H. Liu has studied Gaussian curvature and mean curvature of translation surfaces in 3-dimensional space [16]. D. W. Yoon has studied the differential geometric properties of translation surfaces by applying the Laplace operator to the Gauss transform [17]. Additionally, numerous studies have been conducted on translation surfaces [18–22].

In this study, generating curves of translation surfaces are associated with some special curve pairs. The article investigates the conditions necessary for these translation surfaces to be both developable and minimal surface, while also characterizing the conditions that make this possible.

## 2. Preliminaries

In this section, for parametrized curves and surface elements some basics definitions and theorems are given.

A regular naturally parametrization of class $C^k$, with $k \geq 1$ of a curve in $\mathbb{R}^3$ is a vector function $\mathbf{x} : I \subset \mathbb{R} \to \mathbb{E}^3$, $s \mapsto \mathbf{x}(s) = (\mathbf{x}_1(s), \mathbf{x}_2(s), \mathbf{x}_3(s))$ defined on an interval $I$ which satisfies $\mathbf{x}$ is of class $C^k$ and $\mathbf{x}'(s) \neq 0$ for all $s \in I$. A curve $\mathbf{x}$ is continuously differentiable if $\mathbf{x}'(s)$ exists for all $s \in I$ and the derivative $\mathbf{x}'(s)$ is a continuous function; thinking dynamically, the vector $\mathbf{x}'(s)$ is the velocity of the curve at time $s$. We call $\mathbf{x}(s)$ *naturally parametrized curve* if $\mathbf{x}_i(s)$ $(i = 1, 2, 3)$ is of class $C^k$ and $\|\mathbf{x}'(s)\| = 1$, for each $s \in I$ [23].

Let $\mathbf{x}(s)$ be *biregular*, that is, $\mathbf{x}'(s) \times \mathbf{x}''(s) \neq 0$, for each $s \in I$. We consider a trihedron $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ along $\mathbf{x}(s)$, so-called *Frenet frame*, where [23]

$$\mathbf{t}(s) = \mathbf{x}'(s), \quad \mathbf{n}(s) = \frac{\mathbf{t}'(s)}{\|\mathbf{t}'(s)\|}, \quad \mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s).$$

The curvature $\kappa$, a non-negative scalar field, is defined by setting $\kappa(s) = \|\mathbf{t}'(s)\|$ and torsion is defined by setting $\tau(s) = \langle \mathbf{n}'(s), \mathbf{b}(s) \rangle$. If the naturally parametrized curve $\mathbf{x}$ has unit speed and strictly positive curvature, then the following equations hold [23]

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}.$$

where $\kappa \neq 0$ for the Frenet frame to be defined.

Let $\mathbf{x}$ and $\mathbf{y}$ be naturally parametrized curves in $\mathbb{E}^3$ with parameter $u$ and $v$, respectively. Let $\{t_{\mathbf{x}}(u), n_{\mathbf{x}}(u), b_{\mathbf{x}}(u), \kappa_{\mathbf{x}}(u), \tau_{\mathbf{x}}(u)\}$ and $\{t_{\mathbf{y}}(v), n_{\mathbf{y}}(v), b_{\mathbf{y}}(v), \kappa_{\mathbf{y}}(v), \tau_{\mathbf{y}}(v)\}$ be Frenet elements of $\mathbf{x}$ and $\mathbf{y}$, respectively. Some special curve pairs is studied by S. Yuce and A. Sabuncuoglu and the following results are given [24, 25].

Let's assume that $(\mathbf{x}, \mathbf{y})$ curve pair is Bertrand curve pair. In this sitation, since the normal vectors of the $\mathbf{x}$ and $\mathbf{y}$ have the same direction, they are written as

$$\begin{aligned} \mathbf{t}_{\mathbf{x}}(u) &= \cos\theta \mathbf{t}_{\mathbf{y}}(v) - \sin\theta \mathbf{b}_{\mathbf{y}}(v), && (2) \\ \mathbf{n}_{\mathbf{x}}(u) &= \mathbf{n}_{\mathbf{y}}(v), && (3) \\ \mathbf{b}_{\mathbf{x}}(u) &= \sin\theta \mathbf{t}_{\mathbf{y}}(v) + \cos\theta \mathbf{b}_{\mathbf{y}}(v) && (4) \end{aligned}$$

and

$$\begin{aligned} \kappa_{\mathbf{x}}(u) &= \kappa_{\mathbf{y}}(v)\cos\theta + \tau_{\mathbf{y}}(v)\sin\theta, && (5) \\ \tau_{\mathbf{x}}(u) &= -\kappa_{\mathbf{y}}(v)\sin\theta + \tau_{\mathbf{y}}(v)\cos\theta, && (6) \end{aligned}$$

where $\theta$ is the constant angle between the mutually tangent vectors.

Let's assume that $(\mathbf{x}, \mathbf{y})$ curve pair is Mannheim curve pair. Since the normal vector of the $\mathbf{x}$ and binormal vector of the curve $\mathbf{y}$ have the same direction, they are written as

$$\begin{aligned} \mathbf{t}_{\mathbf{x}}(u) &= \cos\theta \mathbf{t}_{\mathbf{y}}(v) + \sin\theta \mathbf{n}_{\mathbf{y}}(v), && (7) \\ \mathbf{n}_{\mathbf{x}}(u) &= \mathbf{b}_{\mathbf{y}}(v), && (8) \\ \mathbf{b}_{\mathbf{x}}(u) &= -\sin\theta \mathbf{t}_{\mathbf{y}}(v) + \cos\theta \mathbf{n}_{\mathbf{y}}(v) && (9) \end{aligned}$$

and

$$\begin{aligned} \kappa_{\mathbf{x}}(u) &= \tau_{\mathbf{y}}(v)\sin\theta \frac{dv}{du}, && (10) \\ \tau_{\mathbf{x}}(u) &= -\tau_{\mathbf{y}}(v)\cos\theta \frac{dv}{du}, && (11) \end{aligned}$$

where $\theta$ is the constant angle between the mutually tangent vectors.

Let's assume that $(\mathbf{x},\mathbf{y})$ curve pair be involute-evolute partner curve. Since the mutual tangent vectors of the $\mathbf{x}$ and $\mathbf{y}$ curves are perpendicular, the following equations are available

$$\begin{aligned} \mathbf{t_x}(u) &= \mathbf{n_y}(v), \quad (12)\\ \mathbf{n_x}(u) &= \cos\theta\mathbf{t_y}(v)+\sin\theta\mathbf{b_y}(v), \quad (13)\\ \mathbf{b_x}(u) &= -\sin\theta\mathbf{t_y}(v)+\cos\theta\mathbf{b_y}(v) \quad (14)\end{aligned}$$

where $\theta$ is the constant angle between $\mathbf{t_x}$ and $\mathbf{n_y}$, and

$$\kappa_{\mathbf{x}}(u)=\frac{\sqrt{\kappa_{\mathbf{y}}^2+\tau_{\mathbf{y}}^2}}{(c-s)\kappa_{\mathbf{y}}}. \quad (15)$$

Let $M$ be a regular surface in $\mathbb{R}^3$ parameterized by $\chi(u,v)$. Some basic concepts of $M$ surface is studied by M.P. Do Cormo and these concepts are given below [3].

The standart unit normal vector field $\mathbf{n}$ on surface $M$ can be defined by

$$\mathbf{n}=\frac{\chi_u\times\chi_v}{\|\chi_u\times\chi_v\|}. \quad (16)$$

Also, the first and second fundamental forms of the surface $M$ are as follows

$$I=Edu^2+2Fdudv+Gdv^2,$$
$$II=edu^2+2fdudv+gdv^2,$$

where the $E,F$ and $G$ components are called the coefficients of the first fundemental form of the surface, and the $e,f$ and $g$ components are called the coefficients of the second fundemental form, respectively. The following equations are given for the first and second fundamental form coefficients of the surface

$$E=\langle\chi_u,\chi_u\rangle,\quad F=\langle\chi_u,\chi_v\rangle,\quad G=\langle\chi_v,\chi_v\rangle \quad (17)$$

and

$$e=\langle\chi_{uu},\mathbf{n}\rangle,\quad f=\langle\chi_{uv},\mathbf{n}\rangle,\quad g=\langle\chi_{vv},\mathbf{n}\rangle. \quad (18)$$

On the other hand, the Gaussian curvature $K$ and the mean curvature $H$ of the surface $M$ are as follows

$$K=\frac{eg-f^2}{EG-F^2} \quad (19)$$

and

$$H=\frac{Eg+Ge-2Ff}{2(EG-F^2)}. \quad (20)$$

**Theorem 1.** *Let $M$ be a regular surface in $\mathbb{R}^3$. If the Gaussian curvature of the surface $M$ is zero, the surface is called the developable surface [26] .*

**Theorem 2.** *Let $M$ be a regular surface in $\mathbb{R}^3$. If the mean curvature of the surface $M$ is zero, the surface is called the minimal surface [26].*

## 3. Translation Surfaces Created with Curve Pairs

Translation surfaces are formed by the sum of the two curves, from Eq. (1), translation surface is as follows

$$\chi(u,v) = \mathbf{x}(u) + \mathbf{y}(v) \tag{21}$$

where $\mathbf{x}$ and $\mathbf{y}$ are generating curves. If the partial derivatives of the translation surface given above are taken according to $u$ and $v$, we have

$$\chi_u = \mathbf{t_x}, \tag{22}$$

$$\chi_v = \mathbf{t_y}, \tag{23}$$

$$\chi_{uu} = \kappa_\mathbf{x} \mathbf{n_x}, \tag{24}$$

$$\chi_{vv} = \kappa_\mathbf{y} \mathbf{n}_y, \tag{25}$$

$$\chi_{uv} = \frac{d}{dv}\mathbf{t_x}. \tag{26}$$

The unit normal of the translation surface from Eqs. (16), (22) and (23), we get

$$\mathbf{n} = \frac{\mathbf{t_x} \times \mathbf{t_y}}{\|\mathbf{t_x} \times \mathbf{t_y}\|}. \tag{27}$$

The coefficients of the first and second fundamental forms of the translation surface are obtained from Eqs. (17), (18) and Eqs. (22)-(26), as

$$E = \langle \chi_u, \chi_u \rangle = 1, \tag{28}$$

$$F = \langle \chi_u, \chi_v \rangle = \langle \mathbf{t_x}, \mathbf{t_y} \rangle, \tag{29}$$

$$G = \langle \chi_v, \chi_v \rangle = 1 \tag{30}$$

and

$$e = \langle \chi_{uu}, \mathbf{n} \rangle = \frac{\kappa_\mathbf{x}}{\|\mathbf{t_x} \times \mathbf{t_y}\|} \langle \mathbf{n_x}, \mathbf{t_x} \times \mathbf{t_y} \rangle, \tag{31}$$

$$f = \langle \chi_{uv}, \mathbf{n} \rangle = \frac{1}{\|\mathbf{t_x} \times \mathbf{t_y}\|} \langle \frac{d}{dv}\mathbf{t_x}, \mathbf{t_x} \times \mathbf{t_y} \rangle, \tag{32}$$

$$g = \langle \chi_{vv}, \mathbf{n} \rangle = \frac{\kappa_\mathbf{y}}{\|\mathbf{t_x} \times \mathbf{t_y}\|} \langle \mathbf{n_y}, \mathbf{t_x} \times \mathbf{t_y} \rangle. \tag{33}$$

3.1. **Let x and y Bertrand partner curves.** Let the curves $\mathbf{x}$ and $\mathbf{y}$, which are the generating curves of the translation surface parameterized by Eq. (1), be the Bertrand partner curve. In this case, from Eq. (2) and (27) the unit normal of the translation surface is

$$\mathbf{n} = \frac{(\cos\theta \mathbf{t_y} - \sin\theta \mathbf{b_y}) \times \mathbf{t_y}}{\|(\cos\theta \mathbf{t_y} - \sin\theta \mathbf{b_y}) \times \mathbf{t_y}\|} = -\mathbf{n_y}. \tag{34}$$

Since the principal normal vector fields of Bertrand curve pairs are linearly dependent, at the same time $\mathbf{n} = -\mathbf{n_x}$.

The coefficients of the first fundamental form from Eq. (2) and Eqs. (28)-(30), are obtained as

$$\begin{aligned} E &= \langle \chi_u, \chi_u \rangle = 1, \\ F &= \langle \chi_u, \chi_v \rangle = \langle (\cos\theta \mathbf{t_y} - \sin\theta \mathbf{b_y}), \mathbf{t_y} \rangle = \cos\theta, \\ G &= \langle \chi_v, \chi_v \rangle = 1. \end{aligned}$$

The coefficients of the second fundamental form from Eqs. (2),(5) and Eqs. (31)-(33), are as follows

$$\begin{aligned} e &= \langle \kappa_\mathbf{x} \mathbf{n_x}, -\mathbf{n_x} \rangle = -\kappa_\mathbf{x}, \\ f &= \langle (\kappa_\mathbf{y} \cos\theta + \tau_\mathbf{y} \sin\theta)\mathbf{n_y}, -\mathbf{n_y} \rangle = -\kappa_\mathbf{x}, \\ g &= \langle \kappa_\mathbf{y} \mathbf{n_y}, -\mathbf{n_y} \rangle = -\kappa_\mathbf{y}. \end{aligned}$$

The Gaussian and mean curvatures of translation surfaces, whose generating curves are Bertrand partner curves from Eqs. (19) and (20), are calculated as

$$K = \frac{eg - f^2}{EG - F^2} = \frac{\kappa_\mathbf{x}(\kappa_\mathbf{y} - \kappa_\mathbf{x})}{\sin^2\theta} \tag{35}$$

and

$$H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)} = \frac{-\kappa_\mathbf{x} - \kappa_\mathbf{y} + 2\cos\theta\kappa_\mathbf{x}}{2\sin^2\theta}. \tag{36}$$

**Theorem 3.** *Let $\chi(u,v) = \mathbf{x}(u) + \mathbf{y}(v)$ be a translation surface where $\mathbf{x}$ and $\mathbf{y}$ are generating curve. For translation surfaces, whose generating curves are Bertrand partner curves to be developable surfaces the necessary and sufficient condition is that this $\mathbf{y}$ is helix.*

*Proof.* Considering that $\kappa_\mathbf{x} \neq 0$, from Eqs. (5), (35) and Theorem 1, it becomes

$$\kappa_\mathbf{x} = \kappa_\mathbf{y}$$

and

$$\kappa_\mathbf{y} \cos\theta + \tau_\mathbf{y} \sin\theta = \kappa_\mathbf{y}.$$

So, we get

$$\frac{\tau_\mathbf{y}}{\kappa_\mathbf{y}} = \frac{1 - \cos\theta}{\sin\theta}.$$

Since $\theta$ is a constant angle, $\frac{\tau_\mathbf{y}}{\kappa_\mathbf{y}}$ =constant. So generating curve $\mathbf{y}$ is helix. □

**Theorem 4.** *Let $\chi(u,v) = \mathbf{x}(u) + \mathbf{y}(v)$ be a translation surface where $\mathbf{x}$ and $\mathbf{y}$ are generating curves. Suppose that the generating curves are a pair of Bertrand curves. The necessary and sufficient condition for the surface $\chi$ to be a minimal surface is that the curve $\mathbf{x}$ is a helix.*

*Proof.* From Eqs. (5) and (6), we can easily see that

$$\kappa_{\mathbf{y}} = \kappa_{\mathbf{x}}(v)\cos\theta - \tau_{\mathbf{x}}(v)\sin\theta. \tag{37}$$

Using Eqs. (36), (37) and Theorem 2, the following equation can be given

$$\kappa_{\mathbf{x}} - \cos\theta\kappa_{\mathbf{x}} = \tau_{\mathbf{x}}\sin\theta$$

and

$$\frac{\tau_{\mathbf{x}}}{\kappa_{\mathbf{x}}} = \frac{1-\cos\theta}{\sin\theta}.$$

Since $\theta$ is a constant angle, $\frac{\tau_{\mathbf{x}}}{\kappa_{\mathbf{x}}}$ =constant. So generating curve $\mathbf{x}$ is helix. □

**Example 1.** *Let $\mathbf{x} : I \subset \mathbb{R} \to \mathbb{E}^3$ be naturally parametrized curve in $\mathbb{R}^3$ parameterized by*

$$\mathbf{x}(u) = \left(\cos\frac{u}{5}, \sin\frac{u}{5}, \frac{\sqrt{24}}{5}u\right).$$

*The naturally parametrized curve $\mathbf{y}$ which is the Bertrand partner curve of the $\mathbf{x}$ curve is as follows*

$$\mathbf{y}(v) = \left(\frac{24}{25}\cos\frac{v}{5}, \frac{24}{25}\sin\frac{v}{5}, \frac{\sqrt{24}}{5}v\right).$$

*The translation surface generating by the $\mathbf{x}$ and $\mathbf{y}$ Bertrand partner curves is parameterized as follows*

$$\chi(u,v) = \left(\cos\frac{u}{5} + \frac{24}{25}\cos\frac{v}{5}, \sin\frac{u}{5} + \frac{24}{25}\sin\frac{v}{5}, \frac{\sqrt{24}}{5}u + \frac{\sqrt{24}}{5}v\right).$$

*In Fig. (1), we present the graph of the above translation surface and its generating Bertrand partner curves $\mathbf{x}$ and $\mathbf{y}$.*

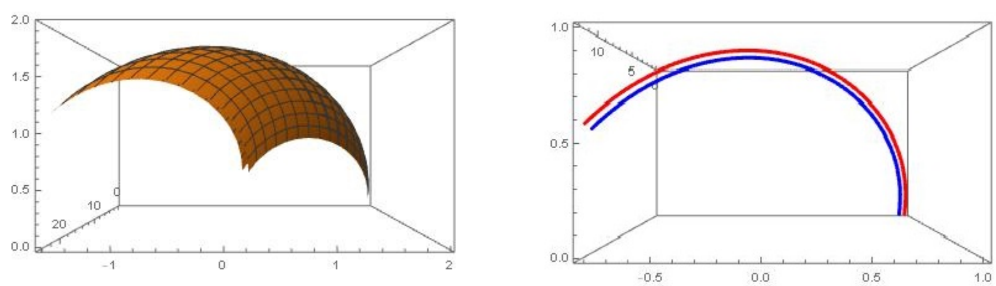


FIGURE 1. Translation surface and its generating curves $\mathbf{x}$(Red) and $\mathbf{y}$(Blue) for Bertrand partner curve.

3.2. **Let x and y Mannheim partner curves.** Let the curves $\mathbf{x}$ and $\mathbf{y}$, which are the generating curves of the translation surface parameterized by Eq. (1), be the Mannheim partner curves. In this case, from Eq. (7) and (27), the unit normal of the translation surface is

$$\mathbf{n} = \frac{(\cos\theta \mathbf{t_y} + \sin\theta \mathbf{n_y}) \times \mathbf{t_y}}{\|(\cos\theta \mathbf{t_y} + \sin\theta \mathbf{n_y}) \times \mathbf{t_y}\|} = -\mathbf{b_y}. \tag{38}$$

Since the principal normal vector and binormal vector fields of Mannheim curve pairs are linearly dependent, at the same time $\mathbf{n} = -\mathbf{n_x}$. The coefficients of the first fundamental form from Eq. (7) and Eqs. (28)-(30), are as follow

$$\begin{aligned}
E &= \langle \chi_u, \chi_u \rangle = 1, \\
F &= \langle \chi_u, \chi_v \rangle = \langle (\cos\theta \mathbf{t_y} + \sin\theta \mathbf{n_y}), \mathbf{t_y} \rangle = \cos\theta, \\
G &= \langle \chi_v, \chi_v \rangle = 1.
\end{aligned}$$

The coefficients of the second fundamental form from Eqs. (7),(10) and Eqs. (31)-(33), are obtained as

$$\begin{aligned}
e &= \langle \kappa_\mathbf{x} \mathbf{n_x}, -\mathbf{n_x} \rangle = -\kappa_\mathbf{x}, \\
f &= \langle -\kappa_\mathbf{y} \sin\theta \mathbf{t_y} + \kappa_\mathbf{y} \cos\theta \mathbf{n_y} + \tau_\mathbf{y} \sin\theta \mathbf{b_y}, -\mathbf{b_y} \rangle = -\tau_\mathbf{y} \sin\theta, \\
g &= \langle \kappa_\mathbf{y} \mathbf{n_y}, \mathbf{b_y} \rangle = 0.
\end{aligned}$$

If we calculate the Gaussian and mean curvatures of translation surfaces, whose generating curves are Mannheim partner curves, from Eqs. (19) and (20), we have

$$K = \frac{eg - f^2}{EG - F^2} = -\tau_\mathbf{y}^2 \tag{39}$$

and

$$H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)} = \frac{-\kappa_\mathbf{x} + \tau_\mathbf{y} \sin 2\theta}{2\sin^2\theta}. \tag{40}$$

**Theorem 5.** *Let $\chi(u,v) = \mathbf{x}(u) + \mathbf{y}(v)$ be a translation surface where $\mathbf{x}$ and $\mathbf{y}$ are generating curve. For translation surfaces, whose generating curves are Mannheim partner curves to be developable surfaces, the necessary sufficient condition is that the curve $\mathbf{y}$ is a planar curve.*

*Proof.* It is easily seen from Eq. (39) and Theorem 1 that $\tau_\mathbf{y} = 0$. This means that the curve $\mathbf{y}$ is a planar curve. □

**Theorem 6.** *Let $\chi(u,v) = \mathbf{x}(u) + \mathbf{y}(v)$ be a translation surface where $\mathbf{x}$ and $\mathbf{y}$ are generating curve. For translation surfaces, whose generating curves are Mannheim partner curves to be minimal surfaces, the necessary sufficient condition is that the curve $\mathbf{y}$ is a planar curve or $v = c_1 u + c_2,\ c_1, c_2 \in \mathbb{R}$.*

*Proof.* From Eqs. (10), (40) and Theorem 2, the following equation can be given

$$\tau_\mathbf{y} \sin 2\theta = \tau_\mathbf{y} \sin\theta \frac{dv}{du}$$

and

$$2\tau_{\mathbf{y}}\cos\theta = \tau_{\mathbf{y}}\frac{dv}{du}.$$

Here $\tau_{\mathbf{y}} = 0$ is an obvious solution. So $\mathbf{y}$ is a planar curve. Let $\tau_{\mathbf{y}} \neq 0$ then, we get

$$2\cos\theta \int du = \int dv.$$

If $2\cos\theta = c_1, c_1 \in \mathbb{R}$ is selected here, we obtain

$$v = c_1 u + c_2, \;\; c_1, c_2 \in \mathbb{R}.$$

□

**Example 2.** *Let $\mathbf{x} : I \subset \mathbb{R} \to \mathbb{E}^3$ be arbitrary parametrized curve in $\mathbb{R}^3$ parameterized by*

$$\mathbf{x}(u) = \left(\frac{8}{5}\cos u, \frac{8}{5}\sin u, \frac{4}{5}u\right).$$

*The arbitrary parametrized curve $\mathbf{y}$ which is the Mannheim partner curve of the curve $\mathbf{x}$ is as follows*

$$\mathbf{y}(v) = \left(-\frac{8}{5}(\sin v + \cos v), \frac{8}{5}(\sin v + \cos v), \frac{4}{5}v\right).$$

*The translation surface generating by the $\mathbf{x}$ and $\mathbf{y}$ Mannheim partner curves is parameterized as follows*

$$\chi(u,v) = \left(\frac{8}{5}\cos u - \frac{8}{5}(\sin v + \cos v), \frac{8}{5}\sin u + \frac{8}{5}(\sin v + \cos v), \frac{4}{5}u + \frac{4}{5}v\right).$$

*In Fig. (2), we present the graph of the above translation surface and its generating Mannheim partner curves $\mathbf{x}$ and $\mathbf{y}$.*

3.3. **Let x and y involute-evolute partner curves.** Let the curves $\mathbf{x}$ and $\mathbf{y}$, which are the generating curves of the translation surface parameterized by Eq. (1), be the involute-evolute partner curves. So, from Eq. (12) and (27), the unit normal of the translation surface is

$$\mathbf{n} = \frac{\mathbf{n_y} \times \mathbf{t_y}}{\|\mathbf{n_y} \times \mathbf{t_y}\|} = -\mathbf{b_y}. \tag{41}$$

The coefficients of the first fundamental form from Eq. (12) and Eqs. (28)-(30), are as follows

$$\begin{aligned} E &= \langle \chi_u, \chi_u \rangle = 1, \\ F &= \langle \chi_u, \chi_v \rangle = \langle \mathbf{n_y}, \mathbf{t_y} \rangle = 0, \\ G &= \langle \chi_v, \chi_v \rangle = 1. \end{aligned}$$

If we calculate the coefficients of the second fundamental form from Eqs. (12),(13),(15) and Eqs. (31)-(33), we can easily see that

$$e = \langle \kappa_{\mathbf{x}}\mathbf{n_x}, -\mathbf{b_y} \rangle = -\kappa_{\mathbf{x}}\sin\theta,$$

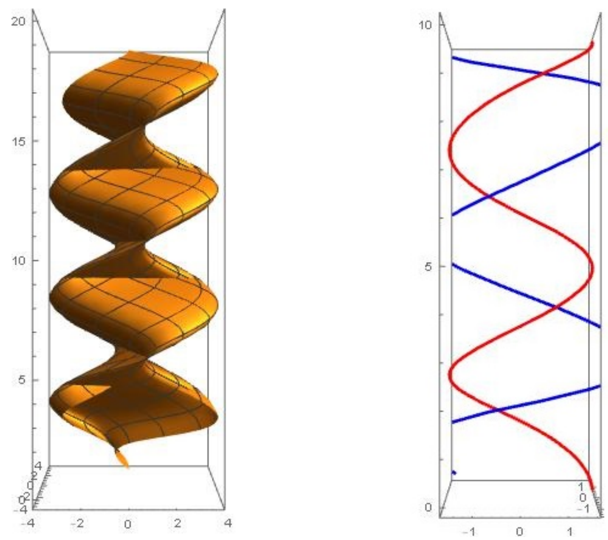


FIGURE 2. Translation surface and its generating curves $\mathbf{x}$(Red) and $\mathbf{y}$(Blue) for Mannhiem partner curve.

$$\begin{aligned} f &= \langle -\kappa_{\mathbf{y}}\mathbf{t}_{\mathbf{y}} + \tau_{\mathbf{y}}\mathbf{b}_{\mathbf{y}}, -\mathbf{b}_{\mathbf{y}}\rangle = -\tau_{\mathbf{y}}, \\ g &= \langle \kappa_{\mathbf{y}}\mathbf{n}_{\mathbf{y}}, \mathbf{b}_{\mathbf{y}}\rangle = 0. \end{aligned}$$

The Gaussian and mean curvatures of translation surfaces, whose generating curves are involute-evolute partner curves are obtained from Eqs. (19) and (20), as follows

$$K = \frac{eg - f^2}{EG - F^2} = -\tau_{\mathbf{y}}^2 \tag{42}$$

and

$$H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)} = \frac{-\kappa_{\mathbf{x}}\sin\theta}{2}. \tag{43}$$

**Theorem 7.** *Let $\chi(u, v) = \mathbf{x}(u) + \mathbf{y}(v)$ be translation surface where $\mathbf{x}$ and $\mathbf{y}$ are generating curves. Suppose that the generating curves are the involute-evolute partner curves. The necessary and sufficient condition for the surface $\chi$ to be developable surface is that the curve $\mathbf{y}$ is a planar curve.*

*Proof.* It is easily seen from Eq. (42) and Theorem 1 that $\tau_{\mathbf{y}} = 0$. This means that the curve $\mathbf{y}$ is a planar curve. □

**Theorem 8.** *Let $\chi(u, v) = \mathbf{x}(u) + \mathbf{y}(v)$ be translation surface where $\mathbf{x}$ and $\mathbf{y}$ are generating curves. Suppose that the generating curves are the involute-evolute partner curves. In this case, the translation surface $\chi$ cannot be a minimal surface.*

*Proof.* Since $\kappa_{\mathbf{x}} \neq 0$, considering Eq. (43), it is seen that $H \neq 0$. Therefore, such translation surfaces cannot be minimal. $\square$

**Example 3.** *Let $\mathbf{x} : I \subset \mathbb{R} \to \mathbb{E}^3$ be arbitrary parametrized curve in $\mathbb{R}^3$ parameterized by*

$$\mathbf{x}(u) = \left(\frac{8}{5}\cos u, \frac{8}{5}\sin u, \frac{4}{5}u\right).$$

*The arbitrary parametrized curve $\mathbf{y}$ involute partner curve of the $\mathbf{x}$ curve is as follows*

$$\mathbf{y}(v) = \left(\frac{8}{5}\cos v - \frac{2}{5}\sin v + \frac{2}{5}v\sin v, \frac{8}{5}\sin v + \frac{2}{5}\cos v - \frac{2}{5}v\cos v, \frac{3}{5}v\right).$$

*The translation surface generating by the $\mathbf{x}$ and $\mathbf{y}$ involute-evolute partner curves is parameterized as follows*

$$\chi(u,v) = \left(\tfrac{8}{5}\cos u + \tfrac{8}{5}\cos v - \tfrac{2}{5}\sin v + \tfrac{2}{5}v\sin v, \tfrac{8}{5}\sin u + \tfrac{8}{5}\sin v + \tfrac{2}{5}\cos v 2\tfrac{2}{5}v\cos v, \tfrac{4}{5}u + \tfrac{3}{5}v\right).$$

*In Fig. (3), we present the graph of the above translation surface and its generating involute-evolute partner curves $\mathbf{x}$ and $\mathbf{y}$.*

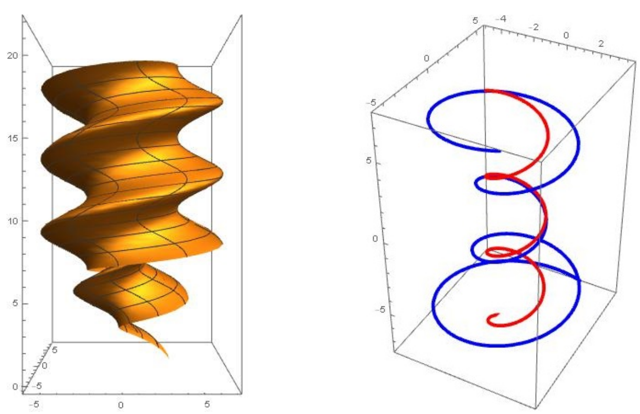


FIGURE 3. Translation surface and its generating curves $\mathbf{x}$(Red) and $\mathbf{y}$(Blue) for involute-evolute partner curves.

**Author Contribution Statements** The authors jointly worked on the results and they read and approved the final manuscript.

**Declaration of Competing Interests** The authors declare that they have no competing interest.

**Acknowledgements** The authors thank the referees for their valuable contributions to the article.

## References

[1] Bertrand, J. M., Mémoire sur la théorie des courbes á double courbure, *Comptes Rendus*, 15 (1850), 332–350.

[2] Mannheim, A., De l'emploi de la courbe représentative de la surface des normales principales d'une courbe gauche pour la démonstration de propriétés relatives à cette courbure, *Comptes Rendus des Séances de l'Académie des Sciences*, 86 (1878), 1254-1256.

[3] Do Cormo, M. P., Differential Geometry of Curves and Surfaces, Prentice-Hall, New Jersey, 1976.

[4] Eren, K., Ayvacı, K. H., Şenyurt, S., On ruled surfaces constructed by the evolution of a polynomial space curve, *Journal of Science and Arts*, 23(1) (2023), 77-90. https://doi.org/10.46939/J.Sci.Arts-23.1-a06

[5] Has, A., Yılmaz, B., Yaylı, Y., Constant angle ruled surfaces in $\mathbb{E}_1^3$, *Fundamental Journal of Mathematics and Applications*, 6 (2023), 78-88. https://doi.org/10.33401/fujma.1218966

[6] Güler, E., Yaylı, Y., Local isometry of the generalized helicoidal surfaces family in 4-space, *Malaya Journal of Matematik*, 11(2) (2023), 210-218. https://doi.org/10.26637/mjm1102/009

[7] Li, Y., Eren, K., Ayvacı, K. H., Ersoy, S., The developable surfaces with pointwise 1-type Gauss map of Frenet type framed base curves in Euclidean 3-space, *AIMS Mathematics*, 8(1) (2023), 2226-2239. https://doi.org/10.3934/math.2023115

[8] Turhan, E., Altay, G., Minimal surfaces in three dimensional Lorentzian Heisenberg group, *Beiträge zur Algebra und Geometrie / Contributions to Algebra and Geometry*, 55 (2014), 1–23. https://doi.org/10.1007/s13366-013-0156-4

[9] Çetin, M., Tunçer, Y., Ekmekçi, N., Translation Surfaces in Euclidean 3-Space, *International Journal of Mathematical and Computational Sciences*, 5(4) (2011), 49-56.

[10] López, R., Perdomo, O., Minimal translation surfaces in Euclidean space, *The Journal of Geometric Analysis*, 27 (2017), 2926–2937. https://doi.org/10.1007/s12220-017-9788-1

[11] Ali, A. T., Abdel Aziz, H. S., Sorour, A. H., On curvatures and points of the translation surfaces in Euclidean 3-space, *Journal of the Egyptian Mathematical Society*, 23 (2015), 167–172. https://doi.org/10.1016/j.joems.2014.02.007

[12] Perez Diaz, S., Shen, L., Parameterization of rational translational surfaces, *Theoretical Computer Science*, 835 (2020), 156-167. https://doi.org/10.1016/j.tcs.2017.08.005

[13] Brauner, H., Lehrbuch der Konstruktiven Geometrie, Springer-Verlag, New York, 2013.

[14] Schober, H., Transparente Schalen From Topologie Tragwerk, Wilhelm Ernst and Sohn, Berlin, 2015.

[15] Verstraelen, L., Walrave, J., Yaprak, S., The minimal translation surfaces in Euclidean space, *Soochow Journal of Mathematics*, 20(1) (1994), 77-82.

[16] Liu, H. L., Translation surfaces with constant mean curvature in 3-dimensional spaces, *Journal of Geometry*, 64 (1999), 141-149. https://doi.org/10.1007/BF01229219

[17] Yoon, D. W., On the Gauss map of translation surfaces in Minkowski 3-space, *Taiwanese Journal of Mathematics*, 6(3) (2002), 389-398. https://www.jstor.org/stable/43833333

[18] Lone, M. S., Karacan, M. K., Tuncer, Y., Es, H., Translation surfaces in affine 3-space, *Hacet.J.Math.Stat.*, 49(6) (2020), 1944–1954. https://doi.org/10.15672/hujms.564921
[19] Munteanu, M., Nistor, A. I., On the geometry of the second fundamental form of translation surfaces in $\mathbb{E}^3$, *Houston J. Math.*, 37(4) (2011), 1087-1102. https://api.semanticscholar.org/CorpusID:15706402
[20] Çetin, M., Kocayiğit, H., Önder, M., Translation surfaces according to Frenet frame in Minkowski 3-space, *Int. J. Phsy. Sci.*, 7(47) (2012), 6135-6143. https://doi.org/10.5897/IJPS12.560
[21] Yuan, Y., Liu, H. L., Some new translation surfaces in 3-Minkowski space, *Journal of Math. Res. and Exp.*, 31(6) (2011), 1123-1128.
[22] Aydın, M. E., Külahçı, M. A., Öğrenmiş, A. O., Constant curvature translation surfaces in Galilean 3-space, *International Electronic Journal of Geometry*, 12(1) (2019), 9-19. https://doi.org/10.36890/iejg.545741
[23] Struik, D. J., Lectures on Classical Differential Geometry, Dover Publications, New York, 1988.
[24] Sabuncuoğlu, A., Diferansiyel Geometri, Nobel Academic Publishing, Ankara, 2014.
[25] Yüce, S., Öklid Uzayında Diferansiyel Geometri, Pegem Press, Ankara, 2022.
[26] O'Neill, B., Semi-Riemannian Geometry, Academic Press, New York, 1983.

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 178–191 (2024)
DOI:10.31801/cfsuasmas.1144430
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

*Research Article; Received: July 17, 2022; Accepted: September 15, 2023*

# COMPOSITIONS OF INTEGERS AND FIBONACCI NUMBERS

Busra AL[1] and Mustafa ALKAN[2]

[1]Department of Computer Technologies and Programming, Manavgat Vocational School, Akdeniz University, TÜRKİYE
[2]Department of Mathematics, Faculty of Science, Akdeniz University, TÜRKİYE

Abstract. In this paper, we deal with the compositions of the integers. We present the decompositions for both the composition sets and the odd composition sets of the integers. Thus the decompositions provide us to have not only an alternative proof of some well known identies but also many new identities for Fibonacci numbers and Lucas numbers. Thus we investigate the generating functions for the product sum of the odd composition sets of the integers and attain some functional equations.

## 1. Introduction

Fibonacci numbers and compositions of a positive integer are simply expressed concepts but has many important features with many applications. Since these concepts were defined, these concepts have attracted the attention of many scientists and the results have made incredible contributions to almost all fields of sciences. These discoveries further increased the importance of mathematical analysis and number theory.

The Fibonacci numbers are numbers in which each number is the sum of the two preceding ones, denoted by $f_n$ with the initial conditions, $f_0 = 0$, $f_1 = 1$. That is, $f_n = f_{n-1} + f_{n-2}$ for $n > 1$. Moreover, in literature, there are many generalizations of Fibonacci numbers and the other special numbers with many applications.

A composition of an integer $n$ is a way of writing $n$ as a sum of positive integers. The individual summands of a composition called its parts. In the combinatorics, a classical result about the number of compositions of $n$ with an integer $k$ parts is given by the coefficient of $x^n$ of the polynomial or power series $\left(\sum\limits_{i=1}^{\infty} x^i\right)^k$ where

2020 *Mathematics Subject Classification.* 05A15, 05A17, 05A18, 11B39, 11B99.
*Keywords.* Compositions of integers, the odd combination set, Fibonacci numbers, Lucas numbers, generating function.

[1] busraal@akdeniz.edu.tr-Corresponding author; 0000-0002-1637-5355
[2] alkan@akdeniz.edu.tr; 0000-0002-4452-4442.

$|x| < 1$. These coefficients exhibit fascinating mathematical properties, closely resembling Binomial coefficients and have many useful applications ( [12], [17], [20], [21], [22]).

By using Binomial properties, Hoggart and Lind ( [22]) showed the relationship between a composition of an integer and Fibonacci numbers and proved that

(i) $f_n$ is the number compositions of an integer $n$ into odd parts

(ii) $f_{2n}$ is the sum of the products of the parts over all compositions of an integer $n$, i.e.

$$f_{2n} = \sum_{a_1+a_2+...+a_k=n} a_1 a_2 ... a_k. \tag{1}$$

Recently, there has been interested $n$–color compositions of an integer $m$ is defined as composition of $m$ for which a part of size $n$ can take on $n$ colors ( [1], [2], [27]). Then by the identity 1, it is clear that the number of $n$–color compositions of an integer $m$ is $f_{2m}$ the $2m$th Fibonacci number. Therefore, we wonder about the sequence of the sum of the products of the parts over all compositions whose parts are either odd or even. The main purpose of this paper is to investigate what the sum of the products of the parts over all compositions with odd parts is and interpret the relations among the generating functions, the set theory, compositions of an integer, Fibonacci numbers and Lucas numbers.

At first, we decompose the set of compositions of an integer and so give some very useful interpretations of the decompositions. Then we obtain an alternative proof of the above result and well- known identity by using these decompositions and reconstruct the connections between the composition of an integer and the Fibonacci numbers. These decompositions also provide us to derive some new identities and relations including the Fibonacci numbers and Lucas numbers. Next, we investigate some generating functions for the sequence of the sum of the products of the parts over all compositions whose parts are odd, the even term of the sequence and the odd term of the sequence.

Then we acquire the sequence of the sum of the products of the parts over all compositions whose each part is odd. Therefore, we focus on the generating functions for the numbers of $n$–color compositions with odd parts and so we work out their properties.

## 2. Decompositions of the Composition Sets of the Integers

In this section, we focus on decomposing the composition sets and the composition sets whose all parts are either odd or even. Then we find out some recurrence relations and also obtain an alternative proof for some well know results by using this decompositions.

We denote the composition set of an integer $n$ as follows

$$P_n = \left\{ (a_1, a_2, ..., a_t) : \ a_1 + a_2 + ... + a_t = n, \quad a_i, t \in \mathbb{Z}^+ \right\}.$$

It is well known that the number of elements of $P_n$ is $2^{n-1}$.

Now we recall the following operations for the element $a = (a_1, a_2, ..., a_t) \in P_n$ and an integer $j$;

$$\begin{aligned}(j \odot a) &= (j, a_1, a_2, ..., a_t)\,,\\ (j \oplus a) &= (a_1 + j, a_2, ..., a_t)\,.\end{aligned}$$

Then we use the notations $j \oplus P_n$ and $j \odot P_n$ for the following sets,

$$\begin{aligned}j \oplus P_n &= \{j \oplus a : a \in P_n\},\\ j \odot P_n &= \{j \odot a : a \in P_n\}.\end{aligned}$$

**Theorem 1.** *[6]Let $n, r$ be positive integers $(r \leq n)$. Then the set $P_n$ is disjoint union of the sets $(r \oplus P_{n-r})$ and $(i \odot P_{n-i})$ for all $i \in \{1, \ldots, r\}$,*

$$P_n = (r \oplus P_{n-r}) \cup \left(\cup_{i=1}^{r}(i \odot P_{n-i})\right).$$

*Proof.* It is sufficient to prove the inclusion $P_n \subseteq (r \oplus P_{n-r}) \cup (\cup_{i=1}^{r}(i \odot P_{n-i}))$.

Let $x = (a_1, \ldots, a_m) \in P_n$. If $a_1 \leq r$ then $x \in \cup_{i=1}^{r}(i \odot P_{n-i})$. Now assume that $r < a_1$. Then $b = a_1 - r$ and so define the element $y = (b, a_2, a_3, \ldots, a_m) \in P_{n-r}$. Then it is clear that $x = r \oplus y \in (r \oplus P_{n-r})$.

It is also clear that $(r \oplus P_{n-r}) \cap (i \odot P_{n-i}) = \varnothing$ for all $i \in \{1, \ldots, r\}$. □

**Corollary 1.** *[3]For a positive integer $n$, we have*

$$P_{n+1} = (1 \oplus P_n) \cup (1 \odot P_n).$$

Let $n$ be a positive integer. It is clear that the number of the elements of both $(1 \oplus P_n)$ and $(1 \odot P_n)$ are equal, i.e. $|1 \oplus P_n| = |1 \odot P_n|$ and it follows that $|P_{n+1}| = 2\,|1 \odot P_n|$ since these sets are disjoint. On the other hand, by $|P_2| = 2$, we have that $|P_n| = 2^{n-1}$ by induction method. Therefore we have completed an alternative proof by using the set theory for the well-known result as a result of the Corollary 1.

Now we point out our attention to the composition sets whose parts are even or odd. Let us use the notions

$$\begin{aligned}O_n &= \{(a_1, ..., a_t) :\ a_1 + ... + a_t = n \text{ and } a_i \text{ is positive odd integer}\}\\ E_{2n} &= \{(2a_1, ...2a_t) :\ 2a_1 + ... + 2a_t = 2n \text{ and } a_i \text{ is positive integer}\}\end{aligned}$$

and we call the set as an odd composition set $O_n$ (even composition set $E_n$) of an integer $n$. It is clear that the even composition set of an even integer $2n$ involved to the composition set of an integer $n$ and so the number of elements of the even composition set of $2n$ is $2^{n-1}$.

At this moment, we focus on to decompose the odd composition set as union of subset of odd combinations set of integers.

**Theorem 2.** *For a positive integer $n$, we decompose the odd composition set of an integer $n$ as a disjoint union of subset of odd combinations set of integers;*

$$O_{2n+1} = \{(2n+1)\} \cup \bigcup_{i=0}^{n-1} \left((2i+1) \odot O_{2(n-i)}\right) \tag{2}$$

$$O_{2n} = \bigcup_{i=0}^{n-1} \left((2i+1) \odot O_{2(n-i)-1}\right). \tag{3}$$

*Proof.* Let $n$ be a positive integer. It is enough to show one side inclusion for the odd number $2n+1$.

Let $x = (2a_1+1, ..., 2a_t+1)$ and assume that $t$ is different from 1. Then $n - 2a_1 - 1 = 2m$ for an integer even and so the element $b = (2a_2+1, ...2a_t+1)$ is $O_{2m}$. Therefore $x = (2a_1+1) \odot O_{2n-2a_2}$ and this complete the proof. □

With the decomposition in Theorem 2, we prove again a well-known result using set theory.

**Corollary 2.** *The number of element of the odd composition set of an integer $n$ is the n.th Fibonacci number.*

*Proof.* Let $k_n$ be the number of element of the odd composition set of an integer $n$. Since the sets in Theorem 2 are disjoint, it is easy to prove that $k_{n+1} = k_n + k_{n-1}$ and $k_1 = 1, k_2 = 1$. □

As a conclusion of Theorem 2, we can reprove the well known identities [25, page 92]

$$f_{2n+1} = 1 + \sum_{i=1}^{n} f_{2i}$$

$$f_{2n} = \sum_{i=0}^{n-1} f_{2i+1}$$

for both the even and odd Fibonacci number.

## 3. Product Sum Function

By the motivation of the identity 1, we interested in the sequence of the sum of the products the parts over all compositions. In this section, we define function from compositions set to integer to obtain some number sequences and then interpret the relations among the set theory, the compositions of an integer, Fibonacci numbers and Lucas numbers. Thus we attain an alternative proof for the identity 1.

### 3.1. The composition set of the integers.

Now we establish the function from the composition sets to positive integers defined by

$$T_n := T(P_n) = \sum_{a \in P_n} \bar{a}.$$

We call $T_n = T(P_n)$ as the product sum of the composition set $P_n$ (or the product sum of the integer $n$). For $n = 0$, we may assume that $T_0 = 1$.

We give an easy numeric example with the new notions;

**Example 1.** *Let* $n = 4$. *Then it follows that*

$$P_4 = \{(4), (1, 1, 1, 1), (1, 1, 2), (1, 2, 1), (2, 1, 1), (2, 2), (3, 1), (1, 3)\}$$

*and* $T_4 = T(P_4) = 21$.*Moreover, it follows*

$$1\odot P_4 = \{(1, 4), (1, 1, 1, 1, 1), (1, 1, 1, 2), (1, 1, 2, 1), (1, 2, 1, 1), (1, 2, 2), (1, 3, 1), (1, 1, 3)\}$$

$$1 \oplus P_4 = \{(5), (2, 1, 1, 1), (2, 1, 2), (2, 2, 1), (2, 3), (3, 1, 1), (3, 2), (4, 1)\}$$

*and so* $P_5 = (1 \odot P_4) \cup (1 \oplus P_4)$. *Then* $T_5 = T(P_5) = 55$.

By using Theorem 1, we develop a recurrence for the product sum of the composition sets.

**Theorem 3.** *For a positive integer* $n$, *we have*

$$T_{n+1} = T_n + \sum_{i=0}^{n} T_{n-i}. \tag{4}$$

*Proof.* For an element $a \in P_{n+1}$, there is $b = (b_1, b_2, ..., b_l) \in P_n$ such that either $\overline{a} = \overline{1 \odot b} = \overline{b}$ or $\overline{a} = \overline{1 \oplus b}$ and so $\overline{a} = \overline{1 \odot b} = \overline{b}$ or $\overline{a} = \overline{1 \oplus b} = (b_2....b_l) + \overline{b}$. Hence we have that

$$T(1 \odot P_n) = \sum_{1\odot b\in 1\odot P_n} \overline{b} = T_n.$$

Moreover, it follows that

$$\begin{aligned} T(1 \oplus P_n) &= \sum_{a\in P_n} (1 + a_1).a_2.a_3...a_t \\ &= \sum_{a\in P_n} (a_1.a_2.a_3...a_t) + \sum_{i=1}^{n} \sum_{(a_2.a_3...a_t)\in P_{n-i}} (a_2a_3...a_t) \\ &= T_n + \sum_{i=1}^{n} T_{n-i} = \sum_{i=0}^{n} T_{n-i}. \end{aligned}$$

Therefore, we have that

$$T_{n+1} = T(P_{n+1}) = T(1 \odot P_n) + T(1 \oplus P_n) = T_n + \sum_{i=0}^{n} T_{n-i}.$$

Hence we have completed the proof. □

By using the recurrence relation Identity 4, we gain the generating function for the product sum of the positive integers. From [25], we recall the generating function for even Fibonacci numbers is that

$$f(x) = \frac{x}{1 - 3x + x^2} = \sum_{n=1}^{\infty} f_{2n}x^n$$

Thus we give an alternative proof of the result of Hoggart and Lind in [22].

**Theorem 4.** *The generating function of the product sum of the positive integer is*

$$\sum_{n=1}^{\infty} T_n x^n = \frac{x}{1-3x+x^2}.$$

*i.e. The product sum of the positive integer n is nth even Fibonacci number*

*Proof.* Let $h(x) = \sum_{n=1}^{\infty} T_n x^n$. Then

$$\begin{aligned} h(x) &= x + \sum_{n=1} T_{n+1} x^{n+1} \\ &= x + x\sum_{n=1}\left(T_n + \sum_{i=0}^{n} T_{n-i}\right) x^n \\ &= x + xh(x) - x^2 h(x) + 2xh(x). \end{aligned}$$

Thus we get the function as

$$h(x) = \frac{x}{1-3x+x^2}.$$

□

As a result of Theorem 3 and Theorem 4, we obtain the known identity [25, Page 92- Identity 5.3] for odd Fibonacci numbers and also prove a new identities for Fibonacci numbers in the following;

**Theorem 5.** *Let $n, m$ be positive integers. Then we have*

$$f_{2n+1} = 1 + \sum_{i=1}^{n} f_{2i} \tag{5}$$

$$f_{2n} = n + \sum_{i=1}^{n-1} (n-i) f_{2i}. \tag{6}$$

*Proof.* By Theorem 3, we have the recurrence

$$T_{n+1} = T_n + \sum_{i=0}^{n} T_{n-i}. \tag{7}$$

and it follows that $T_{n+1} = T_0 + T_n + \sum_{i=1}^{n} T_i$. Thus we gain that

$$f_{2(n+1)} = 1 + f_{2n} + \sum_{i=0}^{n} f_{2(n-i)}$$

Since $f_{2n+1} = f_{2n+2} - f_{2n}$, we have proved the identity 5.

Now we decompose $P_n$ to get some new equations for the Fibonacci numbers. For an integer $i$, we define the set

$$(i \odot P_{n-i}) = \{(i, a_1, a_2, ..., a_t) : a_1 + a_2 + ... + a_t = n - i, \quad a_i, t \in \mathbb{Z}^+\}.$$

Then it is easy to check that

$$P_n = \cup_{i=1}^{n} (i \odot P_{n-i})$$

and also for all $i, j$ with $i \neq j$, it follows that $(i \odot P_{n-i}) \cap (j \odot P_{n-j}) = \varnothing$. Therefore it follows that

$$T(i \odot P_{n-i}) = \sum_{(a_1, a_3 \ldots a_t) \in P_{n-i}} i.a_1.a_3 \ldots a_t = iT(P_{n-i}) = iT_{n-i}$$

and so

$$T_n = T(P_n) = \sum_{i=1}^{n} T(i \odot P_{n-i}) = \sum_{i=1}^{n} iT_{n-i} = \sum_{i=0}^{n-1} (n-i)T_i. \tag{8}$$

Thus the we complete the proof. □

**Theorem 6.** *Let $n, m$ be positive integers $(m \leq n)$. Then we have*

$$f_{2n} - f_{2m} = \sum_{i=1}^{n-m} i f_{2(n-i)} + (n-m) \sum_{i=1}^{m} f_{2(n-i)} \tag{9}$$

$$f_{2n-1} - f_{2m-1} = \sum_{i=1}^{n-m} f_{2(n-i)}. \tag{10}$$

*Proof.* For any integers $n, r$ we get that

$$f_{2n} - f_{2(n-r)} = \sum_{i=1}^{r} i f_{2(n-i)} + r \sum_{i=1}^{n-r} f_{2(n-i)}$$

and so substituting $m = n - r$, we acquire the identity 9.

By Theorem 3, we have the recurrence

$$T_{n+1} = T_n + \sum_{i=0}^{n} T_{n-i}. \tag{11}$$

and it follows that

$$\begin{aligned} \sum_{i=1}^{n-m} T_{n-i} &= \sum_{i=m}^{n-1} T_n + \sum_{i=1}^{m-1} T_n - \sum_{i=1}^{m-1} T_n \\ &= \sum_{i=1}^{n-1} T_n - \sum_{i=1}^{m-1} T_n = f_{2n-1} - f_{2m-1} \end{aligned}$$

Thus we achieve the identity 10. □

By by Theorem 5 we have the following equation

$$f_{2n+2} = 1 + n + f_{2n} + \sum_{i=1}^{n-1} (n-i+1) f_{2i}$$

and we also obtain

$$f_{2n+3} = n + 2 + 2f_{2n} + \sum_{i=1}^{n-1}(n-i+2)f_{2i}.$$

For an integer $r$, we have

$$\begin{aligned} f_{2n+r} &= (f_{2n}+1)f_r + nf_{r-1} + \left[f_{r-1}\sum_{i=1}^{n-1}(n-i)f_{2i} + f_r\sum_{i=1}^{n-1}f_{2i}\right] \\ &= f_r f_{2n+1} + f_{r-1}f_{2n}. \end{aligned}$$

Therefore we just gain the combinatorial proof of the Honsberger's formula by using the compositions of an integer.

**Corollary 3.** *For positive integers $n, m$, we have*

$$\begin{aligned} f_{2n+2m} &= f_{2m}f_{2n+1} + f_{2m-1}f_{2n} \\ f_{2n+2m+1} &= f_{2m+1}f_{2n+1} + f_{2m}f_{2n}. \end{aligned}$$

**Corollary 4.** *Let $n$ be positive integer. Then we have*

$$f_{4n} = f_{2n}f_{2n-1} + f_{2n}f_{2n+1} \tag{12}$$
$$f_{4n+1} = f_{2n}^2 + f_{2n+1}^2 \tag{13}$$
$$\begin{aligned} f_{4n+2} &= f_{2n}f_{2n+1} + f_{2n+1}f_{2n+2} \\ f_{4n+3} &= f_{2n}f_{2n+2} + f_{2n+1}f_{2n+3} \end{aligned}$$

*Proof.* It is clear from Corollary 3. □

Let $l_n$ be the $n$ *th* term of Lucas sequence, defined by $l_0 = 2$, $l_1 = 1$, and $l_n = l_{n-1} + l_{n-2}$, $n > 3$. Also, one of the well-known relation between Fibonacci numbers and Lucas numbers is

$$l_n = f_{n-1} + f_{n+1}. \tag{14}$$

Thus by using the identity 13 and Cassini's formula, we obtain

$$\begin{aligned} f_{4n+1} &= f_{2n}^2 + 1 + f_{2n}f_{2n+2} \\ &= f_{2n}l_{2n+1} + 1 \end{aligned}$$

and it follows that

$$\begin{aligned} f_{4n+2} &= f_{4n} + f_{4n+1} = f_{2n}(l_{2n} + l_{2n+1}) + 1 \\ &= f_{2n}l_{2n+1} + f_2. \end{aligned}$$

Therefore, we just gain the following identity which is the general form of the well known result ( [25, page 90]).

**Corollary 5.** *For positive integers $r, n$, we have the equality*

$$f_{4n+r} = f_{2n}l_{2n+r} + f_r.$$

**Theorem 7.** *For a positive integer $n$, we have the identities for Lucas numbers*

$$l_{2n+1} = 2n+1+\sum_{i=1}^{n-1} l_{2i+1}(n-i), \tag{15}$$

$$l_{2n} = 3n+1+f_{2n-2}+\sum_{i=1}^{n-2}\left(2f_{2i}+(n-1-i)l_{2i+1}\right). \tag{16}$$

*Proof.* By Theorem 5, we get the following result

$$f_{2n} = n+\sum_{i=1}^{n-1}(n-i)f_{2i}.$$

Thus,

$$\begin{aligned} l_{2n+1} &= f_{2n}+f_{2n+2} \\ &= \left[n+\sum_{i=1}^{n-1}(n-i)f_{2i}\right]+\left[n+1+\sum_{i=0}^{n-1}(n-i)f_{2(i+1)}\right] \end{aligned}$$

and so we have proved the identity 15.

For the second the identity, it is known that

$$f_{2n-1} = 1+\sum_{i=1}^{n-1} f_{2i}. \tag{17}$$

and

$$l_{2n-2} = f_{2n-3}+f_{2n-1}\ . \tag{18}$$

Then we gain the equation

$$l_{2n-2} = 2\left[1+\sum_{i=1}^{n-2} f_{2i}\right]+f_{2n-2}.$$

On the other hand, by the identity 15, we get

$$\begin{aligned} l_{2n} &= 2n+1+f_{2n-2}+\sum_{i=1}^{n-2}\left(2f_{2i}+(n-1-i)l_{2i+1}\right) \\ &= 3n+1+2f_{2n-2}+\left(\sum_{i=2}^{n-2}(2n+1-2i)f_{2i}\right). \end{aligned}$$

□

**Corollary 6.** *Let $n, r$ be positive integers. Then we have*

$$l_{4n+r} = f_{2n}l_{2n+(r-1)}+f_{2n+1}l_{2n+r}$$

3.2. **The odd composition set of the integers.** Now we focus on the combinations of an integer whose each part is either odd nor even and we reach to the main goal of the paper which is to investigate the product sum of both an odd and even composition of an integer $n$.

Let us define the number sequence such as

$$o_n \quad : \quad = \sum_{a \in O_n} \bar{a} \tag{19}$$

$$e_n \quad : \quad = \sum_{a \in E_n} \bar{a}. \tag{20}$$

One may compute the sequence as

$$\begin{aligned} o_1 &= 1, o_2 = 1, o_3 = 4, o_4 = 7, \; o_5 = 15, o_6 = 32, o_7 = 65, o_8 = 137 \\ e_2 &= 2, e_4 = 16, e_6 = 48. \end{aligned}$$

By using the decomposition of an odd composition of an integer $n$, we figure out a recurrence relations for the product sum of an odd composition of an integer $n$.

**Theorem 8.** *For a positive integer $n \geq 1$, we have the recurrence relations for both an even and an odd term of the product sum of an odd composition of an integer*

$$o_{2n+2} = o_{2n+1} + 2o_{2n} + o_{2n-1} - o_{2n-2} \tag{21}$$

$$o_{2n+3} = 3o_{2n} + 3o_{2n+1} - o_{2n-2}. \tag{22}$$

*Proof.* Let $n$ be an positive integer. Then we apply the the definition of the product sum function to the decomposition in Theorem 2 and so we get

$$\begin{aligned} o_{2n+1} &= 2n + 1 + \sum_{i=0}^{n-1} \sum_{b \in O_{2(n-i)}} (2i+1)\bar{b} \\ &= 2n + 1 + \sum_{i=0}^{n-1} (2i+1) o_{2(n-i)} \end{aligned}$$

and it also follows that

$$\begin{aligned} o_{2n+3} &= 2 + o_{2n+2} + \left( 2n + 1 + \sum_{i=0}^{n-1} (2i+1) o_{2(n-i)} \right) + 2 \sum_{i=0}^{n-1} o_{2(n-i)} \\ &= 2 + o_{2n+2} + o_{2n+1} + 2 \sum_{i=1}^{n} o_{2i}. \end{aligned}$$

When we compute the difference between $o_{2n+3}$ and $o_{2n+1}$, we get the recurrence for the odd term of the product sum of an odd composition of an integer $n$

$$o_{2n+3} = o_{2n+2} + 2o_{2n+1} + o_{2n} - o_{2n-1}. \tag{23}$$

On the other hand, by the decomposition in Theorem 2, we point out the recurrence for the even term of the product sum of an odd composition of an integer $n$ as

$$o_{2n} = \sum_{i=0}^{n-1}(2i+1)o_{2(n-i)-1}.$$

Then we compute

$$\begin{aligned} o_{2n+2} &= o_{2n+1} + \sum_{i=0}^{n-1}(2i+1+2)o_{2(n-i)-1} \\ o_{2n+2} &= o_{2n+1} + o_{2n} + 2\sum_{i=1}^{n} o_{2i-1}. \end{aligned}$$

By the difference between $o_{2n+2}$ and $o_{2n+2}$, we obtain the recurrence for the even terms

$$o_{2n+2} = o_{2n+1} + 2o_{2n} + o_{2n-1} - o_{2n-2}.$$

By substituting $o_{2n+2}$ in the identity 23, we figure out

$$o_{2n+3} = 3o_{2n} + 3o_{2n+1} - o_{2n-2}$$

This completes the proof. □

**Theorem 9.** *The generating function for the product sum of an odd composition sets is*

$$U(x) = 1 + x^2\,(x+1)\,\frac{-2x + x^2 - 1}{x + 2x^2 + x^3 - x^4 - 1},$$

*where* $|x| < 1$.

*Proof.* For an integer $n$, we have the recurrence relations for either an even or an odd term of the product sum of an odd composition of an integer

$$\begin{aligned} o_{2n+3} &= 3o_{2n} + 3o_{2n+1} - o_{2n-2} \\ o_{2n+2} &= o_{2n+1} + 2o_{2n} + o_{2n-1} - o_{2n-2}. \end{aligned}$$

Let $U(x) = \sum_{n=1}^{\infty} o_n x^n = 1 + \sum_{n=1}^{\infty} o_{2n}x^{2n} + \sum_{n=1}^{\infty} o_{2n+1}x^{2n+1}$ be the generating function for the product sum of an odd composition of integers and so it is enough to investigate

$$\begin{aligned} A(x) &= \sum_{n=1} o_{2n}x^{2n} \\ B(x) &= \sum_{n=1}^{\infty} o_{2n+1}x^{2n+1}. \end{aligned}$$

By using the recurrence identity 22, it is easy to compute that

$$(1-3x^2)B(x) = x^3(3-x^2)A(x) + 4x^3. \tag{24}$$

Similarly it is also easy to compute

$$A(x) = \frac{x\left(x^2+1\right)}{\left(x^2-1\right)^2}B(x) + \frac{x^2\left(x^2+1\right)}{\left(x^2-1\right)^2}, \tag{25}$$

due to the recurrence identity 21. Then combining the equations 24 and 25, we figure out both $A$ and $B$ and so it follows that

$$\begin{aligned} B(x) &= -x^3 \frac{5x^2-6x^4+x^6-4}{\left(x+2x^2+x^3-x^4-1\right)\left(x-2x^2+x^3+x^4+1\right)} \\ A(x) &= x^2 \frac{\left(x^2+1\right)^2}{\left(x-2x^2+x^3+x^4+1\right)\left(-x-2x^2-x^3+x^4+1\right)}. \end{aligned}$$

Therefore we investigate the generating function

$$U(x) = 1 + x^2\left(x+1\right)\frac{-2x+x^2-1}{x+2x^2+x^3-x^4-1}.$$

□

Moreover, we study out the generating function for either an odd or even term of product sum of an odd composition.

**Theorem 10.** *The generating function for the either an odd or an even term of product sum of an odd composition sets are*

$$\begin{aligned} O(x) &= -x\frac{5x-6x^2+x^3-4}{x^4-5x^3+4x^2-5x+1}, \\ E(x) &= x\frac{(x+1)^2}{x^4-5x^3+4x^2-5x+1}, \end{aligned}$$

*where* $|x| < 1$.

*Proof.* Let

$$\begin{aligned} E &= E(x) = \sum_{n=1}^{\infty} o_{2n}x^n \\ O &= O(x) = \sum_{n=1}^{\infty} o_{2n+1}x^n. \end{aligned}$$

be the generating function for the either an odd or an even term of product sum of an odd composition sets. Then by using the recurrence identity 21 and 22, we compute

$$(1-3x)O = x(3-x)E + 4x$$

and due to the recurrences, we compute

$$E = \frac{x\left(x+1\right)\left(O+1\right)}{\left(x-1\right)^2} = \frac{\left(x^2+x\right)}{\left(x-1\right)^2}O + \frac{\left(x^2+x\right)}{\left(x-1\right)^2}.$$

Therefore we figure out the generating function for the either an odd or even term of product sum of an odd composition and this completes the proof. □

**Author Contribution Statements** The authors contributed equally to the study.

**Declaration of Competing Interests** No conflict of interest or common interest has been declared by the authors.

**Acknowledgements** The present paper was supported by the Scientific Research Project Administration of Akdeniz University.

## References

[1] Agarwal, A. K., n-Colour composition, *Indian J. Pure Appl. Math.,* 31(11) (2000), 1421-1427.
[2] Agarwal, A. K., Andrews, G. E., Rogers-Ramanujan identities for partitions with "N copies of N", *J. Combin. Theory Ser. A.,* 45(1) (1987), 40-49.
[3] Al, B., Alkan, M., Some Relations Between Partitions and Fibonacci Numbers, In: Proceedings Book of the 2nd Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2019) (Ed. by Y. Simsek, A. Bayad, M. Alkan, I. Kucukoglu and O. Ones), Antalya, Turkey, August 28-31, 2019, 14-17; ISBN: 978-2-491766-00-9.
[4] Al, B., Alkan, M., On relations for the partitions of numbers, *Filomat,* 34(2) (2020), 567–574. DOI:10.2298/FIL2002567A
[5] Al, B., Alkan, M., A Note on the Composition of a Positive Integer whose Parts are Odd Integers, International Conference on Artificial Intelligence and Applied Mathematics in Engineering Abstract Book (2022), 141. https://icaiame.com/wp-content/uploads/2022/06/ICAIAME-2022-Accepted-Abstracts-E-Book.pdf
[6] Al, B., Alkan, M., A Note on Color Compositions and the Patterns, In: Proceedings Book of the 5th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2022), 2022, 158-161. ISBN: 978-625-00-0917-8
[7] Andrews, G. E., The Theory of Partitions, Addison-Wesley Publishing, New York, 1976.
[8] Andrews, G. E., Erikson, K., Integer Partitions, Cambridge University Press, Cambridge, 2004.
[9] Andrews, G. E., Hirschhorn, M. D., Sellers, J. A., Arithmetic properties of partitions with even parts distinct, *Ramanujan Journal,* 23(1–3) (2010), 169–181. DOI:10.1007/s11139-009-9158-0
[10] Apostol, T. M., On the Lerch Zeta function, *Pacific J. Math.,* 1 (1951), 161–167. DOI:10.2140/pjm.1951.1.161
[11] Apostol, T. M., Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976.
[12] Archibald, M., Blecher, A., Knopfmacher, A., Inversions and parity in compositions of integers, *Journal of Integer Sequences,* 23 (2020). https://cs.uwaterloo.ca/journals/JIS/VOL23/Archibald/arch3.pdf
[13] Birmajer, D., Gil, J. B., Weiner, M. M. D., (an + b)-color compositions, arXiv:1707.07798.
[14] Chen, S. C., On the number of partitions with distinct even parts, *Discrete Math.,* 311 (2011), 940-943. DOI:10.1016/j.disc.2011.02.025
[15] Euler, L., Introduction to Analysis of the Infinite, Vol. 1, Springer-Verlag, 1988.
[16] Ewell, J. A., Recurrences for the partition function and its relatives, *Rocky Mountain Journal of Mathematics,* 34(2) (2004). DOI:10.1216/rmjm/1181069871
[17] Gessel I. M., Li, J., Compositions and Fibonacci identities, *Journal of Integer Sequences,* 16 (2013). DOI:10.48550/arXiv.1303.1366
[18] Gil, B., Tomosko, J. A., Fibonacci colored compositions and applications, arXiv:2108.06462.
[19] Gupta, H., Partitions - A Survey, *Journal of Research of the Notional Bureau of Standards-B. Mathematical Sciences*, 74B(1) (1970).

[20] Heubach, S., Mansour, T., Compositions of n with parts in a set, *Congr. Numer.*, 168 (2004), 127–143.

[21] Heubach, S., Mansour, T., Combinatorics of Compositions and Words, CRC Press, 2010.

[22] Hoggatt, V. E., Lind, D. A., Fibonacci and binomial properties of weighted compositions, *J. Combin. Theory.*, 4 (1968), 121-124. DOI:10.1016/S0021-9800(68)80037-7

[23] Horadam, A. F., Jacobsthal representation numbers, *Fibonacci Quarterly*, 34(1) (1996), 40-54.

[24] Janjic, M., Some formulas for numbers of restricted words, *Journal of Integer Sequences*, 20 (2017).

[25] Koshy, T., Fibonacci and Lucas Numbers with Applications, Canada: Wiley-Interscience Publication, 2001, 6-38.

[26] Merzouka, H., Boussayoudb, A., Chelgham, M., Generating functions of generalized Tribonacci and Tricobsthal polynomials, *Montes Taurus J. Pure Appl. Math.*, 2(2), (2020), 7–37.

[27] Shapcott, C., C-color compositions and palindromes, *Fibonacci Quart.,* 50(4) (2012), 297-303.

[28] Stanley, R. P., Enumerative Combinatorics, Vol 1, 2nd edition, Cambridge University Press, 2011.

[29] Simsek, Y., Generating functions for finite sums involving higher powers of binomial coeffients:Analysis of hypergeometric functions inculudinf new families of polynomilies and numbers, *J.Math. Anal Appl.,* 477 (2019), 2328-1352.

[30] Ozdemir, G., Simsek, Y., Milovanovic, G. V., Generating functions for special polynomials and numbers including Apostos-Type and Humbert-Type polynomials, *Mediterr. J. Math.,* 14(117) (2017). DOI:10.1007/s00009-017-0918-6

[31] Wilf, H. S., Generating Functionology, Academic Press, Inc., 1994.

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 192–210 (2024)
DOI:10.31801/cfsuasmas.1207144
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

*Research Article; Received: November 19, 2022; Accepted: October 25, 2023*

# FRACTIONAL ORDER MATHEMATICAL MODELING OF LUMPY SKIN DISEASE

Yogeeta NARWAL[1] and Savita RATHEE[2]

[1]Government College, Baund Kalan, Charkhi Dadri 127025, Haryana, INDIA
[1,2]Department of Mathematics, Maharshi Dayanand University, Rohtak 124001, Haryana, INDIA

Abstract. In this article, we study the fractional-order SEIR mathematical model of Lumpy Skin Disease (LSD) in the sense of Caputo. The existence, uniqueness, non-negativity and boundedness of the solutions are established using fixed point theory. Using a next-generation matrix, the reproduction number $R_0$ is determined for the disease's prognosis and durability. Using the fractional Routh-Hurwitz stability criterion, the evolving behaviour of the equilibria is investigated. Generalized Adams–Bashforth–Moulton approach is applied to arrive at the solution of the proposed model. Furthermore, to visualise the efficiency of our theoretical conclusions and to track the impact of arbitrary-order derivative, numerical simulations of the model and their graphical presentations are carried out using MATLAB(R2021a).

## 1. Introduction

Lumpy skin disease mainly spread to ruminants such as cattle and water buffaloes (Bubalus bubalis), making it a non-zoonotic viral disease that develop and reproduce entirely in non-human hosts via arthropod vectors such as biting flies, mosquitoes, and ticks. Contagious sustenance such as contaminated fodder, water and animal semen during artificial insemination are also responsible for the spread. It is a trans-boundary disease brought on by the Lumpy skin disease virus (LSDV) which go by names Pseudo-urticaria, Neethling viral disease belonging to the Poxviridae family, and genus Capripoxvirus ([6], [19], [28], [51], [59]).

Zambia marked the presence of LSD in 1929 [38], propagating to Zimbabwe and South Africa in 1949, Ethiopia in 1983, Israel in 1989, and then spreading

2020 *Mathematics Subject Classification.* 00A71, 26A33, 34A08, 47H10, 92D43.
*Keywords.* Lumpy skin disease, SEIR mathematical model, fractional derivative, fixed point theory, Adams-Bashforth-Moulton method.

[1] yogeetawork@gmail.com-Corresponding author; 0000-0003-2956-6836
[2] savitarathee.math@mdurohtak.ac.in; 0000-0002-1540-007X.

throughout the Middle East, West Asia, and Europe. It produced a massive economic calamity in South Africa for about 30 years (1950-1980) [3], [30]. The year 2018-19 recorded infections in Greece, Georgia, and Russia. Cattle in various Asian nations are currently suffering from LSD including Nepal [4], Thailand [7], [37], Malaysia, Laos, Cambodia [8], Mynmar [18], Bangladesh [25], India [23], China [32], Sri Lanka, Bhutan and Vietnam [58].

Though cattle are the prime species to be infected by LSD but experimental infections show that the virus can also infect sheep, goat, giraffe, gazalles and impalas [23]. The name LSD is attributed to the fact that lymph nodes of the infected animal grows and resemble lumps on the skin. Large cutaneous nodules emerge on the head, neck, arms, legs, udder, abdomen, and private parts of the infected cattle subsequently evolve into ulcers and finally convert into skin scabs [51]. According to the FAO [24], it is a high morbidity(2-45 percent) and low mortality disease (less than 10 percent). The disease evolve in 4 to 14 days.

August 2019 marked the initial outbreak of LSD in the Indian states of Odisha and West Bengal [56]. Within a few months, other LSD outbreaks were recorded across the country causing the dairy industry to incur significant financial losses. With the most cow and buffalo in the world, India is the largest milk producer and ranks first in the world, producing twenty-four percent of global milk output in 2021-22. According to government data, lumpy skin disease has infected millions of cattle and killed more than 1,84,000 in India, causing less milk production due to weakness and appetite loss caused by mouth ulcers, inadequate development, decreasing draught power, and reproductive difficulties such as abortions, infertility, and a lack of sperm for artificial insemination. As a result, LSD has been identified in India as a potentially lethal disease for cattle.

1.1. **Motivation and Research Background.** Modelling of epidemic diseases is of utmost importance to understand the behaviour of the ailment across time and to devise appropriate safeguards for the same. Numerous epidemic models have been developed for various diseases, including dengue and chikungunya [1], typhoid [2], cholera [10], HIV/AIDS [11], Covid-19 [14], [15], [57], leptospirosis, H1N1, measles [17], and others. But to our surprise there is not enough research on transmission dynamics and LSD control using a compartmental modelling technique; by the time this study was completed, there had only been one work [46], to examine the effects of vaccination on LSD and the spread of the illness in Ethiopia. Butt et al. [15] had also researched the SVEIR epidemic model and examined it for the presence of a unique positive and bounded solution at the end of initial revision. The authors of both of these studies, however, relied on the traditional integer-order derivatives, which are frequently unable to foresee the remembrance and inheritance characteristics of substances and phenomenons, leading to erroneous depictions of dynamic real-world events. Due to the significant amount of unidentified, uncertainties, and misinformation, developing a mathematical model that accurately captures LSD using classical differentiation is a difficult task. The use of non-local operators is encouraged by coincidences and diminishing retention effects, the argument being supported by plenty of scholarly articles [12], [13], [42], [43], [60].

Fractional derivatives come in a wide range of forms, both with and without singular kernels. For singular kernels, we've got the derivatives of Caputo, Riemann-Liouville, and Katugampola [27], [53]. The Caputo-Fabrizio fractional derivative [16], which has an exponential kernel, and the Atangana-Baleanu fractional derivative [9], which has a Mittag-Leffler kernel, are the two types of fractional derivatives without singular kernels. It is crucial to work with fractional-order derivatives because they provide a more accurate way to describe LSD outbreaks, even while memory and genetic features are implicated. We offer and examine the fractional order SEIR mathematical model in Caputo sense in light of the recent research to comprehend the evaluation, existence, stability, and control of LSD and to the best of our knowledge, this is the first paper to use fractional order derivative for modeling the transmission dynamics of LSD, which is critical for understanding the epidemiology and dynamic nature of exotic disease for timely disease management and planning because of the global character of the fractional derivatives which improves the system's consistency domain. The Caputo derivative serves best as a base model and is preferred over Riemann–Liouville fractional derivative for formulating epidemiological models for the obvious reasons concerning the use of initial and boundary conditions and the differentiation of a constant being zero. For more details one can refer to the following researches [4], [7], [8], [18], [20], [23], [25], [26], [36], [40], [47], [52], [54], [58], [62].

1.2. **Structure of the Paper.** The following is how rest of the paper is set up: Section 2 presents auxiliary results and essential notions from fractional calculus. The LSD propagation model is devised in Section 3, along with a schematic diagram for the same. Section 4 provides us with the insights of the model by providing the existence, uniqueness, positivity, and feasible region for the proposed system's solution, along with the analysis of the equilibrium points, reproduction number, and stability of the proposed model. Computational simulations are executed in Section 5 to backup the qualitative analysis results of the model. The findings and discussions required for the policy implications are covered in Section 6.

## 2. Auxiliary Results

**Definition 1** ([31])**.** *The Caputo fractional derivative of a continuous function $g$ on $[0,T]$ is defined as:*

$$\mathfrak{D}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \frac{d^n}{ds^n} g(s) ds,$$

*where $0 < \alpha \leq 1$, $n = [\alpha] + 1$, and $[\alpha]$ represents the integer part of $\alpha$.*

**Definition 2** ([31]). *The fractional integral of a continuous function $g$ on $L^1([0,T],\mathbb{R})$ of order $0<\alpha\leq 1$ corresponding to $t$ is defined as:*

$$I^{\alpha}g(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds.$$

**Definition 3** ([29]). *The Laplace transform is defined by*

$$F(s)=L[f(t)]=\int_0^{\infty}e^{-st}f(t)dt,$$

*where $f(t)$ is $n$-dimensional vector-valued function.*

**Definition 4** ([49]). *The Mittag-Leffler function in two parameters is defined as*

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\ z\in\mathbb{C},$$

*where $\alpha>0$, $\beta>0$, $\mathbb{C}$ denotes the complex plane.*

**Lemma 1** ([29]). *Let $\mathbb{C}$ be a complex plane, for any $\alpha>0$, $\beta>0$ and $A\in C^{n\times n}$,*

$$L[t^{\beta-1}E_{\alpha,\beta}(At^{\alpha})]=\frac{s^{\alpha-\beta}}{s^{\alpha}-A}$$

*holds for $Re(s)>\|A\|^{\frac{1}{\alpha}}$, where $Re(s)$ represents the real part of the complex number $s$.*

**Lemma 2** ([39]). *Let $F(s)$ be the Laplace transform of the function $f(t)$, $n$ being an integer then the Laplace transform of the Caputo fractional derivative of order $\alpha$ is given by*

$$L(\mathfrak{D}^{\alpha}f(t))=s^{\alpha}F(s)-\sum_{k=1}^{n}s^{\alpha-k}f^{(k-1)}(0),\ n-1<\alpha\leq n.$$

**Lemma 3** ([44], Generalized Mean Value Theorem). *Let $g(t)\in C[a,b]$ and $\mathfrak{D}^{\alpha}g(t)\in C[a,b]$ for $0<\alpha\leq 1$, then*

$$g(t)=g(a)+\frac{1}{\Gamma(\alpha)}(\mathfrak{D}^{\alpha}g)(s)(t-a)^{\alpha}$$

*with $0\leq s\leq t$, $\forall\ t\in(a,b]$. Thus, we can deduce that for $g(t)\in C[0,b]$ and Caputo fractional derivative $\mathfrak{D}^{\alpha}g(t)\in C[0,b]$ for $0<\alpha\leq 1$, if $\mathfrak{D}^{\alpha}g(t)\geq 0$, $\forall\ t\in[0,b]$, then the function $g(t)$ is non-decreasing and if $\mathfrak{D}^{\alpha}g(t)\leq 0$, $\forall\ t\in[0,b]$, then the function $g(t)$ is non-increasing $\forall\ t\in[0,b]$.*

**Theorem 1** ([55]). *Consider the fractional differential equation:*

$$\begin{aligned}&\mathfrak{D}^{\alpha}\boldsymbol{x}(t)=f(t,\boldsymbol{x}(t)),\\&\boldsymbol{x}^{(k)}(t_0)=\boldsymbol{x}_0^{(k)},\ k=0,1,\dots,n-1,\end{aligned}\tag{1}$$

*where $\mathfrak{D}^{\alpha}$ represents the Caputo fractional derivative. Let $L > 0$ and $f : [0, L] \times \mathbb{R} \to \mathbb{R}$ is continuous and suppose that there exists a real number $l > 0$ such that $|f(t, x) - f(t, y)| \leq l|x - y|$ for $t \in [0, L]$ and $x, y \in \mathbb{R}$. Then, the initial value problem has a unique solution in $AC[0, L]$.*

**Theorem 2** ([48])**.** *Consider the following fractional-order system:*

$$\mathfrak{D}^{\alpha} X(t) = \mathcal{F}(X); \tag{2}$$

*with $0 < \alpha < 1$, $X(t) = [x^1(t), x^2(t), \ldots, x^n(t)]$ and $\mathcal{F}(X) : [t_0, \infty) \to \mathbb{R}^{n \times n}$. The equilibrium points of system (2) are evaluated by solving system of equations $\mathcal{F}(X) = 0$. These equilibrium points are locally asymptotically stable if each eigenvalue $\lambda$ of the Jacobian matrix $J(X)$ calculated at the equilibrium points satisfies $|arg(\lambda_i)| > \frac{\alpha\pi}{2}$.*

## 3. LSD Propagation Model

A lumpy skin disease propagation model is proposed by categorising the entire cattle population $\mathcal{N}$ into system four different classes: $\mathcal{S}$, $\mathcal{E}$, $\mathcal{I}$ and $\mathcal{R}$ susceptible, exposed, infected and recovered cattle population respectively. $\mathcal{S}$ reflects the cattle population that is prone to infection, $\mathcal{E}$ displays livestock that have previously been exposed to disease-causing germs (LSDV), $\mathcal{I}$ comprises of those cattle who have been identified and confirmed positive for LSD, and finally, the recovered cattle are placed in the category $\mathcal{R}$. According to the model, cattle enter the susceptible population at the rate of $\Xi$ either by migration from some other state or by birth. Susceptible cattle become infected by interacting with the diseased cattle at a contact rate of $\beta$ per cattle per time(morbidity rate). $\eta$, $\rho$, $\sigma$ denotes the incubation, recovery, mortality rate of the disease respectively.

$$\begin{aligned}
\mathfrak{D}_t^{\alpha} \mathcal{S}_t &= \Xi - \beta \mathcal{S}_t \mathcal{I}_t - \sigma \mathcal{S}_t \,, \\
\mathfrak{D}_t^{\alpha} \mathcal{E}_t &= \beta \mathcal{S}_t \mathcal{I}_t - (\sigma + \eta) \mathcal{E}_t \,, \\
\mathfrak{D}_t^{\alpha} \mathcal{I}_t &= \eta \mathcal{E}_t - (\rho + \sigma) \mathcal{I}_t \,, \\
\mathfrak{D}_t^{\alpha} \mathcal{R}_t &= \rho \mathcal{I}_t - \sigma \mathcal{R}_t
\end{aligned} \tag{3}$$

along with the initial conditions $\mathcal{S}_{t=0} = \mathcal{S}_0$, $\mathcal{E}_{t=0} = \mathcal{E}_0$, $\mathcal{I}_{t=0} = \mathcal{I}_0$, $\mathcal{R}_{t=0} = \mathcal{R}_0$. Here, $\mathfrak{D}_t^{\alpha}$ is the Caputo fractional derivative of order $\alpha$; $0.5 < \alpha < 1$.

Table 1. Meaning of various parameters

| Parameter | Significance |
|---|---|
| $\Xi$ | influx rate or birth/migration rate |
| $\beta$ | morbidity rate/number of bites |
| $\eta$ | incubation rate |
| $\rho$ | recovery rate |
| $\sigma$ | death rate |

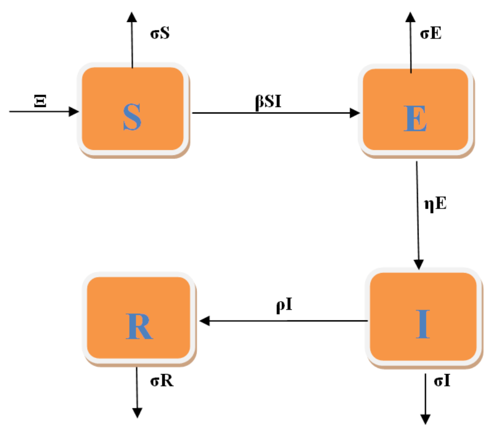



FIGURE 1. An illustration of the model's scheme.

## 4. Model Analysis

This section marks the discussion about the uniqueness of the solution along with its non-negative and bounded nature, the equilibrium points and basic reproduction number are also obtained for the model.

**Theorem 3.** *There is a unique solution* $\mathcal{U}(t) = [\mathcal{S}(t), \mathcal{E}(t), \mathcal{I}(t), \mathcal{R}(t)]^T$ *for the initial value problem given by the system of equations in* (3) *on* $t \geq 0$ *in* $(0, \theta)$ *and the solution will remain in* $\mathbb{R}_+^4$. *Furthermore, the solutions are all bounded.*

*Proof.* Here, Lemma 2 is used to establish the uniqueness of solution for the given system of initial value problems on $(0, \infty)$. Firstly, we shall establish the non-negativity and boundedness of solution. From model (3), we find

$$\begin{aligned} \mathfrak{D}_t^\alpha \mathcal{S}_t|_{\mathcal{S}=0} &= \Xi > 0\,, \\ \mathfrak{D}_t^\alpha \mathcal{E}_t|_{\mathcal{E}=0} &= \beta \mathcal{S}_t \mathcal{I}_t \geq 0\,, \\ \mathfrak{D}_t^\alpha \mathcal{I}_t|_{\mathcal{I}=0} &= \eta \mathcal{E}_t \geq 0\,, \\ \mathfrak{D}_t^\alpha \mathcal{R}_t|_{\mathcal{R}=0} &= \rho \mathcal{I}_t \geq 0\,. \end{aligned}$$

The vector field on each hyperplane enclosing the non-negative orthant points into $\mathbb{R}_+^4$. Furthermore, from system (3)

$$\begin{aligned} &\mathfrak{D}^\alpha \mathcal{N}(t) = \Xi - \sigma \mathcal{N}(t) \geq 0\,, \\ &i.e.\ \mathfrak{D}^\alpha \mathcal{N}(t) + \sigma \mathcal{N}(t) \leq \Xi\,. \end{aligned} \tag{4}$$

Thus, from equation (4) and deduction of Lemma 3, in the case of LSD infection, the total population and hence the sub populations are all bounded. Consequently, the IVP's biologically viable region (3) is

$$\Omega = \left\{ (\mathcal{S}_t, \mathcal{E}_t, \mathcal{I}_t, \mathcal{R}_t) \in \mathbb{R}_+^4 : \mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{R} \geq 0;\ 0 \leq \mathcal{S}_t + \mathcal{E}_t + \mathcal{I}_t + \mathcal{R}_t \leq \frac{\Xi}{\sigma} \right\}. \tag{5}$$

The next step is to demonstrate the uniqueness of solution in $\Omega\ \forall\ t \geq 0$. As we know, $\mathcal{N}(t)$ is the sum $\mathcal{S}(t)$, $\mathcal{E}(t)$, $\mathcal{I}(t)$, $\mathcal{R}(t)$ populations. The Caputo fractional derivative of order $\alpha$ of this equation, gives

$$\mathfrak{D}^\alpha \mathcal{N}(t) = \mathfrak{D}^\alpha \mathcal{S}_t + \mathfrak{D}^\alpha \mathcal{E}_t + \mathfrak{D}^\alpha \mathcal{I}_t + \mathfrak{D}^\alpha \mathcal{R}_t$$

which gives

$$\mathfrak{D}^\alpha \mathcal{N}(t) = \Xi - \sigma \mathcal{N}(t).$$

Now, by taking Laplace transformation using Lemma 2, we have

$$\mathcal{N}(s) = \frac{\Xi\ s^{-1} + s^{\alpha-1}\mathcal{N}(0)}{s^\alpha + \sigma}\,.$$

Using Lemma 1 to obtain inverse Laplace transformation, we get

$$\mathcal{N}(t) = \frac{\Xi}{\sigma}[1 - E_\alpha(-\sigma t^\alpha)] + \mathcal{N}(0) E_\alpha(-\sigma t^\alpha)\,.$$

From the complete monotonicity of $E_\alpha(-t)$ for $t > 0$ and $0 \leq E_\alpha(-\sigma t^\alpha) \leq 1$ on $0 < \alpha \leq 1$ [35], [50], we obtain

$$N(t) \leq \frac{\Xi}{\sigma}\,. \tag{6}$$

To explore the presence of unique solution, we assume the model (3), where all the functions on right hand side of system of equation (3) are continuous and bounded for $t \geq 0$ as $\mathcal{S}(t)$, $\mathcal{E}(t)$, $\mathcal{I}(t)$, $\mathcal{R}(t)$ bounded by equation (6). Also, they satisfy Lipschitz condition. Thus, there exists a bounded and unique solution of the proposed model on $(0, \infty)$ owing to Theorem 1. □

### 4.1. **Equilibrium Points.**

4.1.1. *LSD-free equilibrium.* When there are no infected cattle i.e. $\mathcal{I}_t = 0$. The LSD-free equilibrium point ($E_0$) is attained when we take $\mathcal{E} = 0$, $\mathcal{I} = 0$, $\mathcal{R} = 0$. Thus, the steady state for LSD-free equilibrium is $\left(\frac{\Xi}{\sigma}, 0, 0, 0\right)$.

4.1.2. *Reproduction Number:* The number of cattle infected by a single sick cattle throughout the course of the incubation period in the population of entirely susceptible cattle is known as the reproduction number ($R_0$). The largest eigenvalue of $\mathcal{F}^* {\mathcal{V}^*}^{-1}$ at $E_0$ is used to calculate the reproduction number ($R_0$) of the given model [61].

$$\left[\mathfrak{D}^\alpha \mathcal{S}_t, \mathfrak{D}^\alpha \mathcal{E}_t, \mathfrak{D}^\alpha \mathcal{I}_t, \mathfrak{D}^\alpha \mathcal{R}_t\right]^T = \mathcal{F}(t) - \mathcal{V}(t)\,, \tag{7}$$

where $\mathcal{F}$ represents the rate at which new infections appear in different classes, $\mathcal{V}^-$ is the pace of shifting individual cattle into various classes using all other methods, and $\mathcal{V}^+$ is the pace at which individual cattle are transferred between classes. Also, $\mathcal{V}(t)$=$\mathcal{V}^-(t) - \mathcal{V}^+(t)$ such that

$$\mathcal{F}(t) = \begin{bmatrix} 0 \\ \beta \mathcal{S}_t \mathcal{I}_t \\ 0 \\ 0 \end{bmatrix}, \ \mathcal{V}^+(t) = \begin{bmatrix} \Xi \\ 0 \\ \eta \mathcal{E}_t \\ \rho \mathcal{I}_t \end{bmatrix}, \ \mathcal{V}^-(t) = \begin{bmatrix} \beta \mathcal{S}_t \mathcal{I}_t + \sigma \mathcal{S}_t \\ (\sigma+\eta)\mathcal{E}_t \\ (\rho+\sigma)\mathcal{I}_t \\ \sigma \mathcal{R}_t \end{bmatrix}.$$

At $E_0$, the Jacobian matrix of $\mathcal{F}(t)$ is given by

$$\mathcal{F}^*(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\beta\ \Xi}{\sigma} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The Jacobian matrix of $\mathcal{V}(t)$ is

$$\mathcal{V}^*(t) = \mathcal{V}^{*-}(t) - \mathcal{V}^{*+}(t) = \begin{bmatrix} \sigma & 0 & \frac{\beta\ \Xi}{\sigma} & 0 \\ 0 & (\sigma+\eta) & 0 & 0 \\ 0 & -\eta & (\rho+\sigma) & 0 \\ 0 & 0 & -\rho & \sigma \end{bmatrix}.$$

$\mathcal{F}^*\mathcal{V}^{*-1}$ is the next generation matrix for the model. And, $R_0$ is the spectral radius of this matrix. Now, the eigenvalues of $\mathcal{F}^*\mathcal{V}^{*-1}$ are $0, 0, 0$ and $\frac{\beta\ \Xi\ \eta}{\sigma(\sigma+\eta)(\sigma+\rho)}$. Thus, the reproduction number is given by

$$R_0 = \frac{\beta\ \Xi\ \eta}{\sigma(\sigma+\eta)(\sigma+\rho)}. \tag{8}$$

**Analyzing $R_0$:**
To determine how sensitive each of $R_0$'s parameters is,

$$\frac{\partial R_0}{\partial \beta} = \frac{\Xi\ \eta}{\sigma(\sigma+\eta)(\sigma+\rho)} > 0, \tag{9}$$

$$\frac{\partial R_0}{\partial \Xi} = \frac{\beta \eta}{\sigma(\sigma+\eta)(\sigma+\rho)} > 0, \tag{10}$$

$$\frac{\partial R_0}{\partial \eta} = \frac{\beta\ \Xi\ \sigma}{\sigma(\sigma+\eta)^2(\sigma+\rho)} > 0, \tag{11}$$

$$\frac{\partial R_0}{\partial \rho} = \frac{-\beta\ \Xi\ \sigma}{\sigma(\sigma+\eta)(\sigma+\rho)^2} < 0, \tag{12}$$

$$\frac{\partial R_0}{\partial \sigma} = \frac{-\beta\ \Xi\ \eta}{\sigma(\sigma+\eta)(\sigma+\rho)} \left\{ \frac{1}{\sigma} + \frac{1}{(\sigma+\eta)} + \frac{1}{(\sigma+\rho)} \right\} < 0. \tag{13}$$

Thus, $R_0$ is increasing with $\beta$, $\Xi$, $\eta$ and decreasing with $\rho$ and $\sigma$.

4.1.3. *LSD-Persistent Equilibrium.* When the number of infected cattle i.e. $\mathcal{I}_t \neq 0$. The LSD-persistent equilibrium point ($E_1$) is attained when the number of infected cattle is not zero i.e ($\mathcal{I} \neq 0$). Therefore, the disease persistent equilibrium point is given by $(\mathcal{S}_1, \mathcal{E}_1, \mathcal{I}_1, \mathcal{R}_1)$, where

$$\mathcal{S}_1 = \frac{(\sigma+\eta)(\sigma+\rho)}{\beta\eta},\ \mathcal{E}_1 = \frac{\beta\ \Xi\ \eta - \sigma(\sigma+\eta)(\sigma+\rho)}{\beta\eta(\sigma+\eta)},$$
$$\mathcal{I}_1 = \frac{\sigma(R_0-1)}{\beta} \text{ and } \mathcal{R}_1 = \frac{\rho(R_0-1)}{\beta}$$

which implies $(\mathcal{S}_1, \mathcal{E}_1, \mathcal{I}_1, \mathcal{R}_1) > 0$ iff $R_0 > 1$. So, the LSD-persistent steady state exists iff $R_0 > 1$. For $R_0 = 1$, LSD-persistent steady state becomes LSD-free steady state.

4.2. **Stability Analysis.**

**Theorem 4.** *LSD-free equilibrium point $E_0 = \left(\frac{\Xi}{\sigma}, 0, 0, 0\right)$ of the system is locally asymptotically stable when $R_0 < 1$, unstable otherwise.*

*Proof.* The Jacobian matrix at $E_0$ is

$$\begin{bmatrix} -\sigma & 0 & -\frac{\beta\ \Xi}{\sigma} & 0 \\ 0 & -(\sigma+\eta) & \frac{\beta\ \Xi}{\sigma} & 0 \\ 0 & \eta & -(\rho+\sigma) & 0 \\ 0 & 0 & \rho & -\sigma \end{bmatrix}.$$

Now, two of the eigenvalues are $-\sigma$. The characteristic equation for finding the remaining two eigenvalues is given by

$$P(\lambda) = \lambda^2 + P_1\lambda + P_2\,, \tag{14}$$

where

$$P_1 = (2\sigma+\eta+\rho),$$
$$P_2 = (\eta+\sigma)(\rho+\sigma) - \frac{\beta\ \Xi\ \eta}{\sigma} = (\eta+\sigma)(\rho+\sigma)[1-R_0].$$

Now, $P_1 > 0$ always and $P_2 > 0$ for $R_0 < 1$. Thus, for $R_0 < 1$, by using Routh-Hurwitz criteria [5], all the eigenvalues of the Jacobian matrix at $E_0$ have negative real parts, it implies from Theorem 2 that the LSD-free equilibrium point is locally asymptotically stable when $R_0 < 1$ and unstable otherwise. □

**Theorem 5.** *The LSD-persistent equilibrium point $E_1 = (\mathcal{S}_1, \mathcal{E}_1, \mathcal{I}_1, \mathcal{R}_1)$ exists and is locally asymptotically stable iff $R_0 > 1$.*

*Proof.* The Jacobian matrix at $E_1$ is

$$\begin{bmatrix} -\sigma R_0 & 0 & -\frac{(\eta+\sigma)(\rho+\sigma)}{\eta} & 0 \\ \sigma(R_0-1) & -(\sigma+\eta) & \frac{(\eta+\sigma)(\rho+\sigma)}{\eta} & 0 \\ 0 & \eta & -(\rho+\sigma) & 0 \\ 0 & 0 & \rho & -\sigma \end{bmatrix}.$$

Thus, on observation we see that one of the eigenvalues is $-\sigma$. The characteristic equation to obtain the remaining eigenvalues is

$$P(\lambda) = \lambda^3 + P_1\lambda^2 + P_2\lambda + P_3\,,$$

where

$$\begin{aligned}
P_1 &= (\sigma(R_0 + 2) + \eta + \rho)\,,\\
P_2 &= \sigma R_0(2\sigma + \eta + \rho)\,,\\
P_3 &= (R_0 - 1)\sigma(\eta + \sigma)(\rho + \sigma)\,.
\end{aligned}$$

Clearly, $P_1 > 0$ and $P_3 > 0$ whenever $R_0 > 1$. Also, $P_1P_2 - P_3 > 0$. Thus, by Routh-Hurwitz criterion, all the eigenvalues of the Jacobian matrix of the system of equations defining the model have negative real parts at LSD-persistent equilibrium point $E_1$ for $R_0 > 1$, which ensures the locally asymptotic stability of the LSD-persistent equilibrium point for $R_0 > 1$ and unstable elsewhere using Theorem 2. $\square$

## 5. Numerical Simulations

Computing findings that highlight the fluctuating nature of the lumpy skin disease propagation model and to verify the analytical outcomes for multiple derivative orders are presented in this section. Using a MATLAB programme supplied by Roberto Garappa in [22], the proposed model is solved using the Adams-Bashforth-Moulton predictor-corrector method. Table 2 carries the variables and parameters used for simulation. According to 19th livestock census-2012 and 20th livestock census-2019 all India report the total Cattle population in the country was 190.90 and 192.50 million respectively [41]. This shows that there has been an approximate increase of 0.0114 percent per year giving us the birth rate or the influx rate ($\Xi$). The morbidity rate ($\beta$) can be retrieved from [43] by making a few necessary changes to it. As per the 20th livestock census-2019, the total cattle population in the state of Gujarat is 10,165,000. Therefore, the total susceptible cattle population is 10,165,000/232. Similarly as in the case of COVID-19 (there it was 250 for the Wuhan city with a population of 11 million), the denominator was chosen early in the epidemic and later proven to be a reasonable figure. It is a suitable parameter for limiting the movement of cattle that were imposed by the respective state governments on different dates between July to September, 2022 as reported by various newspapers [34]. Now, assuming the average number of bites per cattle per day to be 5, this gives us $\beta = 5 * 10,165,000/232$ [43]. The incubation period is between 4 to 14 days [33]. Since, there is no or a little information available about the mortality rate and recovery period(reciprocal of the recovery rate), we assume them to be 0.0057 (half of the birth rate) and 7 days (keeping a positive view), respectively.

TABLE 2. Parameter Values

| **Parameter** | **Value** | **Source** |
|---|---|---|
| $\Xi$ | 0.0114 | [41] |
| $\beta$ | $1.1412 \times 10^{-4}$ | [43] |
| $\eta$ | 1/6 | [33] |
| $\rho$ | 1/7 | Assumed |
| $\sigma$ | 0.0057 | Assumed |

| **Population** | $\mathcal{S}$ | $\mathcal{E}$ | $\mathcal{I}$ | $\mathcal{R}$ |
|---|---|---|---|---|
| **Initial Values** | 43815 | 1 | 1 | 0 |

For the initial populations, the initial susceptible population along with restricted cattle movement is assumed to be $\mathcal{S}_0 = 10,165,000/232$, we assume that initially exposed and infected cattle are 1 each, no recovered cattle. In the event that $R_0 > 1$, the cattle population cannot be free of disease. Following the start of the pandemic, the number of susceptible cattle continued to decline, while the exposed and infected cattle classes show a rapid rise in population density, as seen by Figures 2, 3 and 4, respectively. The rapid rise in the number of recovered cattle population in Figure 5 can be attributed to the massive vaccination drive in the state of Gujarat, steps were made to control disease causing vectors and restrict bovine movement. Regardless of the order, the plots in Figure 6 for each class of cattle population indicates that the proposed model is asymptotically stable for the LSD-persistent equilibrium points the population swiftly approaches its equilibria when we increase the value of $\alpha$. Since the susceptible and infected cattle populations are reduced to negative populations, which is something we all know is not conceivable, we can plainly state that the fractional order models are far superior than the conventional integer order model with $\alpha = 1$. The equations (9), (10) and (13) support the findings of Figure 7(a), (b)and (c), respectively. Equation (11) demonstrates that $R_0$ rises with an increase in the incubation rate, $\eta$, and falls with an increase in the incubation duration $(1/\eta)$, as shown by Figure 8(a). In a similar vein, equation (12) reveals that $R_0$ drops as the recovery rate, $(\rho)$ rises. The recovery period $(1/\rho)$ grows as $R_0$ does, as shown by Figure 8(b).

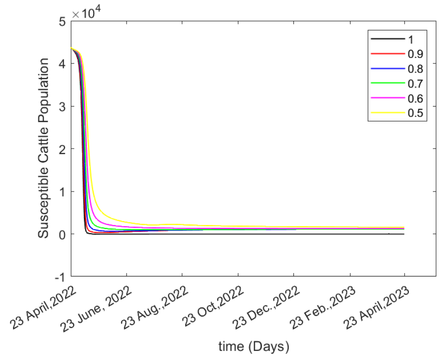



FIGURE 2. Graphical display of the susceptible class at various fractional orders.

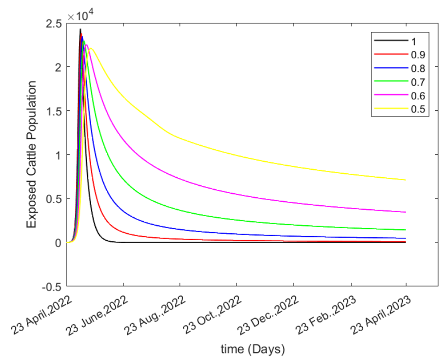



FIGURE 3. Graphical display of the exposed class at various fractional orders.

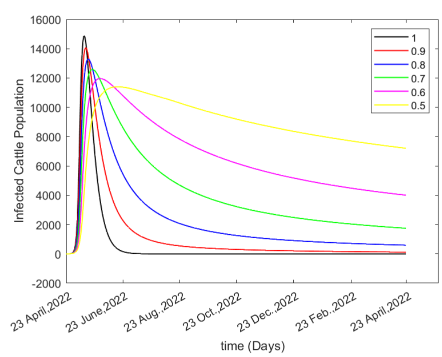



FIGURE 4. Graphical display of the infected class at various fractional orders.

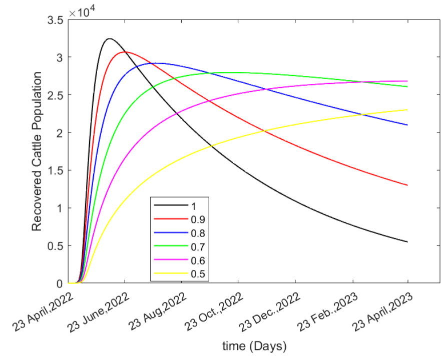
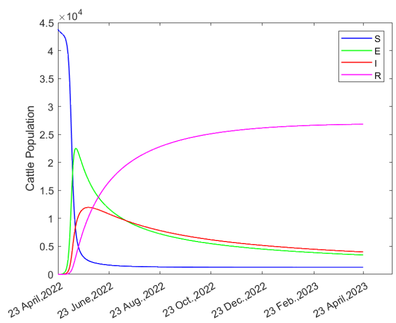
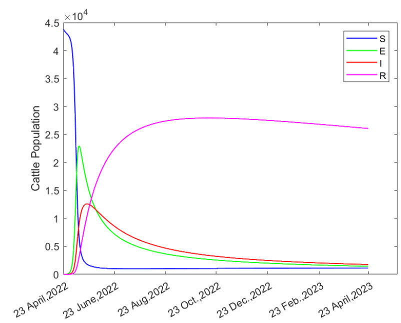


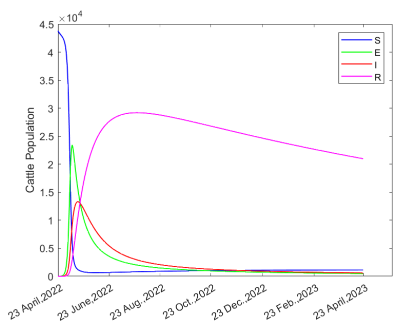

FIGURE 5. Graphical display of the recovered class at various fractional orders.



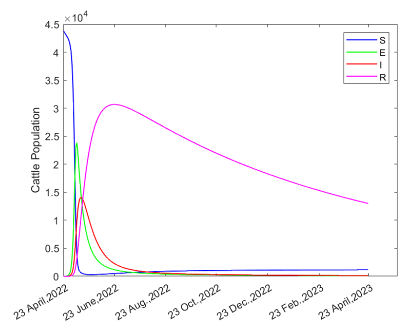

(A) $\alpha = 0.6$




(B) $\alpha = 0.7$

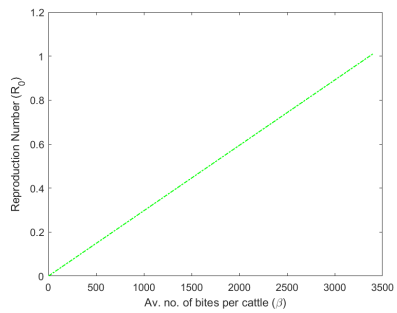



(C) $\alpha = 0.8$

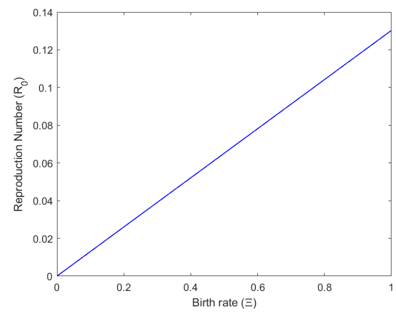



(D) $\alpha = 0.9$

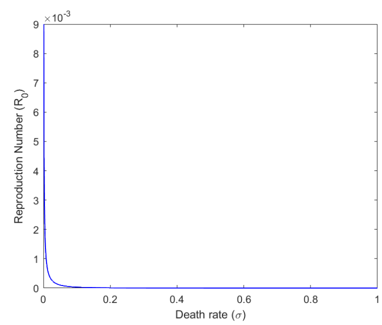
FIGURE 6. Variations of susceptible, exposed, infected and recovered cattle populations with different values of $\alpha$




(A) $R_0$ increases with increase in morbidity rate ($\beta$)

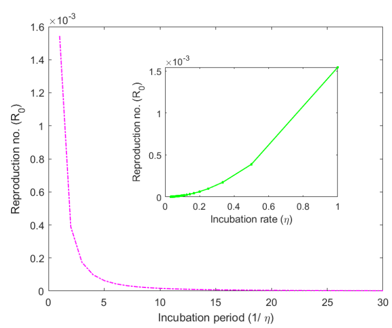



(B) $R_0$ increases with increase in birth rate ($\Xi$)

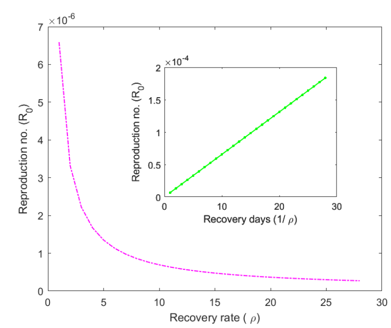



(C) $R_0$ decreases with increase in death rate ($\sigma$)

FIGURE 7. Variation of $R_0$ with $\beta$, $\Xi$, $\sigma$




(A) $R_0$ increases with increase in incubation rate ($\eta$)




(B) $R_0$ decreases with increase in recovery rate ($\rho$)

FIGURE 8. Variation of $R_0$ with $\eta$ and $\rho$

## 6. Concluding Remarks And Future Strategy

For $R_0 < 1$, the diffusion of the virus can be controlled, and the equilibrium free of LSD can be preserved across Gujarat. The susceptible cattle population keeps on decreasing with time. The exposed and infected cattle population regularly rises over time until it reaches a peak, after which it starts to decline until it attains equilibrium. We can see that the best results are shown by taking $\alpha = 0.5$ as it shows the infected cases reach an all-time high in 56 days following the discovery of the first case on April 23 of this year in the hamlet of Kaiyari, located on the Indo-Pak border in the Kutch district's Lakhpat taluka. Mosquito and housefly infestations continue at their peak during the monsoon season, and veterinary scientists and government officials blame a very wet July for the infection's quick spread in Gujarat this year. So far, Gujarat has experienced 1010 mm of rain, which is 20 percent higher than the state normal of 850 mm. The four-month south-west monsoon season began in June and ended in September. There is also an issue with feral cattle in Gujarat, a state where cow slaughter is outlawed, and experts believe these free-roaming cattle may be a factor in the quick spread of LSD. The dearth of knowledge about the sickness may also contribute to the rapid spread of LSD. As can be seen, the peak does not last long, which might be attributed to the state animal husbandry department treating diseased cattle and administering goat pox vaccine to healthy animals in surrounding regions.

This current investigation suggests the following policy changes to assist, isolate, and stop the further spread: import restrictions on domestic cattle and water buffaloes, as well as their products; surveillance beyond the containment zone of goods, trash, and disease spreading vectors; restriction on movement of cattle; pest control measures; incineration; and cleaning and disinfection of the surroundings.

Effective LSD treatment with complete coverage is required. Given that LSD is in close relation to the sheep pox and goat pox viruses, vaccine against same is used to treat LSD. New animals should be inoculated before being introduced to the afflicted farm. Calves reared from vaccinated or naturally infected moms should be inoculated at the age of 3 to 4 months. Bulls used for breeding and pregnant cows can both receive annual vaccinations [21]. The $R_0$ may be used to calculate the amount of vaccine needed to suppress an epidemic (i.e. to reduce $R_0$ below one). The study also emphasised the need of starting immunisation efforts ahead of viral entrance.

**Author Contribution Statements** The authors contributed equally to this article.

**Declaration of Competing Interests** The authors declare that they have no competing interests.

## References


[1] Abboubakar, H., Kamgang, J. C., Nkamba, N. L., Tieudjo, D., Emini, L., Modeling the dynamics of arboviral diseases with vaccination perspective, *Biomath*, 4(1) (2015), ID-1507241. https://doi.org/10.11145/j.biomath.2015.07.241

[2] Abboubakar, H., Kom Regonne, R., Sooppy Nisar, K., Fractional dynamics of typhoid fever transmission models with mass vaccination perspectives, *Fractal and Fractional*, 5(4) (2021), 149. https://doi.org/10.3390/fractalfract5040149

[3] Abdulqa, H. Y., Rahman, H. S., Dyary, H. O., Othman, H. H., Lumpy skin disease, *Reproductive Immunology: Open Access*, 1(4) (2016), 25. https://doi.org/10.21767/2476-1974.100025

[4] Acharya, K. P., Subedi, D., First outbreak of lumpy skin disease in Nepal, *Preventive Veterinary Medicine*, 102(4) (2020), 274–283. https://doi.org/10.1111/tbed.13815

[5] Ahmed, E., El-Sayed, A. M. A., El-Saka, H. A., On some Routh-Hurwitz conditions for fractional order differential equations and their applications in Lorenz, Rössler, Chua and Chen systems, *Physics Letters* A, 358(1) (2006), 1–4. https://doi.org/10.1016/j.physleta.2006.04.087

[6] Al-Salihi, K., Lumpy skin disease: Review of literature, *Mirror of Research in Veterinary Sciences and Animals*, 3(3) (2014), 6–23.

[7] Arjkumpa, O., Suwannaboon, M., Boonrod, M., Punyawan, I., Liangchaisiri, S., Laobannue, P., Punyapornwithaya, V., The first lumpy skin disease outbreak in Thailand (2021): Epidemiological features and spatio-temporal analysis, *Frontiers in Veterinary Science*, 8 (2022), 1580. https://doi.org/10.3389/fvets.2021.799065

[8] Azeem, S., Sharma, B., Shabir, S., Akbar, H., Venter, E., Lumpy skin disease is expanding its geographic range: A challenge for Asian livestock management and food security, *The Veterinary Journal*, 279 (2022), 105785. https://doi.org/10.1016/j.tvjl.2021.105785

[9] Atangana, A., Baleanu, D., New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model, *arXiv preprint* arXiv:1602.03408, (2016).

[10] Baleanu, D., Ghassabzade, F. A., Nieto, J. J., Jajarmi, A., On a new and generalized fractional model for a real cholera outbreak, *Alexandria Engineering Journal*, 61(11) (2022), 9175-9186. https://doi.org/10.1016/j.aej.2022.02.054

[11] Baleanu, D., Hasanabadi, M., Vaziri, A. M., Jajarmi, A., A new intervention strategy for an HIV/AIDS transmission by a general fractional modeling and an optimal control approach, *Chaos, Solitons and Fractals*, 167 (2023), 113078. https://doi.org/10.1016/j.chaos.2022.113078

[12] Bansal, K., Arora, S., Pritam, K. S., Mathur, T., Agarwal, S., Dynamics of crime transmission using fractional-order differential equations, *Fractals*, 30(1) (2022), 2250012 – 1485. https://doi:10.1142/S0218348X22500128

[13] Bansal, K., Mathur, T., Singh, N. S. S., Agarwal, S., Analysis of illegal drug transmission model using fractional delay differential equations, *AIMS Mathematics*, 7(10) (2022), 18173–18193. https://doi.org/10.3934/math.20221000

[14] Butt, A. I. K., Ahmad, W., Rafiq, M., Baleanu, D., Numerical analysis of Atangana-Baleanu fractional model to understand the propagation of a novel corona virus pandemic, *Alexandria Engineering Journal*, 61(9) (2022), 7007–7027. https://doi.org/10.1016/j.aej.2021.12.042

[15] Butt, A. I. K., Imran, M., Batool, S., Nuwairan, M. A., Theoretical analysis of a COVID-19 CF-fractional model to optimally control the spread of pandemic, *Symmetry*, 15(2) (2023), 380. https://doi.org/10.3390/sym15020380

[16] Caputo, M., Fabrizio, M., A new definition of fractional derivative without singular kernel, *Progress in Fractional Differentiation and Applications*, 1(2) (2015), 73–85.

[17] Chen, Y., Liu, F., Yu, Q., Li, T., Review of fractional epidemic models, *Applied Mathematical Modelling*, 97 (2021), 281–307. https://doi.org/10.1016/j.apm.2021.03.044

[18] Das, M., Chowdhury, M. S. R., Akter, S., Mondal, A. K., Uddin, M. J., Rahman, M. M., Rahman, M. M., An updated review on lumpy skin disease: perspective of Southeast Asian countries, *J. Adv. Biotechnol. Exp. Ther*, 4(3) (2021), 322–333. https://doi.org/10.5455/jabet.2021.d133

[19] Davies, G. F., Lumpy skin disease of cattle: A growing problem in Africa and the Near East, *World Ani Rev.*, 68(3) (1991), 37–42.

[20] Djordjevic, J., Silva, C. J., Torres, D. F., A stochastic SICA epidemic model for HIV transmission, *Applied Mathematics Letters*, 84 (2018), 168-175. https://doi.org/10.1016/j.aml.2018.05.005

[21] Gelaye, E., Belay, A., Ayelet, G., Jenberie, S., Yami, M., Loitsch, A., Lamien, C. E., Capripox disease in Ethiopia: Genetic differences between field isolates and vaccine strain, and implications for vaccination failure, *Antiviral Research*, 119 (2015), 28-35.

[22] Garrappa, R., Numerical solution of fractional differential equations: A survey and a software tutorial, *Mathematics*, 6(2) (2018), 16. https://doi.org/10.3390/math6020016

[23] Gupta, T., Patial, V., Bali, D., Angaria, S., Sharma, M., Chahota, R., A review: Lumpy skin disease and its emergence in India, *Veterinary Research Communications*, 44(3) (2020), 111–118. https://doi.org/10.1007/s11259-020-09780-1

[24] https://www.fao.org/home/en.

[25] Hasib, F. M. Y., Islam, M. S., Das, T., Rana, E. A., Uddin, M. H., Bayzid, M., Alim, M. A., Lumpy skin disease outbreak in cattle population of Chattogram, Bangladesh, *Veterinary Medicine and Science*, 7(5) (2021), 1616–1624.

[26] Iqbal, M. S., Yasin, M. W., Ahmed, N., Akgül, A., Rafiq, M., Raza, A., Numerical simulations of nonlinear stochastic Newell-Whitehead-Segel equation and its measurable properties, *Journal of Computational and Applied Mathematics*, 418 (2023), 114618. https://doi.org/10.1016/j.cam.2022.114618

[27] Katugampola, U. N., New approach to a generalized fractional integral, *Applied Mathematics and Computation*, 218(3) (2011), 860–865. https://doi.org/10.1016/j.amc.2011.03.062

[28] Khan, Y. R., Ali, A., Hussain, K., Ijaz, M., Rabbani, A. H., Khan, R. L., Abbas, S. N., Aziz, M. U., Ghaffar, A., Sajid, H. A., A review: surveillance of lumpy skin disease (LSD) a growing problem in Asia, *Microbial Pathogenesis*, 158 (2021), 105050. https://doi.org/10.1016/j.micpath.2021.105050

[29] Kexue, L., Jigen, P., Laplace transform and fractional differential equations, *Applied Mathematics Letters*, 24(12) (2011), 2019–2023. https://doi.org/10.1016/j.aml.2011.05.035

[30] Kononov, A., Prutnikov, P., Shumilova, I., Kononova, S., Nesterov, A., Byadovskaya, O., Sprygin, A., Determination of lumpy skin disease virus in bovine meat and offal products following experimental infection, *Transboundary and Emerging diseases*, 66(3) (2019), 1332–1340. https://doi.org/10.1111/tbed.13158

[31] Kilbas, A. A., Srivastava, H. M., Trujillo, J. J., Theory and Applications of Fractional Differential Equations, Vol. 204, Elsevier, 2006.

[32] Lu, G., Xie, J., Luo, J., Shao, R., Jia, K., Li, S., Lumpy skin disease outbreaks in China, since 3 August 2019, *Transboundary and Emerging Diseases*, 68(2) (2021), 216–219. https://doi.org/10.1111/tbed.13898

[33] *Lumpy Skin Disease(LSD) field manual.* https://www.fao.org/3/i7330e/i7330e.pdf

[34] Lumpy Skin Disease: Ban on livestock transport from 14 districts. The Indian Express. 27 July 2022; Cattle fairs brought to a halt in Rajasthan to control lumpy skin disease. *The Hindu.* 6 August 2022. ISSN 0971-751X, https://www.thehindu.com/news/national/other-states/cattle-fairs-brought-to-a-halt-in-rajasthan-to-control-lumpy-skin-disease/article65737852.ece

[35] Mainardi, F., On some properties of the Mittag-Leffler function $E_\alpha(-t^\alpha)$, completely monotone for $t > 0$ with $0 < \alpha < 1$, *Discrete and Continuous Dynamical Systems Series B (DCDS-B)Series B*, 19(7) (2014), 2267–2278. https://doi.org/10.3934/dcdsb.2014.19.2267

[36] Mehmood, N., Abbas, A., Akgül, A., Abdeljawad, T., Alqudah, M. A., Existence and stability results for coupled system of fractional differential equations involving AB-Caputo derivative, *Fractals*, (2023), 2340023. https://doi.org/10.1142/S0218348X23400236

[37] Moonchai, S., Himakalasa, A., Rojsiraphisal, T., Arjkumpa, O., Panyasomboonying, P., Kuatako, N., Punyapornwithaya, V., Modelling epidemic growth models for lumpy skin disease cases in Thailand using nationwide outbreak data, 2021–2022, *Infectious Disease Modelling*, 8(1) (2023), 282–293. https://doi.org/10.1016/j.idm.2023.02.004

[38] Morris, J. P. A., Pseudo-urticaria, Northern Rhodesia Department of Animal Health, *Annual Report,* 1930(12)(1931).

[39] Miller, K. S., Ross, B., An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, 1993.

[40] Ndaïrou, F., Area, I., Nieto, J. J., Silva, C. J., Torres, D. F., Mathematical modeling of Zika disease in pregnant women and newborns with microcephaly in Brazil, *Mathematical Methods in the Applied Sciences*, 41(18) (2018), 8929–8941. https://doi.org/10.1002/mma.4702

[41] https://www.nddb.coop/information/stats/pop

[42] Nuugulu, S. M., Shikongo, A., Elago, D., Salom, A. T., Owolabi, K. M., Fractional SEIR model for modelling the spread of COVID-19 in Namibia, In *Mathematical Analysis for Transmission of COVID-19* (pp. 161-184). Springer, Singapore, 2021.

[43] Nisar, K. S., Ahmad, S., Ullah, A., Shah, K., Alrabaiah, H., Arfan, M., Mathematical analysis of SIRD model of COVID-19 with Caputo fractional derivative based on real data, *Results in Physics*, 21 (2021), 103772. https://doi.org/10.1016/j.rinp.2020.103772

[44] Odibat, Z. M., Shawagfeh, N. T., Generalized Taylor's formula, *Applied Mathematics and Computation*, 186(1) (2007), 286–293. https://doi.org/10.1016/j.amc.2006.07.102

[45] *OIE Lumpy skin Disease OIE Terrestrial Mannual* 2017 Chapter 2.4.13.

[46] Onyejekwe, O. O., Alemu, A., Ambachew, B., Tigabie, A., Epidemiological study and optimal control for Lumpy Skin Disease (LSD) in Ethiopia, *Advances in Infectious Diseases*, 9(1) (2019), 8–24. https://doi.org/10.4236/aid.2019.91002

[47] Partohaghighi, M., Akgül, A., Akgül, E. K., Attia, N., De la Sen, M., Bayram, M., Analysis of the fractional differential equations using two different methods, *Symmetry*, 15(1) (2023), 65. https://doi.org/10.3390/sym15010065

[48] Petráš , I., Fractional-Order Nonlinear Systems: Modeling, Analysis and Simulation, Springer Science and Business Media, 2011.

[49] Podlubny, I., Fractional Differential Equations, 198 Academic Press. San Diego, California, USA, 1999.

[50] Pollard, H., The completely monotonic character of the Mittag-Leffler function $E_a(-x)$, *Bulletin of the American Mathematical Society*, 54(12) (1948), 1115–1116.

[51] Roche, X., Rozstalnyy, A., TagoPacheco, D., Pittiglio, C., Kamata, A., Beltran Alcrudo, D., Bisht, K., Karki, S., Kayamori, J., Larfaoui, F., Raizman, E., VonDobschuetz, S., Dhingra, M. S., Sumption, K., Introduction and spread of lumpy skin disease in South, East and Southeast Asia: qualitative risk assessment and management, *Food and Agriculture Org.* (2021).

[52] Rachah, A., Torres, D. F., Dynamics and optimal control of Ebola transmission, *Mathematics in Computer Science*, 10(3) (2016), 331–342. https://doi.org/10.1007/s11786-016-0268-y

[53] Samko, S. G., Kilbas, A. A., Marichev, O. I., Fractional Integrals and Derivatives, Gordon and Breach Science Publishers, 1993.

[54] Shahzad, A., Imran, M., Tahir, M., Khan, S. A., Akgül, A., Abdullaev, S., Yahia, I. S., Brownian motion and thermophoretic diffusion impact on Darcy-Forchheimer flow of bioconvective micropolar nanofluid between double disks with Cattaneo-Christov heat flux, *Alexandria Engineering Journal*, 62 (2023), 1-15. https://doi.org/10.1016/j.aej.2022.07.023

[55] Sin, C. S., Well-posedness of general Caputo-type fractional differential equations, *Fractional Calculus and Applied Analysis*, 21(3) (2018), 819–832. https://doi.org/10.1515/fca-2018-0043

[56] Sudhakar, S. B., Mishra, N., Kalaiyarasu, S., Jhade, S. K., Hemadri, D., Sood, R., Bal, G. C., Nayak, M. K., Pradhan, S. K., Singh, V. P., Lumpy skin disease (LSD) outbreaks in cattle in Odisha state, India in August 2019: Epidemiological features and molecular studies, *Transboundary and Emerging Diseases*, 67(6) (2020), 2408–2422. DOI: 10.1111/tbed.13579

[57] Thirthar, A. A., Abboubakar, H., Khan, A., Abdeljawad, T., Mathematical modeling of the COVID-19 epidemic with fear impact, *AIMS Mathematics*, 8(3) (2023), 6447–6465. https://doi.org/10.3934/math.2023326

[58] Tran, H. T. T., Truong, A. D., Dang, A. K., Ly, D. V., Nguyen, C. T., Chu, N. T., Dang, H. V., Lumpy skin disease outbreaks in Vietnam, 2020, *Transboundary and Emerging Diseases*, 68(3) (2021), 977–980. https://doi.org/10.1111/tbed.14022

[59] Tuppurainen, E., Alexandrov, T., Beltrán-Alcrudo, D., Lumpy skin disease field manual–A manual for veterinarians, FAO Animal Production and Health Manual No. 20 (2017), pp. 1–60. Rome: Food and Agriculture Organization of the United Nations (FAO).

[60] Ullah, A., Abdeljawad, T., Ahmad, S., Shah, K., Study of a fractional-order epidemic model of childhood diseases, *J. Funct. Space*, (2020). https://doi.org/10.1155/2020/5895310

[61] Van den Driessche, P., Watmough, J., Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, *Mathematical Biosciences*, 180(1-2) (2002), 29–48. https://doi.org/10.1016/S0025-5564(02)00108-6

[62] Zhang, R., Shah, N. A., El-Zahar, E. R., Akgül, A., Chung, J. D., Numerical analysis of fractional-order Emden-Fowler equations using modified variational iteration method, *Fractals*, (2023), 2340028. https://doi.org/10.1142/S0218348X23400285

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 211–221 (2024)
DOI:10.31801/cfsuasmas.1307158
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

*Research Article; Received: May 30, 2023; Accepted: November 5, 2023*

# SEMIREGULAR, SEMIPERFECT AND SEMIPOTENT MATRIX RINGS RELATIVE TO AN IDEAL

Meltem ALTUN ÖZARSLAN

Department of Mathematics, Hacettepe University, Ankara, TÜRKİYE

Abstract. This paper investigates relative ring theoretical properties in the context of formal triangular matrix rings. The first aim is to study the semiregularity of formal triangular matrix rings relative to an ideal. We prove that the formal triangular matrix ring $T$ is $T'$-semiregular if and only if $A$ is $I$-semiregular, $B$ is $K$-semiregular and $N = M$ for an ideal $T' = \left(\begin{smallmatrix} I & 0 \\ N & K \end{smallmatrix}\right)$ of $T = \left(\begin{smallmatrix} A & 0 \\ M & B \end{smallmatrix}\right)$. We also discuss the relative semiperfect formal triangular matrix rings in relation to the strong lifting property of ideals. Moreover, we have considered the behavior of relative semipotent and potent property of formal triangular matrix rings. Several examples are provided throughout the paper in order to highlight our results.

## 1. Introduction

The celebrated work of Wedderburn and Artin gave a key insight into the structure of a semisimple artinian ring, which makes it an attractive structure to study. Moreover, for a right artinian ring $R$, the Jacobson radical is nilpotent, and the ring $R/\mathrm{J}(R)$ is semisimple, so a main problem would be to "lift" the structure of the factor ring $R/\mathrm{J}(R)$ onto the ring $R$ itself. As a consequence of this, we are led to the concept of lifting idempotents and, consequently, to the notion of a semiperfect ring.

Let $I$ be an ideal in a ring $R$. Recall that an element $a \in R$ is an idempotent modulo $I$ if $a + I \in R/I$ is an idempotent. In this case, we say that $a$ can be lifted to an idempotent (modulo $I$) if there exists an idempotent $e \in R$ with $e - a \in I$. Note that the ideal $I$ in $R$ is called *idempotent lifting* if, whenever $a + I \in R/I$ is an idempotent, then there exists an idempotent $e \in R$ such that $e - a \in I$.

---

2020 *Mathematics Subject Classification.* 16S50; 16D25; 16E50; 16L30.

*Keywords.* Formal triangular matrix ring; strongly lifting ideal; semiregular ring; semiperfect ring; semipotent ring.

meltemaltun@hacettepe.edu.tr; 0000-0001-9926-6770.

Lifting of idempotents is a key method in transferring some structural properties of a factor ring of a ring $R$ up to the ring itself. Several classes of rings are described in terms of the idempotent lifting property of ideals. For example, semiperfect rings are those rings $R$ for which $R/\mathrm{J}(R)$ is semisimple and the Jacobson radical $\mathrm{J}(R)$ of $R$ is idempotent lifting. Some nontrivial generalizations of semiperfect rings, such as semiregular rings and potent rings may be considered as further examples.

As it has been pointed out above, the idempotent lifting property of the Jacobson radical $J(R)$ of $R$ is prominent in the study of semiregular and semiperfect rings. A stronger property than the idempotent lifting property, namely, strong lifting property of ideals, gives rise to a natural generalization of semiregular and semiperfect rings. Semiregular rings relative to an ideal first emerged in a paper [8] by Nicholson and Yousif. Then, Yousif and Zhou [10] studied further semiperfect and perfect rings relative to an ideal in connection with relative semiregular rings. Later, Nicholson and Zhou worked on a natural extension of this work together with strongly lifting ideals to characterize $I$-semiregular and $I$-semiperfect rings for an ideal $I$ of a ring $R$ in [9]. Recall that an ideal $I$ of a ring $R$ is called *strongly lifting* if, whenever $a + I \in R/I$ is an idempotent, then there exists an idempotent $e \in aR$ such that $a - e \in I$. In this work, Nicholson and Zhou further showed that the ring $R$ is $I$-semiregular (semiperfect) if and only if $R/I$ is regular (semisimple) and $I$ is strongly lifting.

Recall that a ring $R$ is called semipotent if each one-sided ideal of $R$ that is not contained in its Jacobson radical $\mathrm{J}(R)$ contains a nonzero idempotent. A semipotent ring $R$ is called potent if, in addition, $\mathrm{J}(R)$ is an idempotent lifting ideal of $R$. Semipotent rings has been generalized to semipotent rings relative to an ideal by Nicholson and Zhou in [9]. It is also important to consider the strong lifting properties of ideals in this setting, and relative potent rings are defined in relation to these ideals as well as relative semipotent rings.

One of important constructions in ring theory is the triangular ring construction. Let $A, B$ be rings and $M$ be a $B$-$A$ bimodule. A *formal triangular matrix ring* is a ring of the form

$$\begin{pmatrix} A & 0 \\ M & B \end{pmatrix} = \left\{ \begin{pmatrix} a & 0 \\ m & b \end{pmatrix} \mid a \in A,\ b \in B,\ \text{and } m \in M \right\}$$

under the usual matrix operations. There are a number of important examples in this class, including lower (upper) triangular matrices over a known ring $R$. Moreover, many surprising examples and counterexamples have emerged via the triangular ring construction in literature by varying the choices of $A, B$ and $M$. By using formal triangular matrix rings, Herstein in [5] provided a counterexample to the Jacobson conjecture, one of the oldest and most well-known conjectures in noncommutative ring theory. In [3], these rings were studied in detail, and in [4], various ring theoretic properties of formal triangular matrix rings were investigated.

This paper aims to unify all these relative properties in the framework of formal triangular matrix rings. In Section 2, we completely give a description of the semiregularity and semiperfectness of formal triangular matrix rings relative to an ideal, proving that $T$ is $T'$-semiregular (resp. semiperfect) if and only if $A$ is $I$-semiregular (resp. semiperfect), $B$ is $K$-semiregular (resp. semiperfect) and $N = M$ for an ideal $T' = \left(\begin{smallmatrix} I & 0 \\ N & K \end{smallmatrix}\right)$ of $T = \left(\begin{smallmatrix} A & 0 \\ M & B \end{smallmatrix}\right)$ (Theorem 1 and Theorem 2). Then, we highlight our results by providing several examples, in particular we show that the "if" part of the above theorems are not in general true if we omit the condition $N = M$. We have further considered the behavior of relative semipotent property of formal triangular matrix rings. Since being a semipotent or a potent ring passes over to formal triangular matrix rings by a result due to Haghany and Varadarajan [4], it is natural to suspect that it may also pass over in the relative case.

Throughout this paper, all rings will be associative rings with an identity element $1 \neq 0$, not necessarily commutative. We will denote by $\mathrm{J}(R)$ the Jacobson radical of a ring $R$.

## 2. Results

Recall that a formal triangular matrix ring $T$ is a ring of the form

$$T = \left\{ \begin{pmatrix} a & 0 \\ m & b \end{pmatrix} \mid a \in A,\ b \in B,\ m \in M. \right\}$$

under the usual matrix addition and multiplication where $A, B$ are two rings and $M$ is a left $B$ right $A$ bimodule. For simplicity, we write

$$T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$$

for the formal triangular matrix ring. The construction of examples and counterexamples for asymmetric ring-theoretic properties is among the major applications of such rings in noncommutative ring theory. In particular, [4] provides a comprehensive resource for various ring-theoretic properties of formal triangular matrix rings.

Moreover, Goodearl covered formal triangular matrix rings in his classic book "Ring Theory: Nonsingular rings and modules" [3]. In order to better understand the ideal structure of a ring of such a type, we must first recall the following fact.

**Proposition 1.** [3] *If $I$ is a two-sided ideal of $A$, $K$ a two-sided ideal of $B$, and $N$ a $B$-$A$ subbimodule of $M$ which contains $MI + KM$, then $\left(\begin{smallmatrix} I & 0 \\ N & K \end{smallmatrix}\right)$ is a two-sided ideal of $T$. Conversely, every two-sided ideal of $T$ has this form.*

We will begin by simplifying the following notation: $T$ denotes the formal triangular matrix ring $\left(\begin{smallmatrix} A & 0 \\ M & B \end{smallmatrix}\right)$, while $T'$ refers to an ideal of the form $\left(\begin{smallmatrix} I & 0 \\ N & K \end{smallmatrix}\right)$ with the additional properties outlined above.

Now we continue with a lemma that is implicit in [8] and proved in [10, Lemma 1.1] and leads us to a number of significant ring-theoretic properties relative to an ideal of a ring $R$.

**Lemma 1.** [10, Lemma 1.1] *Let $I$ be an ideal of the ring $R$. The following conditions are equivalent for a right ideal $I'$ of $R$:*

(1) *There exists $e^2 = e \in I'$ with $(1-e)I' \subseteq I$.*
(2) *There exists $e^2 = e \in I'$ with $I' \cap (1-e)R \subseteq I$.*
(3) *$I' = eR \oplus S$ where $e^2 = e$ and $S \subseteq I$.*

According to Nicholson and Zhou [9] an ideal $I$ of the ring $R$ *respects* a right ideal $I'$ of $R$ if the conditions in Lemma 1 are satisfied. Similarly, $I$ *respects* a left ideal $L \subseteq R$ if $L = Re \oplus S$ where $e^2 = e$ and $S \subseteq I$. It is worth noting that this definition is left-right symmetric, i.e., if $I \triangleleft R$ and $a \in R$, then $I$ respects $aR$ if and only if $I$ respects $Ra$.

Right (left) $I$-semiregular elements and rings first emerged in a paper [8] by Nicholson and Yousif and then were studied in [10] by Yousif and Zhou. Later, Nicholson and Zhou [9] dealt with these elements in terms of respecting a right (left) ideal as defined above and demonstrated that it is not necessary to distinguish between "right $I$-semiregular" and "left $I$-semiregular". Let $I$ be an ideal of the ring $R$. Recall that an element $a \in R$ is called $I$-semiregular if $I$ respects $aR$, i.e., if $e^2 = e \in aR$ exists with $(1-e)a \in I$, or alternatively, if $f^2 = f \in Ra$ exists with $a(1-f) \in I$. As expected, when all elements of the ring $R$ are $I$-semiregular, the ring $R$ is called a $I$-semiregular ring.

It is well known that the topic of lifting of idempotents is a crucial method for identifying the structure of semiregular and semiperfect rings. Nicholson and Zhou studied a natural extension of these notions in connection with strongly lifting ideals in [9]. Recall that an ideal $I$ of a ring $R$ is called *strongly lifting* if, for some $a \in R$, whenever $a^2 - a \in I$, then there exists an idempotent $e \in aR$ with $a - e \in I$. It is possible to replace the conclusion $e \in aR$ by $e \in Ra$ or $e \in aRa$ since this notion is left-right symmetric [9, Lemma 1]. In this work, Nicholson and Zhou further showed that the ring $R$ is $I$-semiregular if and only if $R/I$ is regular and $I$ is strongly lifting. A recent work [1] has shed new light on the question: "What can be said about relative semiregular ideals of the the formal triangular matrix ring?". The author has provided a criterion to decide if a given ideal $T'$ of the formal triangular matrix ring $T$ is strongly lifting.

Our first Theorem is motivated by the above-mentioned results and characterizes the semiregularity of formal triangular matrix rings relative to an ideal.

**Theorem 1.** *Let $T'$ be an ideal of $T$. Then $T$ is $T'$-semiregular if and only if $A$ is $I$-semiregular, $B$ is $K$-semiregular and $N = M$.*

*Proof.* First recall the fact that $T$ is $T'$-semiregular if and only if $T/T'$ is regular and $T'$ is strongly lifting. Now if $T$ is $T'$-semiregular, then $T/T'$ is regular, and so $A/I$ and $B/K$ regular. Further, $\mathrm{J}(T/T') = 0$ implies that $M/N = 0$, that is $N = M$. Moreover, Corollary 2.8 in [1] states that strongly lifting ideals $T'$ of $T$ are those ideals for which $I$ and $K$ are strongly lifting in $A$ and $B$, respectively. Combining these two results with the above-mentioned fact, we get $A$ is $I$-semiregular, $B$ is $K$-semiregular, as desired.

For the converse, first note that $T/T' = \begin{pmatrix} A/I & 0 \\ 0 & B/K \end{pmatrix} \cong A/I \times B/K$. Hence, the regularity of $A/I$ and $B/K$ implies the regularity of $T/T'$. Now, the result is easily seen by again using Corollary 2.8 in [1]. $\square$

As an application, we continue with an illustrative example.

**Example 1.** Let $A = \mathbb{Z}_{30}$, $B = \mathbb{Z}_9$ and $M = \mathbb{Z}_3 \oplus \mathbb{Z}_3$. Let us begin by considering the following ring:

$$T = \begin{pmatrix} \mathbb{Z}_{30} & 0 \\ \mathbb{Z}_3 \oplus \mathbb{Z}_3 & \mathbb{Z}_9 \end{pmatrix}.$$

It is our aim to determine all ideals $T'$ of $T$ with the property that $T$ is $T'$-semiregular by using Theorem 1. To do this, we first need to specify the strongly lifting ideals $I$ ($K$) of $A$ ($B$) for which $A/I$ ($B/K$) is von Neumann regular, respectively. Since $A$ and $B$ are exchange rings, all ideals of these two rings are strongly lifting and an easy computation shows that all factors of the form $A/I$ and $B/K$ are von Neumann regular except for the case $K = 0$ in $B$.

Taking into account the ideal structure of $T$ described in Proposition 1 and letting $N = M$, the following ideals $T'$ are those for which $T$ is $T'$-semiregular:

- $\begin{pmatrix} 0 & 0 \\ M & \mathbb{Z}_9 \end{pmatrix}, \begin{pmatrix} 15\mathbb{Z}_{30} & 0 \\ M & \mathbb{Z}_9 \end{pmatrix}, \begin{pmatrix} 10\mathbb{Z}_{30} & 0 \\ M & \mathbb{Z}_9 \end{pmatrix}, \begin{pmatrix} 6\mathbb{Z}_{30} & 0 \\ M & \mathbb{Z}_9 \end{pmatrix};$
- $\begin{pmatrix} 5\mathbb{Z}_{30} & 0 \\ M & \mathbb{Z}_9 \end{pmatrix}, \begin{pmatrix} 3\mathbb{Z}_{30} & 0 \\ M & \mathbb{Z}_9 \end{pmatrix}, \begin{pmatrix} 2\mathbb{Z}_{30} & 0 \\ M & \mathbb{Z}_9 \end{pmatrix}, \begin{pmatrix} \mathbb{Z}_{30} & 0 \\ M & \mathbb{Z}_9 \end{pmatrix};$
- $\begin{pmatrix} 0 & 0 \\ M & 3\mathbb{Z}_9 \end{pmatrix}, \begin{pmatrix} 15\mathbb{Z}_{30} & 0 \\ M & 3\mathbb{Z}_9 \end{pmatrix}, \begin{pmatrix} 10\mathbb{Z}_{30} & 0 \\ M & 3\mathbb{Z}_9 \end{pmatrix}, \begin{pmatrix} 6\mathbb{Z}_{30} & 0 \\ M & 3\mathbb{Z}_9 \end{pmatrix};$
- $\begin{pmatrix} 5\mathbb{Z}_{30} & 0 \\ M & 3\mathbb{Z}_9 \end{pmatrix}, \begin{pmatrix} 3\mathbb{Z}_{30} & 0 \\ M & 3\mathbb{Z}_9 \end{pmatrix}, \begin{pmatrix} 2\mathbb{Z}_{30} & 0 \\ M & 3\mathbb{Z}_9 \end{pmatrix}, \begin{pmatrix} \mathbb{Z}_{30} & 0 \\ M & 3\mathbb{Z}_9 \end{pmatrix}.$

**Remark 1.** Note that the “if” part of the above theorem is not in general true if we omit the condition $N = M$ as shown in the following example.

Let $A = \mathbb{Z}_4$, $B = \mathbb{Z}_2$ and $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Consider the formal triangular matrix ring

$$T = \begin{pmatrix} \mathbb{Z}_4 & 0 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}.$$

We wish to find an ideal $T'$ of $T$ for which $A$ is $I$-semiregular, $B$ is $K$-semiregular, but $T$ is not $T'$-semiregular due to the fact that $N \neq M$. For this, we first observe that the ideals $I = 2\mathbb{Z}_4$ of $\mathbb{Z}_4$ and $K = 0$ of $\mathbb{Z}_2$ are strongly lifting, respectively. Further, $A/I \cong B/K \cong \mathbb{Z}_2$ is clearly von Neumann regular. Hence, these two together imply that $A$ is $I$-semiregular and $B$ is $K$-semiregular.

Taking into account the ideal structure of $T$ described in Proposition 1, we let $N = \mathbb{Z}_2 \oplus 0$ a $(B, A)$-subbimodule of $M$ and

$$T' = \begin{pmatrix} 2\mathbb{Z}_4 & 0 \\ \mathbb{Z}_2 \oplus 0 & 0 \end{pmatrix}.$$

Then the ring $T$ is not $T'$-semiregular since the ring

$$T/T' \cong \begin{pmatrix} \mathbb{Z}_2 & 0 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}$$

is not von Neumann regular.

It should be noted that the Jacobson radical $\mathrm{J}(R)$ of a ring $R$ is not necessarily idempotent lifting. However, if it is idempotent lifting, it is also strongly lifting. In fact, as is well-known, a ring $R$ is semiregular if and only if $R/\mathrm{J}(R)$ is regular and $\mathrm{J}(R)$ is idempotent lifting. Hence, the $\mathrm{J}(R)$-semiregular rings are just the semiregular rings and we get the following immediate corollary to the above result.

**Corollary 1.** *$T$ is semiregular if and only if $A$ and $B$ are semiregular.*

A right $I$-semiperfect ring is one in which every right ideal $M$ of $R$ fulfills the equivalent conditions as stated in Lemma 1. Left $I$-semiperfect rings can be defined in a similar vein. A short proof of the right-left symmetry of this notion appears in [9] by showing the following equivalence

$$R \text{ is } I\text{-semiperfect} \Leftrightarrow R/I \text{ is semisimple and } I \text{ is strongly lifting.}$$

We now determine a necessary and sufficient condition for the triangular matrix ring $T$ to be $T'$-semiperfect.

**Theorem 2.** *If $T'$ is an ideal of $T$, then the following conditions are equivalent:*

(i) *$T$ is $T'$-semiperfect;*
(ii) *$A$ is $I$-semiperfect, $B$ is $K$-semiperfect and $N = M$.*

*Proof.* To begin with, let us recall that $T$ is $T'$-semiperfect if and only if $T/T'$ is semisimple and $T'$ is strongly lifting. Now if $T$ is $T'$-semiperfect, then $T/T'$ is semisimple, and so are $A/I$ and $B/K$. Further, $\mathrm{J}(T/T') = 0$ implies that $M/N = 0$, that is $N = M$. Moreover, Corollary 2.8 in [1] states that strongly lifting ideals $T'$ of $T$ correspond to those ideals for which $I$ and $K$ are strongly lifting in $A$ and $B$. Combining these two results with the above-mentioned fact, we get $A$ is $I$-semiperfect, $B$ is $K$-semiperfect, as desired.

For the converse, consider the ring isomorphism $T/T' = \left(\begin{smallmatrix} A/I & 0 \\ 0 & B/K \end{smallmatrix}\right) \cong A/I \times B/K$. Hence, the semisimplicity of $A/I$ and $B/K$ implies the semisimplicity of $T/T'$. We get our assertion by putting these and Corollary 2.8 in [1] together. □

**Example 2.** Let $\mathbb{Z}_{(p)}$ be the localization of the ring of integers $\mathbb{Z}$ at a prime ideal $p\mathbb{Z}$, $\mathbb{Z}_{p^\infty}$ be the Prüfer group and let $\hat{\mathbb{Z}}_p$ be $p$-adic integers.

Consider the formal triangular matrix ring

$$T = \begin{pmatrix} \mathbb{Z}_{(p)} & 0 \\ \mathbb{Z}_{p^\infty} & \hat{\mathbb{Z}}_p \end{pmatrix}.$$

Our goal is to determine all ideals $T'$ of $T$ with the property that $T$ is $T'$-semiperfect by using Theorem 2. For this, it is enough to identify all strongly lifting ideals of $\mathbb{Z}_{(p)}$ and $\hat{\mathbb{Z}}_p$ for which every factor ring of these two rings is semisimple Artinian. Since $\mathbb{Z}_{(p)}$ and $\hat{\mathbb{Z}}_p$ are exchange rings, all ideals of these two rings are strongly lifting. It is not difficult to see that factor rings are semisimple Artinian for the ideals $p\mathbb{Z}_{(p)}$ and $\mathbb{Z}_{(p)}$ and the ideals $p\hat{\mathbb{Z}}_p$ and $\hat{\mathbb{Z}}_p$ of the uniserial rings $\mathbb{Z}_{(p)}$ and $\hat{\mathbb{Z}}_p$, respectively.

Hence, the ring $T$ is $T'$-semiperfect for the following list of ideals $T'$:

$$\begin{pmatrix} p\mathbb{Z}_{(p)} & 0 \\ \mathbb{Z}_{p^\infty} & p\hat{\mathbb{Z}}_p \end{pmatrix}, \begin{pmatrix} p\mathbb{Z}_{(p)} & 0 \\ \mathbb{Z}_{p^\infty} & \hat{\mathbb{Z}}_p \end{pmatrix}, \begin{pmatrix} \mathbb{Z}_{(p)} & 0 \\ \mathbb{Z}_{p^\infty} & p\hat{\mathbb{Z}}_p \end{pmatrix}, \begin{pmatrix} \mathbb{Z}_{(p)} & 0 \\ \mathbb{Z}_{p^\infty} & \hat{\mathbb{Z}}_p \end{pmatrix}.$$

In the example below, it can be seen that the assumption "$N = M$" in the "if" part of Theorem 2 cannot simply be dropped.

**Example 3.** As an example of an ideal $T'$ of $T$ for which $A$ is $I$-semiperfect, $B$ is $K$-semiperfect, but $T$ is not $T'$-semiperfect, we would like to recall an example of Berberian that was discussed in detail in [2, Example 1].

Let $\mathbb{C}$ be the complex field and $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ be the division ring of real quaternions.

Take $A = \mathbb{C}$, $B = \mathbb{H}$ and $M = \mathbb{H}$. Let us begin by considering the following ring

$$T = \begin{pmatrix} \mathbb{C} & 0 \\ \mathbb{H} & \mathbb{H} \end{pmatrix}.$$

The ring $T$ is an exchange ring due to the fact that the rings $\mathbb{C}$ and $\mathbb{H}$ are all exchange rings [6, Proposition 2.1].

We further observe that the ideals $I = 0$ of $\mathbb{C}$ and $K = \mathbb{H}$ of $\mathbb{H}$ are strongly lifting as exchange rings are precisely the rings that every one-sided ideal is strongly lifting. Furthermore, $A/I$ and $B/K$ are clearly semisimple. Hence, we have $A$ is $I$-semiperfect and $B$ is $K$-semiperfect.

On the other hand, for the ideal

$$T' = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{H} \end{pmatrix}$$

of the ring $T$, $T$ is not $T'$-semiperfect since the ring

$$T/T' \cong \begin{pmatrix} \mathbb{C} & 0 \\ \mathbb{H} & 0 \end{pmatrix}$$

is not semisimple Artinian.

A ring $R$ is called semipotent if each right ideal of $R$ that is not contained in its Jacobson radical $\mathrm{J}(R)$ contains a nonzero idempotent. Note that this notion is left-right symmetric. A semipotent ring $R$ is called potent if, in addition, $\mathrm{J}(R)$ is an idempotent lifting ideal of $R$. Examples of these rings include exchange rings (see [7, Proposition 1.9]). It is well known that a formal triangular matrix ring $T$ is semipotent (respectively, potent) if and only if $A$ and $B$ are semipotent (respectively, potent) [4, Theorem 6.4].

Semipotent rings has been generalized to semipotent rings relative to an ideal based on the following lemma proposed by Nicholson and Zhou [9].

**Lemma 2.** [9, Lemma 19] *The following are equivalent for $I \lhd R$:*

1. *If $I' \nsubseteq I$ is a right ideal, then there exists $e^2 = e \in I' - I$.*
2. *If $a \notin I$, then there exists $e^2 = e \in aR - I$.*
3. *If $a \notin I$ there exists $x \in R$ such that $xax = x \notin I$.*

Following Nicholson and Zhou, for an ideal $I$ in a ring $R$, $R$ is said to be $I$-semipotent if the above conditions in Lemma 2 are fulfilled, and is said to be $I$-potent if it is $I$-semipotent and $I$ is strongly lifting in $R$. In other words, the semipotent (potent) rings are simply the $\mathrm{J}(R)$-semipotent ($\mathrm{J}(R)$-potent) rings. Since the property of being a semipotent or a potent ring transfers to formal triangular matrix rings by the above-mentioned result due to Haghany and Varadarajan [4, Theorem 6.4], it is natural to suspect that it may also transfer in the relative case.

We now interpret this notion in the language of formal triangular matrix rings. We mimic the proof of Haghany and Varadarajan [4, Theorem 6.4].

**Theorem 3.** *Let $T'$ be an ideal of $T$. If $T$ is $T'$-semipotent then $A$ is $I$-semipotent and $B$ is $K$-semipotent, respectively.*

*Proof.* Assume that $T$ is $T'$-semipotent. We first claim that $A$ is $I$-semipotent. Let $I' \nsubseteq I$ be a right ideal in $A$. Then $I'' = \begin{pmatrix} I' & 0 \\ 0 & 0 \end{pmatrix}$ is a right ideal not contained in $T'$. Hence there exists $e \in I'$ with $\left(\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right) \in I'' - T'$. This implies that $e^2 = e \in I' - I$, as desired. Secondly, we claim that $B$ is $K$-semipotent. Let $K' \nsubseteq K$ be a right ideal in $B$. Then $K'' = \begin{pmatrix} 0 & 0 \\ K'M & K' \end{pmatrix}$ is a right ideal of $T$ not contained in $T'$. Since $T$ is $T'$-semipotent, there exists a nonzero element $\left(\begin{smallmatrix} 0 & 0 \\ m & f \end{smallmatrix}\right) \in K'' - T'$ with $\left(\begin{smallmatrix} 0 & 0 \\ m & f \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ m & f \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ m & f \end{smallmatrix}\right) \in K'' - T'$. This implies that $f^2 = f$ and $fm = m$. Since $\left(\begin{smallmatrix} 0 & 0 \\ m & f \end{smallmatrix}\right)$ is nonzero, we get $f \neq 0$ or $m \neq 0$. By considering $fm = m$, we get $f \neq 0$. Thus, $0 \neq f$ with $f^2 = f \in K' - K$, as desired. □

The converse of Theorem 3 does not hold in general, as can be seen in the following example.

**Example 4.** There exist a formal triangular ring $T$ and an ideal $T'$ of $T$ such that $A$ is $I$-semipotent, $B$ is $K$-semipotent, but $T$ is not $T'$-semipotent.

Let $R = \mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_2\times\cdots$ be the direct product of rings. Take into consideration the following subring of $R$:

$$A = \{(n, \bar{n}_2, \bar{n}_3, \cdots, \bar{n}_k, \bar{n}, \cdots) | n, n_i \in \mathbb{Z},\ k \geqslant 2\}.$$

Putting $I = \{(2m, \bar{0}, \bar{0}, \cdots) \mid m \in \mathbb{Z}\}$, it follows that $A$ is $I$-semipotent by [9, Example 23].

Now, take $B = \mathbb{Z}_4$ and $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Consider the formal triangular matrix ring

$$T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}.$$

We further consider the ideal $K = 2\mathbb{Z}_4$ of $B = \mathbb{Z}_4$. Since $B$ is $K$-semiregular by Remark 1, it is $K$-semipotent, too.

On the other hand, for the ideal

$$T' = \begin{pmatrix} I & 0 \\ \mathbb{Z}_2 \oplus 0 & K \end{pmatrix}$$

of the ring $T$, $T$ is not $T'$-semipotent. To show the last statement, consider the following ideal of the ring $T$

$$\tilde{T} = \begin{pmatrix} I & 0 \\ 0 \oplus \mathbb{Z}_2 & K \end{pmatrix}.$$

Then $\tilde{T}$ is clearly not contained in the ideal $T$. On the other hand, an easy computation shows that the ideal $\tilde{T}$ of $T$ doesn't contain any nonzero idempotent. Thus, there do not exist any idempotent in $\tilde{T}$ which is not in $T'$. By Lemma 2, $T$ is not $T'$-semipotent, as desired.

As we mentioned above, relative potent rings is a proper subclass of the class of relative semipotent rings with the additional strongly lifting condition on the relative ideal. Due to the fact that strongly lifting ideals $T'$ of $T$ are those ideals for which $I$ and $K$ are strongly lifting in $A$ and $B$, respectively [1, Corollary 2.8], Theorem 3 implies the the following immediate corollary.

**Corollary 2.** *Let $T'$ be an ideal of $T$. If $T$ is $T'$-potent, then $A$ is $I$-potent and $B$ is $K$-potent, respectively.*

The converse of Theorem 3 is not true in general, it is natural to ask the question what additional conditions are required for this to happen. We will show below that this question has an affirmative answer for ideals of the form $\left(\begin{smallmatrix} I & 0 \\ M & K \end{smallmatrix}\right)$ in the formal triangular matrix ring $T$.

**Theorem 4.** *Let $T'$ be an ideal of the form $\left(\begin{smallmatrix} I & 0 \\ M & K \end{smallmatrix}\right)$ in $T$. If $A$ is $I$-semipotent and $B$ is $K$-semipotent, then $T$ is $T'$-semipotent.*

*Proof.* Assume that $A$ is $I$-semipotent and $B$ is $K$-semipotent. We will show that $T$ is $T'$-semipotent. Let $\tilde{T}$ be a right ideal of $T$ with $\tilde{T} \nsubseteq T'$. Then there exists an element $t = \left(\begin{smallmatrix} a & 0 \\ m & b \end{smallmatrix}\right) \in \tilde{T} - T'$. This implies that either $a \notin I$ or $b \notin K$. First, assume that $a \notin I$. Then there exist a non-zero idempotent $e$ in $aA - I$. Set $e = ar$ for some $r \in A$. Then $are = e^2 = e \neq 0$. Note that

$$\left(\begin{smallmatrix} e & 0 \\ mre & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} are & 0 \\ mre & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & 0 \\ m & b \end{smallmatrix}\right)\left(\begin{smallmatrix} re & 0 \\ 0 & 0 \end{smallmatrix}\right),$$

and so $\left(\begin{smallmatrix} e & 0 \\ mre & 0 \end{smallmatrix}\right) \in tT \subseteq \tilde{T}$. Also

$$\left(\begin{smallmatrix} e & 0 \\ mre & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} e & 0 \\ mre & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} e & 0 \\ mre & 0 \end{smallmatrix}\right).$$

Thus $\left(\begin{smallmatrix} e & 0 \\ mre & 0 \end{smallmatrix}\right)$ is a non-zero idempotent in $tT - T'$. If $b \notin K$, there exists a non-zero idempotent $f \in bB$ and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & f \end{smallmatrix}\right)$ is a non-zero idempotent in $tT - T'$. This proves that $T$ is $T'$-semipotent. $\square$

Considering the definition of a relative potent (resp. semipotent and potent) ring we end the paper with the following corollaries of Theorem 3 and Theorem 4.

**Corollary 3.** *Let $T'$ be an ideal of of the form $\left(\begin{smallmatrix} I & 0 \\ M & K \end{smallmatrix}\right)$ in $T$. If $A$ is $I$-potent and $B$ is $K$-potent, then $T$ is $T'$-potent.*

**Corollary 4.** [4, Theorem 6.4] *Let $T$ be the formal triangular matrix ring of the form $\left(\begin{smallmatrix} A & 0 \\ M & B \end{smallmatrix}\right)$. Then $T$ is semipotent (resp. potent) if and only if $A$ and $B$ are semipotent (resp. potent).*

**Declaration of Competing Interests** The author declares that there is no competing interest regarding the publication of this paper.

**Acknowledgements** The author would like to thank the referees for their careful considerations.

## References


[1] Altun Özarslan, M, Lifting properties of formal triangular matrix rings, *RACSAM*, 115(3) (2021), Paper No. 104, 13 pp. https://doi.org/10.1007/s13398-021-01044-0

[2] Berberian, S. K., The center of a corner of a ring, *J. Algebra*, 71(2) (1981), 515-523. https://doi.org/10.1016/0021-8693(81)90191-5

[3] Goodearl, K. R., Ring Theory. Nonsingular Rings and Modules, Monographs and Textbooks in Pure and Appl. Math., 33, Marcel Dekker, New York, 1976.

[4] Haghany, A, Varadarajan, K., Study of formal triangular matrix rings, *Comm. Algebra*, 27 (1999), 5507-5525. https://doi.org/10.1080/00927879908826770

[5] Herstein, I. N., A counter example in Noetherian rings, *Proc. Nat. Acad. Sci. U.S.A.*, 54 (1965), 1036-1037. https://doi.org/10.1073/pnas.54.4.1036

[6] Hong, C. Y., Kim, N. K., Lee, Y., Exchange rings and their extensions, *J. Pure Appl. Algebra*, 179 (2003), 117-126. https://doi.org/10.1016/S0022-4049(02)00299-2

[7] Nicholson, W. K., Lifting idempotents and exchange rings, *Trans. Amer. Math. Soc.*, 229 (1977), 269-278. https://doi.org/10.1090/S0002-9947-1977-0439876-2

[8] Nicholson, W. K., Yousif, M.F., Weakly continuous and C2-rings, *Comm. Algebra*, 29 (2001), 2429-2446. https://doi.org/10.1081/AGB-100002399

[9] Nicholson, W. K., Zhou, Y, Strong lifting, *J. Algebra*, 285(2) (2005), 795-818. https://doi.org/10.1016/j.jalgebra.2004.11.019

[10] Yousif, M. F., Zhou, Y, Semiregular, semiperfect and perfect rings relative to an ideal, *Rocky Mountain J. Math.*, 32 (2002), 1651-1671. https://doi.org/10.1216/rmjm/1181070046

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 222–234 (2024)
DOI:10.31801/cfsuasmas.1225525
ISSN 1303-5991 E-ISSN 2618-6470

*Research Article; Received:December 28, 2022; Accepted: November 2, 2023*

# MULTIGRID METHODS FOR NON COERCIVE VARIATIONAL INEQUALITIES

Nour El Houda NESBA[1] and Mohammed BEGGAS[2]

[1]Department of Mathematics, University of Echahid Hamma Lakhder, Eloued, ALGERIA
[2]Department of Mathematics, Exact Sciences Faculty, University of El Oued, El Oued 39000, ALGERIA

Abstract. In this study, our examination centers around the numerical resolution of non-coercive issues using a multi-grid approach. Our particular emphasis is directed towards employing multi-grid methodologies to tackle non-linear variational inequalities. Our primary goal involves confirming the consistent convergence of the multi-grid algorithm. To attain this objective, we make use of fundamental sub-differential calculus and glean insights from the convergence principles of non-linear multi-grid techniques.

## 1. Introduction

Contemporary literature showcases a diverse array of computational technique that are harnessed to address intricate real-world challenges spanning various scientific and engineering domains. These methodologies have been crafted and utilized to confront demanding problems, yielding efficient resolutions within their respective fields. Many researchers have explored these computational strategies to tackle a number of applied problems, propelling comprehension and advance understanding and progress in many scientific fields.

Commonly used numerical methods for solving boundary problems generally lead, after discretisation, to the solution of systems of algebraic equations. These numerical techniques, encompassing iterative methods like Jacobi, Gauss-Seidel iteration, and relaxation methods, are frequently chosen due to their conventional nature.However, they may show a slow convergence of fine mesh sizes and complexity when applied to general ellipticity problems. In contrast, multi-grid methods offer

2020 *Mathematics Subject Classification.* 65M55, 65N30, 49N05, 65K15, 35J86.

*Keywords.* Multigrid method, variational inequality, finite element, iterative method, HJB equation, non coercive operator.

[1] nour17nesba@gmail.com- Corresponding author; 0000-0003-0800-8253;

[2] beggasmr@gmail.com; 0000-0002-7926-7802

.

a clear advantage. These algorithms exhibit linear expenses based on the number of discretization points. These algorithms exhibit linear expenses based on the number of discretization points, regardless of the problem's dimensions. Particularly, these methods are adept at resolving linear and non-linear partial differential equations (PDEs) as well as linear V.Is (Variational inequalities)[12, 10, 7]. Their linear complexity makes them powerful tools for large problems, greatly reducing computational requirements while ensuring accurate solutions. Multi-grid techniques are widely praised as a fast approach to tackling various forms of variational equations and inequalities [11], particularly in the area the discretized elliptic problems that leads to an $M$-matrix [6].

Through a conforming finite element method $P_1$ [4], we will be providing an overview of non-linear variational inequalities (N.V.I) problems and their discretization in the following section. Additionally, The Hoppe multi-grid method [14, 9] served as an inspiration for our algorithm, which views the V.I as stationary Hamilton-Jacobi-Bellman(H.J.B) equations. The iteration matrices are provided for an algorithm known as the M.G.H.J.B, or multi-grid Hierarchy Jacobi.

First, we present original results on the approximation and smoothness properties within the $L^\infty$ norm. We then demonstrate the consistent convergence of the M.G.H.J.B algorithm. Finally, we apply the numerical method to a specific scenario where the operator is linear and unconstrained, and the second element is independent of the solution. In this context, we implemented the Gauss-Seidel method and the multigrid method V and W cycles. Numerical experiments are performed to evaluate the efficiency and performance of these methods in solving the proposed problem.

## 2. Multigrid Method

2.1. **Assumptions and Notations.** Suppose that $\Omega$ is an open in $\mathbb{R}^N$ with a sufficiently regular border $\partial\Omega$.

We define second order operators with $u, v \in H^1(\Omega)$,

$$\mathfrak{A} = \sum_{1 \le j,k \le N} \eth_{jk}(x) \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{k=1}^{N} \mathfrak{b}_k(x) \frac{\partial}{\partial x_k} + \mathfrak{b}_0(x),$$

where $\eth_{jk}(x), \mathfrak{b}_k(x), \mathfrak{b}_0(x)$ are sufficiently regular coefficients such that:

$$\eth_{kj}(x) = \eth_{jk}(x), \quad \mathfrak{b}_0(x) \ge \beta > 0; \quad (x \in \Omega).$$

Also, we define the associated bilinear non-coercive forms

$$\mathfrak{a}(u,v) = \int_\Omega \left( \sum_{1 \le j,k \le N} \eth_{jk} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_k} + \sum_{k=1}^{N} \mathfrak{b}_k \frac{\partial u}{\partial x_k} v + \mathfrak{b}_0(x) uv \right) dx,$$

and the operators

$$\mathcal{B} = \sum_{1\leq j,k\leq N} \eth_{jk}(x)\frac{\partial^2}{\partial x_j \partial x_k} + \sum_{k=1}^{N} \mathfrak{b}_k(x)\frac{\partial}{\partial x_k} + (\mathfrak{b}_0(x) + \lambda), \tag{1}$$

we choose $\lambda > 0$ is sufficiently large so that $\mathcal{B} = \mathfrak{A} + \lambda I$ are strongly elliptic on $H^1(\Omega)$ and

$$\mathfrak{b}(u,v) = \mathfrak{a}(u,v) + \lambda(u,v). \tag{2}$$

Additionally, we consider $f$ a second member as following:

$$f \in L^\infty(\Omega); \quad f \geq 0$$

and obstacle $\psi \in W^{2,\infty}$, where $\psi > 0$.

2.2. **Problem Continuous.** The aim is to find $u$ the solution of the problem presented by the following V.Is:
Find $u$ solution of:

$$\begin{cases} \mathfrak{b}(u, v-u) \geq (f + \lambda u, v-u), \quad \forall v \in H^1(\Omega), \\ u \leq \psi; \quad v \leq \psi. \end{cases} \tag{3}$$

It has been confirmed that this issue has a singular solution, as demonstrated by the theorem of fixed point and from the aforementioned assumptions (see [1]).

2.3. **Discretization.** In order to build a multi-grid loop, we create a sequence of discretization steps referred to as $0 < \mathfrak{h}_{k+1} < \mathfrak{h}_k < 1$ such that the grids are nested $\mathfrak{h}_{k+1} = \frac{\mathfrak{h}_k}{2}$.
Subsequently, we delineate $\Omega_k = \Omega_{\mathfrak{h}_k}, V_k = V_{\mathfrak{h}_k}, \mathfrak{A}_k = \mathfrak{A}_{\mathfrak{h}_k}$ and we establish a series of uniform regular triangulations referred to as $\{T_k, k \in \mathbb{N}_0\}$. For all $T_k$,we have

$$\begin{gathered} \Omega_k \subset \Omega_{k+1} \subset \Omega, \\ \operatorname{dist}(\partial\Omega_k, \partial\Omega) \leq c_0 \mathfrak{h}_k^2, \\ \mathfrak{h}_k \mathfrak{h}_{k+1} \leq c_1. \end{gathered}$$

We introduce $V_{\mathfrak{h}_k} = \left\{v_{\mathfrak{h}_k} \in C(\Omega) \cap H^1 \ ; v_{\mathfrak{h}_k}/_T \in P_1\right\}$, for simplicity we write:

$$V_k = \left\{v_k \in C(\Omega) \cap H^1; \quad v_k/r \in P_1\right\}.$$

The shape function $\varphi_k^i, i \in (1, \ldots, m(\mathfrak{h}_k))$ of the usual basis is defined as: $\varphi_k^i\left(x_k^j\right) = \delta_{ij}$, where $x_k^j$ be a node of the $T_k$ triangulation .
So, the ordinary restriction operator $r_k$ is defined like:

$$r_k v(x) = \sum_{i=1}^{m(\mathfrak{h}_k)} v\left(M_k^i\right) \varphi_k^i(x). \tag{4}$$

If we suppose $U_k = \mathbb{R}^{m_k}$. Then, $r_k : U_k \to V_k$ is a bijection.
$U_k$ is equipped with the scalar product

$$< u, v >= \mathfrak{h}_k^2 \sum_{i=1}^{m(\mathfrak{h}_k)} u_i v_i, \quad \|u\|_k =< u, v >_k^{1/2}.$$

The maximum norms in $U_k$ and $V_k$ are equivalent, we denote them $\|\cdot\|_\infty$. We have the following lemma (see [2]).

**Lemma 1.** *There exists $C_1, C_2$ independent of $k$ such that*

$$\begin{array}{ll} \|r_k(u)\|_\infty & = \|u\|_\infty, \quad \forall u \in U_k. \\ C_1\|v\|_\infty & \leq \|r_k^*(v)\|_\infty \leq C_2\|v\|_\infty, \quad \forall v \in V_k. \end{array} \tag{5}$$

2.4. **Problem Discrete.** Continuing in a logical sequence, we present the discretization matrices $\mathcal{B}_k$ and the bilinear form $b\left(\varphi_k^1, \varphi_k^s\right)$, where $\varphi_s$ the shape functions. With these descriptions established. Now, we are positioned to formulate the discrete problem in the subsequent manner:
Find $u_k \in V_k$ solution of:

$$\left\{ \begin{array}{l} < \mathcal{B}_k u_k, v_k - u_k > \geq < f_k + \lambda u_k, v_k - u_k >, \quad \forall v_k \in V_k \\ u_k \leq r_k\psi, \quad v_k \leq r_k\psi \end{array} \right. \tag{6}$$

We make the assumption that the matrices $\mathcal{B}_k$ are $M$-matrices.(see [3] ).

2.5. **H.J.B form.** The correspondence between the finite-dimensional V.I (3) and a representation in Hamilton-Jacobi-Bellman (H.J.B) form is easily discernible (see [10]). We detail the selected numerical technique for resolving the stationary H.J.B equations.
In the traditional framework, we recollect certain convergence outcomes that will play a crucial role in affirming the M.G.H.J.B algorithm's convergence expounded in the following:
***Iterative diagram:***
Step 1: Choose $u_k^0 \in \mathbb{R}^{n_k}$ as initial vector.
Step 2 : Calculate the solution $u_k^{\nu+1} \in \mathbb{R}^{n_k}$ of the following recurrence equation

$$\mathcal{B}_k^\nu u_k^{\nu+1} - Z_k^\nu = 0, \tag{7}$$

such that

$$Z_k^\nu = F_k^\nu + \lambda u_k^\nu$$

where

$$\mathcal{B}_{k,i}^\nu = \left\{ \begin{array}{ccc} \mathcal{B}_{k,i}\left(u_k\right) & \text{if} & \mathcal{B}_{k,i}u_{k,i}^\nu - Z_{k,i} > u_{k,i}^\nu - \psi_{k,i}, \\ u_{k,i} & \text{if} & 1 \leq i \leq N, \end{array} \right. \tag{8}$$

$$Z_{k,i}^\nu = \left\{ \begin{array}{ccc} Z_{k,i} & \text{if} & \mathcal{B}_{k,i}u_{k,i}^\nu - Z_{k,i} > u_{k,i}^\nu - \psi_{k,i}, \\ u_{k,i} & \text{if} & 1 \leq i \leq N. \end{array} \right. \tag{9}$$

Let the discrete H.J.B equation where $u_k^*$ be the unique solution

$$\max_{1\le i\le N}\left(\mathcal{B}_{k,i}u_k^* - Z_{k,i}, u_{k,i}^* - \psi_{k,i}\right) = 0. \tag{10}$$

We will formulate the subsequent theorem and introduce our problem derived from the (H.J.B) equation, drawing inspiration from Hoppe's [10].

**Theorem 1.** *Let $u_k^\nu$ be the solution in the iteration defined and it satisfies the H.J.B equation. Furthermore, We make that $\mathcal{B}_k$ is continuously differentiable then the sequence $(u_k^\nu)_{\nu\ge 0}$ converges and approaches $u_k^*$.*

Previously moving forward with presenting the findings, it is relevant to revisit the subsequent theorem:

**Theorem 2.** *(see [1] , [5]) If the previous notations and assumptions are satisfies. So , we have:*

$$\|u - u_k^*\|_\infty \le C\mathfrak{h}_k^2\left|\log\mathfrak{h}_k\right|^2\|g(u)\|_\infty. \tag{11}$$

2.6. **Multi-grid ( M.G.H.J.B) algorithm for V.Is.** For the multi-grid method we choose an iteration $u_k^\nu, \nu > 0$.So, we obtain $\bar{u}_k^\nu$, by using an iterative method to solve the system (7) by $\alpha$

$$\bar{u}_k^\nu = S_k^\alpha\left(u_k^\nu\right) \tag{12}$$

where $S_k$ is the smoothing operator and $\alpha$ is the number performed of iterations. The solution of (7) is denoted by $u_k^*$. The error setting $e_k^\nu = \bar{u}_k^\nu - u_k^*$, and the residual $d_k^{(\nu)} = Z_k^\nu - \mathcal{B}_k^\nu\bar{u}_k^\nu$, the equation (7) can be write as

$$\mathcal{B}_k^\nu\left(\bar{u}_k^\nu + e_k^\nu\right) = Z_k^\nu.$$

This leads to the residual equation

$$\mathcal{B}_k^\nu e_k^\nu = Z_k^\nu - \mathcal{B}_k^\nu\bar{u}_k^\nu = d_k^\nu.$$

After the relaxation on $\mathcal{B}_k^\nu\bar{u}_k^\nu = Z_k^\nu$ on the fine grid, the error will display a continuous nature. However, the error on the coarse grid appears to be more oscillatory, leading to the relaxation. At the $(k-1)$ level, we need to compute $e_{k-1}^\nu$ for determine $e_k^\nu$, where $e_{k-1}^\nu$ is the solution of the coarse grid system

$$\mathcal{B}_{k-1}^\nu e_{k-1}^\nu = d_{k-1}^\nu. \tag{13}$$

We can interpret $e_{k-1}^\nu\left(\text{resp}\,\mathcal{B}_{k-1}^\nu, d_{k-1}^\nu\right)$ and $e_k^\nu\left(\text{resp}\,\mathcal{B}_k^\nu, d_k^\nu\right)$ as approximation operator at level $(k-1)$ and $(k)$ respectively. Additionally, we have $\mathcal{R}_k$ the restriction operator and $\mathcal{P}_k$ its reverse .
consecquently, at the $(k)$ level we identify an improved iteration

$$u_k^{\nu+1} = \bar{u}_k^\nu + \mathcal{P}_k\left(e_{k-1}^\nu\right). \tag{14}$$

Because of the nested structure, we employ the well-defined identity operator

$$\pi : V_{k-1} \longrightarrow V_k; \quad \pi v = v,$$

the operators of extension and restriction define like

$$\mathcal{P}_k = r_k^{-1}r_{k-1}, \quad \mathcal{R}_k = \mathcal{P}_k^t. \tag{15}$$

2.7. **Matrix of the M.G.H.J.B Algorithm.** For each iteration, The matrix of the two-grid method with $\alpha_1$ pre-smoothing and $\alpha_2$ post-smoothing iterations at the $(k)$ level is given by

$$TG_k(\alpha_1,\alpha_2) = S_k^{\alpha_2}\left((\mathcal{B}_k^\nu)^{-1} - \mathcal{P}_k\left(\mathcal{B}_{k-1}^\nu\right)^{-1}\mathcal{R}_k\right)(\mathcal{B}_k^\nu)\, S_k^{\alpha_1}. \tag{16}$$

**Theorem 3.** *(see [13] )The multi-grid technique embodies a linear iterative approach, with the iteration matrix referred to as $MG_k$*

$$\begin{aligned} &MG_0 = 0,\\ &MG_k = S_k^{\alpha_2}\left(I_k - \mathcal{P}_k\left(I_k - MG_{k-1}\right)\left(\mathcal{B}_{k-1}^\nu\right)^{-1}\mathcal{R}_k\right)(\mathcal{B}_k^\nu)\, S_k^{\alpha_1}, \\ &\quad\;\; = TG_k + S_k^{\alpha_2}\mathcal{P}_k MG_{k-1}\left(\mathcal{B}_{k-1}^\nu\right)^{-1}\mathcal{R}_k\,(\mathcal{B}_k^\nu)\, S_k^{\alpha_1}, \quad k = 1,2,.. \end{aligned} \tag{17}$$

## 3. Convergence of the Multi-grid algorithm in $L^\infty$-norm

This section is devoted to presenting a unified convergence analysis of multi-grid algorithm. To prove the convergence, we need the following proprieties

3.1. **Approximation property.**

**Theorem 4.** *(see [8] ) The matrix $\Upsilon_k = \left[(\mathcal{B}_k^\nu)^{-1} - \mathcal{P}_k\left(\mathcal{B}_{k-1}^\nu\right)^{-1}\mathcal{R}_k\right]$ has the approximation property*

$$\|\Upsilon_k\|_\infty \le C h_k^2 \left|\ln h_k\right|^2. \tag{18}$$

*Proof.* The proof was proposed by Arnold in [14] on Theorem 1. □

3.2. **Property of Smoothing.** To prove the smoothness property, we consider the decomposition $\mathcal{B}_k^\nu = E_k - N_k$ and using the following assumptions:
for all $k$

$$E_k \text{ is regular and } \left\|E_k^{-1}N_k\right\|_\infty \le 1,. \tag{19}$$

$$\|E_k\|_\infty \le C h_k^{-2}, \text{ with C independent of k.} \tag{20}$$

In the process of smoothing, we utilize a relaxation method with an iterative matrix

$$S_k = I_k - \omega E_k^{-1} N_k, \quad \omega \in (0,1).$$

For the following theorem, the concept of Arnold Reusken [14] is relevant to our work.

**Theorem 5.** *Under the previous assumptions, there exists a constant $C$, which is independent of both $k$ and $\alpha$. Such that:*

$$\|(\mathcal{B}_k^\nu)\, S_k^\alpha\|_\infty \le C\frac{1}{\sqrt{\alpha}} h_k^{-2}. \tag{21}$$

*(smoothness properties)*

By switching to the norm in (14), from (18) and (21) we can proving the following estimation:

$$\exists C_s : \|S_k^\alpha\|_\infty \leq C_s, \text{ for all } k \text{ and } \alpha. \tag{22}$$

From the equation (16) with two lattices iterate (two-grid) and $\alpha_2 = 0$, we have the following estimate:

$$\begin{aligned}\|TG_k(\alpha_1, 0)\|_\infty &= \left\|\left((\mathcal{B}_k^\nu)^{-1} - \mathcal{P}_k \left(\mathcal{B}_{k-1}^\nu\right)^{-1} \mathcal{R}_k\right) (\mathcal{B}_k^\nu)\, S_k^{\alpha_1}\right\|_\infty \\ &\leq \left\|\left((\mathcal{B}_k^\nu)^{-1} - \mathcal{P}_k \left(\mathcal{B}_{k-1}^\nu\right)^{-1} \mathcal{R}_k\right)\right\|_\infty \|(\mathcal{B}_k^\nu)\, S_k^{\alpha_1}\|_\infty .\end{aligned}$$

Typically, we choose a hierarchy of more than two-grids. in this case, we can define the iterative matrices (17) by the recurrence of (16) for all $(k)$ levels.

**Theorem 6.** *( [13] ) Consider a multi-grid method for a given iterative matrix (17). Then under the previous assumption, for the parameter value $\alpha_2 = 0, \alpha_1 = \alpha > 0, \tau \geq 2$. For each $\zeta \in (0, 1)$ there is a$\alpha^*$such that for all $\alpha \geq \alpha^*$*

$$\|MG_k\|_\infty \leq \zeta, k = 0, 1, ... \tag{23}$$

*hold.*

*Proof.* If the previous properties are related with (22), then we can stratify the same steps as in [ [13] , Theorem 7.20]. □

The main result of our study was in the following theorem.

**Theorem 7.** *For two meshes $(k)$ and $(k-1)$ and the previous given the iterated $u_k^\nu, \nu \geq 0$ satisfy:*

$$\left\|u_k^{\nu+1} - u_k^*\right\|_\infty \leq \left(\frac{C}{\sqrt{\alpha}} \left|\mathrm{Logh}_k\right|^2\right) \|u_k^\nu - u_k^*\|_\infty . \tag{24}$$

*Proof.* We have

$$\begin{aligned}\left\|u_k^{\nu+1} - u_k^*\right\|_\infty = &\quad \left\|\left(\left(I_k - \mathcal{P}_k \left(I_k - MG_{k-1}\right) \left(\mathcal{B}_{k-1}^\nu\right)^{-1} \mathcal{R}_k\right) (\mathcal{B}_k^\nu)\, S_k^{\alpha_1}\right) (u_k^\nu - u_k^*)\right\|_\infty \\ \leq &\quad \left\|\left(I_k - \mathcal{P}_k \left(I_k - MG_{k-1}\right) \left(\mathcal{B}_{k-1}^\nu\right)^{-1} \mathcal{R}_k\right)\right\|_\infty \|\mathcal{B}_k^\nu S_k^{\alpha_1}\|_\infty \|u_k^\nu - u_k^*\|_\infty \\ \leq &\quad \left(\frac{C_2}{\sqrt{\alpha}} h_k^{-2}\right) \left(C_1 h_k^2 \left|\log h_k\right|^2\right) \|(u_k^\nu - u_k^*)\|_\infty \\ \leq &\quad \left(\frac{C_1 C_2}{\sqrt{\alpha}}\right) \left|\log h_k\right|^2 \|u_k^\nu - u_k^*\|_\infty\end{aligned}$$

□

## 4. Numerical Simulation

In this part, we applied this method to the numerical example of a non-linear variational inequality.
We suppose that the problem to be sufficiently smooth data and we apply the dynamic programming principle of Bellman, then we solve (3) as we discussed before, using the following datas:

- **Mixed operator**

$$\begin{cases} \mathcal{B}u \geq f, & \text{in } \Omega = [0,1]^2 \\ \langle \mathcal{B}u - f, u - \psi \rangle = 0, \\ u \leq \psi, \\ u = 0, & \text{in } \partial\Omega. \end{cases} \tag{25}$$

Where

$$\begin{aligned} \mathcal{B}u &= -\Delta u - 0.02\frac{\partial^2 u}{\partial x \partial y} + 0.15\tfrac{\partial u}{\partial x} + 0.1\tfrac{\partial u}{\partial y} + (1+\lambda)u, \\ f &= \sin(\pi x)\sin(2\pi y) sin(\pi(x+y)) + \lambda u, \\ \lambda &= 2, \\ \psi &= 0. \end{aligned}$$

We are constrain ourselves to the discretization of finite element method with a uniform triangulation and $P_1$ shape functions. For the domain, we have decretized by Matlab PDE toolbox (Matlab R2017b) for mesh generation. We solve the equation (25) by the M.G with 64 triangle and 41 nodes in the domain. This numerical illustration is performed to showcase the high efficiency of the M.G method. For the pre/post-smoothing of the M.G, we choose the Gauss-Seidel (G.S) method. The degrees of freedom chooses lower than 5 ( recursion number of M.G method). Figure 1 illustrates the convergence behaviour of the M.G solver (green and red curves of M.G (V and W cycle)) with respect to the number of iterations performed. For comparison, the convergence behavior of Gauss-Seidel ( blue curves) are included.

Norm of residual obtained after 100 iterations :

| by Gauss Seidel method | by multi-grid V-cycle | by multi-grid W-cycle |
|---|---|---|
| 4.058087199609872e$^{-12}$ | 4.440892098500626e$^{-16}$ | 4.440892098500626e$^{-16}$ |

We have applied the Matlab-backslash-operator(M.B.O), G.S and the M.G (V and W-cycle) are carried out on the finest grid (41 grids) and on the coarsest one (4 nodes) then we get the solutions in figures 2.

Norm of residual obtained after 20 iterations :

| by Gauss Seidel method | by multi-grid V-cycle | by multi-grid W-cycle |
|---|---|---|
| 0.001165086612534 | 4.440892098500626e$^{-16}$ | 4.440892098500626e$^{-16}$ |

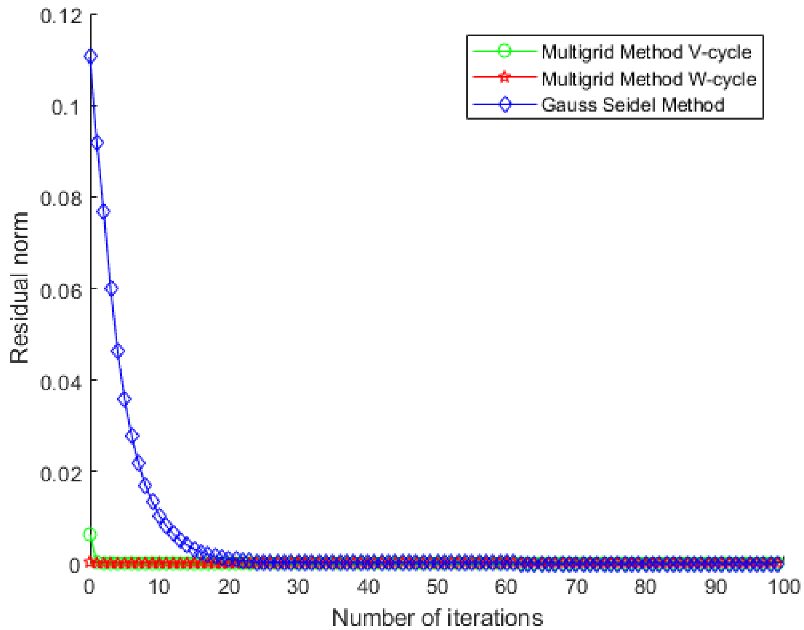



FIGURE 1. Comparison between the convergence of maximum residual norm by M.G and G.S.

- **Simple operator**

$$\begin{cases} \mathcal{B}u \geq f, \quad \text{ in } \Omega = [0,1]^2 \\ \langle \mathcal{B}u - f, u - \psi \rangle = 0, \\ u \leq \psi, \\ u = 0, \quad \text{ in } \partial\Omega. \end{cases} \tag{26}$$

Where

$$\begin{aligned} \mathcal{B}u \quad &= -\Delta u + 0.5x\frac{\partial u}{\partial x} + 0.5y\tfrac{\partial u}{\partial y} + (0.045 + \lambda)u, \\ f \quad &= \sin(2\pi x)\sin(2\pi y) + \lambda u, \\ \lambda \quad &= 1, \\ \psi \quad &= 0. \end{aligned}$$

With the same steps,we have:

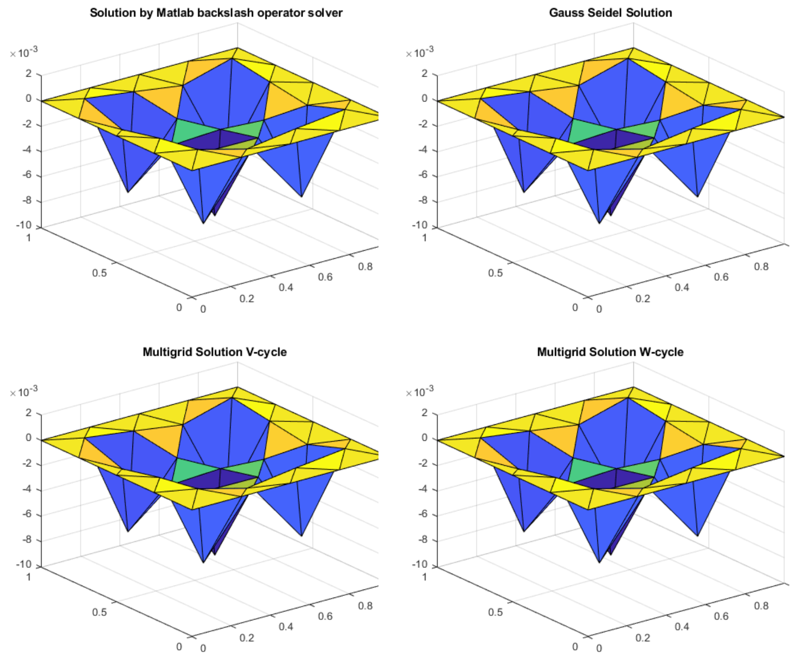



FIGURE 2. Solution of (25) on fine grid using M.B.O, G.S, M.G V-cycle and W-cycle after 100 iterations.

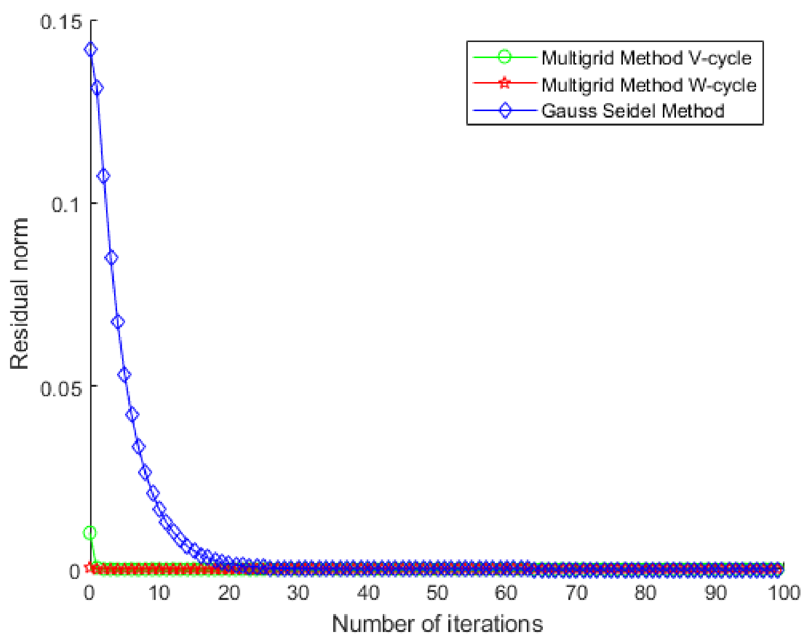



FIGURE 3. Comparison between the convergence of maximum residual norm by M.G and G.S.

Norm of residual obtained after 100 iterations :

| by Gauss Seidel method | by multi-grid V-cycle | by multi-grid W-cycle |
|---|---|---|
| 1.076361222374089e$^{-11}$ | 2.220446049250313e$^{-16}$ | 2.220446049250313e$^{-16}$ |

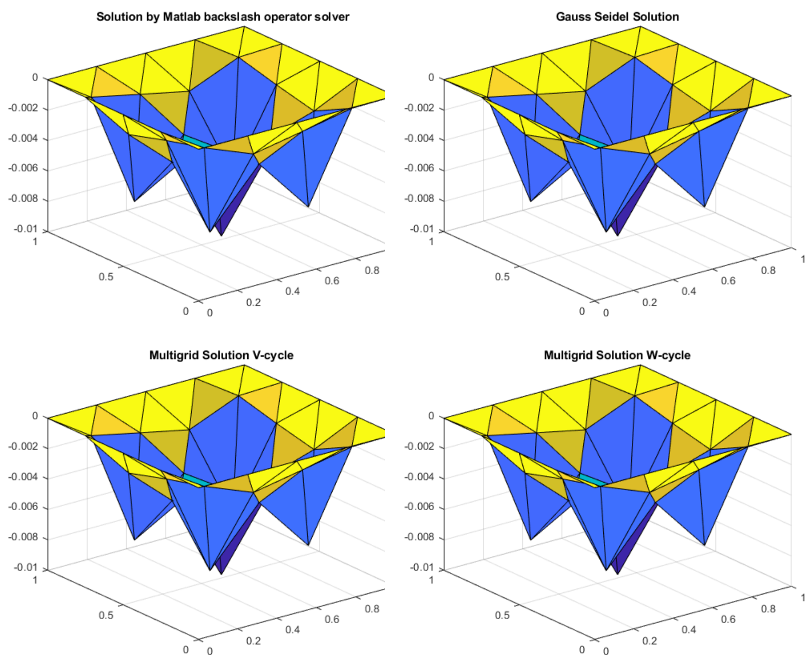



FIGURE 4. Solution of (26) on fine grid using M.B.O, G.S, M.G V-cycle and W-cycle after 100 iterations.

Norm of residual obtained after 10 iterations :

| by Gauss Seidel method | by multi-grid V-cycle | by multi-grid W-cycle |
|---|---|---|
| 0.020709274936256 | 4.884981308350689e$^{-15}$ | 2.220446049250313e$^{-16}$ |

**Remark 1.** *Should we conduct more than 10 iterations, the M.G approach emerges as the optimal method.*

4.1. **Conclusion.** Discretizing elliptic V.I. via efficient iterative solutions is the main focus of our study, employing algebraic M.G. The goal is to tackle loop domains' discretization using adaptive finite element approximation. Once discretization is complete, we successfully apply M.G to address the discrete problems at hand. Our main objective is to establish uniform convergence through our approach, and our research demonstrates the M.G's significant reduction in iteration count compared to the maximum norm method.

By means of numerical experimentation, we have constructed an example of a variational inequality. Our results indicate that the G.S. method, despite a substantial number of iterations, is unsuccessful in producing satisfactory outcomes. On the other hand, through the use of an error-damping mechanism that reduces high-frequency errors and transfers low-frequency errors to a coarser grid for alleviation, M.G. significantly enhances convergence and achieves it within a limited number of iterations. Our team recognizes the exceptional potential for further development using these methodologies.
Our numerical solution could be even more efficient and scalable if we explore the prospect of applying a parallel full M.G to surmount unconstrained elliptical inequalities. This avenue presents an interesting opportunity to cater to a broader range of problem domains.

**Author Contribution Statements** All authors contributed equally in the writing of this paper. All authors read and approved the final manuscript.

**Declaration of Competing Interests** The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## References


[1] Boulbrachene, M., Haiour, M., The finite element approximation of Hamilton-Jacobi-Bellman equations, *Computers & Mathematics with Applications*, 41(7-8) (2001), 993–1007. https://www.sciencedirect.com/science/article/pii/S0898122100003345

[2] Brezzi, F., Caffarelli, L. A., Convergence of the discrete free boundary for finite element approximations, *R.A.I.R.O Anal. Numer.*, 17 (1983), 385-395. https://eudml.org/doc/193422

[3] Ciarlet, P-G., Raviart, P-A., Maximum principle and uniform convergence for the finite element method, *Computer Methods in Applied Mechanics and Engineering*, 2(1) (1973), 17–31. https://doi.org/10.1016/0045-7825(73)90019-4

[4] Cortey-Dumont, P., On the finite element approximation in the $L^\infty$ norm of variational inequalities with nonlinear operators, *Numer.Num.*, 47 (1985), 45-57. https://eudml.org/doc/133022

[5] Cortey-Dumont, P., Sur I'analyse numérique des equations de Hamilton-Jacobi-Bellman, *Mathematical Methods in The Applied Sciences*, 198-209 (1987). https://onlinelibrary.wiley.com/doi/10.1002/mma.1670090115

[6] Hackbusch, W., Multi-grid Methods and Applicatons, Springer, Berlin- HeidelBerg- New York, 1985. https://link.springer.com/book/10.1007/978-3-662-02427-0

[7] Hackbusch, W., Mittelmann, H. D., On Multi-grid methods for variational inequalities, *Numerische Mathematik*, 42 (1983), 65-75 (1983). http://resolver.sub.uni-goettingen.de/purl?PPN362160546_0042

[8] Haiour, M., Etude de la convergence uniforme de la méthode multigrilles appliquées aux problèmes frontières libres, PhD thesis, Université de Annaba-Badji Mokhtar. https://www.pnst.cerist.dz/detail.php?id=19817/

[9] Hoppe, R. H. W., Multi-grid methods for Hamilton-Jacobi-Bellman equations, *Numerische Mathematik*, 49(2) (1986), 239–254. https://link.springer.com/article/10.1007/BF01389627

[10] Hoppe, R. H. W., Multigrid algorithms for variational inequalities, *SIAM Journal on Numerical Analysis*, 24(5) (1987), 1046–1065. https://www.jstor.org/stable/2157638

[11] Kinderlehrer, D., Stampacchia, G., An Introduction to Variational Inequalities and their Applications, Academic Press, New York, 1980. https://lib.ugent.be/en/catalog/ebk01:1000000000551554

[12] Nesba, N. E. H., Beggas, M., Belouafi, M. E., Ahmad, I., Ahmad, H., Askar, S., Multigrid methods for the solution of nonlinear variational inequalities, *European Journal of Pure and Applied Mathematics Published by New York Business Global*, 16(3) (2023), 1956-1969. https://ejpam.com/index.php/ejpam/article/view/4835

[13] Reusken, A., Introduction to multigrid methods for elliptic boundary value problems, *Inst, f¨ur Geometrie und Praktische Mathematik*, (2008). https://www.igpm.rwth-aachen.de/Download/reports/reusken/ARpaper53.pdf

[14] Reusken, A., On maximum norm convergcnce of multigrids methods for elliptic boundary value problems, *SIAM J. Numer. Anal*, 29(6) (1992), 1569-1578. https://epubs.siam.org/doi/abs/10.1137/0731020.

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 235–258 (2024)
DOI:10.31801/cfsuasmas.1210841
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

*Research Article; Received:November 28, 2022; Accepted: November 15, 2023*

# A WEIGHTED GOMPERTZ-G FAMILY OF DISTRIBUTIONS FOR RELIABILITY AND LIFETIME DATA ANALYSIS

Joseph Thomas EGHWERIDO[1] and Lawrence Chukwudumebi NZEI[2]

[1]Department of Statistics, Federal University of Petroleum Resources Effurun, Delta State, NIGERIA
[2]Department of Statistics, University of Benin, Edo State, NIGERIA

Abstract. This article is set to push new boundaries with leading-edge innovations in statistical distribution for generating up-to-the-minute contemporary distributions by a mixture of the second record value of the Gompertz distribution and the classical Gompertz model (weighted Gompertz model) using T-X characterization, especially used for two-sided schemes that provide an accurate model. The quantile, ordinary, and complete moments, order statistics, probability, and moments generating functions, entropies, probability weighted moments, Lin's condition random variable, reliability in multicomponent stress strength system, reversed, and moments of residuals life and other reliability characteristics in engineering, actuarial, economics, and environmental technology were derived in their closed form. To investigate and test the flexibility, viability, tractability, and performance of the proposed Weighted Gompertz-G (WGG) generated model, the shapes of some sub-models of the WGG model were examined. The shapes of the sub-models indicated J-shapes, increasing, decreasing, and bathtub hazard rate functions. The maximum likelihood estimation of the WGG-generated model parameters was examined. An illustration with simulation and real-life data analysis indicated that the WGG-generated model provides consistently better goodness-of-fit statistics than some competitive models in the literature.

## 1. Introduction

Modeling real-life data set requires a distribution that has a true reflection of the character of that data. However, to unravel the interest of some important Poisson scenarios, a parsimonious statistical distribution is required. Hence, new statistical

2020 *Mathematics Subject Classification.* Primary 62E10; Secondary 62E15, 62E20.
*Keywords.* Gompertz distribution, Poisson processes, record values, weighted Gompertz, weighted model.

[1] eghwerido.joseph@fupre.edu.ng- Corresponding author; 0000-0001-8986-753X;
[2] nzeilawrence@ymail.com; 0000-0002-4441-805X

.

models are often introduced to harness salient factors responsive for good decision making.

Oftentimes, change-point models are characterized by abrupt behavioural structures that may be very complicated to handle by the usual classical statistical distributions. These events are but not limited to macroeconomic events characterized by abrupt increases interest rates and inflation. The abrupt behavioural shift might also be the case in extreme events like the storm and rainfall events that have ravaged some countries in recent time. The advent of the novel epidemic COVID 19 is also not exempted. Another example is the lifetime scenario that are subjected to unexpected and rapid shocks. Hence, this study is proposed to deal with such change-point by constructing an appropriate weighted generated distribution called Weighted Gompertz-G (WGG) distribution that can address the differentials. Though the method of generating new distribution is not new, using the weighted generator concept to generate new models is a new approach targeted at change-point problems. Thus, this article will use the weighted Gompertz generator approach to generate new continuous distributions that are more flexible, and viable in their goodness-of-fit test statistics.

The Gompertz model has played a vital role in modeling scenarios that deal with survival times, reliability, human mortality, and actuarial data with exponential increase outcomes. Thus, it has received considerable attention from demographers, economics, and actuaries. This includes [13], and [12] who proposed the shifted Gompertz-G and alpha power Teissier distributions. A flexible alpha power Gompertz distribution was proposed in [14]. [27] emphasised on some applications of the Gompertz distribution in Poisson process. A negative rate of aging parameter with Gompertz distribution was proposed in [22]. [8] proposed the Teissier distribution. [16] proposed the Marshall-Olkin Teissier distribution. The gamma–Gompertz distribution was proposed in [29]. [15] developed the alpha power Marshall-Olkin-G model. [23]developed the Topp-Leone Gompertz distribution with application to glass data. The reliability properties and applications of the alpha power Topp-Leone-G distribution was considered in [17]. However, some researches have been contributed to generating newer classical statistical distributions include [5] and [2] who proposed exponentiated T-X and T-X family of distributions. The type I half-logistic family of distributions proposed by [10]. The beta and generalized gamma-generated distributions by [30]. A tetration distribution developed by [11]. Odd Truncated Inverse Exponential Weibull Exponential by [1]. [24] proposed a New Member from the T-X Family of distribution. A New Odd Log-Logistic Lindley Distribution was proposed in [3]. The Bivariate Lack-of-Memory Distributions was developed in [21]. [20] proposed the U family of distributions. A new estended Weibull distribution was developed in [26]. [18] proposed the alpha power Teissier-G Distribution and its Applications in reliability analysis. Exponentiated Gumbel Weibull Logistic model was developed in [25]. Weighted Weibull-G was introduced by [19].

Let $T$ be a nonnegative random variable with a probability density function (pdf) $f(t)$ such that for a suppose $t > 0$, weight function $w(t) = \beta + exp(\beta t) - 1$ ,and expectation $E[w(t)] = \frac{\beta\lambda+1}{\lambda}$. Then, [7] defined the pdf and cumulative distribution function (cdf) $F(t)$ of the weighted Gompertz distribution as

$$f(t) = \frac{\beta\lambda^2}{(1+\beta\lambda)}(\beta - 1 + e^{\beta t})e^{(\beta t - \lambda(e^{\beta t}-1))},\ t > 0\ \beta\ ,\lambda\ > 0, \tag{1}$$

and

$$F(t) = 1 - \left[1 + \frac{\lambda(e^{\beta t}-1)}{(1+\beta\lambda)}\right]e^{-\lambda(e^{\beta t}-1)},\ t > 0\ \beta\ ,\lambda\ > 0, \tag{2}$$

with $\lambda$ and $\beta$ as the shape and scale parameters.

Modeling abrupt behavioural structure and scenarios has become more complicated as a result of their change-point. Though the method of generating new distribution is not new, using the weighted generator concept to generate new models is a new approach. Hence, this study is motivated to propose a model with a true reflection of the character of the data obtained. Thus, the WGG generated model tends to improve the goodness-of-fit, and the test statistics of the existing distributional models using weighted distribution characterization.

The study aim at introducing a class of generator with the aid of the weighted Gompertz model called the weighted Gompertz generator. This generated model will improve the performance, flexibility and the viability of the goodness-of-fit of the abrupt behaviourial change-point scenarios in lifetime modeling.

## 2. The Weighted Gompertz-G Distribution

Suppose a nonnegative random variable $T$ is defined on the interval $T \in [m, n]$ for $-\infty < m < n < \infty$ with pdf $r(G(t))$ such that $r(G(t)) = -\log[1 - G(t)]$ is monotonically non-decreasing; $r(G(t))$ is closed in the interval $[m, n]$; and $r(G(t))$ approaches $m$ as $t$ tends to negative infinity, and $r(G(t))$ approaches $n$ as $t$ tends to positive infinity. Thus, by [4] the cdf and the pdf of the WGG generated class of distribution can be expressed as

$$F(t) = 1 - \left[1 + \frac{\lambda[(1-G(t))^{-\beta} - 1]}{1+\beta\lambda}\right]e^{-\lambda[(1-G(t))^{-\beta}-1]}\ t > 0,\ \lambda, \beta > 0, \tag{3}$$

and

$$f(t) = \frac{\lambda^2\beta}{(1+\beta\lambda)(1-G(t))^{(1+\beta)}}g(t)((1-G(t))^{-\beta} + \beta - 1)e^{-\lambda[(1-G(t))^{-\beta}-1]}, \tag{4}$$

for $t > 0,\ \lambda, \beta > 0$, where $g(t)$, and $G(t)$ are the parents pdf and cdf.

The WGG generated reliability model can be expressed as

$$S_{WGG}(t) = \left[1 + \frac{\lambda[(1-G(t))^{-\beta} - 1]}{1+\beta\lambda}\right]e^{-\lambda[(1-G(t))^{-\beta}-1]}\ t > 0,\ \lambda, \beta > 0. \tag{5}$$

The hazard rate function that corresponds to the WGG generated model is defined as

$$h_{WGG}(t) = \frac{\frac{\lambda^2\beta}{(1+\beta\lambda)(1-G(t))^{(1+\beta)}} g(t)((1-G(t))^{-\beta}+\beta-1)}{\left[1+\frac{\lambda[(1-G(t))^{-\beta}-1]}{1+\beta\lambda}\right]} \quad t>0,\ \lambda,\beta>0. \tag{6}$$

The reversed hazard rate function is obtained as

$$r_{WGG}(t) = \frac{\frac{\lambda^2\beta}{(1+\beta\lambda)(1-G(t))^{(1+\beta)}} g(t)((1-G(t))^{-\beta}+\beta-1)e^{-\lambda[(1-G(t))^{-\beta}-1]}}{1-\left[1+\frac{\lambda[(1-G(t))^{-\beta}-1]}{1+\beta\lambda}\right]e^{-\lambda[(1-G(t))^{-\beta}-1]}}, \tag{7}$$

for $t>0,\ \lambda,\beta>0$.

The cumulative hazard rate function of the WGG generated function is give as:

$$H_{WGG}(t) = \log(1+\beta\lambda)-\log([1+\beta\lambda]+\lambda[(1-G(t))^{-\beta}-1])+\lambda([1-G(t)]^{-\beta}-1). \tag{8}$$

## 3. The Quantile Function

Quantile is fundamental for the simulation and estimation of a distribution parameter(s). Hence, it is a function that associates the probability distribution function of the WGG generated model of a random variable $T$ such that the probability of the variable being less than or equal to that value equals the probability for a uniform interval $q \in (0,1)$ is defined as

$$t = G^{-1}\left[1-\left[\frac{W_{-1}((q-1)(1+\beta\lambda)e^{(1+\beta\lambda)})-(1+\beta\lambda)}{\lambda}+1\right]^{-\frac{1}{\beta}}\right], \tag{9}$$

where $W_{-1}$ is the Lambert-W or omega function as defined in [13] and [16] such that $W(t)=e^{(W(t))}=t\in[-1,\infty)$.

In particular, the median is obtained when $q=0.5$.

**Theorem 1.** *The shape, characteristics, and behaviour of the WGG generated model can be examined by investigating the first and second derivatives of the* log *of the WGG generated pdf model. Thus, for $f'(t)<0$. Then, then cdf $F(t)$ will be decreasing monotonically for all values of $t$. The WGG generated model will be bimodal if $f''(t)$ changes its signs from negative to non-negative, viz-a-viz.*

*Proof.* The $\log f(t)$ is give as

$$\begin{aligned}\log f(t) =& 2\log\lambda+\log\beta-\log(1+\beta\lambda)+\log g(t)-(1+\beta\lambda)\log(1-G(t))\\ &+\log([1-G(t)]^{-\beta}+\beta-1)-\lambda([1-G(t)]^{-\beta}-1).\end{aligned}$$

Thus, taking the derivative with respect to the variable,we have

$$\frac{\partial\log f(t)}{\partial t} = \frac{g'(t)}{g(t)}+(1+\beta\lambda)\frac{g(t)}{S(t)}+\frac{\beta g(t)S^{-\beta-1}(t)}{S^{\beta}(t)+\beta-1}-\lambda g(t)S^{-\beta-1}(t),$$

where $S(t)=1-G(t)$. Hence, $f'(t)<0$ if $g(t)<0$.

The second derivative was implemented to determine if the model was bimodal. Thus, the second derivative is given as

$$\begin{aligned}\frac{\partial^2 \log f(t)}{\partial t^2} =& \frac{g''(t)}{g(t)} - \frac{1}{g'(t)} + (1+\beta\lambda)[\frac{g'(t)}{S(t)} + \frac{g^2(t)}{S^2(t)}] - \frac{\lambda g'(t)}{S^{(\beta+1)}(t)} \\ &+ \lambda(1+\beta\lambda)\frac{g^2(t)}{S^{(\beta+2)}(t)} + \frac{\beta g'(t) S^{-\beta-1}(t)}{S^{\beta}(t)+\beta-1} \\ &+\beta(\beta+1)\frac{g^2(t)S^{-(\beta+2)}(t)}{S^{\beta}(t)+\beta-1} + \frac{\beta^2 g^2(t) S^{-2(\beta+1)}(t)}{(S^{\beta}(t)+\beta-1)^2}.\end{aligned}$$

□

## 4. Order statistics

Order statistics are useful tools to improve the robustness of sampling plans by variables, and shorten test times of Poisson processes.

Let $T_{(1)}, T_{(2)}, T_{(3)}, \ldots, T_{(k)}$ be the order statistics for a random variable $T_1, T_2, T_3, \ldots, T_k$ with WGG distribution. Then, the WGG density of the $u^{th}$ order statistics is given as

$$f_u(t) = \frac{k!}{(u-1)!(k-u)!} F^{u-1}(t) S^{k-u}(t) f(t) \quad -\infty < t < \infty. \tag{10}$$

However, using the binomial expansion, and noting that $S = 1 - G(t)$, we have the order statistics as

$$\begin{aligned}f_u(t) =& \frac{\beta\lambda^2 S^{-(\beta+1)} k!}{(u-1)!(k-u)!}(S^{-\beta}+\beta-1)\sum_{j=0}^{u-1}(-1)^{u-j-1}\binom{u-1}{j} \\ &\times\Big[(1+\beta\lambda)+\lambda(S^{-\beta}-1)\Big]^{k+j-u+1} e^{-\lambda(S^{-\beta}-1)(k+j-u+1)}.\end{aligned} \tag{11}$$

The minimum order statistics is obtained when $u = 1$, and the maximum order statistics is obtained when $u = k$ respectively.

### 4.1. **Record value distributions of the WWG model.**

Let $T_i$ for $i = 1, 2, 3, \ldots, k$ be a finite sequence of independently identically distributed random variables with WGG generated cdf $F(t)$ and a record times given as $U(1) = 1$ and $U(k+1) = min\{j > U(k);\ T_j > T_{u(k)}\};\ k \in \mathbb{N}$ with the random variable $T_{u(k)}$ $(k \in \mathbb{N})$ as the upper record values. Then, the pdf of the $i$ upper record value $UR_i = T_{u(k)}$ with a special case of $UR_1 = T_1$ is given as

$$\begin{aligned} f_{UR_i}(t) =& \frac{f(t)}{\Gamma(i)}\{-\log[1-F(t)]\}^{i-1} \\ =& \frac{\lambda^2\beta g(t)((1-G(t))^{-\beta}+\beta-1)e^{-\lambda[(1-G(t))^{-\beta}-1]}}{(1+\beta\lambda)(1-G(t))^{(1+\beta)}\Gamma(i)} \\ &\times\left\{-\log\left[\left[1+\frac{\lambda[(1-G(t))^{-\beta}-1]}{1+\beta\lambda}\right]e^{-\lambda[(1-G(t))^{-\beta}-1]}\right]\right\}^{i-1} \end{aligned} \tag{12}$$

## 5. Sub-models

Some special sub-models were considered for flexibility, viability, and tractability using the proposed WGG generated model. We present some special cases of the WGG generated family of distributions since it extends several useful distributions in the literature. For all cases listed next, we consider $t, \lambda, \beta > 0$. Especially sub-models with increasing, decreasing shaped data with or without a flat region in modeling. These special sub-models include Burr-XII, Lomax, and Frechet distributions.

5.1. **Weighted Gompertz-G Burr-XII (WGG-B) distribution.** Consider the Burr XII distribution with positive parameters $\theta$ and $\rho$, and cdf and pdf given as $G(t) = 1-(1+t^\theta)^{-\rho}$ and $g(t) = \theta\rho t^{\theta-1}(1+t^\theta)^{-\rho-1}$ . Then, inserting these expressions into Equations (3) and (4) gives the WGG-B density function with the cdf and pdf given as

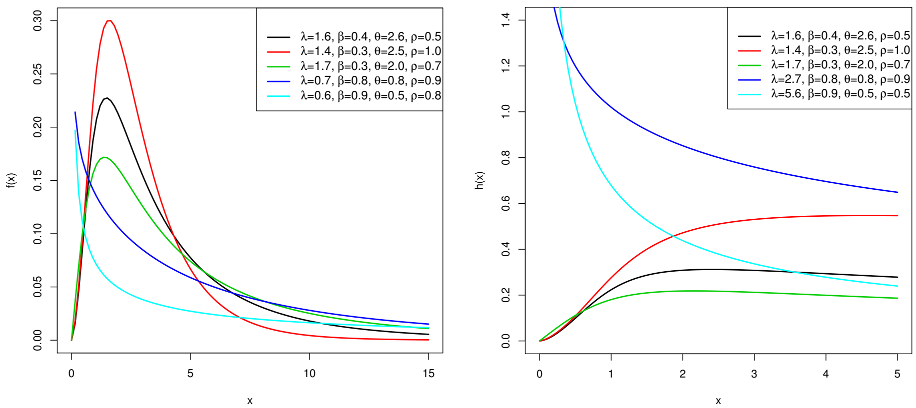
$$F(t) = 1-\left[1+\frac{\lambda[(1+t^\theta)^{\beta\rho}-1]}{1+\beta\lambda}\right]e^{-\lambda[(1+t^\theta)^{\beta\rho}-1]},\ t>0,\ \lambda,\beta,\theta,\rho>0, \tag{13}$$

and

$$\begin{aligned} f(t) =& \frac{\lambda^2\beta(1+t^\theta)^{\rho(1+\beta)}}{(1+\beta\lambda)}((1+t^\theta)^{\beta\rho}+\beta-1)e^{-\lambda[(1+t^\theta)^{\beta\rho}-1]} \\ &\times\theta\rho t^{\theta-1}(1+t^\theta)^{-\rho-1},\ t>0,\ \lambda,\beta,\theta,\rho>0. \end{aligned} \tag{14}$$

Plots of the WGG-B density function for the selected parameter values are displayed in Figure 1a. Figure 1b displays the corresponding hazard rate function (hrfs) for particular values of the parameters. The shapes of the hazard rate function indicated increasing, and decreasing.

5.2. **Weighted Gompertz-G Lomax (WGG-L) distribution.** Consider the Lomax distribution with positive shape parameters $\theta$ and scale parameter $\rho$, and cdf and pdf given as $G(t) = 1-(1+\frac{t}{\rho})^{-\rho}$ and $g(t) = \frac{\theta}{\rho}[1+\frac{t}{\rho}]^{-(\theta+1)}$ . Then, inserting these expressions into Equations (3) and (4) gives the WGG-L density function with the cdf and pdf given as




(A) WGG-B density

(B) WGG-B hazard rate function

FIGURE 1. The plots WGG-B model for selected values of parameters

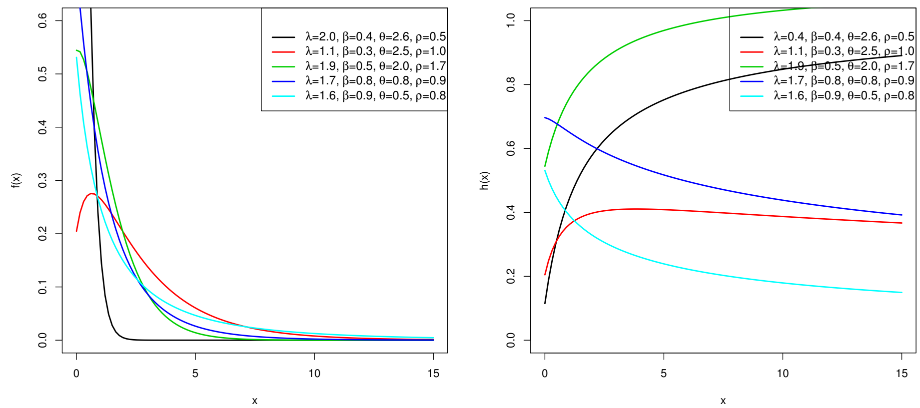
$$F(t) = 1 - \left[1 + \frac{\lambda[(1+\frac{t}{\rho})^{\beta\rho} - 1]}{1+\beta\lambda}\right] e^{-\lambda[(1+\frac{t}{\rho})^{\beta\rho}-1]},\ t > 0,\ \lambda, \beta, \theta, \rho > 0, \tag{15}$$

and

$$\begin{aligned} f(t) =& \frac{\lambda^2\beta(1+\frac{t}{\rho})^{\rho(1+\beta)}}{(1+\beta\lambda)}((1+\frac{t}{\rho})^{\beta\rho} + \beta - 1)e^{-\lambda[(1+\frac{t}{\rho})^{\beta\rho}-1]}, \\ &\times\frac{\theta}{\rho}[1+\frac{t}{\rho}]^{-(\theta+1)},\ t > 0,\ \lambda, \beta, \theta, \rho > 0. \end{aligned} \tag{16}$$

Plots of the WGG-L density function for the selected parameter values are displayed in Figure 2a. Figure 2b displays the corresponding hrfs for some particular values of the parameters. The shapes of the hazard rate function indicated increasing, and decreasing.

5.3. **Weighted Gompertz-G Frechet (WGG-F) distribution.** Consider the Frechet distribution with positive shape parameters $\theta$ and scale parameter $\rho$, and cdf and pdf given as $G(t) = e^{-(\frac{\theta}{t})^{\rho}}$ and $g(t) = \rho\theta^{\rho}t^{-\rho-1}e^{-(\frac{\theta}{t})^{\rho}}$ . Then, inserting these expressions into Equations (3) and (4) gives the WGG-F density function with the cdf and pdf given as




(A) WGG-L density

(B) WGG-L hazard rate function

Figure 2. The plots WGG-L model for selected values of parameters

$$F(t) = 1 - \left[1 + \frac{\lambda[(1 - e^{-(\frac{\theta}{t})^\rho})^{-\beta} - 1]}{1 + \beta\lambda}\right] e^{-\lambda[(1-e^{-(\frac{\theta}{t})^\rho})^{-\beta}-1]},\ t > 0,\ \lambda, \beta, \theta, \rho > 0, \tag{17}$$

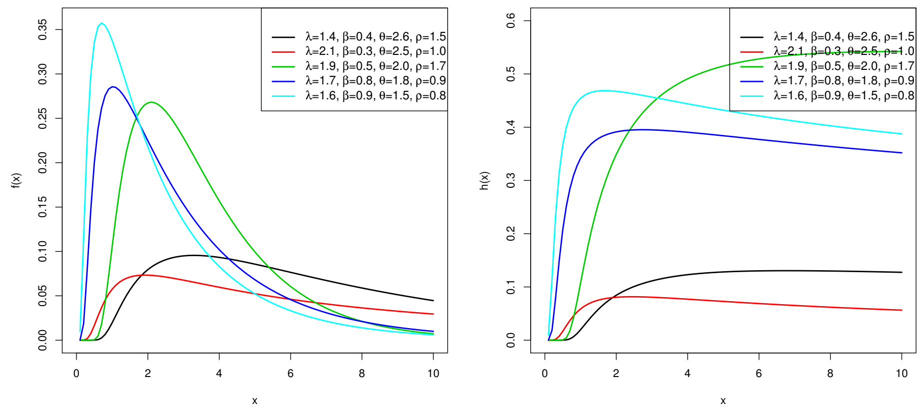
and

$$\begin{aligned} f(t) =& \frac{\lambda^2\beta}{(1+\beta\lambda)(1 - e^{-(\frac{\theta}{t})^\rho})^{(1+\beta)}}((1 - e^{-(\frac{\theta}{t})^\rho})^{-\beta} + \beta - 1)e^{-\lambda[(1-e^{-(\frac{\theta}{t})^\rho})^{-\beta}-1]} \\ &\times \rho\theta^\rho t^{-\rho-1} e^{-(\frac{\theta}{t})^\rho},\ t > 0,\ \lambda, \beta, \theta, \rho > 0. \end{aligned} \tag{18}$$

Plots of the WGG-F density function for the selected parameter values are displayed in Figure 3a. Figure 3b displays the corresponding hrfs for some particular values of the parameters. The shapes of the hazard rate function indicated an increase.

## 6. Mathematical Expression

To examine the productivity of the WGG generated model, mathematical expansion of the pdf and cdf is carried out. The exponential term in (3) and (4) can




(A) WGG-F density

(B) WGG-F hazard rate function

FIGURE 3. The plots WGG-F model for selected values of parameters

be expressed as

$$e^{-\lambda((1-G(t))^{-\beta}-1)} = \sum_{w=0}^{\infty} \frac{(-1)^w \lambda^w ((1-G(t))^{-\beta}-1)^w}{w!}.$$

Also, by binomial expansion, we have

$$((1-G(t))^{-\beta}-1)^w = \sum_{u=0}^{w} (-1)^{w-u} \binom{w}{u} (1-G(t))^{-u\beta}.$$

Hence, the WGG generated pdf can be expressed as power function as

$$f(t) = \sum_{i,w=0}^{\infty} \sum_{u=0}^{w} \alpha \mu_{(i,w,u)} g(t) G^i(t), \tag{19}$$

where

$$\alpha = \frac{\Gamma(u\beta+2\beta+i+1)}{\Gamma(u\beta+2\beta+1)} + (\beta-1)\frac{\Gamma(u\beta+\beta+i+1)}{\Gamma(u\beta+\beta+1)},$$

and

$$\mu_{(i,w,u)} = (-1)^{2w-u+i} \binom{w}{u} \frac{\lambda^{w+2}}{i!w!} \frac{\beta}{(1+\beta\lambda)},$$

where $\Gamma(\cdot)$ is a gamma function.

## 7. Statistical Properties

The viability and performance of the proposed model will be investigated by examining some general statistical properties of the WGG generated model in this section.

Oftentimes, the expectation, variance, and moments of random variables can be obtained from some characteristics of the distribution function. Some of these functions are the probability generating function and the moment generating function.

*Lin's condition random variable* The Lin's function for a pdf $f$ of a random variable $T$ with a support $t > 0$ is defined as

$$L_{f(t)} = -t\frac{f'(t)}{f(t)} = -t\sum_{i,w=0}^{\infty}\sum_{u=0}^{w}\alpha\mu_{(i,w,u)}\frac{ig^2(t)G^{i-1}(t)+g'(t)G^i(t)}{g(t)G^i(t)}.$$

*Incomplete moments*

The incomplete moments of the WGG generated model allow the shape of the moments of WGG generated distribution, which is of interest for many areas, including econometrics, finance, and reliability, to be visible.

The $k^{th}$ incomplete moment, say $\tau_k(t)$ of the WGG generated moment is given as

$$\tau_k(y) = \sum_{i,w=0}^{\infty}\sum_{u=0}^{w}\alpha\mu_{(i,w,u)}\eta_{k,i}(y),$$

where $\eta_{k,i} = \int_0^y t^k g(t)G^i(t)dt$.

*Probability generating function*

This is a useful mechanism for characterizing the distribution of the random variable $T$ with the WGG generated model. It can succinctly be used to describe the sequence of the probability of the random variable $T$ with the WGG distribution. Hence, a random variable $T$ with a WGG distribution has the probability generating function defined as

$$\begin{aligned} P(z) &= \sum_{i,w=0}^{\infty}\sum_{u=0}^{w}\int_0^{\infty} z^t\alpha\mu_{(i,w,u)}g(t)G^i(t)dt \\ &= \sum_{i,w,a=0}^{\infty}\sum_{u=0}^{w}\frac{(\log z)^a\alpha\mu_{(i,w,u)}}{a!}\int_0^{\infty} t^a g(t)G^i(t)dt \qquad (20) \\ &= \sum_{i,w,a=0}^{\infty}\sum_{u=0}^{w}\frac{(\log z)^a\alpha\mu_{(i,w,u)}}{a!}p(z), \end{aligned}$$

where

$$p(z) = \int_0^{\infty} t^a g(t)G^i(t)dt \ |z| \leq 1.$$

*Moment generating function*

The probability density function of the random variable $T$ can be identified using the moment generating function instrument. This is, however, possible since the moment generating function is a non-negative integral of measurable function. Thus, for a random variable $T$ with a WGG distribution, the moment generating function is given as

$$\begin{aligned} M_T(z) &= \sum_{i,w=0}^{\infty}\sum_{u=0}^{w}\int_0^{\infty} e^{zt}\alpha\mu_{(i,w,u)}g(t)G^i(t)dt \\ &= \sum_{i,w,a=0}^{\infty}\sum_{u=0}^{w}\frac{z^a\alpha\mu_{(i,w,u)}}{a!}\int_0^{\infty} t^a g(t)G^i(t)dt \\ &= \sum_{i,w,a=0}^{\infty}\sum_{u=0}^{w}\frac{z^a\alpha\mu_{(i,w,u)}}{a!}p(z) \end{aligned} \tag{21}$$

*Probability weighted moments*

One of the widely used characteristics of a distribution is called L-moments or probability weighted moments. This characteristic is used in hydrology to estimate the parameters of flood distributions. This might be because it is less sensitive to outliers, lower sampling variability, and fast convergence to asymptotic normality. The shape of the WGG generated probability distribution can also be summarized using the L-moments. Thus, L-moments are defined as:

$$P_{wm}(w,v) = \int_0^{\infty} t^w F^v(t)f(t)dt. \tag{22}$$

However, $F^v$ can be expressed as

$$F^v = \sum_{i,w=0}^{\infty}\sum_{u=0}^{w+p}\sum_{p=0}^{v}(-1)^{2w+v+i-u}\binom{v}{p}\binom{w+p}{u}\frac{\lambda^{w+p}p^w\Gamma(k\beta+i)}{w!i!\Gamma(k\beta)(1+\beta\lambda)^p}G^i(t)$$

where $\Gamma(\cdot)$ is a gamma function. Hence, L-moments is given as

$$P_{wm}(w,v) = \sum_{i,w=0}^{\infty}\sum_{u=0}^{w+p}\sum_{p=0}^{v}R_{(i,w,u,p)}\alpha\mu_{(i,w,u)}T_i \tag{23}$$

where

$$T_i = \int_0^{\infty} t^w g(t)G^{2i}(t)dt$$

and

$$R_{(i,w,u,p)} = (-1)^{2w+v+i-u}\binom{v}{p}\binom{w+p}{u}\frac{\lambda^{w+p}p^w\Gamma(k\beta+i)}{w!i!\Gamma(k\beta)(1+\beta\lambda)^p}.$$

*Entropies*

The heterogeneity or impurity of the target variable of Poisson process can be measured by the amount of uncertainty associated in the value of a random variable.

Thus, the Shannon entropy of WGG generated random variable $T$ is defined as

$$S_e(T) = E\Big[-\sum_{i,w=0}^{\infty}\sum_{u=0}^{w}\Big(\log\mu_{(i,w,u)} + \log\mu + \log g(t) + i\log G(t)\Big)\Big] \tag{24}$$

The Renyi entropy is a measure that increasingly weighs all WGG generated random events with nonzero probability. As $\theta$ approaches zero, the WGG generated Renyi entropy is given as

$$R_\theta = \frac{1}{(1-\theta)}\log\int_0^\infty f^\theta(t)dt \;\; \theta > 0, \theta \neq 0. \tag{25}$$

This implies

$$\begin{aligned}
R_\theta =& \frac{1}{(1-\theta)}\log\int_0^\infty\Big(\sum_{i,w=0}^{\infty}\sum_{u=0}^{w}\mu_{(i,w,u)}\alpha g(t)G^i(t)\Big)^\theta dt\\
=& \frac{1}{(1-\theta)}\log\Big[\Big(\sum_{i,w=0}^{\infty}\sum_{u=0}^{w}\mu_{(i,w,u)}\alpha\Big)^\theta\int_0^\infty g(t)^\theta G^i(t)^\theta\Big]dt\\
=& \frac{1}{(1-\theta)}\log\Big[\Big(\sum_{i,w=0}^{\infty}\sum_{u=0}^{w}\mu_{(i,w,u)}\alpha D_i\Big)^\theta\Big],
\end{aligned} \tag{26}$$

where

$$D_i = \int_0^\infty g(t)G^i(t)dt. \; i = 1, 2, 3, \ldots$$

*Moment of the residual* In reliability theory, and life testing scenarios, the additional lifetime a process or a product that a component or chain has survived up to time $t$ is called the vitality function or residual life function or truncated moment. It can also be used to obtain the distribution function $F(t)$. Thus, the $k^{th}$ moment of the residual life defined as $M_{rs}(x) = E[(T-x)^k \mid T \geq t]$. Hence, it is expressed as

$$\begin{aligned}
M_{rs}(x) =& \frac{1}{1-F(x)}\int_x^\infty (T-x)^k f(t)dt = \frac{1}{1-F(x)}\sum_{a=1}^{k}(-1)^{k-a}x^{k-a}\int_x^\infty t^a f(t)dt\\
=& \frac{\alpha}{1-F(x)}\sum_{i,w=0}^{\infty}\sum_{u=0}^{w}\sum_{a=1}^{k}(-1)^{k-a}x^{k-a}\mu_{(i,w,u)}\Im_i,
\end{aligned} \tag{27}$$

where

$$\Im_i = \int_x^\infty t^a g(t)G^i(t)dt.$$

**Theorem 2.** *Let $T$ be a random variable with a WGG generated probability distribution function $F(t)$. Let $S(t) = 1 - F(t)$ and $M_k(y) = E[(T-y)^k \mid T > y],\ y \geq 0$.*

*Then,*

$$\frac{M_k'(y) + kM_k(y)}{M_k(y)} = \frac{M_{k-1}'(y) + (k-1)M_{k-1}(y)}{M_{k-1}(y)} \text{ or}$$

*equivalently,*

$$M_{k-1}'(y) = -(k-1)M_{k-2}'(y) + \frac{M_k'(y)}{M_k(y)}M_{k-1}(y) + \frac{kM_{k-1}^2(y)}{M_k(y)}.$$

*Proof.* Let

$$M_k(y) = \frac{1}{S(y)} \int_y^{\infty} k(t-y)^{k-1} S(t)dt.$$

Then,

$$\log M_k(y) = \log \int_y^{\infty} k(t-y)^{k-1} S(t)dt - \log S(y).$$

Thus, differentiating with respect to $y$, we have

$$\frac{M_k'(y)}{M_k(y)} = \frac{\int_y^{\infty} -k(k-1)(t-y)^{k-2} S(t)dt}{\int_y^{\infty} k(t-y)^{k-1} S(t)dt} - \frac{S'(y)}{S(y)} = \frac{-kM_{k-1}(y)}{M_k(y)} - \frac{S'(y)}{S(y)}.$$

Hence,

$$\frac{M_k'(y) + kM_{k-1}(y)}{M_k(y)} = -\frac{S'(y)}{S(y)} = \frac{M_{k-1}'(y) + (k-2)M_{k-2}(y)}{M_{k-1}(y)}.$$

□

## 8. Parameter Estimation

It is intuitive to note that the parameters of the WGG generated model are descriptive measures of the entire population that determine the shape and location of the curve on the plot of the WGG generated distribution. Hence, for a better forecasting and regression analysis of the proposed WGG model to be efficient, there is a need to obtain the parameter estimates of the WGG generated model. Thus, in this section, the parameters of the WGG generated model are estimated using the maximum likelihood estimation (MLE) method.

8.1. **Maximum Likelihood.** Let $\mathbf{T} = (T_1, T_2, \ldots, T_k)$ be a random sample obtained from the WGG generated distribution with unknown parameter vector $\Theta = (\beta, \lambda, \psi)^T$. Let $\mathbf{t} = (t_1, t_2, \ldots, t_k)$ be a sample value of a random sample $\mathbf{T}$. Then, we can obtain the log-likelihood as

$$\begin{aligned}\ell =&2k\log\lambda + k\log\beta + \sum_{a=1}^{k}\log g(t_a,\psi) - k\log(1+\beta\lambda)\\ &-(1+\beta\lambda)\sum_{a=1}^{k}\log(1-G(t_a,\psi)) + \sum_{a=1}^{k}\log((1-G(t_a,\psi))^{-\beta}+\beta-1)\\ &-\sum_{a=1}^{k}\lambda((1-G(t_a,\psi))^{-\beta}-1).\end{aligned} \tag{28}$$

The parameters of the WGG generated model are obtained by taking the first partial derivative of the log-likelihood of the WGG model with respect to each of the parameters and equate to zero. Thus, we have

$$\frac{\partial\ell}{\partial\lambda} = \frac{2k}{\lambda} - \frac{k\beta}{1+\beta\lambda} - \beta\sum_{a=1}^{k}\log(1-G(t_a,\psi)) - \sum_{a=1}^{k}((1-G(t_a,\psi))^{-\beta}-1) = 0, \tag{29}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\psi} =&\sum_{a=1}^{k}\frac{g'(t_a,\psi)}{g(t_a,\psi)} + \beta\sum_{a=1}^{k}\frac{g(t_a,\psi)(1-G(t_a,\psi))^{-\beta-1}}{((1-G(t_a,\psi))^{-\beta}+\beta-1)}\\ &+(1+\beta\lambda)\sum_{a=1}^{k}\frac{g(t_a,\psi)}{1-G(t_a,\psi)} - \beta\lambda\sum_{a=1}^{k}g(t_a,\psi)(1-G(t_a,\psi))^{-\beta-1} = 0,\end{aligned} \tag{30}$$

and

$$\begin{aligned}\frac{\partial\ell}{\partial\beta} =&\frac{k}{\beta} - \lambda\sum_{a=1}^{k}\log(1-G(t_a,\psi)) + \sum_{a=1}^{k}\frac{(1-G(t_a,\psi))^{-\beta}\log(1-G(t_a,\psi))}{(1-G(t_a,\psi))^{-\beta}+\beta-1}\\ &-\frac{k\lambda}{1+\beta\lambda} - \lambda\sum_{a=1}^{k}(1-G(t_a,\psi))^{-\beta}\log(1-G(t_a,\psi)) = 0.\end{aligned} \tag{31}$$

However, the solutions to the nonlinear equations (29), (30), and (31) are obtained in closed form using numerical methods. These numerical methods are beyond the scope of this article.

## 9. Applications

The viability, tractability, and performance of the WGG generated model is examined by first performing a Monte Carlo simulation of some sub-models of the proposed model. The real-life applications of some of the sub-models of the proposed model were investigated and compared to some competitive-related models in the literature. The WGG sub-models were compared with some existing models based on their mean squared errors in the simulation cases and goodness-of-fit test statistics in life applications.

9.1. **Simulation study.** A Monte Carlo simulation was carried out to test the flexibility and efficiency of the proposed distribution. The simulation was achieved using the quantile function in (9) to generate random data for the proposed model with $0 < q < 1$ for various values of $\lambda = 1.0, \beta = 1.0, \theta = 0.2$ and $\rho = 1.0$ for the Burr XII sub-model. $\lambda = 0.9, \beta = 2.3, \theta = 0.1$ and $\rho = 0.01$ for the Lomax sub-model, and $\lambda = 0.1, \beta = 0.1, \theta = 0.3$ and $\rho = 0.8$ for the Frechet sub-model for 1000 replicated trials.

The sample size $n$ are taken as $n = 5, 10, 20, 50, 100, 150, 200,$ 250,300, $350, 400, 450,$ and 500 The simulation studied the mean estimated (ME), biases, and mean squared errors (MSE). The result of the simulation is as shown in Table 1. In Table one, we observed that the biases converge to zero as sample sizes increase. The estimated mean also converges to the true value as the sample sizes increases. The mean square errors converge to zero.

The bias is obtained for $(W = \lambda, \beta, \theta, \rho)$ as

$$B\hat{ias}_W = \frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{W}_i - W\right).$$

Also, the MSE is obtained as

$$M\hat{S}E_W = \frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{W}_i - W\right)^2.$$

9.2. **Life applications.** In most cases of statistical modeling, the interest is to estimate the model parameters and evaluate their test statistics goodness-of-fit. Thus, in this section, the viability, tractability, and effectiveness of the proposed model is investigated with the illustration of real-life data sets. The measures of the test statistics' goodness-of-fit were examined with some existing neighbourhood models in the literature. These models in the literature include, but are not limited to, the class of Weibull, Gompertz, Kumaraswamy, and Frechet distributions. The test statistics considered include the Akaike information criterion (AIC), Anderson-Darling (A), Cramer-von Mises (W), Kolmogrov-Smirnov (KS), and p-value (p-val). The larger the p-value and the smaller the test statistics the better the model fits the data.

9.2.1. *Obesity Data.* The first data consist of 22 obesity among children and adolescents aged 12-19 by selected characteristics: United States, selected between 2015 - 2018 as reported by [9]. The data are available in https://www.cdc.gov/nchs/hus /contents.htm-Table-027. The data were measured based on height and weight. The data are as follows:

18.9,15.1,23.1,9.8,25.7,26.9,19.8,16.0,19.2,12.0,28.0,29.2,
17.9,14.2,27.0,7.4,23.3,24.6,23.9,21.7,18.4,10.6.

The descriptive statistics of the data are given in Table 2.

Table 1. The mean estimates (ME), biases and mean squared errors (MSE) for $\lambda, \beta, \theta$ and $\rho$ with WGG generated sub-models

| Distribution | Parameters | n | ME | Bias | MSE |
|---|---|---|---|---|---|
| Burr XII | | 05 | 1.0602 0.8288 0.1762 1.0033 | 0.3603 0.3288 0.8306 0.7033 | 0.3597 0.1811 0.6907 0.6271 |
| | | 10 | 1.1340 0.8115 0.1718 0.9558 | 0.3592 0.3157 0.2286 0.6779 | 0.3544 0.1581 0.6873 0.5358 |
| | | 20 | 1.1951 0.7708 0.1694 0.9466 | 0.3592 0.3157 0.1285 0.6740 | 0.3474 0.1303 0.1871 0.5326 |
| | $\lambda = 1.0$ | 50 | 1.1601 0.7416 0.1715 0.9556 | 0.2284 0.1115 0.1283 0.0667 | 0.3349 0.1255 0.0370 0.5226 |
| | | 100 | 1.0615 0.7558 0.1724 0.9740 | 0.1244 0.0192 0.1283 0.0648 | 0.1345 0.0240 0.0269 0.4176 |
| | $\beta = 1.0$ | 150 | 1.0219 0.7796 0.1714 0.9648 | 0.1159 0.0178 0.1282 0.0582 | 0.0400 0.0221 0.0269 0.3150 |
| | | 200 | 0.9716 0.7961 0.1717 0.9779 | 0.1089 0.0130 0.0278 0.0575 | 0.0331 0.0211 0.0261 0.2107 |
| | $\theta = 0.2$ | 250 | 0.9756 0.7893 0.1729 0.9582 | 0.0181 0.0101 0.0176 0.0418 | 0.0317 0.0180 0.0256 0.1012 |
| | | 300 | 0.9397 0.8078 0.1722 0.9667 | 0.0028 0.0098 0.0173 0.0406 | 0.0259 0.0164 0.0152 0.0196 |
| | $\rho = 1.0$ | 350 | 0.9408 0.8157 0.1717 0.9575 | 0.0021 0.0096 0.0101 0.0466 | 0.0158 0.0149 0.0148 0.0190 |
| | | 400 | 0.9408 0.8157 0.1717 0.9575 | 0.0010 0.0078 0.0062 0.0298 | 0.0028 0.0142 0.0134 0.0106 |
| | | 450 | 0.9841 0.8092 0.1727 0.9269 | 0.0001 0.0058 0.0058 0.0269 | 0.0009 0.0084 0.0127 0.0094 |
| | | 500 | 0.9911 0.8030 0.1742 0.9152 | 0.0001 0.0016 0.0038 0.0152 | 0.0006 0.0063 0.0101 0.0092 |
| Lomax | | 05 | 1.0055 2.1054 0.0956 0.0143 | 0.7055 0.4735 1.3044 0.0918 | 0.7714 0.4765 1.3050 0.0919 |
| | | 10 | 0.9631 2.1852 0.0959 0.0109 | 0.6631 0.4724 1.3041 0.0917 | 0.7078 0.4740 1.3046 0.0917 |
| | | 20 | 0.9085 2.2429 0.0963 0.0098 | 0.6085 0.4718 1.3037 0.0916 | 0.6287 0.4727 1.3042 0.0917 |
| | $\lambda = 0.9$ | 50 | 0.8801 2.2735 0.0951 0.0084 | 0.6071 0.1010 1.3049 0.0416 | 0.1091 0.1720 0.1043 0.0216 |
| | | 100 | 0.8806 2.2724 0.0958 0.0082 | 0.0232 0.0208 0.0142 0.0391 | 0.0254 0.0420 0.0138 0.0116 |
| | $\beta = 2.3$ | 150 | 0.8870 2.2708 0.0977 0.0083 | 0.0196 0.0192 0.0123 0.0313 | 0.0238 0.0300 0.0125 0.0113 |
| | | 200 | 0.8870 2.2718 0.0991 0.0084 | 0.0133 0.0184 0.0109 0.0212 | 0.0147 0.0292 0.0114 0.0112 |
| | $\theta = 0.1$ | 250 | 0.8896 2.2710 0.0987 0.0084 | 0.0132 0.0181 0.0093 0.0211 | 0.0136 0.0198 0.0111 0.0111 |
| | | 300 | 0.8962 2.2692 0.1010 0.0087 | 0.0096 0.0173 0.0082 0.0210 | 0.0122 0.0180 0.0092 0.0110 |
| | $\rho = 0.01$ | 350 | 0.8993 2.2684 0.1020 0.0088 | 0.0070 0.0159 0.0080 0.0199 | 0.0100 0.0166 0.0082 0.0109 |
| | | 400 | 0.9016 2.2681 0.1028 0.0089 | 0.0030 0.0129 0.0072 0.0195 | 0.0097 0.0121 0.0073 0.0098 |
| | | 450 | 0.9002 2.2673 0.1014 0.0090 | 0.0006 0.0057 0.0066 0.0191 | 0.0055 0.0089 0.0067 0.0094 |
| | | 500 | 0.9001 2.2659 0.1005 0.0091 | 0.0001 0.0054 0.0055 0.0157 | 0.0038 0.0075 0.0056 0.0069 |
| Frechet | | 05 | 0.0488 0.3418 0.0439 1.0031 | 0.1515 0.4418 0.1193 0.8031 | 0.1978 0.6305 0.1201 0.9035 |
| | | 10 | 0.0485 0.3282 0.0472 0.9886 | 0.1512 0.4282 0.1191 0.7886 | 0.1899 0.6233 0.1195 0.8846 |
| | | 20 | 0.0556 0.2775 0.0467 0.9320 | 0.1444 0.3775 0.1185 0.7320 | 0.1740 0.5772 0.1193 0.8254 |
| | $\lambda = 0.1$ | 50 | 0.0830 0.1931 0.0453 0.8610 | 0.1170 0.2931 0.1183 0.0610 | 0.1355 0.2597 0.1189 0.1302 |
| | | 100 | 0.0973 0.1332 0.0384 0.8298 | 0.1027 0.2332 0.1175 0.0435 | 0.0227 0.0289 0.0188 0.0141 |
| | $\beta = 0.1$ | 150 | 0.1020 0.1129 0.0352 0.8287 | 0.0980 0.2129 0.1174 0.0423 | 0.0205 0.0269 0.0182 0.0113 |
| | | 200 | 0.1054 0.1054 0.0334 0.8234 | 0.0946 0.2024 0.1166 0.0402 | 0.0191 0.0225 0.0179 0.0104 |
| | $\theta = 0.3$ | 250 | 0.1069 0.1046 0.0326 0.8244 | 0.0935 0.1976 0.1148 0.0368 | 0.0177 0.0138 0.0176 0.0094 |
| | | 300 | 0.1084 0.1036 0.0325 0.8352 | 0.0916 0.2036 0.1116 0.0352 | 0.0154 0.0131 0.0167 0.0083 |
| | $\lambda = 0.8$ | 350 | 0.1102 0.1024 0.0317 0.8068 | 0.0898 0.2024 0.1061 0.0298 | 0.0124 0.0110 0.0160 0.0074 |
| | | 400 | 0.1041 0.1016 0.0315 0.8012 | 0.0889 0.2036 0.1047 0.0287 | 0.0113 0.0102 0.0153 0.0066 |
| | | 450 | 0.1021 0.1014 0.0309 0.8003 | 0.0879 0.2037 0.1033 0.0244 | 0.0100 0.0099 0.0137 0.0053 |
| | | 500 | 0.1006 0.1007 0.0307 0.8005 | 0.0874 0.2034 0.1028 0.0234 | 0.0092 0.0091 0.0127 0.0029 |

Table 2. The Descriptive statistics of obesity among children and adolescents data set to 2 decimal points

| Mean | Median | $\sigma$ | IQR | Variance | Kurtosis | Skewness | 25% | 75% | 99% |
|---|---|---|---|---|---|---|---|---|---|
| 19.67 | 19.50 | 6.30 | 9.10 | 39.66 | -1.12 | -0.29 | 15.33 | 24.43 | 28.95 |

We observed from Table 2 that the a negative kurtosis and skewness were obtained. This implies that the distribution of the obesity data is flatter than a normal curve with the same mean and standard deviation. Hence, the data are left skewed.

Table 3 shows the test statistics of the goodness-of-fit measure of comparison adopted for comprehensive comparison.

Table 3: The goodness-of-fit measure of obesity among children and adolescents data set (standard errors in parentheses)

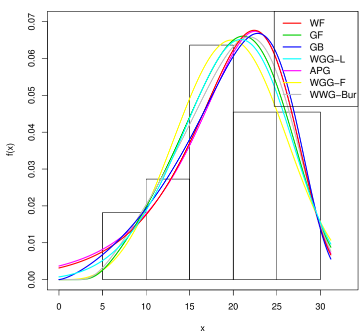
| Distribution | p-value | AIC | KS | W | A | Estimates |
|---|---|---|---|---|---|---|
| WGG-B | 0.9997 | 97.9173 | 0.0219 | 0.0012 | 0.0528 | $\hat{\lambda} = 0.2201(0.1022)$ |
| | | | | | | $\hat{\beta} = 1.2315(0.0898)$ |
| | | | | | | $\hat{\rho} = 0.0872(0.0098)$ |
| | | | | | | $\hat{\theta} = 0.2124(0.0252)$ |
| WGG-L | 0.9390 | 102.3145 | 0.1071 | 0.0278 | 0.2021 | $\hat{\lambda} = 0.0075(0.0020)$ |
| | | | | | | $\hat{\beta} = 1.2912(0.0125)$ |
| | | | | | | $\hat{\rho} = 0.1142(0.0967)$ |
| | | | | | | $\hat{\theta} = 0.8494(0.3254)$ |
| WGG-F | 0.9324 | 109.8906 | 0.2095 | 0.0452 | 0.3183 | $\hat{\lambda} = 0.0022(0.0004)$ |
| | | | | | | $\hat{\beta} = 2.4857(0.8351)$ |
| | | | | | | $\hat{\rho} = 0.8792(0.2743)$ |
| | | | | | | $\hat{\theta} = 1.0516(0.2778)$ |
| KB | 0.7640 | 153.2259 | 0.1356 | 0.0811 | 0.5349 | $\hat{\alpha} = 33.4661(17.9125)$ |
| | | | | | | $\hat{\beta} = 47.4488(46.2083)$ |
| | | | | | | $\hat{\rho} = 0.0331(1.8429)$ |
| | | | | | | $\hat{\theta} = 21.8947(7.1942)$ |
| KL | 0.6961 | 154.7281 | 0.1443 | 0.1006 | 0.6475 | $\hat{\alpha} = 14.5201(14.8943)$ |
| | | | | | | $\hat{\beta} = 1.3267(1.7158)$ |
| | | | | | | $\hat{\rho} = 0.0079(0.0030)$ |
| | | | | | | $\hat{\theta} = 20.3753(16.9103)$ |
| KF | 0.7788 | 152.0259 | 0.1336 | 0.0642 | 0.4347 | $\hat{\alpha} = 5.1639(6.2917)$ |
| | | | | | | $\hat{\beta} = 166.4803(246.7566)$ |
| | | | | | | $\hat{\rho} = 0.6187(0.1771)$ |
| | | | | | | $\hat{\theta} = 20.9740(34.9274)$ |
| KW | 0.6844 | 147.3603 | 0.0915 | 0.0194 | 0.1346 | $\hat{\alpha} = 0.2099(0.2886)$ |
| | | | | | | $\hat{\beta} = 1.1818(1.3944)$ |
| | | | | | | $\hat{\rho} = 0.0356(0.0084)$ |
| | | | | | | $\hat{\theta} = 12.5159(17.5384)$ |
| APG | 0.6866 | 147.2333 | 0.0901 | 0.0296 | 0.2029 | $\hat{\alpha} = 1.8905(2.7086)$ |
| | | | | | | $\hat{\beta} = 0.0051(0.0037)$ |
| | | | | | | $\hat{\rho} = 0.1627(0.0287)$ |
| GB | 0.2511 | 156.8906 | 0.2095 | 0.0452 | 0.3183 | $\hat{\alpha} = 0.0022(0.0004)$ |
| | | | | | | $\hat{\beta} = 2.4857(0.8351)$ |

Table 3 – *Continued from previous page*

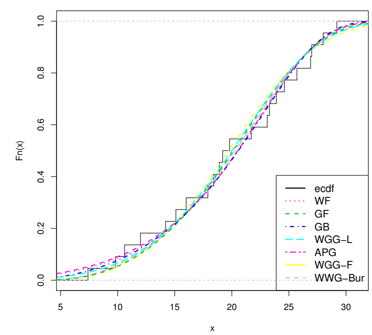
| Distribution | p-value | AIC | KS | W | A | Estimates |
|---|---|---|---|---|---|---|
| | | | | | | $\hat{\rho} = 0.8792(0.2743)$ |
| | | | | | | $\hat{\theta} = 1.0516(0.2778)$ |
| GF | 0.5457 | 148.5009 | 0.1055 | 0.0249 | 0.1808 | $\hat{\alpha} = 0.7959(3.4859)$ |
| | | | | | | $\hat{\beta} = 6.7388(13.2331)$ |
| | | | | | | $\hat{\rho} = 1.0350(0.8096)$ |
| | | | | | | $\hat{\theta} = 27.6166(54.6914)$ |
| GL | 0.5390 | 149.3145 | 0.1071 | 0.0278 | 0.2021 | $\hat{\alpha} = 0.0075(0.0080)$ |
| | | | | | | $\hat{\beta} = 6.2912(4.8123)$ |
| | | | | | | $\hat{\rho} = 0.1142(0.0967)$ |
| | | | | | | $\hat{\theta} = 0.8494(0.7254)$ |
| WL | 0.4568 | 146.4643 | 0.1024 | 0.0245 | 0.1779 | $\hat{\alpha} = 1.0199(2.1532)$ |
| | | | | | | $\hat{\beta} = 7.5745(3.4940)$ |
| | | | | | | $\hat{\rho} = 31.1218(20.4511)$ |
| | | | | | | $\hat{\theta} = 5.2402(0.7263)$ |
| WF | 0.8703 | 149.7794 | 0.1203 | 0.0348 | 0.2507 | $\hat{\alpha} = 0.0413(0.2325)$ |
| | | | | | | $\hat{\beta} = 7.7834(2.0028)$ |
| | | | | | | $\hat{\rho} = 7.7834(8.9362)$ |
| | | | | | | $\hat{\theta} = 3.4925(4.9272)$ |
| WB | 0.7543 | 149.9324 | 0.1228 | 0.0371 | 0.2652 | $\hat{\alpha} = 0.0073(0.0079)$ |
| | | | | | | $\hat{\beta} = 6.9216(3.2352)$ |
| | | | | | | $\hat{\rho} = 0.3107(0.5863)$ |
| | | | | | | $\hat{\theta} = 1.1514(2.4617)$ |
| GE | 0.4868 | 147.0767 | 0.0900 | 0.0284 | 0.1958 | $\hat{\alpha} = 0.0093(0.0111)$ |
| | | | | | | $\hat{\beta} = 0.5355(0.6937)$ |
| | | | | | | $\hat{\rho} = 0.3373(0.4140)$ |
| GW | 0.6824 | 148.0124 | 0.0926 | 0.0226 | 0.1569 | $\hat{\alpha} = 0.0335(0.1008)$ |
| | | | | | | $\hat{\beta} = 0.0745(0.1903)$ |
| | | | | | | $\hat{\rho} = 0.1381(0.1830)$ |
| | | | | | | $\hat{\theta} = 2.4173(0.3935)$ |
| TF | 0.5892 | 149.5043 | 0.0883 | 0.0293 | 0.2053 | $\hat{\alpha} = 0.0086(0.0091)$ |
| | | | | | | $\hat{\beta} = 0.3939(2.0093)$ |
| | | | | | | $\hat{\rho} = 0.6124(1.6391)$ |
| | | | | | | $\hat{\theta} = -0.0118(0.0251)$ |

Figure 5 shows the empirical histogram and cdfs of the obesity real-life data applications.

9.2.2. *Precipitations in Karachi city, Pakistan Data.* The second data examined comprises 59 annual maximum precipitations in Karachi city, Pakistan, for the




(A) Empirical densities




(B) Empirical cdfs

FIGURE 4. The Empirical densities and cdfs of obesity among children and adolescents data set

years 1950-2009 as used in [6]. The precipitation records help water management studies and flood defense systems to predict floods and droughts. The precipitation data also help to minimize the risk of large hydraulic structures. The values of the data are:

11.8, 6.5, 54.9, 39.9, 16.8, 30.2, 38.4, 76.9, 73.4, 117.6, 157.7, 148.6, 11.4, 5.6, 63.6, 62.4, 85, 256.3, 24.9, 148.6, 160.5, 131.3, 77, 155.2, 217.2, 105.5, 166.8, 157.9, 73.6, 291.4, , 30, 270.4, 160, 96.3, 185.7, 429.3, 184.9, 262.5, 80.6, 138.2, 28, 39.3, 210.3, 315.7, 107.7, 33.3, 302.6, 159.1, 78.7, 33.2, 52.2, 92.7,150.4, 43.7, 68.3, 20.8, 179.4, 245.7, 19.5.

The descriptive statistics of the data are given in Table 4.

TABLE 4. The Descriptive statistics of annual maximum precipitations in Karachi city, Pakistan data set to 2 decimal points

| Mean | Median | $\sigma$ | IQR | Variance | Kurtosis | Skewness | 25% | 75% | 99% |
|---|---|---|---|---|---|---|---|---|---|
| 118.40 | 92.70 | 93.21 | 120.65 | 8688.99 | 0.64 | 0.99 | 39.60 | 160.25 | 363.41 |

We observed from Table 4 that the a positive kurtosis and skewness indicated that distribution is peaked and possesses thick tails, and most values are clustered around the left tail of the distribution while the right tail of the distribution is longer.

Table 5: The goodness-of-fit measure of maximum precipitations in Karachi city, Pakistan data set (standard errors in parentheses)

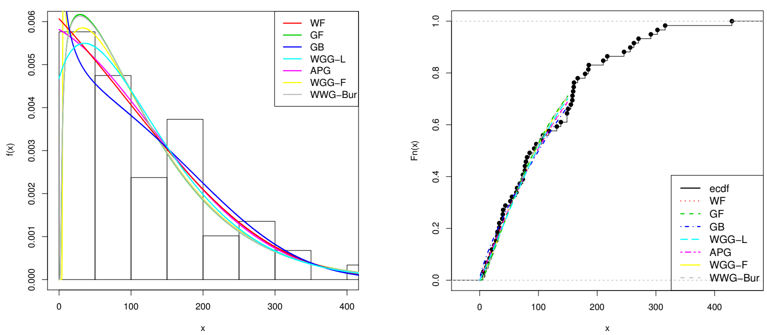
| Distribution | p-value | AIC | KS | W | A | Estimates |
|---|---|---|---|---|---|---|
| WGG-B | 0.9470 | 383.7050 | 0.0961 | 0.0454 | 0.2577 | $\hat{\lambda} = 0.0951(0.1763)$ |
| | | | | | | $\hat{\beta} = 1.2401(0.1875)$ |
| | | | | | | $\hat{\rho} = 1.7186(0.2943)$ |
| | | | | | | $\hat{\theta} = 1.9784(0.6677)$ |
| WGG-L | 0.9376 | 391.7440 | 0.0968 | 0.0461 | 0.2616 | $\hat{\lambda} = 0.1689(0.0778)$ |
| | | | | | | $\hat{\beta} = 1.2239(0.0682)$ |
| | | | | | | $\hat{\rho} = 1.0199(0.0332)$ |
| | | | | | | $\hat{\theta} = 1.5536(0.0941)$ |
| WGG-F | 0.8617 | 401.1031 | 0.0989 | 0.0688 | 0.3093 | $\hat{\lambda} = 1.4858(0.4944)$ |
| | | | | | | $\hat{\beta} = 1.2023(0.7353)$ |
| | | | | | | $\hat{\rho} = 1.1434(0.1052)$ |
| | | | | | | $\hat{\theta} = 1.3496(0.9372)$ |
| KB | 0.2911 | 691.8905 | 0.1276 | 0.1372 | 0.8463 | $\hat{\alpha} = 8.3342(2.2157)$ |
| | | | | | | $\hat{\beta} = 56.1819(92.7683)$ |
| | | | | | | $\hat{\rho} = 0.0182(0.0000)$ |
| | | | | | | $\hat{\theta} = 11.1408(1.0780)$ |
| KL | 0.4207 | 687.9069 | 0.1145 | 0.0848 | 0.4997 | $\hat{\alpha} = 1.7166(0.2951)$ |
| | | | | | | $\hat{\beta} = 3.3847(2.8572)$ |
| | | | | | | $\hat{\rho} = 0.0040(0.0010)$ |
| | | | | | | $\hat{\theta} = 1.5341(1.0421)$ |
| KF | 0.3786 | 687.6918 | 0.1185 | 0.0883 | 0.5257 | $\hat{\alpha} = 6.8464(2.1692)$ |
| | | | | | | $\hat{\beta} = 161.821(229.22)$ |
| | | | | | | $\hat{\rho} = 0.2188(0.0564)$ |
| | | | | | | $\hat{\theta} = 30.025(31.898)$ |
| KW | 0.7467 | 684.7171 | 0.0883 | 0.0467 | 0.2692 | $\hat{\alpha} = 0.8755(0.4893)$ |
| | | | | | | $\hat{\beta} = 0.5662(0.6176)$ |
| | | | | | | $\hat{\rho} = 0.0112(0.0098)$ |
| | | | | | | $\hat{\theta} = 1.3454(0.3905)$ |
| APG | 0.8959 | 682.9092 | 0.0748 | 0.0438 | 0.2641 | $\hat{\alpha} = 1.5772(2.1911)$ |
| | | | | | | $\hat{\beta} = 0.0073(0.0040)$ |
| | | | | | | $\hat{\rho} = 0.0023(0.0022)$ |
| GB | 0.6326 | 684.8519 | 0.0972 | 0.0491 | 0.2803 | $\hat{\alpha} = 0.0075(0.0045)$ |
| | | | | | | $\hat{\beta} = 2.7856(1.9958)$ |
| | | | | | | $\hat{\rho} = 0.3543(0.3103)$ |
| | | | | | | $\hat{\theta} = 1.2401(0.9676)$ |

Table 5 – *Continued from previous page*

| Distribution | p-value | AIC | KS | W | A | Estimates |
|---|---|---|---|---|---|---|
| GF | 0.6774 | 683.6124 | 0.0937 | 0.0435 | 0.2460 | $\hat{\alpha} = 0.1587(0.4962)$ |
| | | | | | | $\hat{\beta} = 1.8235(2.5702)$ |
| | | | | | | $\hat{\rho} = 0.7248(0.7488)$ |
| | | | | | | $\hat{\theta} = 22.8034(61.9942)$ |
| GL | 0.7704 | 685.1274 | 0.0864 | 0.0488 | 0.2849 | $\hat{\alpha} = 0.1380(2.0119)$ |
| | | | | | | $\hat{\beta} = 1.7962(38.1464)$ |
| | | | | | | $\hat{\rho} = 0.0437(0.3829)$ |
| | | | | | | $\hat{\theta} = 0.7748(16.8750)$ |
| WF | 0.7042 | 681.9136 | 0.0916 | 0.0434 | 0.2462 | $\hat{\alpha} = 0.0358(0.0180)$ |
| | | | | | | $\hat{\beta} = 0.2947(0.1467)$ |
| | | | | | | $\hat{\rho} = 4.1927(2.0935)$ |
| | | | | | | $\hat{\theta} = 8.6209(1.7749)$ |
| WB | 0.5757 | 684.6917 | 0.1016 | 0.0519 | 0.2954 | $\hat{\alpha} = 0.0073(0.0049)$ |
| | | | | | | $\hat{\beta} = 2.4377(1.0282)$ |
| | | | | | | $\hat{\rho} = 0.4476(0.4885)$ |
| | | | | | | $\hat{\theta} = 0.9901(1.3096)$ |
| WL | 0.1985 | 751.7122 | 0.1398 | 0.0730 | 0.4267 | $\hat{\alpha} = 3.6920(0.7601)$ |
| | | | | | | $\hat{\beta} = 0.0923(0.0253)$ |
| | | | | | | $\hat{\rho} = 0.6424(0.0600)$ |
| | | | | | | $\hat{\theta} = 0.1421(0.5247)$ |
| GE | 0.3220 | 682.9042 | 0.0717 | 0.0420 | 0.2562 | $\hat{\alpha} = 0.0857(0.0178)$ |
| | | | | | | $\hat{\beta} = 0.0438(0.0473)$ |
| | | | | | | $\hat{\rho} = 0.0707(0.6964)$ |
| GW | 0.4824 | 687.6262 | 0.0604 | 0.0582 | 0.3764 | $\hat{\alpha} = 0.0341(0.0082)$ |
| | | | | | | $\hat{\beta} = 0.0787(0.0160)$ |
| | | | | | | $\hat{\rho} = 0.3342(0.0000)$ |
| | | | | | | $\hat{\theta} = 0.7105(0.0072)$ |
| TF | 0.3617 | 701.1031 | 0.1201 | 0.2688 | 1.6393 | $\hat{\alpha} = 28.4858(29.4944)$ |
| | | | | | | $\hat{\beta} = 31.2023(13.7353)$ |
| | | | | | | $\hat{\rho} = 1.1434(0.1052)$ |
| | | | | | | $\hat{\theta} = 0.9372(4.2815)$ |

Figure 6 shows the empirical histogram and cdfs of the obesity real-life data applications.

9.3. **Discussion.** In Tables 3 and 5, we observed that the p-values of the WGG generated models are the highest with the lowest AIC test statistic in Burr XII, Lomax, and Frechet sub-models. Hence, the WGG model has provided a better alternative to making statistical distributions more flexible, and viable compared




(A) Empirical densities

(B) Empirical cdfs

FIGURE 5. The Empirical densities and cdfs of maximum precipitations in Karachi city, Pakistan data set

to the model generated by Gompertz, Weibull, Kumaraswamy, and Alpha power models.

## 10. Conclusion

Intuitively, a two-parameter weighted Gompertz-G generated distribution was examined and introduced by making use of a weighted Gompertz and the T-X characterizations. The newly developed model has found its uses in cases where two-sided abrupt changes schemes occurred in applications. The WGG model has provided a better alternative to making statistical distributions more flexible, and viable compared to the model generated by Gompertz, Weibull, Kumaraswamy, and Alpha power models. The statistical properties and estimations of the model parameters were obtained. The viability and flexibility of the WGG-generated model were demonstrated by illustration of a simulation and real data sets using their goodness-of-fit statistics. The outcomes of the WGG-generated test statistics indicated a better viable, tractable, flexible, and parsimonious generator compared to some competitive models in the literature. Hence, it can be used as a better alternative in reliability theory and extreme value theory.

**Author Contribution Statements** The authors contributed equally to this paper. All authors read and approved this paper's final form.

**Declaration of Competing Interests** The authors wish to state clearly that there is no conflict of interest.

## References


[1] Abd-AL-Motalib, R. S., Abed AL-Kadim, K., The odd truncated inverse exponential Weibull exponential distribution, *Journal of Positive School Psychology* , 6(2) (2022), 5361-5375.

[2] Aljarrah, M. A., Lee, C., Famoye, F., On generating T-X family of distributions using quantile functions, *Journal of Statistical Distributions and Applications*, 1(1) (2014), 1-17. https://doi.org/10.1186/2195-5832-1-2

[3] Alizadeh, M., Altun, E., Ozel, G., Afshari, M., Eftekharian, A., A new odd log-logistic Lindley distribution with properties and applications, *Sankhya A*, 81 (2019), 323-346. https://doi.org/10.1007/s13171-018-0142-x

[4] Alzaatreh, A., Famoye, F., Lee, C., A new method for generating families of continuous distributions, *Metron*, 71 (2013), 63-79. https://doi.org/10.1007/s40300-013-0007-y

[5] Alzaghal, A., Famoye, F., Lee, C., Exponentiated T-X family of distributions with some applications, *International Journal of Statistics and Probability*, 2(3) (2013), 31–49. https://doi.org/10.5539/ijsp.v2n3p31

[6] Arshad, M. Z., Iqbal, M. Z., Mutairi, A. A., A comprehensive review of datasets for statistical research in probability and quality control, *Journal of Mathematical and Computational Science*, 11(3) (2021), 3663-3728. https://doi.org/10.28919/jmcs/5692

[7] Bakouch, S. H., Ahmed, M. T., El-Bar, A., A new weighted Gompertz distribution with applications to reliability data, *Applications of Mathematics*, 3 (2017), 269–296. https://doi.org/10.21136/AM.2017.0277-16

[8] Bemmaor, A. C., Modelling the Diffusion of New Durable Goods: Word-of-mouth Effect Versus Consumer Heterogeneity: In G. Laurent, G.L. Lilien, B. Pras eds., Research 98Traditions in Marketing. Boston, Kluwer Academic Publishers, 1, 201-229, 1992. https://doi.org/10.1007/978-94-011-1402-8-6

[9] CDC, Obesity among children and adolescelent between the aged 2- 19 years by selected characteristics, https://www.cdc.gov/nchs/hus/contents.htm-Table-027, (2020).

[10] Cordeiro, G. M., Alizadeh, M., Diniz Marinho, P. R., The type I half-logistic family of distributions, *Journal of Statistical Computation and Simulation*, 86(4) (2016), 707-728. https://doi.org/10.1080/00949655.2015.1031233

[11] Dijoux, Y., Construction of the tetration distribution based on the continuous iteration of the exponential-minus-one function, *Applied Stochastic Models in Business and Industry*, (2020), 1-26. https://doi.org/10.1002/asmb.2538

[12] Eghwerido, J. T., The alpha power Teissier distribution and its applications, *Afr. Stat.*, 16(2) (2021), 2731-2745. http://dx.doi.org/10.16929/as/2021.2731.181

[13] Eghwerido, J. T., Agu, F. I., The shifted Gompertz-G family of distributions: properties, and applications, *Mathematica Slovaca*, 71(5) (2021a), 1291-1308. http://dx.doi.org/10.1515/ms-2021-0053

[14] Eghwerido, J. T., Nzei, L. C, Agu, F. I., Alpha power Gompertz distribution: properties and applications, *Sankhya A - The Indian Journal of Statistics*, 83(1) (2021b), 449-475. http://dx.doi.org/10.1007/s13171-020-00198-0

[15] Eghwerido, J. T., Oguntunde, P. E., Agu, F. I., The alpha power Marshall-Olkin-G family of distributions: properties, and applications, *Sankhya A*, 85 (2023), 172-197. http://dx.doi.org/10.1007/s13171-020.00235-y

[16] Eghwerido, J. T., The Marshall-Olkin Teissier generated model for lifetime data, *Journal of the Belarusian State University: Mathematics and Informatics*, 1 (2022), 46-65. https://doi.org/10.33581/2520-6508-2022-1-46-65

[17] Eghwerido, J. T., Agu, F. I., The statistical properties and applications of the alpha power Topp-Leone-G distribution, *Heliyon*, 8(6) (2020a), 1-10. https://doi.org/10.1016/j.heliyon.2022.e09775

[18] Eghwerido, J. T., Agu, F. I., The alpha power Muth-G distribution and its applications in survival and reliability analyses, *Mathematica Slovaca*, 73(6) (2023), 1-18.

[19] Hassana, A. S., Shawkia, A. W., Muhammeda, A. H., Weighted Weibull-G family of distributions: theory and application in the analysis of renewable energy sources,*Journal of Positive School Psychology*, 6(3) (2022), 9201-9216.

[20] Jamal, F., Chesneau, C., Saboor, A., Aslam, M., Tahir, M. H., Mashwani, W. K., The U family of distributions: properties and applications, *Mathematica Slovaca*, 72(1) (2022), 217-240. https://doi.org/10.1515/ms-2022-0015

[21] Lin, G. D., Dou, X., Kuriki, S., The bivariate lack-of-memory distributions, *Sankhya A*, 81 (2019), 273-297. https://doi.org/10.1007/s13171-017-0119-1

[22] Marshall, A., Olkin, I., Life Distributions. Structure of Nonparametric, Semiparametric, and Parametric Families. Springer Series in Statistics, Springer, New York, 2007.

[23] Nzei, L. C., Eghwerido, J. T., Ekhosuehi, N., Topp-Leone Gompertz distribution: properties and application, *Journal of Data Science*, 18(4) (2020), 782-794. https://doi.org/10.6339/JDS.202010-18(4)-0012

[24] Osatohanmwen, P., Oyegue, F. O., Ogbonmwan, S. M., A new member from the T-X family of distributions: the Gumbel-Burr XII distribution and its properties, *Sankhya A*, 81 (2019), 298-322. https://doi.org/10.1007/s13171-017-0110-x

[25] Osatohanmwen, P., Efe-Eyefia, E., Oyegue, F. O., Osemwenkhae, J. E., Ogbonmwan, S. M., Afere, B. A., The exponentiated Gumbel-Weibull(logistic) distributions with application to Nigeria's COVID-19 infections data, *Annal of Data Science*, 9 (2022), 909-943. https://doi.org/10.1007/s40745-022-00373-0

[26] Peng, X., Yan, Z., Estimation and application for a new extended Weibull distribution, *Reliab. Eng. Syst. Saf.*, 121 (2014), 34-42. https://doi.org/10.1016/j.ress.2013.07.007

[27] Pollard, J. H., Valkovics, E. J., The Gompertz distribution and its applications, *Genus*, 48 (1992), 15-28. https://doi.org/10.2307/29789100

[28] Rastogi, M. K., Merovci, F., Bayesian estimation for parameters and reliability characteristic of the Weibull Rayleigh distribution, *Journal of King Saud University-Science*, 30 (2018), 472-478. https://doi.org/10.1016/j.jksus.2017.05.008

[29] Shama,M. S., Dey, S., Altun, E., Afify, A. Z., The gamma–Gompertz distribution: theory and applications, *Mathematics and Computers in Simulation*, 193 (2022), 689-712. https://doi.org/10.1016/j.matcom.2021.10.024

[30] Zografos, K., Balakrishnan, N., On families of beta and generalized gamma-generated distributions and associated inference, *Statistical Methodology*, 6(4) (2009), 344–362. https://doi.org/10.1016/j.stamet.2008.12.003

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 259–273 (2024)
DOI:10.31801/cfsuasmas.1316623
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

*Research Article; Received:June 19, 2023; Accepted: October 6, 2023*

# FRACTIONAL APPROACH FOR DIRAC OPERATOR INVOLVING M-TRUNCATED DERIVATIVE

Ahu ERCAN

Department of Mathematics, Firat University, 23119 Elazig, TÜRKİYE

Abstract. In this study, we examine the basic spectral information for systems governed by the Dirac equation with distinct boundary conditions, utilizing a modified form of local derivatives known as M-truncated derivative (MTD). The spectral information discussed includes the representation of solutions in the form of integral equations, the asymptotics vector-valued eigenfunctions and eigenvalues, and their normalized forms, all within the context of the MTD method that incorporates truncated Mittag-Leffler functions. This type of MTD provides the features of integer-order operator theory. Also, by virtue of the parameters $\alpha$ and $\gamma$ , we analyze and compare the solutions with graphs in terms of different potentials, different eigenvalues and different orders. Thus, the aim of this article is to consider spectral structure of Dirac system in frame of M-truncated derivative by proping with visual analysis.

## 1. Introduction

Studies related to several types of differential equations are always attracted by scientists. Because the differential equations have the speciality to model more complex natural systems. Also, the main advantage of fractional derivatives is that it allows us to achieve better results in modeling. Many fractional integral and derivatives like Liouville-Caputo, Riemann-Liouville, Hilfer, Atangana-Baleanu, Caputo-Fabrizio, etc. has been introduced and studied by scientists in [4, 12, 13, 28]. In recently, Khalil et al. has described the local derivative, which is also referred to as the conformable derivative depending on the basic limit definitions of the derivative firstly in [20]. The conformable derivative is very useful in applied mathematics because it shows parallel features to the ordinary derivative like quotient of two functions and the derivative of the product. Also it enables changing of order between $0 < \alpha \leq 1$. Because of this reason, many scientists have applied this

2020 *Mathematics Subject Classification.* 34A08, 34A45, 34L30.

*Keywords.* M-truncated derivative, Dirac operator, spectral data, visual results.

ahuduman24@gmail.com; 0000-0001-6290-2155.

derivative to their studies like [1–3, 5, 6, 8, 9, 16–18, 20]. Proportional $\alpha-$ derivative has similar features with conformable derivative but it differs in its limit definition which presented by Katugampola [19] . It was studied in [6, 7]. In recently, M-fractional derivative containing a Mittag-Leffler function with one parameter has been introduced by Sousa and Oliveira in [25–27]. Benefiting from the definition of these four local derivatives as mentioned above, M-truncated derivative is introduced by Sousa and Oliveira in [25] and it represents a generalization of the other four local derivatives because of the additional parameter inside definition. All other definitions of local derivative such as Katugampola, M-truncated derivative are adaptations of conformable derivative. In these derivatives, basic formulas such as quotient of two function, derivative of the product, chain rule, Leibniz rule etc. shares similarities with conformable derivatives. Spectral analysis of M-truncated derivative for Sturm-Liouville problem and some applications containing truncated Mittag- Leffler function are studied in [24, 29–31].

Dirac equation has a big importance in the modern field of atomic physics. The deepest meaning of the Dirac equation was that any relative definition of a particle necessarily includes not only the wave function of a single particle, but also multiple wave functions representing the potential of other particles. Dirac equation systems have applications in many branches of science like electrical engineering, mathematics and physics. New applications of conclusions and opinions from this topic shed light on future problems such as inverse problems of spectral theory. A first-order matrix linear differential equation whose solution is a 4-component wave function (a spinor) is so important in physics and mathematics [10, 14, 15].

Let $L$ be a matrix operator defined by

$$\begin{pmatrix} V(x)+m & 0 \\ 0 & V(x)-m \end{pmatrix}$$

where $V(x)$ is a potential function, $m$ is the mass of a particle and $y(x)$ denote a two component vector function $y(x)=\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$. Then let's consider the equation

$$\left(B\frac{d}{dx}+L-\lambda I\right)y=0$$

where $\lambda$ is a parameter and

$$B=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\; I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is equivalent to a system

$$\begin{aligned} \frac{dy_2(x,\lambda)}{dx}+(V(x)+m)\,y_1(x,\lambda) &= \lambda y_1(x,\lambda), \\ -\frac{dy_1(x,\lambda)}{dx}+(V(x)-m)\,y_2(x,\lambda) &= \lambda y_2(x,\lambda). \end{aligned}$$

The basic analysis of the spectral structure for the Dirac operator means that finding asymptotic behaviors of the eigenvalues, the vector-valued eigenfunctions and the norming constants and showing the reality of the eigenvalues and the ortogonality of the eigen-vector-functions, etc. This type of analysis is called a direct problem. In this article, the reality of the eigenvalues, and the orthogonality of the eigenfunctions have been shown and the asymptotic formulas for the eigenvalues, eigen-vector-functions, the normalized eigen-vector-functions and the norming constants have been obtained in terms of M-truncated derivative for Dirac system having separated boundary conditions. The studies on the direct and inverse eigenvalue problems can be viewed from [11, 21–23]. Basic spectral features of linear differential operators including conformable derivatives, which inspired our work, were studied by [2,3,5,6,25]. Authors have established an existence and uniqueness theorem for a conformable fractional Dirac system in study [2]. Also they have addresses the existence of a spectral function for a singular conformable Dirac system in [3]. The M-truncated derivative can be employed in studies related to eigenvalue problems and spectral analysis. It is particularly beneficial in such analyses related to Dirac operators based on fractional derivatives. Our primary reason for selecting this particular local derivative is its inclusivity of other local derivatives, owing to the presence of an additional parameter associated with the Mittag-Leffler function. Differing from the literature, our results are more comprehensive compared to other local derivatives due to the presence of the parameter associated with the Mittag-Leffler function.

The layout of this research is presented in the following way: in section 2, we present to definitions and fundamental properties of MTD. In section 3, the spectral structure of Dirac system is studied. This main part of our study involves the reality of the eigenvalues, the orthogonality of the eigenvector-functions, and asymptotic formulas for the vector-valued eigen-functions, the eigenvalues, the norming constants and their normalized forms. Section 4 prerents detailed discussion about simulation anaysis by supporting with the graphs for different values of $\alpha, \gamma$ and $\lambda$. In part 5, the remarks of main results close the paper.

## 2. Preliminaries

In this part, we assign some necessary definitions, theorems and lemmas related to MTD.

**Definition 1.** *[25] The concept of the truncated Mittag-Leffler function with a single parameter is introduced through,*

$$ {}_{i}E_{\gamma}\left(z\right)=\sum_{k=0}^{i}\frac{z^{k}}{\Gamma\left(\gamma k+1\right)}. \tag{1} $$

**Definition 2.** *[25] Let $f : [0,\infty) \to \mathbb{R}$ be a function for $t > 0$, then MTD of $f$ with order $0 < \alpha \leq 1$ is defined by*

$$ {}_iT_M^{\alpha,\gamma} f(t) = \lim_{\varepsilon \to 0} \frac{f\left(t_i E_\gamma\left(\varepsilon t^{-\alpha}\right)\right) - f(t)}{\varepsilon} \tag{2} $$

*where ${}_iE_\gamma(.)$ is the truncated Mittag-Leffler function defined in (1) for $\gamma > 0$.*

**Definition 3.** *[25] The M- integral is defined as follows*

$$ \left({}_MI_a^{\alpha,\gamma} f\right)(\tau) = \int_a^\tau f(t)\, d_{\alpha,\gamma}t = \Gamma(\gamma+1) \int_a^\tau \frac{f(t)}{t^{1-\alpha}} dt $$

*where $\gamma > 0$, $\alpha \in (0,1]$, and $f$ is defined in $(a,\tau]$ .*

**Lemma 1.** *[25] Let $\alpha \in (0,1]$, $\gamma > 0$ and $f, g$ be $\alpha-$differentiable at a point $t > 0$. Then,*

1. ${}_iT_M^{\alpha,\gamma}(af + bg) = a_i T_M^{\alpha,\gamma} f + b_i T_M^{\alpha,\gamma}(g)$ *for* $a, b \in \mathbb{R}$;
2. ${}_iT_M^{\alpha,\gamma}(t^n) = nt^{n-\alpha}$ *for all* $n \in \mathbb{R}$;
3. ${}_iT_M^{\alpha,\gamma}(fg) = f_i T_M^{\alpha,\gamma}(g) + g_i T_M^{\alpha,\gamma}(f)$;
4. ${}_iT_M^{\alpha,\gamma}\left(\dfrac{f}{g}\right) = \dfrac{g_i T_M^{\alpha,\gamma}(f) - f_i T_M^{\alpha,\gamma}(g)}{g^2}$;
5. ${}_iT_M^{\alpha,\gamma}(c) = 0$, *$c$ is a constant*;
6. ${}_iT_M^{\alpha,\gamma}(fog)(t) = f'(g(t))_i\, T_M^{\alpha,\gamma} g(t)$, *for $f$ is differentiable at $g(t)$*;
7. *If $f$ is differentiable, thus* ${}_iT_M^{\alpha,\gamma}(f)(t) = \frac{t^{1-\alpha}}{\Gamma(\gamma+1)} \frac{df(t)}{dt}$.

**Theorem 1.** *Assume that $f$,$g : [a,b] \to \mathbb{R}$ and $fg$ is differentiable. Then, we have*

$$ \Gamma(\gamma+1) \int_a^b s^{\alpha-1} f(s)_i\, T_M^{\alpha,\gamma} g(s)\, ds = f(t)\, g(t)\big|_a^b - \Gamma(\gamma+1) \int_a^b s^{\alpha-1} g(s)_i\, T_M^{\alpha,\gamma} f(s)\, ds. $$

The $L^2_{\alpha,\gamma}(0,\pi)$ is a Hilbert space with inner product

$$ (y,z) = \int_0^\pi y^T(x,\lambda_1)\, z(x,\lambda_2)\, d_{\alpha,\gamma}x, $$

where $y^T = (y_1, y_2)$ and $d_{\alpha,\gamma}x = \Gamma(\gamma+1)\, x^{\alpha-1} dx$.

In the next section, we will analyze the Dirac systems in terms of the MTD and we are able to obtain general representations of solutions that involve parameters $\alpha$ and $\gamma$. Additionally, using the MTD approach, we can also present asymptotic formulas for eigen-vector-functions and eigenvalues. The general results which found in main results correspond to classical Dirac systems when $\alpha = 1$ and $\gamma = 1$.

## 3. Main Results

Let us consider Dirac system containing M-tuncated derivative as follows:

$$\begin{array}{r} {}_iT_M^{\alpha,\gamma} y_2(x) + p(x) y_1(x) = \lambda y_1(x), 0 < \alpha \leq 1, x \in [0,\pi] \\ - {}_iT_M^{\alpha,\gamma} y_1(x) + r(x) y_2(x) = \lambda y_2(x), \end{array} \tag{3}$$

where ${}_iT_M^{\alpha,\gamma}$ is MTD operator, $p(x)$ and $r(x)$ are continuous and real-valued functions on $[0,\pi]$, $y(x)$ is $2\alpha-$continuously differentiable on $[0,\pi]$, ${}_iT_M^{\alpha,\gamma} y(x)$ is continuous on $[0,\pi]$. Deal with the system (3) subject to boundary conditions

$$y_1(0)\sin a + y_2(0)\cos a = 0, \tag{4}$$

$$y_1(\pi)\sin b + y_2(\pi)\cos b = 0, \tag{5}$$

where $a$ and $b$ are real constants.

Let symbolize the solution of (3) by $\varphi(x,\lambda) = \begin{pmatrix} \varphi_1(x,\lambda) \\ \varphi_2(x,\lambda) \end{pmatrix}$ satisfying the following initial conditions

$$\varphi_1(0,\lambda) = \cos a,\ \varphi_2(0,\lambda) = -\sin a. \tag{6}$$

**Theorem 2.** *Let $\lambda_1$ and $\lambda_2$ be two distinct eigenvalues of the problem (3) − (5). Then the corresponding eigen-vector-functions $y(x,\lambda_1)$ and $z(x,\lambda_2)$ are orthogonal on $L^2_{\alpha,\gamma}(0,\pi)$ Hilbert space, that is,*

$$\int_0^\pi y^T(x,\lambda_1) z(x,\lambda_2)\, d_{\alpha,\gamma}x = 0, \qquad \lambda_1 \neq \lambda_2. \tag{7}$$

*Proof.* Since the $y(x,\lambda_1)$ and $z(x,\lambda_2)$ satisfy the system (3), we have

$$\begin{aligned} {}_iT_M^{\alpha,\gamma} y_2(x,\lambda_1) + p(x) y_1(x,\lambda_1) &= \lambda_1 y_1(x,\lambda_1), \\ -{}_iT_M^{\alpha,\gamma} y_1(x,\lambda_1) + r(x) y_2(x,\lambda_1) &= \lambda_1 y_2(x,\lambda_1), \\ {}_iT_M^{\alpha,\gamma} z_2(x,\lambda_2) + p(x) z_1(x,\lambda_2) &= \lambda_2 z_1(x,\lambda_2), \\ -{}_iT_M^{\alpha,\gamma} z_1(x,\lambda_2) + r(x) z_2(x,\lambda_2) &= \lambda_2 z_2(x,\lambda_2). \end{aligned}$$

If we multiply these equations by $z_1(x,\lambda_2)$, $z_2(x,\lambda_2)$, $-y_1(x,\lambda_1)$ and $-y_2(x,\lambda_1)$, respectively, and sum together, we get

$$\begin{aligned} &(\lambda_1 - \lambda_2)(z_1(x,\lambda_2) y_1(x,\lambda_1) + z_2(x,\lambda_2) y_2(x,\lambda_1)) \\ &\qquad = {}_iT_M^{\alpha,\gamma} \{ z_1(x,\lambda_2) y_2(x,\lambda_1) - z_2(x,\lambda_2) y_1(x,\lambda_1) \}. \end{aligned}$$

Applying the integral ${}_M I_0^{\alpha,\gamma}$ from 0 to $\pi$ on both side of the last equality, one can find

$$(\lambda_1 - \lambda_2) \int_0^\pi y^T(x,\lambda_1) z(x,\lambda_2)\, d_{\alpha,\gamma}x = \left( z_1(x,\lambda_2) y_2(x,\lambda_1) - z_2(x,\lambda_2) y_1(x,\lambda_1) \right)\Big|_0^\pi.$$

By virtue of boundary conditions (4) and (5) , one can obtain

$$(\lambda_1 - \lambda_2) \int_0^{\pi} y^T_{\lambda_1}(x)\, z_{\lambda_2}(x)\, d_{\alpha,\gamma}x = 0.$$

□

**Theorem 3.** *All eigenvalues of the problem defined by* (3) − (5) *are real.*

*Proof.* Let $\lambda_1 = a + ib$ be an eigenvalue with eigenfunction $y(x, \lambda_1)$. Since $p(x)$ and $r(x)$ real-valued functions, $\lambda_2 = \bar{\lambda}_1 = a - ib$ is also an eigenvalue with the eigenfunctions $\bar{y}(x, \lambda_2)$. By considering Theorem 2, we have

$$\begin{aligned}\left(\lambda - \bar{\lambda}\right) \int_0^{\pi} y^T(x, \lambda_1)\, \bar{y}(x, \lambda_2)\, d_{\alpha,\gamma}x &= 0,\\ \left(\lambda - \bar{\lambda}\right) \int_0^{\pi} \left\{y_1^2(x, \lambda_1) + y_2^2(x, \lambda_1)\right\} d_{\alpha,\gamma}x &= 0,\end{aligned}$$

and since $y(x) \neq 0$ , we have $\lambda = \bar{\lambda}$. □

**Theorem 4.** *The solution of the system* (3) *satisfying the initial conditions* (6) *provides the following integral equation system,*

$$\begin{aligned}\varphi_1(x, \lambda) &= \cos\left(\frac{\lambda\Gamma(\gamma+1)\, x^{\alpha}}{\alpha} - a\right) - \int_0^x \sin\left(\lambda\Gamma(\gamma+1)\left(\frac{t^{\alpha} - x^{\alpha}}{\alpha}\right)\right) p(t)\, \varphi_1(t, \lambda)\, d_{\alpha,\gamma}t\\ &\quad + \int_0^x \cos\left(\lambda\Gamma(\gamma+1)\left(\frac{t^{\alpha} - x^{\alpha}}{\alpha}\right)\right) r(t)\, \varphi_2(t, \lambda)\, d_{\alpha,\gamma}t, \qquad (8)\end{aligned}$$

$$\begin{aligned}\varphi_2(x, \lambda) &= \sin\left(\frac{\lambda\Gamma(\gamma+1)\, x^{\alpha}}{\alpha} - a\right) - \int_0^x \cos\left(\lambda\Gamma(\gamma+1)\left(\frac{t^{\alpha} - x^{\alpha}}{\alpha}\right)\right) p(t)\, \varphi_1(t, \lambda)\, d_{\alpha,\gamma}t\\ &\quad - \int_0^x \sin\left(\lambda\Gamma(\gamma+1)\left(\frac{t^{\alpha} - x^{\alpha}}{\alpha}\right)\right) r(t)\, \varphi_2(t, \lambda)\, d_{\alpha,\gamma}t. \qquad (9)\end{aligned}$$

*Proof.* By using the variation of parameters method given in [17], we express the representation of the solutions as follow:

$$\begin{aligned}\varphi_1(x, \lambda) &= -c_1(x)\, y_1(x) + c_2(x)\, y_2(x)\\ \varphi_2(x, \lambda) &= c_1(x)\, y_2(x) + c_2(x)\, y_1(x)\end{aligned}$$

where

$$\begin{aligned}
c_1(x) &= -\int_0^x \left(p(t)\, y_1(t)\, \varphi_1(t,\lambda) - r(t)\, y_2(t)\, \varphi_2(t,\lambda)\right) d_{\alpha,\gamma} t + c_1, \\
c_2(x) &= \int_0^x \left(p(t)\, y_1(t)\, \varphi_2(t,\lambda) + r(t)\, y_2(t)\, \varphi_1(t,\lambda)\right) d_{\alpha,\gamma} t + c_2, \\
y_1(x) &= \sin \frac{\lambda \Gamma(\gamma+1)\, x^\alpha}{\alpha} \text{ and } y_2(x) = \cos \frac{\lambda \Gamma(\gamma+1)\, x^\alpha}{\alpha}.
\end{aligned}$$

If we benefit from the initial conditions (6), it can be easily seen (8) and (9). □

**Theorem 5.** *As* $|\lambda| \to \infty$, *the estimates are provided as follows:*

$$\varphi_1(x,\lambda) = \cos\left(\xi(x,\lambda) - a\right) + O\left(\frac{1}{\lambda}\right), \tag{10}$$

$$\varphi_2(x,\lambda) = \sin\left(\xi(x,\lambda) - a\right) + O\left(\frac{1}{\lambda}\right), \tag{11}$$

$$\frac{\partial \varphi_1(x,\lambda)}{\partial \lambda} = -\Gamma(\gamma+1)\frac{x^\alpha}{\alpha} \sin\left(\xi(x,\lambda) - a\right) + O(1), \tag{12}$$

$$\frac{\partial \varphi_2(x,\lambda)}{\partial \lambda} = \Gamma(\gamma+1)\frac{x^\alpha}{\alpha} \cos\left(\xi(x,\lambda) - a\right) + O(1), \tag{13}$$

*for* $0 \le x \le \pi$ *where*

$$\xi(x,\lambda) = \frac{\lambda \Gamma(\gamma+1)}{\alpha} x^\alpha + \frac{1}{2}\int_0^x \left(p(t) + r(t)\right) d_{\alpha,\gamma} t. \tag{14}$$

*Proof.* Let us introduce by $\varphi(x,\lambda)$ the solution of the system (3) satisfying the initial conditions (6). If the problem (3), (6) is considered for $p(x) = r(x) \equiv 0$, the solution of this problem stand for $\psi(x,\lambda)$. Therby, one can easily obtain that

$$\psi_1(x,\lambda) = \cos\left(\frac{\lambda \Gamma(\gamma+1)}{\alpha} x^\alpha - a\right), \tag{15}$$

$$\psi_2(x,\lambda) = \sin\left(\frac{\lambda \Gamma(\gamma+1)}{\alpha} x^\alpha - a\right). \tag{16}$$

If the solution of the problem (3), (6) is applied to the transformation matrix operator, we have [22]

$$\varphi(x,\lambda) = R(x)\, \psi(x,\lambda) + \int_0^x K(x,s)\, \psi(s,\lambda)\, d_{a,\gamma} s \tag{17}$$

in here $R(x)$ and $K(x,s)$ are matrices of second-order that can be continuously differentiated twice,

$$R(x)=\begin{pmatrix}\gamma(x) & \beta(x)\\ -\beta(x) & \gamma(x)\end{pmatrix} \tag{18}$$

and $\gamma(x)$ and $\beta(x)$ can be computed as below:

$$\begin{aligned}\gamma(x) &= \cos\left(\frac{1}{2}\int_0^x (p(t)+r(t))\, d_{\alpha,\gamma}t\right),\\ \beta(x) &= -\sin\left(\frac{1}{2}\int_0^x (p(t)+r(t))\, d_{\alpha,\gamma}t\right),\end{aligned}$$

for $\kappa=1$. Thereby considering by (17) and (18), we find the formulas

$$\begin{aligned}\varphi_1(x,\lambda) &= \cos(\xi(x,\lambda)-a)+\int_0^x K_{11}(x,s)\cos\left(\tfrac{\lambda\Gamma(\gamma+1)}{\alpha}s^\alpha-a\right)d_{\alpha,\gamma}s\\ &\quad+\int_0^x K_{12}(x,s)\sin\left(\tfrac{\lambda\Gamma(\gamma+1)}{\alpha}s^\alpha-a\right)d_{\alpha,\gamma}s\\ \varphi_2(x,\lambda) &= \sin(\xi(x,\lambda)-a)+\int_0^x K_{11}(x,s)\cos\left(\tfrac{\lambda\Gamma(\gamma+1)}{\alpha}s^\alpha-a\right)d_{\alpha,\gamma}s\\ &\quad+\int_0^x K_{12}(x,s)\sin\left(\tfrac{\lambda\Gamma(\gamma+1)}{\alpha}s^\alpha-a\right)d_{\alpha,\gamma}s\end{aligned} \tag{19}$$

where $K_{ij}(x,s)$ are the components of the matrix $K(x,s)$ for $i,j=1,2$ from (17). To gain the asymptotics in (10) and (11), it is enough to integrate by parts the integrals including in (19), because of the differentiability of the functions $K_{ij}(x,s)$. In a similar manner, if we differentiate (19) in terms of $\lambda$, we obtain the asymptotics in (12) and (13).

Additionally, we demonstrate the asymptotic behaviors of the eigenvalues using the MTD approach, enabling us to observe how the formulas change as $\alpha$ and $\gamma$ vary.

□

**Theorem 6.** *The eigenvalues of the problem outlined by equations (3) to (5) in their asymptotic forms are given as follows:*

$$\lambda_{\pm n}=\frac{\alpha}{\Gamma(\gamma+1)\pi^\alpha}(\pm n\pi+c)+O\left(\frac{1}{n}\right), \tag{20}$$

*where*

$$c=a-b-\frac{1}{2}\int_0^x (p(t)+r(t))\, d_{\alpha,\gamma}t.$$

*Proof.* The eigenvalues of the given problem overlap with the roots of the characteristic function

$$\triangle(\lambda) = \varphi_1(\pi, \lambda)\sin b + \varphi_2(\pi, \lambda)\cos b.$$

If we put asymptotics of the eigen-vector-functions $\varphi_1(\pi, \lambda)$ and $\varphi_2(\pi, \lambda)$ from the estimates (11) into $\triangle(\lambda)$, we obtain

$$\cos(\xi(x, \lambda) - a)\sin b + \sin(\xi(x, \lambda) - a)\cos b + O\left(\lambda^{-1}\right) = 0.$$

After some calculation with the aid of trigonometric functions, we reach

$$\sin\left(\frac{\lambda\Gamma(\gamma+1)\pi^{\alpha}}{\alpha} + c\right) + O\left(\lambda^{-1}\right) = 0. \tag{21}$$

It is clearly seen that the equation (21), for large $|\lambda|$, has solutions in the form

$$\frac{\lambda\Gamma(\gamma+1)\pi^{\alpha}}{\alpha} + c = n\pi + \delta_n,$$

it is obvious that $sin\delta_n = O\left(n^{-1}\right)$, i.e. $\delta_n = O\left(n^{-1}\right)$. Therefore the asymptotic formula for eigenvalues is obtained in (20). □

**Theorem 7.** *The asymptotic formula for the norming constants is given by*

$$\rho_n = \sqrt{\frac{\pi^{\alpha}\Gamma(\gamma+1)}{\alpha}} + O\left(\frac{1}{n}\right).$$

*Proof.* By utilizing the asymptotic formula for eigenvalues given in (20), we can reobtain the asymptotics for eigen-vector-functions as follows:

$$\varphi_1(x, \lambda_n) = \cos(\xi_n - a) + O\left(n^{-1}\right) \tag{22}$$

$$\varphi_2(x, \lambda_n) = \sin(\xi_n - a) + O\left(n^{-1}\right) \tag{23}$$

where $\xi(x, \lambda_n) = \xi_n = \frac{\lambda_n\Gamma(\gamma+1)}{\alpha}x^{\alpha} + \frac{1}{2}\int\limits_0^x (p(t) + r(t))\, d_{\alpha,\gamma}t.$

To reach at the asymptotic expression for the norming constants, take in consideration the following integral

$$\begin{aligned} \rho_n^2 &= \int\limits_0^{\pi} \left\{\varphi_1^2(x, \lambda_n) + \varphi_2^2(x, \lambda_n)\right\} d_{\alpha,\gamma}x, \\ &= \int\limits_0^{\pi} \left\{\cos^2(\xi_n - a)\, x + \sin^2(\xi_n - a)\right\} d_{\alpha,\gamma}x + O\left(\frac{1}{n}\right), \\ &= \frac{\pi^{\alpha}\Gamma(\gamma+1)}{\alpha} + O\left(\frac{1}{n}\right). \end{aligned}$$

Hence, the proof is completed. □

**Theorem 8.** *Asymptotic expression of the normalized vector-valued eigenfunctions is given in the form,*

$$\widetilde{\varphi}(x,\lambda_n)=\begin{pmatrix}\sqrt{\dfrac{\alpha}{\pi^{\alpha}\Gamma(\gamma+1)}}\cos(\xi_n-a)+O\left(n^{-1}\right)\\ \sqrt{\dfrac{\alpha}{\pi^{\alpha}\Gamma(\gamma+1)}}\sin(\xi_n-a)+O\left(n^{-1}\right)\end{pmatrix}.$$

*Proof.* The proof can be easily seen with the help of Theorem 7.

□

## 4. Illustrative Results

In the current section, the representation of the solutions $y_1(x)$ and $y_2(x)$ for Dirac equation is offered by means of MTD under different orders of $\alpha$, different potentials and different values of $\lambda$. If the values of $\alpha$ increases while the value of $\gamma$ is constant, Figure 1 (a) and (b) have showed a right-sided shift for the solution curves. If the values of $\gamma$ increases while the value of $\alpha$ is constant, Figure 2 (a) and (b) have showed a smaller right-sided shift in the solutions than Figure 1. Figure 3 demonstrates the acting for the solutions when $q=0,1,2,3$. Also, the roots of the characteristic function are computed detailed under different values of $\alpha$ in Table 1. If one pays attention to Figure 4 (a), (b) and (c), it can be easily seen that the value of $\alpha$ increases which is equal to 0.1, 0.3, 0.5, respectively, the frequency of the oscillation interval decreases. That is as the value of $\alpha$ decreases the number of eigenvalues of considered problem increases. Thereby $\alpha$ is changed the mobility of the solutions curves increases and it provides important advantage in applications of spectral analysis. Lastly, in Figure 5 (a) and (b), the graphs of eigenfunctions corresponding to different eigenvalues were plotted according to the changing the value of $\alpha$ and $\gamma$, respectively. Also Figure 5 (b) shows that eigenfunctions overlap for different values of $\gamma$. The main purpose in drawing graphs with different values is that one can observe the behavior of representations of solutions curves for Dirac equation in light of MTD. Also the approximate eigenvalues are given for different orders of $\alpha$ and $\gamma$ in Table 1. Assume that $a=1$, $b=\frac{\pi}{4}$ for all figures.

Table 1. The roots of $\triangle(\lambda)$ for $x=\pi$

| $\alpha$ | $\lambda_1$ | $\lambda_2$ | $\gamma$ | $\lambda_1$ | $\lambda_2$ |
|---|---|---|---|---|---|
| 0.1 | $-0.2945$ | 0.0215 | 0.1 | $-0.8679$ | 0.0636 |
| 0.3 | $-0.7028$ | 0.0515 | 0.3 | $-0.9200$ | 0.0674 |
| 0.5 | $-0.9316$ | 0.0683 | 0.5 | $-0.9316$ | 0.0683 |
| 0.7 | $-1.0374$ | 0.0760 | 0.7 | $-0.9087$ | 0.0666 |
| 0.9 | $-1.0609$ | 0.0777 | 0.9 | $-0.8585$ | 0.0629 |
| 0.99 | $-1.0527$ | 0.0771 | 0.99 | $-0.8291$ | 0.0607 |

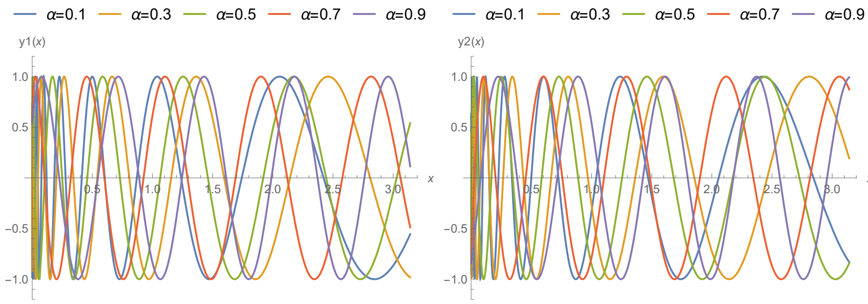



FIGURE 1. Comparative analysis for different orders of $\alpha, \lambda = 10, \quad p(x) = r(x) = 0, \gamma = 0.5$

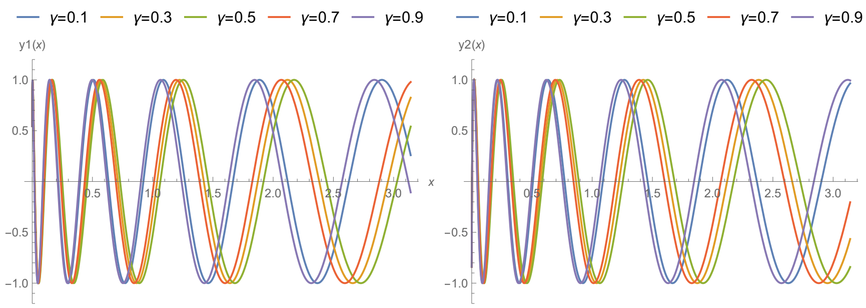



FIGURE 2. Comparative analysis for different orders of $\gamma, \ \lambda = 10, \quad p(x) = r(x) = 0, \alpha = 0.5.$

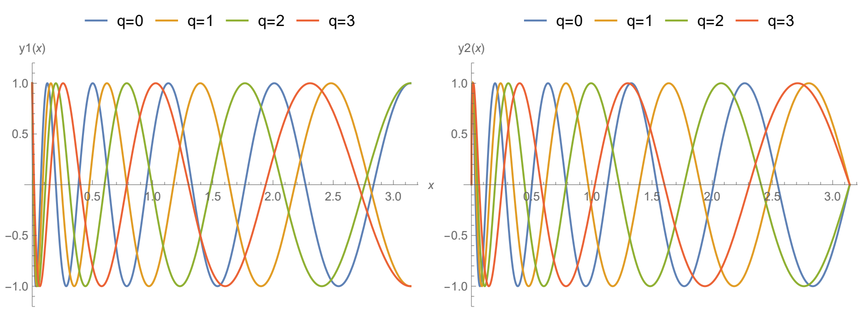



FIGURE 3. Comparative analysis for different values of the potentials, $\lambda = 10, \quad p(x) = r(x) = q, \gamma = 0.5, \alpha = 0.5$

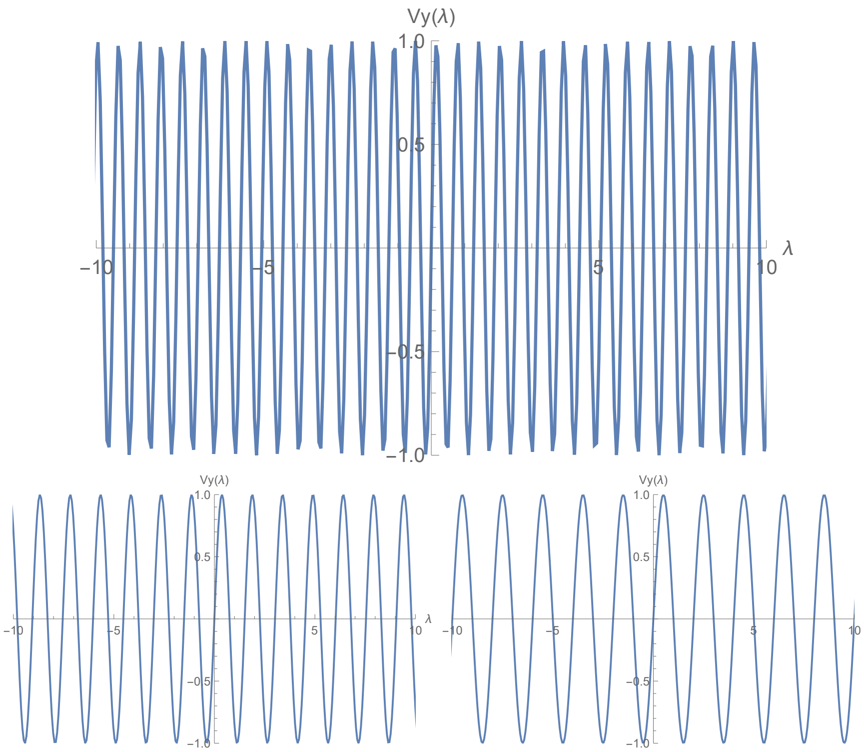



FIGURE 4. Comparisons of the roots of the characteristic function under different orders, $\lambda = 10, \quad p(x) = r(x) = 0, \gamma = 0.5$

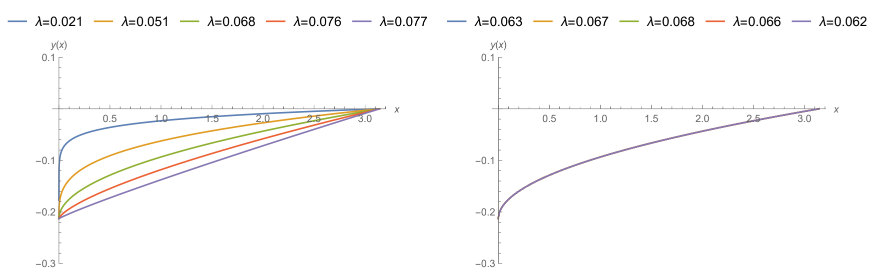



FIGURE 5. Comparisons of the eigenfunctions benefit from Table 1 under different order of $\alpha$ and $\gamma$

## 5. Conclusion

In here, we analyzed spectral structure of Dirac systems which has been studied by Levitan and Sargsjan [22] for integer order case in light of MTD. For one-dimensional Dirac operator in sense of MTD, its fundamental spectral theory is given systematically and behaviours of eigen-vector-functions are observed with graphics under different orders, potentials and eigenvalues. We obtain the representations of the solutions and asymptotics for the norming constants, the eigenvalues, eigen-vector-functions, and the normalized eigen-vector-functions. To gain these important results, certain calculations like variation of parameters method, Leibniz rule, and so forth are made in sense of MTD. The most important advantage of MTD is that this definition offers the features of the integer-order calculus. MTD give us the change to examine derivatives of infinite order . Also, we give comparative analysis of the solutions by graphs with different orders $\alpha$ and $\gamma$, different eigenvalues and different potentials. Thereby, we observe the behaviours of the mobility of the solutions. Thus, we have supplied a large amount of spectral theory for the considered problem in terms of MTD.

**Author Contribution Statements** As the sole author, the work is entirely the author's own.

**Declaration of Competing Interests** The author declares that there is no competing interest regarding the publication of this paper.

**Acknowledgements** The author would like to thank the editor and the reviewers for their valuable comments and suggestions which resulted in substantial improvement in the presentation of the paper.

## References

[1] Abdeljawad, T., On conformable fractional calculus, *J. Comput. Appl. Math.*, 279 (2015), 57-66. https://doi.org/10.1016/j.cam.2014.10.016

[2] Allahverdiev, B. P., Tuna, H. One-dimensional conformable fractional Dirac system, *Bol. Soc. Mat. Mex.*, 26(1) (2020), 121-146. https://doi.org/10.1007/s40590-019-00235-5

[3] Allahverdiev, B. P., Tuna, H., Spectral expansion for singular conformable fractional Dirac systems, *Rend. Circ. Mat. Palermo*, 69(3) (2020), 1359–1372. https://doi.org/10.1007/s12215-019-00476-3

[4] Allahverdiev, B. P., Tuna, H., Regular fractional Dirac type systems, *Facta Univ. Ser. Math. Inform.*, 36(3) (2021), 489-499. https://doi.org/10.22190/FUMI200318036A

[5] Al-Refai, M., Abdeljawad, T., Fundamental results of conformable Sturm-Liouville eigenvalue problems, *Complexity*, (2017), 1-7. https://doi.org/10.1155/2017/3720471

[6] Anderson, D. R., Ulness, D. J., Newly defined conformable derivatives, *Adv. Dyn. Syst. Appl.*, 10(2) (2015), 109-137.

[7] Anderson, D. R., Ulness, D. J., Properties of the Katugampola fractional derivative with potential application in quantum mechanics, *J. Math. Phys.*, 56(6) (2015), 063502. https://doi.org/10.1063/1.4922018.

[8] Atangana, A., Baleanu, D., Alsaedi, A., New properties of conformable derivative, *Open Math.*, 13(1) (2015), 889-898. https://doi.org/10.1515/math-2015-0081
[9] Baleanu, D., Jarad, F., Uğurlu, E., Singular conformable sequential differential equations with distributional potentials. *Quaest. Math.*, 42(3) (2019), 277-287. https://doi.org/10.2989/16073606.2018.1445134
[10] Bjorken, J. D., Drell, S. D., Relativistic Quantum Mechanics, McGraw-Hill, New York, 1964.
[11] Ercan, A., Panakhov, E. S., Stability of the spectral problem for Dirac operators, *Aip Conf. Proc.*, 1738 (2016), 290010.
[12] Ercan, A., On the fractional Dirac systems with non-singular operators, *Thermal Sci.*, 23(6) (2019), 2159-2168. https://doi.org/10.2298/TSCI190810405E
[13] Ercan, A., Bas, E., Regular spectral problem for conformable Dirac system with simulation analysis, *J. Interdiscip. Math.*, 24(6) (2021), 1497-1514. https://doi.org/10.1080/09720502.2020.1827507
[14] Greiner, W., Miller, B., Rafelski, J., Quantum Electrodynamics of Strong Fields, Springer, Berlin, 1985.
[15] Greiner, W., Relativistic Quantum Mechanics: Wave Equations, Springer, Berlin, 1994.
[16] Hammad, M. A., Khalil, R., Abel's formula and wronskian for conformable fractional differential equations, *Int. J. Differ. Equ. Appl.*, 13(3) (2014), 177-183. http://dx.doi.org/10.12732/ijdea.v13i3.1753
[17] Horani, M. A., Hammad, M. A., Khalil, R., Variation of parameters for local fractional nonhomogenous linear-differential equations, *J. Math. Comput. Sci.*, 16 (2016), 147-153.
[18] Jarad, F., Uğurlu, E., Abdeljawad, T., Baleanu, D., On a new class of fractional operators, *Adv. Difference Equ.*, 247 (2017), 16. https://doi.org/10.1186/s13662-017-1306-z
[19] Katugampola, U. N., A new fractional derivative with classical properties, *arXiv preprint*, (2014), arXiv:1410.6535v2.
[20] Khalil, R., Horani, M. A., Yousef, A., Sababheh, M., A new definition of fractional derivative, *J. Comput. Appl. Math.*, 264 (2014), 65-70. https://doi.org/10.1016/j.cam.2014.01.002
[21] Levitan, B. M., Sargsjan, I. S., Introduction to Spectral Theory: Selfadjoint Ordinary Differential Operators, American Mathematical Society, Providence, R.I., 1975.
[22] Levitan, B. M., Sargsjan, I. S., Sturm-Liouville and Dirac Operators, Kluwer Academic Publishers Group, Dordrecht, 1991.
[23] Mamedov, K. R., Akcay, O., Inverse problem for a class of Dirac operators by the Weyl function, *Dynam. Systems Appl.*, 26(1) (2017), 183-195.
[24] Ozarslan, R., Bas, E., Baleanu, D., Acay, B., Fractional physical problems including wind-influenced projectile motion with Mittag-Leffler kernel, *AIMS Math.*, 5(1) (2019), 467-481. https://doi.org/10.3934/math.2020031
[25] Vanterler da C. Sousa J., Capelas de Oliveira, E., A new truncated M-fractional derivative type unifying some fractional derivative types with classical properties, *Int. J. Anal. Appl.*, 16(1) (2018), 83-96.
[26] Vanterler da C. Sousa J., Capelas de Oliveira, E., Leibniz type rule: $\psi$-Hilfer fractional operator, *Nonlinear Sci. Numer. Simul.*, 77 (2019), 305–311. https://doi.org/10.1016/j.cnsns.2019.05.003
[27] Vanterler da C. Sousa J., Capelas de Oliveira, E., Mittag–Leffler functions and the truncated V-fractional derivative, *Mediterr. J. Math.*, 14(6) (2017), 244. https://doi.org/10.1007/s00009-017-1046-z
[28] Yalçınkaya, Y., Some fractional Dirac systems, *Turkish J. Math.*, 47(1) (2023), 110-122. https://doi.org/10.55730/1300-0098.3349
[29] Yusuf, A., Inc, M., Aliyu, A. I., Two-strain epidemic model involving fractional derivative with Mittag-Leffler kernel, *Chaos*, 28(12) (2018), 123121, 11. https://doi.org/10.1063/1.5074084

[30] Yusuf, A., Sulaiman, T. A., Mirzazadeh, M., Hosseini, K., M-truncated optical solitons to a nonlinear Schrödinger equation describing the pulse propagation through a two-mode optical fiber, *Opt. Quant. Electron*, 53(10) (2021), 558. https://doi.org/10.1007/s11082-021-03221-2

[31] Yusuf, A., Sulaiman, T. A., Inc, M., Abdel-Khalek, S., Mahmoud, K. H., $M$-truncated optical soliton and their characteristics to a nonlinear equation governing the certain instabilities of modulated wave trains, *AIMS Math.*, 6(9) (2021), 9207–9221. https://doi.org/10.3934/math.2021535

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 274–284 (2024)
DOI:10.31801/cfsuasmas.1313970
ISSN 1303-5991 E-ISSN 2618-6470

*Research Article; Received: June 13, 2023; Accepted: October 31, 2023*

# APPLICATION OF THE GKM TO SOME NONLINEAR PARTIAL EQUATIONS

Seyma TULUCE DEMIRAY,[1] Ugur BAYRAKCI[2] and Vehpi YILDIRIM[3]

[1,2]Department of Mathematics, Osmaniye Korkut Ata University, Osmaniye, TÜRKİYE
[3]Department of Mathematics, Erzurum Technical University, Erzurum, TÜRKİYE

Abstract. In this manuscript, the strain wave equation, which plays an important role in describing different types of wave propagation in microstructured solids and the (2+1) dimensional Bogoyavlensky Konopelchenko equation, is defined in fluid mechanics as the interaction of a Riemann wave propagating along the y-axis and a long wave propagating along the x-axis, were studied. The generalized Kudryashov method (GKM), which is one of the solution methods of partial differential equations, was applied to these equations for the first time. Thus, a series of solutions of these equations were obtained. These found solutions were compared with other solutions. It was seen that these solutions were not shown before and were presented for the first time in this study. The new solutions of these equations might have been useful in understanding the phenomena in which waves are governed by these equations. In addition, 2D and 3D graphs of these solutions were constructed by assigning certain values and ranges to them.

## 1. Introduction

Nonlinear evolution equations (NLEEs) have been utilized to make mathematical models of encountered problems in various scientific circles. A number of solution methods have been developed by various scientists to solve NLEEs, which have very important areas of use [1–10]. In this study, one of these methods, GKM, has been taken into consideration and applied to the strain wave and (2+1)-dimensional Bogoyavlensky-Konopelchenko (BK) equations.

2020 *Mathematics Subject Classification.* 35A25,35C07,35C08.

*Keywords.* Generalized Kudryashov method, strain wave equation, (2+1)-dimensional Bogoyavlensky-Konopelchenko equation.

[1] seymatuluce@gmail.com; 0000-0002-8027-7290;
[2] ubayrakci42@gmail.com - Corresponding author; 0000-0002-1765-2318
[3] vehpi.yildirim@erzurum.edu.tr; 0000-0003-3837-4756

.

Strain wave equation is given as [11]:

$$u_{tt} - u_{xx} - \epsilon\alpha_1(u^2)_{xx} - k\alpha_2 u_{xxt} + \delta\alpha_3 u_{xxxx} \\ -(\delta\alpha_4 + k^2\alpha_7)u_{xxtt} + k\delta(\alpha_5 u_{xxxxt} + \alpha_6 u_{xxttt}) = 0, \tag{1}$$

where $u(x,t)$ is the micro-strain wave function. $\epsilon$ indicates elastic strain, $\delta$ shows the elastic stresses and the rate between the wavelength and size of the microstructure, $k$ reflects the dissipative effect and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$ are arbitrary constants. Assuming $\delta = O(\epsilon)$ on Eq. (1), an equilibration takes place between dispersion and nonlinearity. If $k = 0$ is selected in this equation, the undistributed state of the micro-stress wave is obtained. In this way, the following equation for the bi-dispersion in microstructured solids is obtained [12–16]:

$$u_{tt} - u_{xx} - \epsilon(\alpha_1(u^2)_{xx} - \alpha_3 u_{xxxx} + \alpha_4 u_{xxtt}) = 0. \tag{2}$$

Recently, the solutions of strain wave equation investigated by various researchers with different methods. Seadawy et al. used the modified extended mapping method for strain wave equation [11]. Ayati et al. applied the functional variable method and Kudryashov method to strain wave equation [12]. Arshad et al. practiced the modified direct algebraic method to strain wave equation [13]. Gao et al. used the F-expansion method for strain wave equation [14]. Irshad et al. practiced the generalized Jacobi elliptic function method to strain wave equation [15]. Kumar et al. used the generalized exponential rational function method for strain wave equation [16]. Joseph implemented the new rational F-expansion method to strain wave equation [17].

(2+1)-dimensional BK equation is given as [18]:

$$u_{xt} + h_1 u_{xxxx} + h_2 u_{xxxy} + h_3 u_{xx} u_x + h_4(u_{xy} u_x + u_{xx} u_y) = 0, \tag{3}$$

where $h_1, h_2, h_3$ and $h_4$ are arbitrary constants. If $h_1 = a, h_2 = \beta, h_3 = 6a, h_4 = 4\beta$ values are selected for the $h_1, h_2, h_3, h_4$ constants in Eq. (3), Eq. (3) can be written as.

$$u_{xt} + \alpha u_{xxxx} + \beta u_{xxxy} + 6\alpha u_{xx} u_x + 4\beta u_{xy} u_x + 4\beta u_{xx} u_y = 0. \tag{4}$$

The resulting Eq. (4) is handled as a two-dimensional generalization of the KdV equation, and under favorable conditions, it can be converted to the KdV equation [19]. This equation provides the Calogero-Bogoyavlensky-Schiff equation for $\alpha = 0$ and is also defined as the interplay of a Riemann wave spreading along the y-axis and a long wave spreading along the x-axis in fluid mechanics [20, 21]. For Eq. (4) $u_y = v_x$ is transformed and integrated, and the following equation is found:

$$u_t + \alpha u_{xxx} + \beta v_{xxx} + 3\alpha(u_x)^2 + 4\beta u_x v_x = 0. \tag{5}$$

Accordingly, Eq. (4) can be expressed as a system as follows:

$$u_t + \alpha u_{xxx} + \beta v_{xxx} + 3\alpha(u_x)^2 + 4\beta u_x v_x = 0,$$
$$u_y = v_x. \tag{6}$$

When looking at the past works on (2+1)-dimensional BK equation. Zhou et al. gave based on its bilinear form, the N th-order breather solutions of the (2+1)-dimensional generalized BK equation [21]. Ray got infinitesimal generators of (2+1)-dimensional BK equation by using Lie group analysis method and investigated symmetry analysis and similarity reduction of (2+1)-dimensional BK equation [18,22]. Chen and Ma obtained the symbolic solutions of the (2+1)-dimensional BK equation that own a Hirota bilinear form [23].

The purpose of this article is to detect soliton solutions of strain wave equation and (2+1)-dimensional BK equation using GKM [24–27]. First of all, the features of GKM, which is the method we used in our study, are explained. Subsequently, some soliton solutions of the strain wave equation and (2+1)-dimensional BK equation were found using this method.

## 2. Analysis of the Method

Consider a general nonlinear partial differential equation for a function $v$ that depends on three variables, as follows:

$$K(v, v_t, v_y, v_x, v_{xx}, ...) = 0. \tag{7}$$

Step 1: First, the traveling wave transform is discussed in the following form;

$$v(x, y, t) = v(\eta), \eta = x + y - mt. \tag{8}$$

Eq. (7) is transformed into an ordinary differential equation using the transformations in Eq. (8) as follows:

$$L(t, y, x, v, v^{'}, v^{''}, \cdots) = 0, \tag{9}$$

where superscripts demonstrate ordinary derivatives according $\eta$
Step 2: Assume that the solutions of Eq. (9) are treated as follows:

$$v(\eta) = \frac{\sum_{i=0}^{\sigma} a_i Q^i(\eta)}{\sum_{j=0}^{\rho} b_j Q^j(\eta)} = \frac{P[Q(\eta)]}{S[Q(\eta)]}, \tag{10}$$

where $Q$ is $\frac{1}{1 \pm e^{\eta}}$. It is stated that $Q$ is the solution of the following equation

$$Q_\eta = Q^2 - Q. \tag{11}$$

Step 3: The solution of Eq. (9) is sought according to this method as follows:

$$v(\eta) = \frac{a_0 + a_1 Q + a_2 Q^2 + \cdots + a_\sigma Q^\sigma}{b_0 + b_1 Q + b_2 Q^2 + \cdots + b_\rho Q^\rho}. \tag{12}$$

The values of $\sigma$ and $\rho$ in Eq. (10) can be determined through the homogeneous balance principle. For this, a balance is established between the highest-order

derivative and the highest-order nonlinear term in Eq. (9).
Step 4: Eq. (10) is inserted into Eq. (9). Thus, a polynomial $R(Q)$ of $Q$ is obtained. Thereafter all coefficients of $R(Q)$ are set equal to zero, to obtain a system of algebraic equations. Solving the resulting system determines $c$ and the coefficients $a_0, a_1, a_2, \ldots, a_\sigma, b_0, b_1, b_2, \ldots, b_\rho$. Finally, the soliton solutions of Eq. (9) are obtained.

## 3. Application of GKM to the equations

**Example 1.** Initially, the following transformation is considered.

$$u(x,t) = u(\eta), \eta = x - ct. \tag{13}$$

Substituting Eq. (13) into Eq. (2) yields the following equation.

$$(c^2-1)u - \epsilon\alpha_1 u^2 + \epsilon(\alpha_3 - c^2\alpha_4)u'' = 0. \tag{14}$$

If the balance principle is applied to Eq. (14), the following equation is obtained

$$\sigma = \rho + 2$$

If $\rho = 1$, then $\sigma = 3$. Thus the following equations are found.

$$u(\eta) = \frac{a_0 + a_1 Q + a_2 Q^2 + a_3 Q^3}{b_0 + b_1 Q}, \tag{15}$$

$$\begin{aligned} u'(\eta) = {} & \left(Q^2 - Q\right) \\ & \times \left[\frac{(a_1 + 2a_2 Q + 3a_3 Q^2)(b_0 + b_1 Q) - b_1\left(a_0 + a_1 Q + a_2 Q^2 + a_3 Q^3\right)}{(b_0 + b_1 Q)^2}\right], \end{aligned}$$

$$\begin{aligned} u''(\eta) = {} & \frac{(Q^2 - Q)(2Q - 1)}{(b_0 + b_1 Q)^2} \\ & \times \left[(a_1 + 2a_2 Q + 3a_3 Q^2)(b_0 + b_1 Q) - b_1\left(a_0 + a_1 Q + a_2 Q^2 + a_3 Q^3\right)\right] \\ & + \frac{(Q^2 - Q)^2}{(b_0 + b_1 Q)^3}\left[(2a_2 + 6a_3 Q)(b_0 + b_1 Q)^2 - 2b_1(a_1 + 2a_2 Q + 3a_3 Q^2)(b_0 + b_1 Q)\right] \\ & + \frac{(Q^2 - Q)^2}{(b_0 + b_1 Q)^3}\left[2b_1^2\left(a_0 + a_1 Q + a_2 Q^2 + a_3 Q^3\right)\right]. \end{aligned}$$

The soliton solutions of the strain wave equation are obtained in different cases as follows;

Case 1.

$$\begin{aligned} a_0 = 0, a_1 = \frac{6b_0(\alpha_3 - \alpha_4)}{\alpha_1(-1 + \epsilon\alpha_4)}, a_3 = \frac{6b_1(-\alpha_3 + \alpha_4)}{\alpha_1(-1 + \epsilon\alpha_4)}, \\ a_2 = \frac{6(-b_0 + b_1)(\alpha_3 - \alpha_4)}{\alpha_1(-1 + \epsilon\alpha_4)}, c = -\frac{\sqrt{-1 + \epsilon\alpha_3}}{\sqrt{-1 + \epsilon\alpha_4}}. \end{aligned}$$

By substituting the above equalities into Eq. (15), the following solution of Eq. (2) is found.

$$u_1(x,t) = \frac{3(\alpha_3-\alpha_4)}{\left(1+cosh\left[x+t\frac{\sqrt{-1+\epsilon\alpha_3}}{\sqrt{-1+\epsilon\alpha_4}}\right]\right)\alpha_1(-1+\epsilon\alpha_4)}. \tag{16}$$

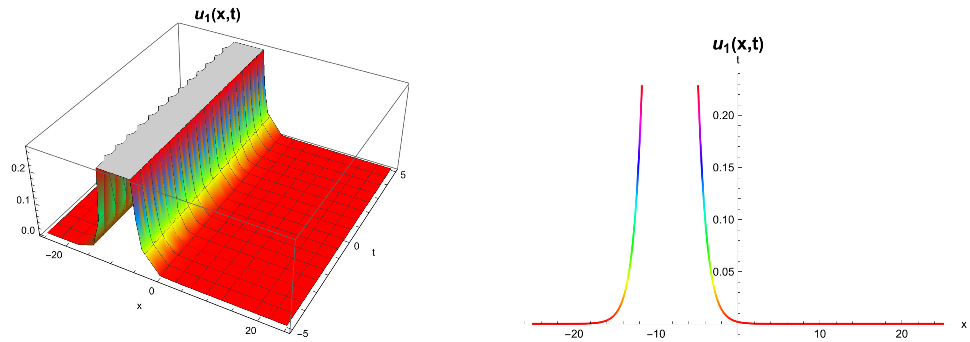



FIGURE 1. 3D and 2D plots of $u_1(x,t)$ solution.

Case 2.

$$a_0 = \frac{b_0(\alpha_3-\alpha_4)}{\alpha_1(1+\epsilon\alpha_4)}, a_1 = \frac{(-6b_0+b_1)(\alpha_3-\alpha_4)}{\alpha_1(1+\epsilon\alpha_4)},$$
$$a_2 = \frac{6(b_0-b_1)(\alpha_3-\alpha_4)}{\alpha_1(1+\epsilon\alpha_4)},$$
$$a_3 = \frac{6b_1(\alpha_3-\alpha_4)}{\alpha_1(1+\epsilon\alpha_4)}, c = \frac{\sqrt{1+\epsilon\alpha_3}}{\sqrt{1+\epsilon\alpha_4}}.$$

By substituting the above equalities into Eq. (15), the following solution of Eq. (2) is found.

$$u_2(x,t) = \frac{\left(-2+cosh\left[x-t\frac{\sqrt{1+\epsilon\alpha_3}}{\sqrt{1+\epsilon\alpha_4}}\right]\right)(\alpha_3-\alpha_4)}{\left(1+cosh\left[x-t\frac{\sqrt{1+\epsilon\alpha_3}}{\sqrt{1+\epsilon\alpha_4}}\right]\right)\alpha_1(1+\epsilon\alpha_4)}. \tag{17}$$

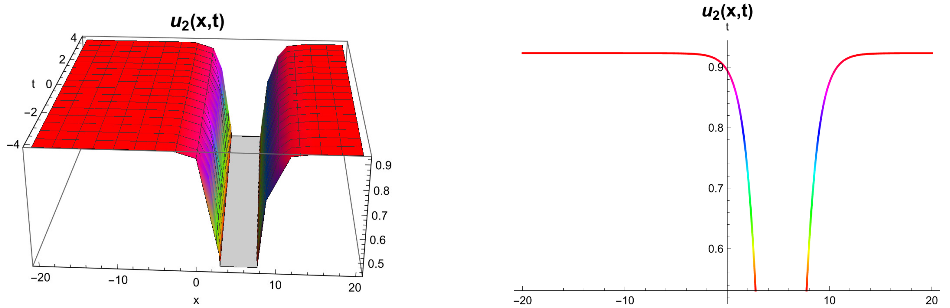



FIGURE 2. 3D and 2D plots of $u_2(x,t)$ solution.

**Example 2.** First, he following transformation is taken into account.

$$u(x,y,t) = u(\eta), v(x,y,t) = v(\eta), \eta = kx + my - ct. \tag{18}$$

Substituting Eq. (18) into system (6) yields the following equation.

$$-cu' + (\alpha k^3 + m\beta k^2)u''' + (3\alpha k^2 + 4m\beta k)(u')^2 = 0. \tag{19}$$

The following equation is obtained by transformation $u' = g$ in Eq. (19).

$$-cg + (\alpha k^3 + m\beta k^2)g'' + (3\alpha k^2 + 4m\beta k)g^2 = 0. \tag{20}$$

As a result of applying (18) transformation to this system,
$v = \frac{m}{k}u$ equality is obtained. If the balance principle is applied to Eq. (20), the following equation is obtained.

$$\sigma = \rho + 2$$

If $\rho = 1$, then $\sigma = 3$. Thus the following equations are found.

$$u(\eta) = \frac{a_0 + a_1 Q + a_2 Q^2 + a_3 Q^3}{b_0 + b_1 Q}, \tag{21}$$

$$\begin{aligned} u'(\eta) = {} & \left(Q^2 - Q\right) \\ & \times \left[ \frac{(a_1 + 2a_2 Q + 3a_3 Q^2)(b_0 + b_1 Q) - b_1\left(a_0 + a_1 Q + a_2 Q^2 + a_3 Q^3\right)}{(b_0 + b_1 Q)^2} \right], \end{aligned}$$

$$\begin{aligned} u''(\eta) = {} & \frac{(Q^2 - Q)(2Q - 1)}{(b_0 + b_1 Q)^2} \\ & \times \left[(a_1 + 2a_2 Q + 3a_3 Q^2)(b_0 + b_1 Q) - b_1\left(a_0 + a_1 Q + a_2 Q^2 + a_3 Q^3\right)\right] \\ & + \frac{(Q^2 - Q)^2}{(b_0 + b_1 Q)^3}\left[(2a_2 + 6a_3 Q)(b_0 + b_1 Q)^2 - 2b_1(a_1 + 2a_2 Q + 3a_3 Q^2)(b_0 + b_1 Q)\right] \\ & + \frac{(Q^2 - Q)^2}{(b_0 + b_1 Q)^3}\left[2b_1^2\left(a_0 + a_1 Q + a_2 Q^2 + a_3 Q^3\right)\right] . \end{aligned}$$

The soliton solutions of the (2+1)-dimensional BK equation are obtained in different cases as follows;;

Case 1.

$$a_0 = 0, a_1 = -\frac{a_2}{6}, a_3 = -a_2, b_0 = 0, c = \frac{k^2 m\beta a_2}{3a_2 - 6kb_1},$$
$$\alpha = -\frac{2m\beta\left(2a_2 - 3kb_1\right)}{3k\left(a_2 - 2kb_1\right)}.$$

Replacing the above equations in Eq. (21), the following solution of system (6) is reached.

$$u_1(x,y,t) = \frac{a_2}{2b_1}\left(tanh\left[\frac{kx}{2} + \frac{my}{2} - \frac{k^2 mt\beta a_2}{6a_2 - 12kb_1}\right] - \frac{kx}{3} - \frac{my}{3} + \frac{k^2 mt\beta a_2}{9a_2 - 18kb_1}\right). \quad (22)$$
$$v_1(x,y,t) = \frac{ma_2}{2kb_1}\left(tanh\left[\frac{kx}{2} + \frac{my}{2} - \frac{k^2 mt\beta a_2}{6a_2 - 12kb_1}\right] - \frac{kx}{3} - \frac{my}{3} + \frac{k^2 mt\beta a_2}{9a_2 - 18kb_1}\right).$$

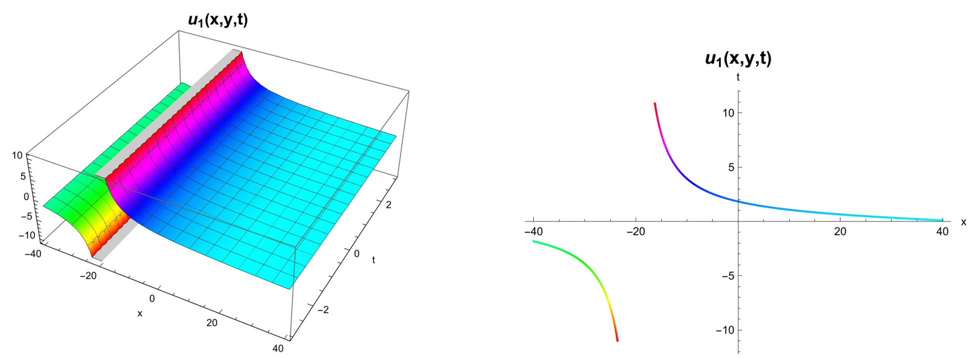



FIGURE 3. 3D and 2D plots of $u_1(x,y,t)$ solution.

Case 2.

$$a_0 = 0, a_1 = -\frac{k\left(k\alpha + m\beta\right)b_1}{3k\alpha + 4m\beta}, a_2 = \frac{6k\left(k\alpha + m\beta\right)b_1}{3k\alpha + 4m\beta},$$
$$a_3 = -\frac{6k\left(k\alpha + m\beta\right)b_1}{3k\alpha + 4m\beta}, b_0 = 0, c = -k^2\left(k\alpha + m\beta\right).$$

Replacing the above equations in Eq. (21), the following solution of system (6) is reached.

$$u_2(x,y,t) = -\frac{k\left(k\alpha + m\beta\right)\left(kx + my + k^2 t\left(k\alpha + m\beta\right) - 3tanh\left[\frac{1}{2}\left(kx + my + k^2 t\left(k\alpha + m\beta\right)\right)\right]\right)}{3k\alpha + 4m\beta}$$
$$(23)$$
$$v_2(x,y,t) = -\frac{m\left(k\alpha + m\beta\right)\left(kx + my + k^2 t\left(k\alpha + m\beta\right) - 3tanh\left[\frac{1}{2}\left(kx + my + k^2 t\left(k\alpha + m\beta\right)\right)\right]\right)}{3k\alpha + 4m\beta}$$

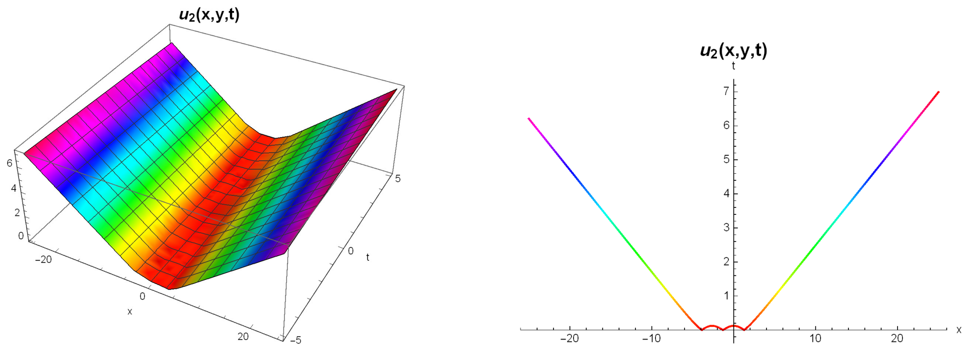



FIGURE 4. 3D and 2D plots of $u_2(x, y, t)$ solution.

## 4. Results and Discussion

In this study, strain wave and (2+1)-dimensional BK equations are studied. Hyperbolic solutions for the strain wave equation and dark soliton solutions for the (2+1)-dimensional BK equation are obtained. When these solutions are compared with previous studies in the literature, it is seen that the solutions are new and presented for the first time in this study. The graphical representations of the obtained solutions are made for the following values.

Figure 1, depicts singular kink soliton for 3D plot of solution (16) for $\alpha_1 = 2, \alpha_3 = 3, \alpha_4 = 0.5, \epsilon = 4, -25 \leq x \leq 25$ values with $-5 \leq t \leq 5$ range and 2D plot of solution for $t = 2.5$ with these values. Figure 2, shows singular kink soliton for 3D plot of solution (17) for $\alpha_1 = 1.5, \alpha_3 = 2, \alpha_4 = 0.2, \epsilon = 1.5, -20 \leq x \leq 20$ values with $-4 \leq t \leq 4$ range and 2D plot of solution for $t = 3$ with these values. Figure 3, represents soliton solution for 3D plot of solution (22) for $a_2 = 2, b_1 = 1, k = 0.05, m = 1, \beta = 1, y = 1, -40 \leq x \leq 40$ values with $-3 \leq t \leq 3$ range and 2D plot of solution for $t = 2$ with these values. Figure 4, depicts smooth soliton for 3D plot of solution (23) for $k = 1, m = 0.2, \alpha = 0.2, \beta = 0.5, y = 2, -25 \leq x \leq 25$ values with $-5 \leq t \leq 5$ range and 2D plot of solution for $t = 3$ with these values.

## 5. Conclusions

In this study, GKM was considered. GKM was applied to the strain wave equation and (2+1)-dimensional BK equations. Thus, hyperbolic soliton solutions of the strain wave equation and dark soliton solutions of the (2+1)-dimensional BK equation were obtained using this method. These solutions were different from the found solutions in other studies and were presented for the first time in this study. The accuracy of the results was confirmed by putting the obtained solutions back into the original equation. The new solutions of these equations studied could have helped to understand the phenomena in which waves are governed by these equations. In addition, some special values and intervals were given to the results obtained using Wolfram Mathematic 2D and 3D graphical representations of the solutions were made.

The considered method can also be applied to other nonlinear partial differential equations. The most important advantage of this method is that all solutions are obtained from a single algebraic equation. This means that it is sufficient to set up a single algorithm and there is no unnecessary computational overhead.

**Author Contribution Statements** The authors contributed equally and they read and approved the final manuscript.

**Declaration of Competing Interests** The authors report that they have no competing interests.

## References

[1] Ahmad, H., Seadawy, A. R., Khan, T. A., Thounthong P., Analytic approximate solutions for some nonlinear parabolic dynamical wave equations, *Journal of Taibah University for Science*, 14(1) (2020), 346–358. https://doi.org/10.1080/16583655.2020.1741943

[2] Arshed, S., New soliton solutions to the perturbed nonlinear Schrödinger equation by exp $(-\Phi(\xi))$-expansion method, *Optik-International Journal for Light and Electron Optics*, 220(165123) (2020), 1–12. https://doi.org/10.1016/j.ijleo.2020.165123

[3] Dusunceli, F., Celik, E., Askin, M., Bulut, H., New exact solutions for the doubly dispersive equation using the improved Bernoulli sub-equation function method, *Indian Journal of Physics*, 95(2) (2021), 309–314. https://doi.org/10.1007/s12648-020-01707-5

[4] Ekici, M., Unal, M., Application of the rational (G'/G)-expansion method for solving some coupled and combined wave equations, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat*, 71(1) (2022), 116–132. https://doi.org/10.31801/cfsuasmas.884025

[5] Rahman, M., Habiba, U., Salam, M., Datta, M., The traveling wave solutions of space-time fractional partial differential equations by modified Kudryashov method., *Journal of Applied Mathematics and Physics*, 8(11) (2020), 2683–2690. https://doi.org/10.4236/jamp.2020.811198

[6] Taşbozan, O., Kurt, A., The new travelling wave solutions of time fractional Fitzhugh-Nagumo equation with Sine-Gordon expansion method, *Adıyaman University Journal of Science*, 10(1) (2020), 256–263. https://doi.org/10.37094/adyujsci.515011

[7] Arnous, A. H., Zhou, Q., Biswas, A., Guggilla, P., Khan, S., Yıldırım, Y., Alshomrani, A. S., Alshehri, H. M., Optical solitons in fiber Bragg gratings with cubic–quartic dispersive reflectivity by enhanced Kudryashov's approach, *Physics Letters A*, 422 (2022), 127797. https://doi.org/10.1016/j.physleta.2021.127797

[8] Zayed, E. M. E., Gepreel, K. A., Shohib, R. M. A., Alngar, M. E. M., Yıldırım, Y., Optical solitons for the perturbed Biswas-Milovic equation with Kudryashov's law of refractive index by the unified auxiliary equation method, *Optik*, 230 (2021), 166286. https://doi.org/10.1016/j.ijleo.2021.166286

[9] Yıldırım, Y., Topkara, E., Biswas, A., Triki, H., Ekici, M., Guggilla, P., Khan, S., Belic, M. R., Cubic–quartic optical soliton perturbation with Lakshmanan–Porsezian–Daniel model by sine-Gordon equation approach, *Journal of Optics*, 50 (2021), 322–329. https://doi.org/10.1007/s12596-021-00685-z

[10] Yıldırım, Y., Biswas, A., Asma, M., Ekici, M., Ntsime, B. P., Zayed, E. M. E., Moshokoa, S. P., Alzahrani, A. K., Belic, M. R., Optical soliton perturbation with Chen–Lee–Liu equation, *Optik*, 220 (2020), 165177. https://doi.org/10.1016/j.ijleo.2020.165177

[11] Seadawy, A. R., Arshad, M., Lu, D., Dispersive optical solitary wave solutions of strain wave equation in micro-structured solids and its applications, *Physica A: Statistical Mechanics and its Applications*, 540 (2020), 1–13. https://doi.org/10.1016/j.physa.2019.123122

[12] Ayati, Z., Hosseini, K., Mirzazadeh, M., Application of Kudryashov and functional variable methods to the strain wave equation in microstructured solids, *Nonlinear Engineering*, 6(1) (2017), 25–29. https://doi.org/10.1515/nleng-2016-0020

[13] Arshad, M., Seadawy, A. R., Lu, D., Study of bright–dark solitons of strain wave equation in micro-structured solids and its applications, *Modern Physics Letters B*, 33(33) (2019), 1–12. https://doi.org/10.1142/S0217984919504177

[14] Gao, W., Silambarasan, R., Baskonus, H. M., Anand, R. V., Rezazadeh, H., Periodic waves of the non dissipative double dispersive micro strain wave in the micro structured solids, *Physica A: Statistical Mechanics and its Applications*, 545 (2020), 1–30. https://doi.org/10.1016/j.physa.2019.123772

[15] Irshad, A., Ahmed, N., Nazir, A., Khan, U., Mohyud-Din, S. T., Novel exact double periodic soliton solutions to strain wave equation in micro structured solids, *Physica A: Statistical Mechanics and its Applications*, 550 (2020), 1–15. https://doi.org/10.1016/j.physa.2019.124077

[16] Kumar, S., Kumar, A., Wazwaz, A. M., New exact solitary wave solutions of the strain wave equation in microstructured solids via the generalized exponential rational function method, *The European Physical Journal Plus*, 135(870) (2020), 1–17. https://doi.org/10.1140/epjp/s13360-020-00883-x

[17] Joseph, S. P., New traveling wave rational form exact solutions for strain wave equation in micro structured solids, *IOP SciNotes*, 2(1) (2021), 1–7. https://doi.org/10.1088/2633-1357/abec2a

[18] Ray, S. S., Lie symmetry analysis and reduction for exact solution of (2+1)-dimensional Bogoyavlensky–Konopelchenko equation by geometric approach, *Modern Physics Letters B*, 32(11) (2018), 1–9. https://doi.org/10.1142/S0217984918501270

[19] Yan, H., Tian, S. F., Feng, L. L., Zhang, T. T., Quasi-periodic wave solutions, soliton solutions, and integrability to a (2+1)-dimensional generalized Bogoyavlensky-Konopelchenko equation, *Waves in Random and Complex Media*, 26(4) (2016), 1–14. https://doi.org/10.1080/17455030.2016.1166289

[20] Xiang-Peng, X., Xi-Qiang, L., Lin-Lin, Z., Explicit solutions of the Bogoyavlensky–Konoplechenko equation, *Applied Mathematics and Computation*, 215(10) (2010), 3669–3673. https://doi.org/10.1016/j.amc.2009.11.005

[21] Zhou, X. M., Tian, S. F., Zhang, L. D., Zhang, T. T., General high-order breather, lump, and semi-rational solutions to the (2+1)-dimensional generalized Bogoyavlensky–Konopelchenko equation. *Modern Physics Letters B*, 35(3) (2021), 1–12. https://doi.org/10.1142/S0217984921500573

[22] Ray, S. S., On conservation laws by Lie symmetry analysis for (2+1)-dimensional Bogoyavlensky–Konopelchenko equation in wave propagation, *Computers and Mathematics with Applications*, 74(6) (2017), 1158–1165. https://doi.org/10.1016/j.camwa.2017.06.007

[23] Chen, S. T., Ma, W. X., Lump solutions to a generalized Bogoyavlensky-Konopelchenko equation, *Frontiers of Mathematics in China*, 13(3) (2018), 525–534. https://doi.org/10.1007/s11464-018-0694-z

[24] Tuluce Demiray, S., New solutions of Biswas-Arshed equation with beta time derivative, *Optik-International Journal for Light and Electron Optics*, 222(165405) (2020a), 1–5. https://doi.org/10.1016/j.ijleo.2020.165405

[25] Tuluce Demiray, S, Bayrakci, U., Soliton solutions of generalized third-order nonlinear Schrödinger equation by using GKM, *Journal of the Institute of Science and Technology*, 11(2) (2021), 1481–1488. https://doi.org/10.21597/jist.861864

[26] Tuluce Demiray, S, Bayrakci, U., Soliton solutions for space-time fractional Heisenberg ferromagnetic spin chain equation by generalized Kudryashov method and modified exp $(-\Omega(\eta))$ -expansion function method, *Revista Mexicana de Fisica*, 67(3) (2021), 393–402. https://doi.org/10.31349/RevMexFis.67.393

[27] Gurefe, Y., The generalized Kudryashov method for the nonlinear fractional partial differential equations with the beta-derivative, *Revista Mexicana de Fisica*, 66(6) (2020), 771–781. https://doi.org/10.31349/RevMexFis.66.771

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 285–305 (2024)
DOI:10.31801/cfsuasmas.1258454
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

*Research Article; Received:March 1, 2023; Accepted: October 10, 2023*

# ON THE STABILITY ANALYSIS OF A FRACTIONAL ORDER EPIDEMIC MODEL INCLUDING THE GENERAL FORMS OF NONLINEAR INCIDENCE AND TREATMENT FUNCTION

Esra KARAOĞLU

Department of Electrical and Electronics Engineering, University of Turkish Aeronautical Association, 06790, Ankara, TÜRKİYE

Abstract. In this paper, we propose to study a SEIR model of fractional order with an incidence and a treatment function. The incidence and treatment functions included in the model are general nonlinear functions that satisfy some meaningful biological hypotheses. Under these hypotheses, it is shown that the disease free equilibrium point of the proposed model is locally and globally asymptotically stable when the reproduction number $R_0$ is smaller than 1. When $R_0 > 1$, it is established that the endemic equilibrium of the studied system is uniformly asymptotically stable. Finally, some numerical simulations are provided to illustrate the theory.

## 1. Introduction

Studying the spread process of infectious diseases has been a very important and popular topic since outbreaks have serious impacts on the economy, daily lives and the future. For finding intervention strategies or treatments and reducing the deaths understanding this mechanism is very important. Mathematical models help us understand the dynamics of epidemiological diseases and talk about the future of the epidemics. Until today, a vast number of mathematical models have been developed for diseases such as rabies, measles, malaria, chickenpox, tuberculosis, cancer, HIV/AIDS and COVID-19 [1, 7–9, 16, 18, 26, 36–38, 40, 46]. When modeling disease transmission, compartmental models such as *SIR* (Suscepted-Infected-Recovered), *SIS*, *SEIR* and *SEIS* models are mostly used in the literature (for more detail, see [8]). The general idea behind these compartmental models is to divide the total population into compartments and describe the transfer from one compartment to

2020 *Mathematics Subject Classification.* 34D23, 26A33, 92D25, 92D30.

*Keywords.* Fractional order SEIR model, uniform asymptotic stability, nonlinear incidence function, treatment function.

ekaraoglu@thk.edu.tr; 0000-0003-2939-1829.

another under some meaningful assumptions [21]. For example, for an *SIR* model, the population is divided into three compartments: susceptible individuals $S$ who are not infected yet, infected individuals $I$ who are infectious and can spread the disease to the susceptible class and recovered (removed) individuals $R$ who recovered the infection and already gained the immunity.

In epidemic models, incidence functions are used in the transmission progress between the susceptible population and the infected or exposed population. In literature, there are different incidence rates and using different incidence rates can affect the dynamic behaviour of the system [27,31]. The most common incidence rates are the bilinear incidence rate $\beta SI$ [45,50], where $\beta$ is the average number of contacts per individual per day, the standard incidence rate $\beta SI/N$ [14,25] where $N = S + I + R$, the saturated incidence rate $\beta SI/(1 + \alpha_1 S + \alpha_2 I)$ [16,19] where $\alpha_1$, $\alpha_2$ are positive constants. Even though it is very hard to fit real data values in infectious disease transmission, nonlinearity is inevitable in the incidence rates. These nonlinear incidence rates seem more realistic because they may include saturation effects, heterogeneous mixing populations, environmental factors, media effects or behavioural changes of individuals, etc. [23]. In 2005, Korobeinikov and Maini [24] studied the stability properties of infectious disease models with a general, arbitrary nonlinear incidence rate $f(S, I, N)$ and obtained the global stability by constructing a Lyapunov function under a more specific incidence rate of the form $g(I)h(S)$. Following this work, Korobeinikov studied the global dynamics of infectious disease models with nonlinear incidence rates in [22] and [23], respectively. In 2014, Li et al. [27] considered an SIR epidemic model with a nonlinear arbitrary incidence function $f(S, I)$ and they improved their model by incorporating a time delay representing the latent period. Recently, in [48], the authors analyzed the stability of a fractional order SEIR model with general incidence rate $F(S)G(I)$ and in [20], the authors studied the local and global stability of the disease free and endemic equilibrium points of a fractional SIR model with a general incidence function $f(S, I)$.

Treatment is an important strategy to reduce the number of infected people during outbreaks. Vaccination can be thought one form of treatment to protect against infection before the outbreak [6]. One of the most efficient methods of treatment is, of course, hospitalisation, but sometimes the resources of countries are not adequate as in the COVID-19 pandemic. In mathematical models, to reflect and analyse the effect of treatment, scientists have incorporated another treatment class into the SIR model [6,42,47], or they used some treatment functions in the models [11,15,28]. In 1991, Anderson and May [4] proposed that the treatment function is proportional to the number of infectious people. Following this work, some different modified treatment functions are used in order to reflect the treatment capacity of communities (For further details, see [11,28]).

In recent decades, fractional differential equations (FDEs) have gained great attention in the stability analysis of ecological or epidemiological models. Besides

FDEs being generalizations of classical differential equations, using FDEs in epidemiology helps to model in a more realistic way since they reflect history (memory) [16,39]. There are different kinds of fractional operators in the literature and these operators may answer distinct real world problems [32]. Moreover, because of the memory effect, FDEs will be more suitable for epidemic models. For example, in 2020, Naik et al. [32], proposed and studied the stability of a FDE system that models COVID-19 pandemic with Atangana-Baleanu or Caputo derivative. They divided the total population into eight groups such as suspected, exposed, symptomatic (infected), asymptomatic, quarantined, treated classes etc. and used the real data from Pakistan. In 2020, Yavuz and Sene [49] studied the stability analysis of a fractional predator–prey model with a harvesting rate. In another paper, Naik et al. [33] has established the global dynamics of a fractional order model for the transmission of HIV epidemic with optimal control in 2020. In 2023, Joshi et al. [16] studied COVID-19 pandemic with Atangana-Baleanu derivative. They used an SIR model with nonlinear Beddington-DeAngelis infection rate and Holling type II treatment rate in their paper. Recently, a vast number of papers containing FDEs for different research areas have been published in the literature [5,16,17,32,34,41].

Motivated by the aforementioned works [20,23,27], in this paper, we have proposed an SEIR model including FDEs with a general nonlinear incidence function $f(S,I)$ and a treatment function $T(I)$:

$$\begin{aligned} D_t^\alpha S(t) &= \lambda - f(S,I) - \mu S, \\ D_t^\alpha E(t) &= f(S,I) - (\beta+\mu+r)E + pT(I), \\ D_t^\alpha I(t) &= \beta E - (\theta+\mu)I - T(I), \\ D_t^\alpha R(t) &= rE + qT(I) - \mu R, \end{aligned} \tag{1}$$

where $D_t^\alpha u$ represents Caputo fractional derivative of the function $u$ with the following initial conditions:

$$\begin{aligned} &S(0) = S_0 > 0,\ E(0) = E_0 > 0, \\ &I(0) = I_0 > 0 \text{ and } R(0) = R_0 > 0. \end{aligned} \tag{2}$$

In this model (1), $S$, $E$, $I$ and $R$ denote the suscepted, exposed, infected, and recovered individuals, respectively. To the best of the author's knowledge, a fractional SEIR model with a general incidence function and treatment function has not been studied yet. We have chosen this model with exposed individuals compartment, as most infectious diseases have an incubation period. Before explaining parameters, we need to emphasize that the originality of this paper comes from the choice of the incidence and treatment functions, $f(S,I)$ and $T(I)$ functions in model (1), respectively. These functions have not been determined specifically so that depending on the studied disease, one may choose his/her function according to the spread of the disease and treatment type. Moreover, considering these functions in a general way increases the complexity of the proofs.

In model (1), parameter $\lambda$ is the recruitment rate which represents the total change in the population and assumed as a positive number, $\beta$ is the rate at which exposed individuals become infectious (incubation rate), $\mu$ is the natural death rate, $\theta$ is the death rate depending on the infection, and $r$ is the recovery rate of exposed individuals. The function $T(I)$ represents the general treatment function. In the model, it is assumed that unsuccessfully treated infectious individuals re-enter the exposed compartment proportional to parameter $p$ and the parameter $q$ denotes the fraction of infectious individuals whose treatments are successful ($p = 1 - q$). The flowchart for the model (1) is given in Figure 1.

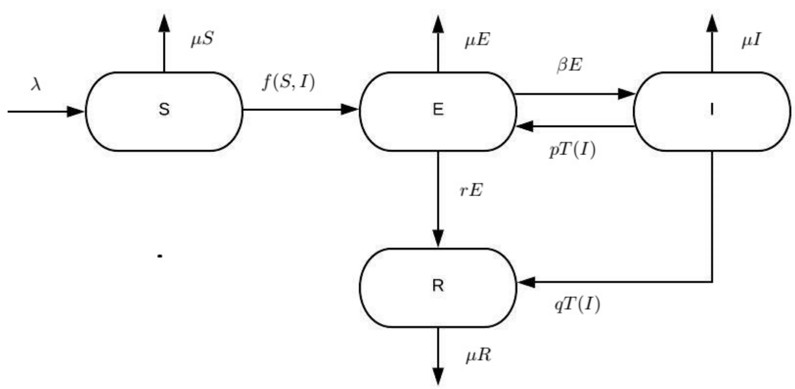



FIGURE 1. The flowchart for the model (1)

The paper is organised as follows: In Section 2, the definition of Caputo fractional derivative is presented and some lemmas are given for the proofs needed for stability analyses. In Section 3, the properties of the incidence function and the treatment function are analysed and the positivity of the solution of system (1) is proved. After that, in Section 4, the equilibrium points of system (1) are determined and the global stability analysis of disease free equilibrium point and the uniform asymptotic stability of endemic equilibrium point are established. Following these theorems, some numerical simulations are carried out to show some examples in Section 5. Finally, we finish this paper with a conclusion part in Section 6.

## 2. PRELIMINARIES ON THE CAPUTO FRACTIONAL CALCULUS

We begin by introducing the definition of Caputo fractional derivative.

**Definition 1** ( [39]). *Let $t_0 > 0$, $t > t_0$, $\alpha, t_0, t \in \mathbb{R}$. The Caputo fractional derivative of order $\alpha$ of a function $f \in \mathbb{C}^n$ is given by*

$$Ct_0 D_t^\alpha = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, \tag{3}$$

*where $0 < n-1 < \alpha < n$ and $\Gamma$ is the Gamma function.*

To prove the nonnegativity of the solutions of model (1) we will need the following lemma.

**Lemma 1** (Generalized Mean Value Theorem [35]). *Suppose that $f \in \boldsymbol{C}[0,a]$ and $D^\alpha f \in (0,a]$, $0 < \alpha \leq 1$. Then one has*

$$f(x) = f(0) + \frac{1}{\Gamma(\alpha)} (D^\alpha f)(\xi) x^\alpha, \tag{4}$$

*with $0 \leq \xi \leq x$, $\forall x \in (0,a]$.*

**Corollary 1** ( [35]). *Suppose that $f \in \boldsymbol{C}[0,a]$ and $D^\alpha f \in \boldsymbol{C}(0,a]$ for $0 < \alpha \leq 1$. If $D^\alpha f(x) \geq 0$ $\forall x \in (0,a)$, then $f(x)$ is non-decreasing for each $x \in [0,a]$. If $D^\alpha f(x) \leq 0$ $\forall x \in (0,a)$, then $f(x)$ is non-increasing for each $x \in [0,a]$.*

**Lemma 2** ( [44]). *Let $x(t) \in \mathbb{R}^+$ be a continuous and derivable function. Then, for any time instant $t \geq t_0$*

$$Ct_0 D_t^\alpha \left[ x(t) - x^* - x^* ln \frac{x(t)}{x^*} \right] \leq \left( 1 - \frac{x^*}{x(t)} \right) Ct_0 D_t^\alpha x(t),$$

$x^* \in \mathbb{R}^+, \quad \forall \alpha \in (0,1)$.

## 3. Basic Properties of the Model

In model (1), we assume that the functions $S$, $E$, $I$ and $R$ and their Caputo fractional derivatives are continuous when $t > 0$.

The general nonlinear incidence function $f(S,I)$ and the treatment function $T(I)$ are considered positive, continuously differentiable functions and they satisfy the following hypotheses:

H1) $f(S,I) > 0$, $f(0,I) = 0$, $f(S,0) = 0$ for all $S, I > 0$.

H2) $\dfrac{\partial f(S,I)}{\partial S} > 0$ and $\dfrac{\partial f(S,I)}{\partial I} > 0$ for all $S, I > 0$.

H3) $\dfrac{\partial f(S,0)}{\partial S} = 0$ and $\dfrac{\partial f(S,0)}{\partial I} > 0$ for all $S > 0$.

H4) $\dfrac{f(S,I)}{I} \leq \dfrac{\partial f(S,0)}{\partial I}$ for all $I > 0$.

H5) $T(0) = 0$ and $T'(I) > 0$ for $I \geq 0$.

H6) The function $\dfrac{T(I)}{I}$ is monotone increasing function, that is,

$\dfrac{T(I)}{I} - T'(I) \leq 0$. (See [11])

These conditions are consistent with biological assumptions and in accordance with literature (see [11,20,23]). For example, for (*H1*),we can think that there will be no transmission when there are no susceptible or infected people. For (*H2*), we understand that the incidence function is a monotonically growing function for all $S, I > 0$. In the absence of an infected person, susceptible individuals will become stagnant and transmission will begin to increase in the case of an infected person. In addition, in the absence of an infected person, there is no need for treatment. When there is an increase in the rate of transmission, that is, the number of infected people increases, we need to apply more treatment strategies (*H5*).

Now, let $\mathbb{R}^4_+ = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_i \geq 0 \textit{ for all } i = 1, 2, 3, 4\}$. We will prove the existence, uniqueness and positivity of the solutions with the following theorem.

**Theorem 1.** *There exists a unique solution of the model (1) with initial conditions under the hypothesis (H1) is satisfied. Moreover, the solution will remain in* $\mathbb{R}^4_+$ *for all* $t \geq 0$.

*Proof.* From Theorem 3.1 and Remark 3.2 of [29], one can see the existence and uniqueness of the solution of system (1) with the initial conditions (2). Now, we will prove that $\mathbb{R}^4_+$ is a positively invariant region. For this, let

$$\begin{aligned} D_t^\alpha S(t)_{|S=0} &= \lambda \geq 0, \\ D_t^\alpha E(t)_{|E=0} &= f(S, I) + pT(I) \geq 0, \\ D_t^\alpha I(t)_{|I=0} &= \beta E \geq 0. \end{aligned}$$

We can make a similar discussion as in Theorem 2 of [3] and with the help of Corollary 1, one can observe that the solution will remain in $\mathbb{R}^4_+$ for all $t \geq 0$. □

## 4. Equilibrium Points and Their Stability

In this section, we study the stability of the equilibrium points of system (1). Since the right hand sides of the first three equations of model (1) do not include $R(t)$ we will deal with the first three variables $S$, $E$ and $I$. System (1) has two possible equilibrium points. There is always a disease-free equilibrium point $E^0 = (S^0, 0, 0)$ where $S^0 = \dfrac{\lambda}{\mu}$ provided that (H1) and (H5) is satisfied.

Using the notations in [43], the matrices $F$ and $V$ for system (1) are given as follows:

$$F = \begin{bmatrix} 0 & f_I(S_0, 0) \\ 0 & 0 \end{bmatrix},$$

$$V = \begin{bmatrix} (\beta + \mu + r) & -pT'(0) \\ -\beta & (\theta + \mu) + T'(0) \end{bmatrix}.$$

The basic reproduction number can be found as

$$R_0 = \frac{\beta f_I(S_0, 0)}{\eta(\theta + \mu) + (\beta q + \mu + r)T'(0)}, \tag{5}$$

where $\eta = \beta + \mu + r$.

**Lemma 3.** *Consider a function $g$ as $g(I) = \lambda\beta - \eta F(I) + \beta p T(I)$. The equation $g(I) = 0$ has a unique positive solution $I_0$ where $F(I) = (\theta + \mu)I + T(I)$.*

*Proof.* One can observe that $g(0) = \lambda\beta > 0$ under the condition $T(0) = 0$ and for the positive value $\frac{\lambda\beta}{\eta(\theta+\mu)}$, we find $g\left(\frac{\lambda\beta}{\eta(\theta+\mu)}\right) = -(\mu + r + q)T(\frac{\lambda\beta}{\eta(\theta+\mu)}) < 0$ since $T(I)$ is an increasing function. Moreover,

$$g'(I) = -\eta(\theta + \mu) - (\beta q + \mu + r)T'(I) < 0.$$

Clearly, the equation $\lambda\beta - \eta F(I) + \beta p T(I) = 0$ has a unique positive solution $I_0$. □

**Theorem 2.** *When $R_0 > 1$, the system (1) has a unique endemic equilibrium point.*

*Proof.* For finding an equilibrium point, let
$D_t^\alpha S(t) = 0$, $D_t^\alpha E(t) = 0$ and $D_t^\alpha I(t) = 0$. Now, we need to solve the following system

$$\begin{aligned} &\lambda - f(S, I) - \mu S = 0, \\ &f(S, I) - \eta E + pT(I) = 0, \\ &\beta E - (\theta + \mu)I - T(I) = 0. \end{aligned} \tag{6}$$

It is easy to see that $E^0 = (S^0, 0, 0)$ where $S^0 = \frac{\lambda}{\mu}$ is always disease-free equilibrium point of system (1). From system (6), we have

$$\lambda - \mu S = f(S, I) = \eta E - pT(I) \tag{7}$$

and

$$E = \frac{F(I)}{\beta} \tag{8}$$

where $F(I)$ is defined as in Lemma 3. By using the equations (7) and (8) one can obtain $S$ as in the following form

$$S = \frac{\lambda}{\mu} - \frac{\eta}{\beta\mu}F(I) + \frac{p}{\mu}T(I). \tag{9}$$

On the other hand, we need to find a positive root for the equation $f(S, I) = \eta E - pT(I)$. Let us define the continuous function

$$H(I) = f\left(\frac{\lambda}{\mu} - \frac{\eta}{\beta\mu}F(I) + \frac{p}{\mu}T(I), I\right) - \frac{\eta}{\beta}F(I) + pT(I). \tag{10}$$

Clearly, $H(0) = 0$ under the hypotheses (H1) and (H5). By using Lemma 3, one can calculate a positive $I_0$ value such that $\lambda\beta - \eta F(I_0) + \beta p T(I_0) = 0$. At this positive $I_0$ value, $H(I_0) = -\lambda < 0$ is obtained.

Moreover, if we look at the derivative of the function $H(I)$, we can see that

$$H'(I) = f_I(\frac{\lambda}{\mu} - \frac{\eta}{\beta\mu}F(I) + \frac{p}{\mu}T(I), I) - \frac{\eta}{\beta}F'(I) + pT'(I) \tag{11}$$

and

$$H'(0) = \frac{[\eta(\theta+\mu) + (\beta q + \mu + r)T'(0)]}{\beta}(R_0 - 1) > 0, \tag{12}$$

which implies that there exist some $I^* \in (0, I_0)$ such that $H(I^*) = 0$. Also, at this positive $I^*$ value, $S^* = \frac{1}{\mu}\left(\lambda - \frac{\eta}{\beta}F(I^*) + pT(I^*)\right) > 0$ since $\frac{\mu}{\beta}(\theta+\mu)I^* + \frac{\mu + r}{\beta}T(I^*) + qT(I^*) < \frac{\mu}{\beta}(\theta+\mu)I_0 + \frac{\mu+r}{\beta}T(I_0) + qT(I_0) = \lambda$ and $T(I)$ is an increasing function and $E^* = \frac{F(I^*)}{\beta} > 0$. This guarantees the existence of a positive endemic equilibruim point $\Sigma^* = (S^*, E^*, I^*)$. For uniqueness of the endemic equilibrium point, assume that we have another positive equilibrium point $\tilde{I}$. On the other hand, one can observe $H'(I^*) < 0$ which says that at every root of the function $H(I)$, the function is strictly decreasing. But, this will be a contradiction. This completes the proof. Next, we will prove the local and global asymptotic stability of the disease-free equilibrium point $E^0$. □

**Theorem 3.** *The disease-free equilibrium point* $E^0 = (S^0, 0, 0)$ *is locally asymptotically stable if* $R_0 < 1$.

*Proof.* The disease-free equilibrium point $E^0$ is locally asymptotically stable if all of the eigenvalues $\lambda_i$ of the Jacobian matrix evaluated at the equilibrium point $E^0$, that is $J(E^0)$, satisfy the Matignon's conditions [30], that is,

$$|arg(\lambda_i)| > \frac{\alpha\pi}{2}. \tag{13}$$

The Jacobian matrix $J(E^0)$ can be evaluated as

$$J(E^0) = \begin{bmatrix} -\mu & 0 & -f_I(S_0, 0) \\ 0 & -\eta & f_I(S_0,0) + pT'(0) \\ 0 & \beta & -(\theta + \mu + T'(0)) \end{bmatrix}. \tag{14}$$

The eigenvalues of the Jacobian matrix $J(E^0)$ are $\lambda_1 = -\mu$ and $\lambda_{2,3} = \frac{-(\eta+\sigma) \pm \sqrt{(\eta+\sigma)^2 - 4(\eta\sigma - \gamma)}}{2}$ where $\sigma = \theta + \mu + T'(0)$ and $\gamma = \beta(f_I(S_0, 0) + pT'(0))$. It is easy to see that all the eigenvalues are real and if $\eta\sigma - \gamma > 0$ all the eigenvalues $\lambda_i$ will be negative. Hence, $|arg(\lambda_i)| > \frac{\alpha\pi}{2}$. Also, note that the inequality $\eta\sigma - \gamma > 0$ implies that $R_0 < 1$. Therefore, if $R_0 < 1$ the disease-free equilibrium point $E^0$ is locally asymptotically stable and if $R_0 > 1$ the trivial equilibrium $E^0$ becomes unstable. □

**Theorem 4.** *Let*

$$\lim_{I \mapsto 0^+} \frac{f(S^0, I)}{f(S, I)} > 1 \qquad for \quad S \in [0, S^0). \tag{15}$$

*The disease-free equilibrium point* $E^0 = (S^0, 0, 0)$ *is globally asymptotically stable if* $R_0 < 1$.

*Proof.* To establish the global stability of the disease-free equilibrium point, consider the following Lyapunov function

$$V(t) = \beta E(t) + \eta I(t). \tag{16}$$

Calculating the time fractional order derivative in Caputo sense of both sides of Eq. 16 and using the hypothesis of this theorem and (H4) we get

$$\begin{aligned}
D_t^\alpha V(t) &= \beta D_t^\alpha E(t) + \eta D_t^\alpha I(t) \\
&= \beta I \left( \frac{f(S,I) - \eta E + pT(I)}{I} \right) \\
&+ \eta(\beta E - (\theta+\mu)I - T(I) \\
&\leq \beta I \left( f_I(S,0) - \frac{\eta E}{I} + \frac{pT(I)}{I} \right) \\
&+ \eta(\beta E - (\theta+\mu)I - T(I) \\
&= I\left(\beta f_I(S,0) - \eta(\theta+\mu)\right) + (\beta p - \eta)T(I) \\
&\leq I(\beta f_I(S^0,0) - \eta(\theta+\mu)) + (\beta p - \eta)T(I) \\
&\leq I(\eta(\theta+\mu) + (\beta q + \mu + r)T'(0))(R_0 - 1)
\end{aligned} \tag{17}$$

Clearly, if $R_0 < 1$ then $D_t^\alpha V(t)$ is negative. Therefore, the equilibrium point $E^0$ is globally asymptotically stable. This completes the proof. $\square$

**Theorem 5.** *Let* $\alpha \in (0,1)$ *and* $R_0 > 1$. *Then the unique endemic equilibrium point* $\Sigma^* = (S^*, E^*, I^*)$ *of system (1) is uniformly asymptotically stable if the following conditions hold*

$$\begin{aligned}
&\frac{f(S,I)}{f(S^*,I^*)} < 1 \qquad \textit{for} \quad S < S^*, \\
&\frac{f(S,I)}{f(S^*,I^*)} > 1 \qquad \textit{for} \quad S > S^*
\end{aligned} \tag{18}$$

*and*

$$\begin{aligned}
&\frac{E}{E^*} < \frac{f(S,I)}{f(S^*,I^*)} < 1 \qquad \textit{for} \quad E < E^*, \\
&\frac{E}{E^*} > \frac{f(S,I)}{f(S^*,I^*)} > 1 \qquad \textit{for} \quad E > E^*
\end{aligned} \tag{19}$$

*and*

$$\begin{aligned}
&\frac{E}{E^*} \leq \frac{T(I)}{T(I^*)} \qquad \textit{for} \quad \frac{E}{E^*} \leq \frac{I}{I^*}, \\
&\frac{E}{E^*} \geq \frac{T(I)}{T(I^*)} \qquad \textit{for} \quad \frac{E}{E^*} \geq \frac{I}{I^*}.
\end{aligned} \tag{20}$$

*Proof.* We consider the following Lyapunov function

$$L(t) = L_1(S(t)) + L_2(E(t)) + pL_3(I(t)), \tag{21}$$

where

$$L_1(S(t)) = S(t) - S^* - S^* ln \frac{S(t)}{S^*},$$
$$L_2(E(t)) = E(t) - E^* - E^* ln \frac{E(t)}{E^*},$$
$$L_3(I(t)) = I(t) - I^* - I^* ln \frac{I(t)}{I^*}.$$

Function $L$ is defined, continuous and positive definite for all $S(t) > 0$, $E(t) > 0$ and $I(t) > 0$. With the help of Lemma 2, we have

$$\begin{aligned} Ct_0 D_t^\alpha L(t) \leq & \left(1 - \frac{S^*}{S}\right) Ct_0 D_t^\alpha S(t) + \left(1 - \frac{E^*}{E}\right) Ct_0 D_t^\alpha E(t) \\ & + p\left(1 - \frac{I^*}{I}\right) Ct_0 D_t^\alpha I(t). \end{aligned}$$

Using the equations in system (1), one has

$$\begin{aligned} Ct_0 D_t^\alpha L(t) \leq & \left(1 - \frac{S^*}{S}\right)(\lambda - f(S, I) - \mu S) \\ & + \left(1 - \frac{E^*}{E}\right)(f(S, I) - \eta E + pT(I)) \\ & + p\left(1 - \frac{I^*}{I}\right)(\beta E - (\theta + \mu)I - T(I)). \end{aligned}$$

By the equilibrium conditions
$\lambda = f(S^*, I^*) + \mu S^*$, $\eta E^* = f(S^*, I^*) + pT(I^*)$ and $\beta E^* = (\theta + \mu)I^* + T(I^*)$ one can write

$$\begin{aligned} Ct_0 D_t^\alpha L(t) \leq & \left(1 - \frac{S^*}{S}\right) f(S^*, I^*) \\ & - \left(1 - \frac{S^*}{S}\right) f(S, I) + \mu S^* \left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right) \\ & + \left(1 - \frac{E^*}{E}\right)\left(f(S, I) - f(S^*, I^*)\frac{E}{E^*}\right) \\ & + \left(1 - \frac{E^*}{E}\right)\left(-pT(I^*)\frac{E}{E^*} + pT(I)\right) \\ & + p\left(1 - \frac{I^*}{I}\right)((\theta + \mu)I^* + T(I^*))\frac{E}{E^*} \\ & - p\left(1 - \frac{I^*}{I}\right)(\theta + \mu)I - p\left(1 - \frac{I^*}{I}\right)T(I). \end{aligned}$$

After some computations, we obtain

$$
\begin{aligned}
Ct_0 D_t^{\alpha} L(t) \leq\ & f(S^*, I^*) \left( (1 - \frac{S^*}{S}) - (1 - \frac{S^*}{S}) \frac{f(S,I)}{f(S^*, I^*)} \right) \\
& + f(S^*, I^*) \left( (1 - \frac{E^*}{E}) \frac{f(S,I)}{f(S^*, I^*)} \right) \\
& - f(S^*, I^*) \left( (1 - \frac{E^*}{E}) \frac{E}{E^*} \right) \\
& + \mu S^* \left( 2 - \frac{S}{S^*} - \frac{S^*}{S} \right) \\
& - \left( 1 - \frac{E^*}{E} \right) pT(I^*) \frac{E}{E^*} + \left( 1 - \frac{E^*}{E} \right) pT(I) \\
& + p \left( 1 - \frac{I^*}{I} \right) ((\theta + \mu) I^* + T(I^*)) \frac{E}{E^*} \\
& - p \left( 1 - \frac{I^*}{I} \right) (\theta + \mu) I - p \left( 1 - \frac{I^*}{I} \right) T(I).
\end{aligned}
$$

Moreover, we get

$$
\begin{aligned}
Ct_0 D_t^{\alpha} L(t) \leq\ & f(S^*, I^*) \left( 1 - \frac{S^*}{S} \right) \left( 1 - \frac{f(S,I)}{f(S^*, I^*)} \right) \\
& + f(S^*, I^*) \left( 1 - \frac{E^*}{E} \right) \left( \frac{f(S,I)}{f(S^*, I^*)} - \frac{E}{E^*} \right) \\
& + \mu S^* \left( 2 - \frac{S}{S^*} - \frac{S^*}{S} \right) \\
& + pT(I^*) \left( \frac{E^*}{E} - \frac{I^*}{I} \right) \left( \frac{E}{E^*} - \frac{T(I)}{T(I^*)} \right) \\
& + p(\theta + \mu) I^* \left( 2 - \frac{I}{I^*} - \frac{I^*}{I} \right).
\end{aligned}
$$

By Theorem hypotheses,

$$\left( 1 - \frac{S^*}{S} \right) \left( 1 - \frac{f(S,I)}{f(S^*, I^*)} \right) \leq 0$$

and

$$\left( 1 - \frac{E^*}{E} \right) \left( \frac{f(S,I)}{f(S^*, I^*)} - \frac{E}{E^*} \right) \leq 0,$$

where strict equality holds when $S = S^*$, $E = E^*$. Moreover,

$$\left( \frac{E^*}{E} - \frac{I^*}{I} \right) \left( \frac{E}{E^*} - \frac{T(I)}{T(I^*)} \right) \leq 0$$

is satisfied under the assumptions of the theorem. On the other hand,

$$\left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right) \leq 0$$

and

$$\left(2 - \frac{I}{I^*} - \frac{I^*}{I}\right) \leq 0,$$

for all $S, I > 0$, since the arithmetic mean-geometric mean inequality is satisfied. So, by Theorem 3.1 in [10], the positive endemic equilibrium point $\Sigma^* = (S^*, E^*, I^*)$ of system (1) is uniformly asymptotically stable. □

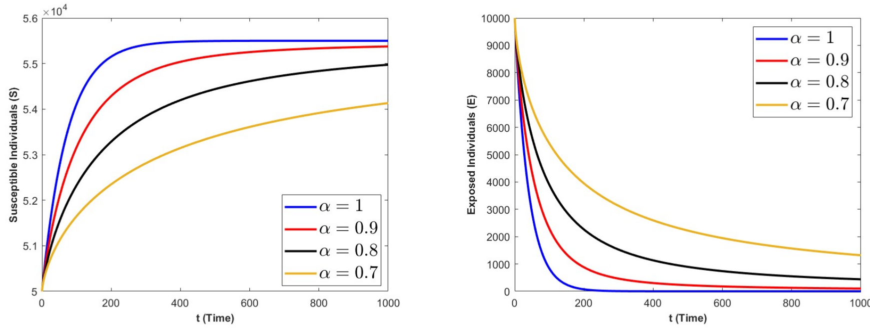
## 5. Numerical Simulations

In this section, we perform some numerical simulations to observe the results obtained in Section 4. We study system (1) for different values of the noninteger order derivative $\alpha$. For the numerical simulations, the Adams-Bashforth-Moulton scheme [12] is used in Matlab. The following Table 1 shows the parameters that are used in system (1). The assumed values $p$ and $q$ have been chosen randomly. The value of $r$ has taken as small due to the recovery rate of an exposed person should be small. The death rate is again randomly selected that one of the 5 patients died from the disease. The value of the parameter $w$ is also considered as small and variable in our numerical examples.

Table 1. Parameter values used in numerical simulations

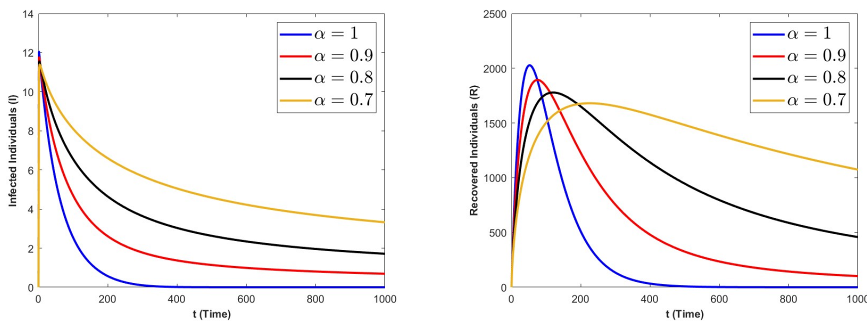
| **Parameter** | **Explanation** | **Value** |
|---|---|---|
| $\lambda$ | Recruitment rate | 792.8571 [47] |
| $\mu$ | Natural death rate | 1/70 [47] |
| $\beta$ | Incubation rate | 0.00368 [47] |
| $r$ | Recovery rate of exposed individuals | 0.01 (Assume) |
| $p$ | Unsuccessfully treated individuals | 0.8 (Assume) |
| $q$ | Successfully treated individuals | 0.2 (Assume) |
| $\theta$ | Death rate depending on the infection | 0.2 (Assume) |
| $w$ | Transmission coefficient | $5 \times 10^{-3}$ (Assume) |
| $r_1$ | Treatment function coefficient | 0.5 [11] |
| $r_2$ | Treatment function coefficient | 0.1 [11] |
| $\gamma$ | Transmission coefficient | 0.1 [19] |
| $\alpha_1$ | Incidence function parameter | 0.01 [19] |
| $\alpha_2$ | Incidence function parameter | 0.01 [19] |

As a first example, we have chosen bilinear incidence function $f(S, I) = wSI$ and the treatment function $T(I) = \dfrac{r_1 I^2}{1 + r_2 I}$ where $w$, $r_1$ and $r_2$ are positive parameters. The chosen incidence and treatment functions satisfy our hypotheses. The bilinear




$(a)$ $S(t)$. $(b)$ $E(t)$.

FIGURE 2. Trajectories for system (1) with parameters given in Table 1 and the initial conditions $S(0) = 50000$, $E(0) = 10000$, $I(0) = 0$ and $R(0) = 0$.

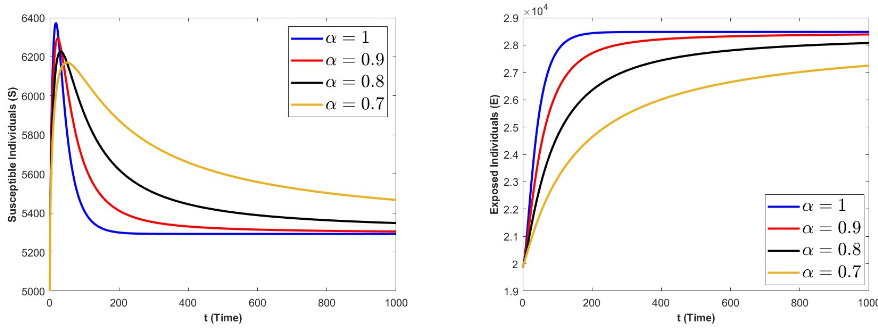



$(a)$ $I(t)$. $(b)$ $R(t)$.

FIGURE 3. Trajectories for system (1) with parameters given in Table 1 and the initial conditions $S(0) = 50000$, $E(0) = 10000$, $I(0) = 0$ and $R(0) = 0$.

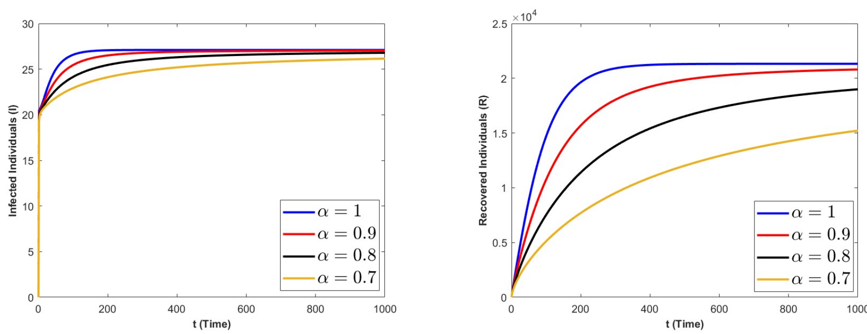
incidence function is mostly common in literature and it indicates that the rate of transmission increases as the number of infected people increases, that is, as the connection between the susceptible population and the infected population increases. On the other hand, the chosen treatment function is a monotonically increasing function for $I > 0$. On the next page, the readers can see another numerical example with a saturated treatment function $T(I) = \dfrac{r_1 I}{1 + r_2 I}$.

To obtain the value of $R_0$ smaller than one, first we choose the transmission coefficient $w = 5 \times 10^{-6}$ and find $R_0 = 0.1704 < 1$. In this case, the disease-free equilibrium point $E^0 = (S^0, 0, 0)$ where $S^0 = 55500$ is globally asymptotically stable which is depicted in Figure 2 and 3.




$(a)$ $S(t)$. $(b)$ $E(t)$.

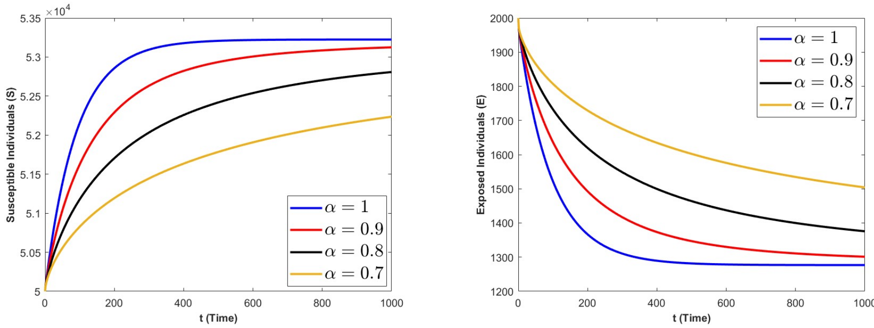
FIGURE 4. Trajectories for system (1) with parameters given in Table 1 and the initial conditions $S(0) = 5000$, $E(0) = 20000$, $I(0) = 0$ and $R(0) = 0$.




$(a)$ $I(t)$. $(b)$ $R(t)$.

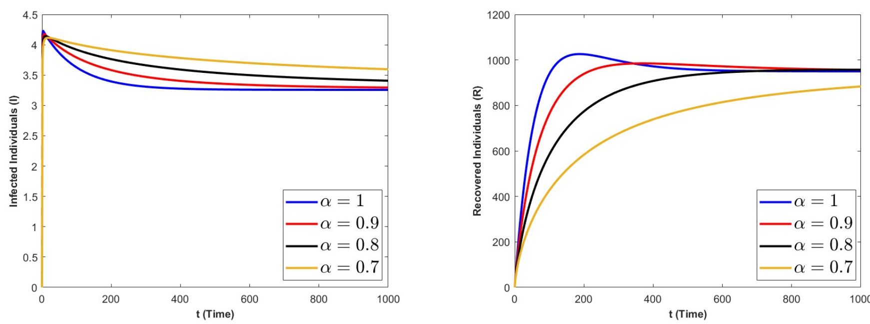
FIGURE 5. Trajectories for system (1) with parameters given in Table 1 and the initial conditions $S(0) = 5000$, $E(0) = 20000$, $I(0) = 0$ and $R(0) = 0$.

If we choose $w = 5 \times 10^{-3}$, we calculate $R_0 = 170.4087 > 1$. Then, with respect to the calculations in Theorem 2, one can find the approximate value of $I_0$ as 29.368 and the root of the function $H(I)$ as $I^* = 27.0856$. After that, $S^* = 5335.5$




$(a)$ $S(t)$. $(b)$ $E(t)$.

FIGURE 6. Trajectories for system (1) with parameters given in Table 1 and the initial conditions $S(0) = 50000$, $E(0) = 2000$, $I(0) = 0$ and $R(0) = 0$.

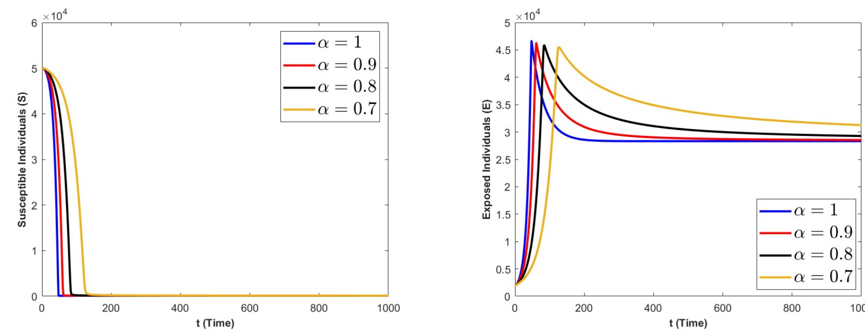



$(a)$ $I(t)$. $(b)$ $R(t)$.

FIGURE 7. Trajectories for system (1) with parameters given in Table 1 and the initial conditions $S(0) = 50000$, $E(0) = 2000$, $I(0) = 0$ and $R(0) = 0$.

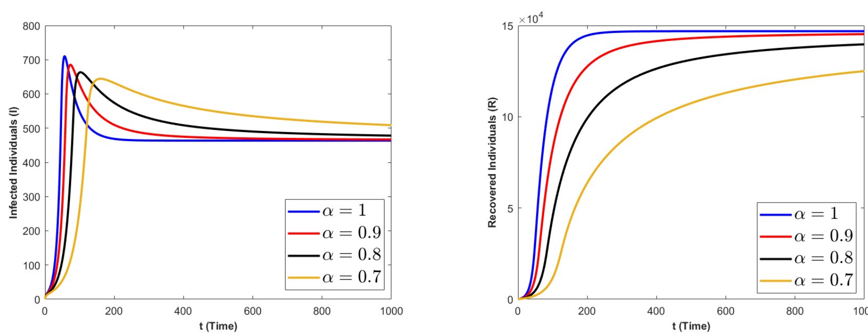
and $E^* = 28455$ can be easily obtained. According to Theorem 5 conditions the equilibrium point $\Sigma^* = (S^*, E^*, I^*)$ of system (1) is uniformly asymptotically stable which can be seen in Figure 4 and 5.

As a second example, we have changed the incidence function as the Beddington-DeAngelis infection rate, that is, $f(S, I) = \dfrac{\gamma SI}{1 + \alpha_1 S + \alpha_2 I}$. The parameters $\gamma$, $\alpha_1$ and $\alpha_2$ can be found in Table 1. As in the paper [16], the parameters $\gamma$, $\alpha_1$ and $\alpha_2$ can be thought as the transmission rate of the disease, a measure of inhibition for the susceptible population, and a measure of inhibition for the infected population, respectively. In this example, using these parameters, the positive equilibrium point is evaluated as $\Sigma^* = (S^*, E^*, I^*) = (53221, 1278.8, 3.26)$ with $R_0 = 6.1298 > 1$. The trajectories can be seen in Figure 6 and 7.




$(a)$ $S(t)$. $(b)$ $E(t)$.

FIGURE 8. Trajectories for system (1) with parameters given in Table 1 and the initial conditions $S(0) = 50000$, $E(0) = 2000$, $I(0) = 0$ and $R(0) = 0$.




$(a)$ $I(t)$. $(b)$ $R(t)$.

FIGURE 9. Trajectories for system (1) with parameters given in Table 1 and the initial conditions $S(0) = 50000$, $E(0) = 2000$, $I(0) = 0$ and $R(0) = 0$.

Additionally, let us take the saturated treatment function, $T(I) = \dfrac{r_1 I}{1 + r_2 I}$ in model (1) with the Beddington-DeAngelis infection function. In this case, the parameters $r_1$ and $r_2$ represent the treatment rate of disease and the limitation in treatment availability. This treatment function has a horizontal asymptote which shows the limitations in the capacity of treatment facilities. The numerical simulations can be found in Figure 8 and 9. Due to limited medical resources, after a while there will be almost no susceptible individuals left, and almost all people can be exposed, infected, or recovered from the disease. (Remember that we have a unique positive equilibrium point and in Figure 8 case (a), S(t) values are approximately 115 not zero.) If we increase the $r_2$ parameter, i.e., the limitations of medical resources, we can see that the number of infected people increases as we expected in Figure 10.

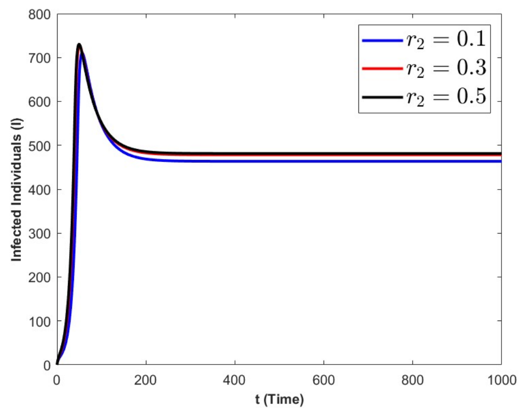


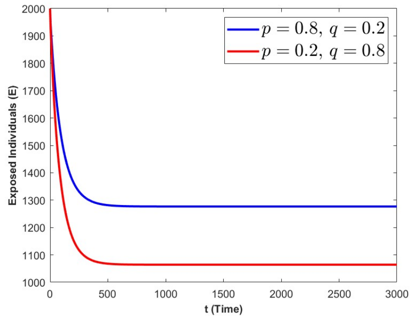

FIGURE 10. Trajectories for system (1) with parameters given in Table 1 and the initial conditions $S(0) = 50000$, $E(0) = 2000$, $I(0) = 0$ and $R(0) = 0$.

Following this second example, we also wondered the effect of the treatment success. As we expect, if the rate of successfully treated individuals $q$ increase, we observe a decrease in the number of the exposed and infected individuals in Figure 11.

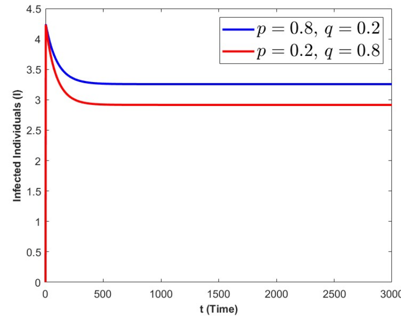
Finally, we need to mention our observation about the parameter $\alpha$. If we look carefully at all the figures, we will see that the system will be stable over a longer period of time if the parameter $\alpha$ decreases.

## 6. Concluding Remarks

In this paper, we have introduced a fractional order SEIR model with a general incidence function $f(S, I)$ and a general treatment function $T(I)$. By analysing the equilibrium points of system (1), we have shown that the disease free equilibrium point is locally asymptotically stable if the basic reproduction number $R_0 < 1$ and globally asymptotically stable if the inequality (15) is satisfied when $R_0 < 1$. We have also constructed a Lyapunov function and found that the endemic




(a) $E(t)$.




(b) $I(t)$.

FIGURE 11. Trajectories for system (1) with parameters given in Table 1 and changing parameters $p$ and $q$ and the initial conditions $S(0) = 50000$, $E(0) = 2000$, $I(0) = 0$ and $R(0) = 0$.

equilibrium point is uniformly asymptotically stable under the conditions (18)-(20). Moreover, in our model, based on the treatment model in [43], unsuccessfully treated infectious individuals re-enter the exposed compartment proportional to parameter $p$. Changing these parameters $p$ and $q$, we also have a chance to compare treatment success.

To the best of the author's knowledge, a fractional SEIR model with a general incidence function and treatment function has not been studied yet. In 2017, Elkhaiar et al. [11] studied the stability analysis of an ordinary differential equation system of the SEIR model with treatment. In that paper, they used a general incidence function and a general treatment function $T^*(I)$. In our paper, if the parameters are changed as $\mu = d$, $p = 0$, $q = 1$, $r = 0$, $\beta = \sigma$, $\theta = 0$ and $T(I) = \gamma I + T^*(I)$, then system (1) becomes to the SEIR model studied in [11] when $\alpha = 1$. Another paper including a stability analysis of a fractional order SEIR model is published in 2020 by Yang et. al. [48]. In our system, if the parameters are chosen as $\lambda = \Lambda^\alpha$, $\mu = d^\alpha$, $r = 0$, $\beta = \sigma^\alpha$, $\theta = 0$ and $qT(I) = \gamma^\alpha I$, $f(S, I) = \beta^\alpha F(S)G(I)$ then system (1) turns into the system that studied in [48]. In 2018, analysis of a fractional order SEIR model with treatment is established by Almeida [2]. If the parameters in our system (1) is chosen as $\lambda = bN$ ($N$ is assumed as fixed population size), $\mu = b$, $r = 0$, $\beta = \sigma$, $p = 0$, $q = 1$, $\theta = 0$ and $f(S, I) = \dfrac{\beta IS}{N}$ and $T(I) = (\mu + q)I$ ($\mu$ and $q$ are the parameters used in [2]) then again system (1) turns into that system used in [2]. Finally, the last example studied in [13] is related to outbreaks of influenza A(H1N1). In [13], the authors proposed a fractional order SEIR model to explain and understand the outbreaks of ifluenza A(H1N1). They used real data values and tested and simulated these data values for their model and chose the best fitted order $\alpha$ of fractional differentiation. In system (1), if the parameters are changed as $\lambda = \mu^\alpha$, $\mu = \mu^\alpha$, $\beta = \Omega^\alpha$, $p = q = \theta = 0$, $q = 1$, $f(S, I) = \beta^\alpha SI$ and $T(I) = \rho^\alpha I$, then one can obtain the system studied in this paper.

As a conclusion, the proposed model in this paper is rather general and as pointed out in [13], in the applications, one can choose the best fitted order of $\alpha$ depending on the real data values. If we look carefully at all the figures, we will see that the system will be stable over a longer period of time if the parameter $\alpha$ decreases.

**Declaration of Competing Interests** The author has no competing interests to declare.

## References


[1] Ahmed, E., El-Sayed, A. M. A., El-Saka, H. A. A., Equilibrium points, stability and numerical solutions of fractional-order predator-prey and rabies models, *Journal of Mathematical Analysis and Applications*, 325 (1) (2007), 542–553. https://doi: 10.1016/j.jmaa.2006.01.087

[2] Almeida, R., Analysis of a fractional SEIR model with treatment, *Applied Mathematics Letters*, 84 (2018), 56–62. https://doi: 10.1016/j.aml.2018.04.015

[3] Almeida, R., Brito da Cruz, A. M. C., Martins, N., Monteiro, M. T. T., An epidemiological MSEIR model described by the Caputo fractional derivative, *International Journal of Dynamics and Control*, 7 (2)(2019), 776–784. https://doi: 10.1007/s40435-018-0492-1

[4] Anderson, R. M., May, R. M., Infectious Diseases of Humans, Oxford: Oxford University Press, 1991.

[5] Atede, A. O., Omame, A., Inyama, S. C., A fractional order vaccination model for COVID-19 incorporating environmental transmission: a case study using Nigerian data, *Bulletin of Biomathematics*, 1 (1) (2023), 78—110. https://doi.org/10.59292/bulletinbiomath.2023005

[6] Bhattacharya, P., Paul, S., Biswas, P., Mathematical modeling of treatment SIR model with respect to variable contact rate, *International Proceedings of Economics Development and Research*, 83 (2015), 34–41.

[7] Brauer, F., Compartmental Models in Epidemiology. In: Brauer F., van den Driessche P., Wu J. (eds) Mathematical Epidemiology. Lecture Notes in Mathematics, vol 1945, Springer, Berlin, Heidelberg. https://doi: 10.1007/978-3-540-78911-6-2

[8] Brauer, F., Mathematical epidemiology: past, present, and future, *Infectious Disease Modelling*, 2 (2) (2017), 113–127. https://doi: 10.1016/j.idm.2017.02.001

[9] Castillo-Chavez, C., Feng, Z., To treat or not to treat: the case of tuberculosis transmission, *Journal of Mathematical Biology*, 35 (6) (1997), 629–656.

[10] Delavari, H., Baleanu, D., Sadati, J., Stability analysis of Caputo fractional-order nonlinear systems revisited, *Nonlinear Dynamics*, 67 (1) (2012), 2433–2439. https://doi: 10.1007/s11071-011-0157-5

[11] Elkhaiar, S., Kaddar, A., Stability analysis of an SEIR model with treatment, *Research in Applied Mathematics*, 1 (2017) article id 101266. https://doi: 10.11131/2017/101266

[12] Garrapa, R., Predictor-corrector PECE method for fractional differential equations, *MATLAB Central File Exchange*, 2012, File ID:32918.

[13] González-Parra, G., Arenas, A. J., Chen-Charpentier, B. M., A fractional order epidemic model for the simulation of outbreaks of influenza A(H1N1), *Mathematical Methods in Applied Sciences*, 37 (15) (2014), 2218–2226. https://doi: 10.1002/mma.2968

[14] Hethcote, H. W., Qualitative analyses of communicable disease model, *Mathematical Biosciences*, 28 (1976), 335–356.

[15] Hu, Z., Ma, W., Ruan, S., Analysis of SIR epidemic models with nonlinear incidence rate and treatment, *Mathematical Biosciences*, 238 (1) (2012), 1–20. https://doi: 10.1016/j.mbs.2012.03.010

[16] Joshi, H., Yavuz, M., Townley, S., Jha, B. K., Stability analysis of a non-singular fractional-order COVID-19 model with nonlinear incidence and treatment rate, *Physica Scripta*, 98 (4) (2023), 045216. https://doi: 10.1088/1402-4896/acbe7a

[17] Joshi, H., Jha, B. K., Chaos of calcium diffusion in Parkinson's infectious disease model and treatment mechanism via Hilfer fractional derivative, *Mathematical Modelling and Numerical Simulation with Applications*, 1 (2) (2021), 84–94. https://doi: 110.53391/mmnsa.2021.01.008

[18] Joshi, H., Jha, B. K., Yavuz, M., Modelling and analysis of fractional-order vaccination model for control of COVID-19 outbreak using real data, *Mathematical Biosciences and Engineering*, 20 (1) (2023), 213–240. https://doi: 10.3934/mbe.2023010

[19] Kaddar, A., Stability analysis in a delayed SIR epidemic model with a saturated incidence rate, *Nonlinear Analysis: Modelling and Control*, 15 (3) (2010), 299–306.

[20] Karaji, P. T., Nyamoradi, N., Analysis of a fractional SIR model with general incidence function, *Applied Mathematics Letters*, 108 (2020) 106499. https://doi: 10.1016/j.aml.2020.106499

[21] Kermack, W. O., McKendrick, A. G., A contribution to the mathematical theory of epidemics, *Proceedings of the Royal Society A*, 115 (772) (1927), 700–721. https://doi: 10.1098/rspa.1927.0118

[22] Korobeinikov, A., Lyapunov functions and global stability for SIR and SIRS epidemiological models with non-linear transmission, *Bulletin of Mathematical Biology*, 30 (2006), 615–626. https://doi: 10.1007/s11538-005-9037-9

[23] Korobeinikov, A., Global properties of infectious disease models with nonlinear incidence, *Bulletin of Mathematical Biology*, 69 (2007), 1871–1886. https://doi: 10.1007/s11538-007-9196-y

[24] Korobeinikov, A., Maini, P. K., Nonlinear incidence and stability of infectious disease models, *Mathematical Medicine and Biology: A Journal of the IMA*, 22 (2) (2005), 113–128.

[25] Korobeinikov, A., Wake, G. C., Lyapunov functions and global stability for SIR, SIRS and SIS epidemiological models, *Applied Mathematics Letters*, 15 (8) (2002), 955–961. https://doi: 10.1016/S0893-9659(02)00069-1

[26] Kuddus, M. A., Rahman, A., Analysis of COVID-19 using a modified SLIR model with nonlinear incidence, *Results in Physics*, 27 (2021), 104478. https://doi: 10.1016/j.rinp.2021.104478

[27] Li, M., Liu, X., An SIR epidemic model with time delay and general nonlinear incidence rate, *Abstract and Applied Analysis*, 2014 (2014), 131257. https://doi: 10.1155/2014/131257

[28] Li, X. Z., Li, W. S., Ghosh, M., Stability and bifurcation of an SIR epidemic model with nonlinear incidence and treatment, *Applied Mathematics and Computation*, 210 (1) (2009), 141–150. https://doi: 10.1016/j.amc.2008.12.085

[29] Lin, W., Global existence theory and chaos control of fractional differential equations, *Journal of Mathematical Analysis and Applications*, 332 (1) (2007), 709–726. https://doi: 10.1016/j.jmaa.2006.10.040

[30] Matignon, D., Stability results for fractional differential equations with applications to control processing, *In: Computational Engineering in Systems Applications*, 2 (1996), 963–968.

[31] McCluskey, C. C., Global stability of an SIR epidemic model with delay and general nonlinear incidence, *Mathematical Biosciences and Engineering*, 7 (4) (2010), 837–850. https://doi: 10.3934/mbe.2010.7.837

[32] Naik, P.A., Yavuz, M., Qureshi, S., Zu, J., Townley, S., Modeling and analysis of COVID-19 epidemics with treatment in fractional derivatives using real data from Pakistan, *The European Physical Journal Plus*, 135(795) (2020). https://doi:10.1140/epjp/s13360-020-00819-5

[33] Naik, P. A., Zu J., Owolabi K. M., Global dynamics of a fractional order model for the transmission of HIV epidemic with optimal control, *Chaos Solitons Fractals* 138, (2020), 109826.

[34] Naim, M., Sabbar, Y., Zeb, A., Stability characterization of a fractional-order viral system with the non-cytolytic immune assumption, *Mathematical Modelling and Numerical Simulation with Applications*, 2(3) (2022), 164–176. https://doi:10.53391/mmnsa.2022.013

[35] Odibat, Z. M., Momani, S., An algorithm for the numerical solution of differential equations of fractional order, *Journal of Applied Mathematics & Informatics*, 26(1-2) (2008), 15-27.

[36] Özköse, F., Yavuz, M., Investigation of interactions between COVID-19 and diabetes with hereditary traits using real data: A case study in Turkey, *Computers in Biology and Medicine*, 141 (2022), 105044, https://doi.org/10.1016/j.compbiomed.2021.105044

[37] Pinto, C. M. A., Carvalho, A. R. M., A latency fractional order model for HIV dynamics, *Journal of Computational and Applied Mathematics, ICMCMST 2015*, 312(2017) (2015), 240–256. https://doi: 10.1016/j.cam.2016.05.019

[38] Pinto, C. M. A., Tenreiro, Machado J. A., Fractional model for malaria transmission under control strategies, *Computers & Mathematics with Applications, Special issue: Fractional Differentiation and its Applications*, 66(5) (2013), 908–916. https://doi: 10.1016/j.camwa.2012.11.017

[39] Podlubny, I., Fractional Differential Equations, New York, Academic Press, 1999.

[40] Silva, C. J., Torres, D. F., Stability of a fractional HIV/AIDS model, *Mathematics and Computers in Simulation*, 164 (2019), 180–190. https://doi: 10.1016/j.matcom.2019.03.016

[41] Uçar, S., Existence and uniqueness results for a smoking model with determination and education in the frame of non-singular derivatives, *Discrete and Continuous Dynamical Systems-S*, 14(7) (2021), 2571–2589. https://doi: 10.3934/dcdss.2020178

[42] Ullah, S., Khan, M. A., Farooq, M., A fractional model for the dynamics of TB virus, *Chaos, Solitons & Fractals*, 116 (2018), 63–71. https://doi: 10.1016/j.chaos.2018.09.001

[43] van den Driessche, P., Watmough, J., Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, *Mathematical Biosciences*, 180(1) (2002), 29–48. https://doi: 10.1016/S0025-5564(02)00108-6

[44] Vargas-De-León, C., Volterra-type Lyapunov functions for fractional-order epidemic systems, *Communications in Nonlinear Science and Numerical Simulation*, 24(1) (2015), 75–85. https://doi: 10.1016/j.cnsns.2014.12.013

[45] Wang, J., Zhang, J,. Jin, Z., Analysis of an SIR model with bilinear incidence rate, *Nonlinear Analysis*, 11(4) (2010), 2390–2402. https://doi: 10.1016/j.nonrwa.2009.07.012

[46] Xu, C., Yu, Y., Chen, Y., Lu, Z., Forecast analysis of the epidemics trend of COVID-19 in the USA by a generalized fractional-order SEIR model, *Nonlinear Dynamics*, 101(3) (2020), 1621–1634. https://doi: 10.1007/s11071-020-05946-3

[47] Yang, Y., Li, J., Ma, Z., Liu, L., Global stability of two models with incomplete treatment for tuberculosis, *Chaos, Solitons & Fractals*, 43(1) (2010), 79–85. https://doi: 10.1016/j.chaos.2010.09.002

[48] Yang, Y., Xu, L., Stability of a fractional order SEIR model with general incidence, *Applied Mathematics Letters*, 105 (2020), 106303. https://doi: 10.1016/j.aml.2020.106303

[49] Yavuz, M., Sene, N., Stability analysis and numerical computation of the fractional predator–prey model with the harvesting rate. *Fractal and Fractional*, 4(3), (2020), 35. https://doi.org/10.3390/fractalfract4030035

[50] Zhang, F., Li, Z., Zhang, F., Global stability of an SIR epidemic model with constant infectious period, *Applied Mathematics and Computation*, 199(1) (2008), 285–291. https://doi: 10.1016/j.amc.2007.09.053

## INSTRUCTIONS TO CONTRIBUTORS

**Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics** (Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.) is a single-blind peer reviewed open access journal which has been published biannually since 1948 by Ankara University, accepts original research articles written in English in the fields of Mathematics and Statistics. It will be published four times a year from 2022. Review articles written by eminent scientists can also be invited by the Editor.

The publication costs for Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics are covered by the journal, so authors do not need to pay an article-processing and submission charges. The PDF copies of accepted papers are free of charges and can be downloaded from the website. Hard copies of the paper, if required, are due to be charged for the amount of which is determined by the administration each year.

All manuscripts should be submitted via our online submission system https://dergipark.org.tr/en/journal/2457/submission/step/manuscript/new. Note that only two submissions per author per year will be considered. Once a paper is submitted to our journal, all co-authors need to wait 6 months from the submission date before submitting another paper to Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. Manuscripts should be submitted in the PDF form used in the peer-review process together with the COVER LETTER and the TEX file (Source File). In the cover letter the authors should suggest the most appropriate Area Editor for the manuscript and potential four reviewers with full names, universities and institutional email addresses. Proposed reviewers must be experienced researchers in your area of research and at least two of them should be from different countries. In addition, proposed reviewers must not be co-authors, advisors, students, etc. of the authors. In the cover letter, the author may enter the name of anyone who he/she would prefer not to review the manuscript, with detailed explanation of the reason. Note that the editorial office may not use these nominations, but this may help to speed up the selection of appropriate reviewers.

Manuscripts should be typeset using the LATEX typesetting system. Authors should prepare the article using the Journal's templates (commun.cls and commun.cst). Manuscripts written in AMS LaTeX format are also acceptable. A template of manuscript can be downloaded in tex form from the link https://dergipark.org.tr/en/download/journal-file/22173(or can be reviewed in pdf form). The title page should contain the title of the paper, full names of the authors, affiliations addresses and e-mail addresses of all authors. Authors are also required to submit their Open Researcher and Contributor ID (ORCID)'s which can be obtained from http://orcid.org as their URL address in the format http://orcid.org/xxxx-xxxx-xxxx-xxxx. Please indicate the corresponding author. Each manuscript should be accompanied by classification numbers from the Mathematics Subject Classification 2020 scheme. The abstract should state briefly the purpose of the research. The length of the Abstract should be between 50 to 5000 characters. At least 3 keywords are required. Formulas should be numbered consecutively in the parentheses. All tables must have numbers (TABLE 1) consecutively in accordance with their appearance in the text and a legend above the table. Please submit tables as editable text not as images. All figures must have numbers (FIGURE 1) consecutively in accordance with their appearance in the text and a caption (not on the figure itself) below the figure. Please submit figures as EPS, TIFF or JPEG format. Authors Contribution Statement, Declaration of Competing Interests and Acknowledgements should be given at the end of the article before the references. Authors are urged to use the communication.bst style in BibTeX automated bibliography. If manual entry is preferred for bibliography, then all citations must be listed in the references part and vice versa. Number of the references (numbers in squared brackets) in the list can be in alphabetical order or in the order in which they appear in the text. Use of the DOI is highly encouraged. Formal abbreviations of the journals can be used. The Editor may seek the advice of two, or three referees, depending on the response of the referees, chosen in consultation with appropriate members of the Editorial Board, from among experts in the field of specialization of the paper. The reviewing process is conducted in strict confidence and the identity of a referee is not disclosed to the authors at any point since we use a single-blind peer review process.

 It is a fundamental condition that articles submitted to COMMUNICATIONS have not been previously published and will not be simultaneously submitted or published elsewhere. After the manuscript has been accepted for publication, the author will not be permitted to make any new additions to the manuscript. Before publication the galley proof is always sent to the author for correction. Thus it is solely the author's responsibility for any typographical mistakes which occur in their article as it appears in the Journal. The contents of the manuscript published in the COMMUNICATIONS are the sole responsibility of the authors.

Declarations/Ethics:
With the submission of the manuscript authors declare that:

- All authors of the submitted research paper have directly participated in the planning, execution, or analysis of study;
- All authors of the paper have read and approved the final version submitted;
- The contents of the manuscript have not been submitted, copyrighted or published elsewhere and the visual-graphical materials such as photograph, drawing, picture, and document within the article do not have any copyright issue;
- The contents of the manuscript will not be copyrighted, submitted, or published elsewhere, while acceptance by the Journal is under consideration.
- The article is clean in terms of plagiarism, and the legal and ethical responsibility of the article belongs to the author(s). Author(s) also accept that the manuscript may go through plagiarism check using iThenticate software;
- The objectivity and transparency in research, and the principles of ethical and professional conduct have been followed. Authors have also declared that they have no potential conflict of interest (financial or non-financial), and their research does not involve any human participants and/or animals.

Research papers published in **Communications Faculty of Sciences University of Ankara** are archived in the Library of Ankara University and in Dergipark immediately following publication with no embargo.

Editor in Chief
http://communications.science.ankara.edu.tr
Ankara University, Faculty of Sciences
06100, Besevler - ANKARA TURKEY

# C O M M U N I C A T I O N S

FACULTY OF SCIENCES
UNIVERSITY OF ANKARA

DE LA FACULTE DES SCIENCES
DE L'UNIVERSITE D'ANKARA

Series A1: Mathematics and Statistics

**Volume: 73** **Number: 1** **Year:2024**


## Research Articles


Bilender ALLAHVERDİEV, Hüseyin TUNA, Fractional Dirac systems with Mittag-Leffler kernel.................................... 1

Merve ÖZKAN, Murat ÖZDEMİR, İsa YİLDİRİM, On θ-convex contractive mappings with application to integral equations.................................................................................................................................................................... 13

Jiarul HOQUE, Shyamapada MODAK, A stronger form of locally closed set and its homeomorphic image........................ 25

Sure KÖME, Zeynep KUMTAS, Generalized bivariate conditional Fibonacci and Lucas hybrinomials.............................. 37

Ferhat TAŞ, Trajectory curves and surfaces: A new perspective via projective geometric algebra......................................... 64

Yasin ALTİNİSİK, Emel ÇANKAYA, Exponentiated generalized Ramos-Louzada distribution with properties and applications.................................................................................................................................................................... 76

Gutti V. R. BABU, Madugula VINOD KUMAR, PPF Dependent common fixed points of generalized weakly contractive type multi-valued mappings.................................................................................................................................... 104

Pranay GOSWAMI, Murli Manohar GOUR, Som Prakash GOYAL, Lokesh Kumar YADAV, Majorization property for certain classes of analytic functions associated with general operator.................................................................................. 122

Semih YILMAZ, Betül ERDOĞAN, Cholesky algorithm of a Lucas type matrix................................................................. 131

Zeynep YAPTI ÖZKURT, Normal automorphisms of free metabelian Leibniz algebras....................................................... 147

Rüya ÜSTER, Affine mappings and multipliers for weighted Orlicz spaces over an affine group $R_+ \times R$.............................. 153

Aykut HAS, Beyhan YILMAZ, Translation surfaces generating with some partner curve.................................................... 165

Busra AL, Mustafa ALKAN, Compositions of integers and Fibonacci numbers.................................................................... 178

Yogeeta NARWAL, Savita RATHEE, Fractional order mathematical modeling of lumpy skin disease................................ 192

Meltem ALTUN ÖZARSLAN, Semiregular, semiperfect and semipotent matrix rings relative to an ideal.......................... 211

Nour El Houda NESBA, Mohammed BEGGAS, Multigrid methods for non coercive variational inequalities.................... 222

Joseph Thomas EGHWERIDO, Lawrence NZEI, A weighted Gompertz-G family of distributions for reliability and lifetime data analysis.................................................................................................................................................. 235

Ahu ERCAN, Fractional approach for Dirac operator involving M-truncated derivative........................................................ 259

Şeyma TÜLÜCE DEMİRAY, Uğur BAYRAKCI, Vehpi YILDIRIM, Application of the GKM to some nonlinear partial equations.................................................................................................................................................................... 274

Esra KARAOĞLU, On the stability analysis of a fractional order epidemic model including the general forms of nonlinear incidence and treatment function............................................................................................................. 285