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Oscillation Criteria of Impulsive Partial Difference Equations

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Abstract. In this paper, some oscillation criteria of certain impulsive partial difference equations with continuous variables are established.

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1. INTRODUCTION

The impulsive differential equations are form a mathematical apparatus for modelling of processes which at certain moments of their development undergo rapid changes. There are many good monographs on the impulsive differential equations[2, 3, 4, 6, 7]. Moreover partial difference equations arise from considerations of random walk problems, the study of molecular orbits, mathematical physics problems and finite-difference schemes. In the recent years, the investigation of the oscillation of partial difference equations with continuous variables has attracted more and more attention in the literature, see e.g. [5, 8, 10, 11].
Let $0 = x_0 < x_1 < \ldots < x_n < x_{n+1} < \ldots$, be fixed points with $\lim_{n\to\infty} x_n = \infty$, and let for $n \in \mathbb{N}$ $x_{n+r} = x_n + \tau$, where r is a fixed natural number, and $\tau > 0$ is a constant. Define $J_{imp} = \{x_n\}_{n=1}^{\infty}$, $\mathbb{R}^+ = [0, \infty)$, $J = \{(x, y) : \ x \in J_{imp}, y \in \mathbb{R}^+\}$.
In 2009 Agarwal and Karakoc studied the oscillation of solutions of impulsive partial difference equations with continuous variables of the type

$$p_1 z(x+a, y+b) + p_2 z(x+a, y) + p_3 z(x, y+b) - p_4 z(x, y) + P(x, y) z(x-\tau, y-\sigma) = 0,$$

$$(x, y) \in (\mathbb{R}^+ \times \mathbb{R}^+) \setminus J,$$

$$z(x_n^+, y) - z(x_n^-, y) = L_n z(x_n^-, y), \qquad (x_n, y) \in J,$$

where $z(x^+,y)=\lim_{\substack{(q,s)\to(x,y)\\ q>x}} z(q,s)$, $z(x^-,y)=\lim_{\substack{(q,s)\to(x,y)\\ q<x}} z(q,s)$[2].
This paper is concerned with the impulsive partial difference equations with continuous variables of the form

$$\begin{aligned} & c_1u(x+a,y+b)+c_2u(x+a,y)+c_3u(x,y+b)-c_4u(x,y) \\ & +\sum_{i=1}^{\upsilon} P_i(x,y)u(x-\tau_i,y-\sigma_i)=0, \qquad (x,y)\in(\mathbb{R}^+\times\mathbb{R}^+)\setminus J, \end{aligned} \tag{1.1}$$

$$u(x_n^+,y)-u(x_n^-,y)=L_nu(x_n^-,y), \qquad (x_n,y)\in J, \tag{1.2}$$

where $u(x^+,y)=\lim_{\substack{(q,s)\to(x,y)\\ q>x}} u(q,s)$, $u(x^-,y)=\lim_{\substack{(q,s)\to(x,y)\\ q<x}} u(q,s)$ and $P_i\in C(\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+-\{0\})$, a,b,τ_i,σ_i are positive constants, c_i, $i=1,2,3,4$, are nonnegative.
A function $u(x,y)$ in $[-\tau,\infty)\times[-\sigma,\infty)$ is said to be solution of (1.1)-(1.2) if

(i) for $(x,y)\in(\mathbb{R}^+\times\mathbb{R}^+)\setminus J$, u is continuous and satisfies (1.1),
(ii) for $(x,y)\in J$, $u(x^+,y)$ and $u(x^-,y)$ exist, $u(x^-,y)=u(x,y)$, and satisfy (1.2).

A solution $u(x,y)$ of (1.1)-(1.2) said to be eventually positive (or negative) if $u(x,y)>0$ (or $u(x,y)<0$) for all large x and y. It is said to be oscillatory if it is neither eventually positive nor eventually negative otherwise, it is called nonoscillatory.

2. MAIN RESULTS

Throughout this paper we shall assume that

(A1) $\{L_n\}_{n=1}^{\infty}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} L_n<\infty$,
(A2) $c_2,c_3\geq c_4>0$,
(A3) $\tau_i=k_ia+\theta_i$, $\sigma_i=l_ib+\eta_i$, where k_i,l_i are nonnegative integers, $\theta_i\in[0,a)$ and $\eta_i\in[0,b)$.
(A4) $Q_i(x,y)=\min\{P_i(s,t): x\leq s\leq x+a, y\leq t\leq y+b\}$ and

$$\inf Q_i(x,y)=q_i\geq 0, \quad i=1,2,\ldots,\upsilon$$

Lemma 2.1. *Assume that $u(x,y)$ be an eventually positive solution of (1.1)-(1.2). Then for $(x,y)\in\mathbb{R}^+\times\mathbb{R}^+$ the function*

$$\omega(x,y)=\int_x^{x+a}\int_y^{y+b}\Big(\prod_{x_0<x_m<s}\big(1+L_m\big)^{-1}\Big)u(s,t)\mathrm{d}t\mathrm{d}s \tag{2.1}$$

is an eventually positive solution of the partial difference inequality

$$\begin{aligned} & c_1\omega(x+a,y+b)+c_2\omega(x+a,y)+c_3\omega(x,y+b) \\ & -c_4\prod_{x_0<x_m<x+a}(1+L_m)\omega(x,y)+\sum_{i=1}^{\upsilon}Q_i(x,y)\omega(x-k_ia,y-l_ib)\leq 0. \end{aligned} \tag{2.2}$$

Here the symbol $\prod_{x_0<x_m<s}a_m$ denotes the product of the members of the sequence $\{a_m\}$ over m such that $x_m\in(x_0,s)\cap J_{imp}$. If $(x_0,s)\cap J_{imp}=\varnothing$, or $x_0\geq s$, then we assume that $\prod_{x_0<x_m<s}a_m=1$.

Proof. So that

$$\prod_{x_0<x_m<x} \left(1+L_m\right)^{-1} u(x,y)$$

is continuous for $(x,y) \in \mathbb{R}^+ \times \mathbb{R}^+$,

$$\begin{aligned} \frac{\partial \omega}{\partial x} &= \int_y^{y+b} \Big[\prod_{x_0<x_m<x+a} \left(1+L_m\right)^{-1} u(x+a,t) - \prod_{x_0<x_m<x} \left(1+L_m\right)^{-1} u(x,t) \Big] \mathrm{d}t \\ &= \int_y^{y+b} \prod_{x_0<x_m<x} \left(1+L_m\right)^{-1} \Big[\prod_{x\le x_m<x+a} \left(1+L_m\right)^{-1} u(x+a,t) - u(x,t) \Big] \mathrm{d}t, \end{aligned} \tag{2.3}$$

$$\frac{\partial \omega}{\partial y} = \int_x^{x+a} \prod_{x_0<x_m<s} \left(1+L_m\right)^{-1} \Big[u(s,y+b) - u(s,y) \Big] \mathrm{d}s. \tag{2.4}$$

Since $u(x,y)$ is an eventually positive solution of (1.1)-(1.2),

$$c_1 u(x+a,y+b) + c_2 u(x+a,y) + c_3 u(x,y+b) - c_4 u(x,y) < 0 \tag{2.5}$$

for $(x,y) \in (\mathbb{R}^+ \times \mathbb{R}^+) \setminus J$. From (A2) and (2.5)

$$u(x+a,y) - u(x,y) < 0 \qquad \text{and} \qquad u(x,y+b) - u(x,y) < 0,$$

eventually. Moreover since $0 < \prod_{x\le x_m<x+a} \left(1+L_m\right)^{-1} \le 1$ we obtain,

$$\prod_{x\le x_m<x+a} \left(1+L_m\right)^{-1} u(x+a,y) - u(x,y) < 0 \tag{2.6}$$

Let $(x,y) \in J$ and say $x = x_n$. From (1.2) and (2.6), we have

$$\begin{aligned} u(x_n,y) &= \frac{1}{1+L_n} u(x_n^+,y) \\ &\ge \frac{1}{1+L_n} \prod_{x_n^+\le x_m<x_n^++a} \left(1+L_m\right)^{-1} u(x_n^++a,y) \\ &= \prod_{x_n\le x_m<x_n+a} \left(1+L_m\right)^{-1} u(x_n+a,y). \end{aligned}$$

Thus for all $(x,y) \in (\mathbb{R}^+ \times \mathbb{R}^+)$ $\frac{\partial \omega}{\partial x} \le 0$ and $\frac{\partial \omega}{\partial y} \le 0$. Therefore

$$\begin{aligned} \omega(x-\tau,y-\sigma) =& \omega(x-(ka+\theta), y-(lb+\eta)) \\ \ge& \omega(x-ka, y-lb). \end{aligned} \tag{2.7}$$

Since $0 < \prod_{x_0<x_m<x+a} \left(1+L_m\right)^{-1} \leq 1$ from (1.1) we obtain

(2.8)

$$\begin{aligned}
0 =& c_1 \int_x^{x+a} \int_y^{y+b} u(s+a,t+b)\mathrm{d}t\mathrm{d}s + c_2 \int_x^{x+a} \int_y^{y+b} u(s+a,t)\mathrm{d}t\mathrm{d}s \\
&+ c_3 \int_x^{x+a} \int_y^{y+b} u(s,t+b)\mathrm{d}t\mathrm{d}s - c_4 \int_x^{x+a} \int_y^{y+b} u(s,t)\mathrm{d}t\mathrm{d}s \\
&+ \int_x^{x+a} \int_y^{y+b} \sum_{i=1}^{\upsilon} P_i(s,t)u(s-\tau_i,t-\sigma_i)\mathrm{d}t\mathrm{d}s \\
\geq& c_1 \int_x^{x+a} \int_y^{y+b} \prod_{x_0<x_m<s} \left(1+L_m\right)^{-1} u(s+a,t+b)\mathrm{d}t\mathrm{d}s \\
&+ c_2 \int_x^{x+a} \int_y^{y+b} \prod_{x_0<x_m<s} \left(1+L_m\right)^{-1} u(s+a,t)\mathrm{d}t\mathrm{d}s \\
&+ c_3 \int_x^{x+a} \int_y^{y+b} \prod_{x_0<x_m<s} \left(1+L_m\right)^{-1} u(s,t+b)\mathrm{d}t\mathrm{d}s - c_4 \int_x^{x+a} \int_y^{y+b} u(s,t)\mathrm{d}t\mathrm{d}s \\
&+ \int_x^{x+a} \int_y^{y+b} \sum_{i=1}^{\upsilon} P_i(s,t) \prod_{x_0<x_m<s} \left(1+L_m\right)^{-1} u(s-\tau_i,t-\sigma_i)\mathrm{d}t\mathrm{d}s.
\end{aligned}$$

Using the definition of ω and Q_i from (2.7), (2.8) we have eventually

$$\begin{aligned}
c_1\omega(x+a,y+b) + c_2\omega(x+a,y) + c_3\omega(x,y+b) - c_4 \prod_{x_0<x_m<x+a} (1+L_m)\omega(x,y) \\
+ \sum_{i=1}^{\upsilon} Q_i(x,y)\omega(x-k_i a, y-l_i b) \leq 0.
\end{aligned}$$

Therefore $\omega(x,y)$ is an eventually positive solution of inequality (2.2). $\square$

Since $\omega(x,y) > 0$ from (2.2), for sufficiently large x and y, we have

$$\begin{aligned}
c_2\omega(x+a,y) &\leq c_4 \prod_{x_0<x_m<x+a} (1+L_m)\omega(x,y), \\
c_3\omega(x,y+b) &\leq c_4 \prod_{x_0<x_m<x+a} (1+L_m)\omega(x,y).
\end{aligned}$$

Let $\lambda_1 = 0$. Then for sufficiently large x and y, we find

$$\begin{aligned}
\omega(x-a,y) &\geq e^{-\lambda_1} \frac{c_2}{c_4} \prod_{x_0<x_m<x} \left(1+L_m\right)^{-1} \omega(x,y) \\
\omega(x-k_i a,y) &\geq e^{-k_i\lambda_1} \left(\frac{c_2}{c_4}\right)^{k_i} \prod_{j=1}^{k_i} \left(\prod_{x_0<x_m<x-(j-1)a} \left(1+L_m\right)^{-1} \right) \omega(x,y)
\end{aligned}$$

and

$$\omega(x, y-b) \geq e^{-\lambda_1}\frac{c_3}{c_4}\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-1}\omega(x,y)$$
$$\omega(x, y-l_ib) \geq e^{-l_i\lambda_1}\big(\frac{c_3}{c_4}\big)^{l_i}\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l_i}\omega(x,y).$$

Therefore

$$\begin{aligned}\omega(x-k_ia, y-l_ib) &\geq e^{-k_i\lambda_1}\big(\frac{c_2}{c_4}\big)^{k_i}\prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\omega(x,y-l_ib)\\ &\geq e^{-(k_i+l_i)\lambda_1}\big(\frac{c_2}{c_4}\big)^{k_i}\big(\frac{c_3}{c_4}\big)^{l_i}\prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\\ &\times\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l_i}\omega(x,y).\end{aligned}$$

From (2.2) and ((A4)) we obtain

(2.9)

$$\begin{aligned}c_1\omega(x+a,y+b)+c_2\omega(x+a,y)+c_3\omega(x,y+b)-c_4\prod_{x_0<x_m<x+a}(1+L_m)\omega(x,y)\\ +\sum_{i=1}^{\upsilon}q_i\omega(x-k_ia,y-l_ib)\leq 0.\end{aligned}$$

Hence

$$\begin{aligned}c_2\omega(x+a,y) \leq c_4\prod_{x_0<x_m<x+a}(1+L_m)\Big[1-\frac{1}{c_4}\sum_{i=1}^{\upsilon}q_ie^{-(k_i+l_i)\lambda_1}\big(\frac{c_2}{c_4}\big)^{k_i}\big(\frac{c_3}{c_4}\big)^{l_i}\\ \times\prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-(l_i+1)}\Big]\omega(x,y)\end{aligned}$$

and

$$\begin{aligned}c_3\omega(x,y+b) \leq c_4\prod_{x_0<x_m<x+a}(1+L_m)\Big[1-\frac{1}{c_4}\sum_{i=1}^{\upsilon}q_ie^{-(k_i+l_i)\lambda_1}\big(\frac{c_2}{c_4}\big)^{k_i}\big(\frac{c_3}{c_4}\big)^{l_i}\\ \times\prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-(l_i+1)}\Big]\omega(x,y),\end{aligned}$$

eventually.
Let

$$\begin{aligned}e^{\lambda_2}=1-\frac{1}{c_4}\sum_{i=1}^{\upsilon}q_ie^{-(k_i+l_i)\lambda_1}\big(\frac{c_2}{c_4}\big)^{k_i}\big(\frac{c_3}{c_4}\big)^{l_i}\prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\\ \times\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-(l_i+1)}.\end{aligned}$$

It is clear that $\lambda_2 \le \lambda_1 = 0$.
Thus
$$\frac{c_2}{c_4}\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-1}\omega(x+a,y) \le e^{\lambda_2}\omega(x,y)$$
and
$$\frac{c_3}{c_4}\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-1}\omega(x,y+b) \le e^{\lambda_2}\omega(x,y),$$
eventually.
Hence
$$\omega(x-k_ia,y) \ge e^{-k_i\lambda_2}\big(\frac{c_2}{c_4}\big)^{k_i}\prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\omega(x,y)$$
and
$$\omega(x,y-l_ib) \ge e^{-l_i\lambda_2}\big(\frac{c_3}{c_4}\big)^{l_i}\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l_i}\omega(x,y).$$
By induction, for $n\ge 1$,
$$\omega(x-a,y) \ge e^{-\lambda_n}\frac{c_2}{c_4}\prod_{x_0<x_m<x}\big(1+L_m\big)^{-1}\omega(x,y)$$
$$\omega(x-k_ia,y) \ge e^{-k_i\lambda_n}\big(\frac{c_2}{c_4}\big)^{k_i}\prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\omega(x,y)$$
and
$$\omega(x,y-b) \ge e^{-\lambda_n}\frac{c_3}{c_4}\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-1}\omega(x,y)$$
$$\omega(x,y-l_ib) \ge e^{-l_i\lambda_n}\big(\frac{c_3}{c_4}\big)^{l_i}\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l_i}\omega(x,y),$$
where
$$e^{\lambda_n} = 1-\frac{1}{c_4}\sum_{i=1}^{\upsilon}q_ie^{-(k_i+l_i)\lambda_{n-1}}\big(\frac{c_2}{c_4}\big)^{k_i}\big(\frac{c_3}{c_4}\big)^{l_i}\prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)$$
$$\times\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-(l_i+1)}.$$
Remark that $-\infty<\lambda_n\le\lambda_{n-1}\le\ldots\le\lambda_1=0$. Hence $\lim_{n\to\infty}\lambda_n=\lambda^*<0$ exists and

(2.10)
$$e^{\lambda^*} = 1-\frac{1}{c_4}\sum_{i=1}^{\upsilon}q_ie^{-(k_i+l_i)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i}\big(\frac{c_3}{c_4}\big)^{l_i}\prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)$$
$$\times\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-(l_i+1)}.$$

Thus

(2.11)

$$\omega(x-k_ia,y-l_ib)\geq e^{-(k_i+l_i)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i}\big(\frac{c_3}{c_4}\big)^{l_i}\prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big) \times\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l_i}\omega(x,y)$$

From (2.2)

(2.12)

$$c_4\prod_{x_0<x_m<x+a}(1+L_m)\omega(x,y)\geq c_1\omega(x+a,y+b)+c_2\omega(x+a,y)+c_3\omega(x,y+b) +\sum_{i=1}^{\upsilon}Q_i(x,y)\omega(x-k_ia,y-l_ib).$$

From (2.11), we obtain

$$\omega(x-k_ia,y-l_ib)\geq e^{-(k_i+l_i-1)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i-1}\big(\frac{c_3}{c_4}\big)^{l_i} \times\prod_{j=1}^{k_i-1}\Big(\prod_{x_0<x_m<x-ja}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l_i}\omega(x-a,y)$$

and

$$\omega(x,y)\geq\frac{1}{c_4}\sum_{i=1}^{\upsilon}Q_i(x,y)e^{-(k_i+l_i-1)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i-1}\big(\frac{c_3}{c_4}\big)^{l_i} \times\prod_{j=1}^{k_i-1}\Big(\prod_{x_0<x_m<x-ja}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-(l_i+1)}\omega(x-a,y).$$

Therefore

$$\omega(x+a,y)\geq\frac{1}{c_4}\sum_{i=1}^{\upsilon}Q_i(x+a,y)e^{-(k_i+l_i-1)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i-1}\big(\frac{c_3}{c_4}\big)^{l_i} \times\prod_{j=1}^{k_i-1}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+2a}\big(1+L_m\big)^{-(l_i+1)}\omega(x,y).$$

By the same way we have

$$\omega(x,y+b)\geq\frac{1}{c_4}\sum_{i=1}^{\upsilon}Q_i(x,y+b)e^{-(k_i+l_i-1)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i}\big(\frac{c_3}{c_4}\big)^{l_i-1} \times\prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l_i}\omega(x,y)$$

and

$$\omega(x+a,y+b) \geq \frac{1}{c_4}\sum_{i=1}^{\upsilon} Q_i(x+a,y+b)e^{-(k_i+l_i-2)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i-1}\big(\frac{c_3}{c_4}\big)^{l_i-1} \times \prod_{j=1}^{k_i-1}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+2a}\big(1+L_m\big)^{-l_i}\omega(x,y).$$

Substituting the above inequalities into (2.12), we obtain

(2.13)

$$\begin{aligned} c_4\prod_{x_0<x_m<x+a}\big(1+L_m\big)\omega(x,y) \geq{}& \frac{c_1}{c_4}\sum_{i=1}^{\upsilon} Q_i(x+a,y+b)e^{-(k_i+l_i-2)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i-1}\big(\frac{c_3}{c_4}\big)^{l_i-1} \\ &\times \prod_{j=1}^{k_i-1}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+2a}\big(1+L_m\big)^{-l_i}\omega(x,y) \\ &+\frac{c_2}{c_4}\sum_{i=1}^{\upsilon} Q_i(x+a,y)e^{-(k_i+l_i-1)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i-1}\big(\frac{c_3}{c_4}\big)^{l_i} \\ &\times \prod_{j=1}^{k_i-1}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+2a}\big(1+L_m\big)^{-(l_i+1)}\omega(x,y) \\ &+\frac{c_3}{c_4}\sum_{i=1}^{\upsilon} Q_i(x,y+b)e^{-(k_i+l_i-1)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i}\big(\frac{c_3}{c_4}\big)^{l_i-1} \\ &\times \prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l_i}\omega(x,y) \\ &+\sum_{i=1}^{\upsilon} Q_i(x,y)e^{-(k_i+l_i-1)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i-1}\big(\frac{c_3}{c_4}\big)^{l_i} \\ &\times \prod_{j=1}^{k_i-1}\Big(\prod_{x_0<x_m<x-ja}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l_i}\omega(x-a,y). \end{aligned}$$

Thus

$$\Big(c_4\prod_{x_0<x_m<x+a}(1+L_m)-\frac{c_1}{c_4}\sum_{i=1}^{\upsilon}Q_i(x+a,y+b)e^{-(k_i+l_i-2)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i-1}\big(\frac{c_3}{c_4}\big)^{l_i-1}$$
$$\times\prod_{j=1}^{k_i-1}\Big(\prod_{x_0<x_m<x-(j-1)a}(1+L_m)^{-1}\Big)\prod_{x_0<x_m<x+2a}(1+L_m)^{-l_i}$$
$$-\frac{c_2}{c_4}\sum_{i=1}^{\upsilon}Q_i(x+a,y)e^{-(k_i+l_i-1)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i-1}\big(\frac{c_3}{c_4}\big)^{l_i}$$
$$\times\prod_{j=1}^{k_i-1}\Big(\prod_{x_0<x_m<x-(j-1)a}(1+L_m)^{-1}\Big)\prod_{x_0<x_m<x+2a}(1+L_m)^{-(l_i+1)}$$
$$-\frac{c_3}{c_4}\sum_{i=1}^{\upsilon}Q_i(x,y+b)e^{-(k_i+l_i-1)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i}\big(\frac{c_3}{c_4}\big)^{l_i-1}$$
$$\times\prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}(1+L_m)^{-1}\Big)\prod_{x_0<x_m<x+a}(1+L_m)^{-l_i}\Big)\omega(x,y)$$
$$\geq\sum_{i=1}^{\upsilon}Q_i(x,y)e^{-(k_i+l_i-1)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i-1}\big(\frac{c_3}{c_4}\big)^{l_i}$$
$$\times\prod_{j=1}^{k_i-1}\Big(\prod_{x_0<x_m<x-ja}(1+L_m)^{-1}\Big)\prod_{x_0<x_m<x+a}(1+L_m)^{-l_i}\omega(x-a,y).$$

Set

(2.14)

$$R(x,y)=c_4\prod_{x_0<x_m<x+a}(1+L_m)-\frac{c_1}{c_4}\sum_{i=1}^{\upsilon}Q_i(x+a,y+b)e^{-(k_i+l_i-2)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i-1}\big(\frac{c_3}{c_4}\big)^{l_i-1}$$
$$\times\prod_{j=1}^{k_i-1}\Big(\prod_{x_0<x_m<x-(j-1)a}(1+L_m)^{-1}\Big)\prod_{x_0<x_m<x+2a}(1+L_m)^{-l_i}$$
$$-\frac{c_2}{c_4}\sum_{i=1}^{\upsilon}Q_i(x+a,y)e^{-(k_i+l_i-1)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i-1}\big(\frac{c_3}{c_4}\big)^{l_i}$$
$$\times\prod_{j=1}^{k_i-1}\Big(\prod_{x_0<x_m<x-(j-1)a}(1+L_m)^{-1}\Big)\prod_{x_0<x_m<x+2a}(1+L_m)^{-(l_i+1)}$$
$$-\frac{c_3}{c_4}\sum_{i=1}^{\upsilon}Q_i(x,y+b)e^{-(k_i+l_i-1)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i}\big(\frac{c_3}{c_4}\big)^{l_i-1}$$
$$\times\prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}(1+L_m)^{-1}\Big)\prod_{x_0<x_m<x+a}(1+L_m)^{-l_i}.$$

Then from (2.13), we obtain

(2.15)

$$\omega(x,y) \geq \frac{1}{R(x,y)} \sum_{i=1}^{\upsilon} Q_i(x,y) e^{-(k_i+l_i-1)\lambda^*} \Big(\frac{c_2}{c_4}\Big)^{k_i-1} \Big(\frac{c_3}{c_4}\Big)^{l_i} \times \prod_{j=1}^{k_i-1} \Bigg(\prod_{x_0<x_m<x-ja} \big(1+L_m\big)^{-1} \Bigg) \prod_{x_0<x_m<x+a} \big(1+L_m\big)^{-l_i} \omega(x-a,y).$$

Similarly, we obtain

(2.16)

$$\omega(x,y) \geq \frac{1}{R(x,y)} \sum_{i=1}^{\upsilon} Q_i(x,y) e^{-(k_i+l_i-1)\lambda^*} \Big(\frac{c_2}{c_4}\Big)^{k_i} \Big(\frac{c_3}{c_4}\Big)^{l_i-1} \times \prod_{j=1}^{k_i} \Bigg(\prod_{x_0<x_m<x-(j-1)a} \big(1+L_m\big)^{-1} \Bigg) \prod_{x_0<x_m<x+a} \big(1+L_m\big)^{-(l_i-1)} \omega(x,y-b)$$

and

(2.17)

$$\omega(x,y) \geq \frac{1}{R(x,y)} \sum_{i=1}^{\upsilon} Q_i(x,y) e^{-(k_i+l_i-2)\lambda^*} \Big(\frac{c_2}{c_4}\Big)^{k_i-1} \Big(\frac{c_3}{c_4}\Big)^{l_i-1} \times \prod_{j=1}^{k_i-1} \Bigg(\prod_{x_0<x_m<x-ja} \big(1+L_m\big)^{-1} \Bigg) \prod_{x_0<x_m<x+a} \big(1+L_m\big)^{-(l_i-1)} \omega(x-a,y-b).$$

From (2.12), we have

$$
\begin{aligned}
1 \geq & \frac{c_1}{c_4 \prod_{x_0<x_m<x+a}(1+L_m)} \frac{\omega(x+a,y+b)}{\omega(x,y)} + \frac{c_2}{c_4 \prod_{x_0<x_m<x+a}(1+L_m)} \frac{\omega(x+a,y)}{\omega(x,y)} \\
& + \frac{c_3}{c_4 \prod_{x_0<x_m<x+a}(1+L_m)} \frac{\omega(x,y+b)}{\omega(x,y)} \\
& + \frac{1}{c_4 \prod_{x_0<x_m<x+a}(1+L_m)} \sum_{i=1}^{\upsilon} Q_i(x,y) \frac{\omega(x-k_i a, y-l_i b)}{\omega(x,y)} \\
\geq & \frac{c_1}{c_4} \frac{1}{R(x+a,y+b)} \sum_{i=1}^{\upsilon} Q_i(x+a,y+b) e^{-(k_i+l_i-2)\lambda^*} \left(\frac{c_2}{c_4}\right)^{k_i-1} \left(\frac{c_3}{c_4}\right)^{l_i-1} \\
& \times \prod_{j=1}^{k_i} \left(\prod_{x_0<x_m<x-(j-2)a} (1+L_m)^{-1} \right) \prod_{x_0<x_m<x+2a} (1+L_m)^{-(l_i-1)} \\
& + \frac{c_2}{c_4} \frac{1}{R(x+a,y)} \sum_{i=1}^{\upsilon} Q_i(x+a,y) e^{-(k_i+l_i-1)\lambda^*} \left(\frac{c_2}{c_4}\right)^{k_i-1} \left(\frac{c_3}{c_4}\right)^{l_i} \\
& \times \prod_{j=1}^{k_i} \left(\prod_{x_0<x_m<x-(j-2)a} (1+L_m)^{-1} \right) \prod_{x_0<x_m<x+2a} (1+L_m)^{-l_i} \\
& + \frac{c_3}{c_4} \frac{1}{R(x,y+b)} \sum_{i=1}^{\upsilon} Q_i(x,y+b) e^{-(k_i+l_i-1)\lambda^*} \left(\frac{c_2}{c_4}\right)^{k_i} \left(\frac{c_3}{c_4}\right)^{l_i-1} \\
& \times \prod_{j=1}^{k_i} \left(\prod_{x_0<x_m<x-(j-1)a} (1+L_m)^{-1} \right) \prod_{x_0<x_m<x+a} (1+L_m)^{-l_i} \\
& + \frac{1}{c_4} \sum_{i=1}^{\upsilon} Q_i(x,y) e^{-(k_i+l_i)\lambda^*} \left(\frac{c_2}{c_4}\right)^{k_i} \left(\frac{c_3}{c_4}\right)^{l_i} \\
& \times \prod_{j=1}^{k_i} \left(\prod_{x_0<x_m<x-(j-1)a} (1+L_m)^{-1} \right) \prod_{x_0<x_m<x+a} (1+L_m)^{-(l_i+1)} = T(x,y),
\end{aligned} \tag{2.18}
$$

for all large x and y.
From (2.18), we obtain the following main result.

Theorem 2.2. *Suppose that*

$$
\limsup_{x,y\to\infty} T(x,y) > 1. \tag{2.19}
$$

Then every solution of (1.1)-(1.2) is oscillatory.

From (2.10), we obtain for all large x and y,

$$
e^{\lambda^*} = 1 - \frac{1}{c_4} \sum_{i=1}^{\upsilon} q_i e^{-(k_i+l_i)\lambda^*} \left(\frac{c_2}{c_4}\right)^{k_i} \left(\frac{c_3}{c_4}\right)^{l_i} L^{-(k_i+l_i+1)},
$$

where $L=\prod_{m=1}^{\infty}\big(1+L_m\big)$.
Therefore

$$c_4L\big(1-e^{\lambda^*}\big)=\sum_{i=1}^{\upsilon}q_ie^{-(k_i+l_i)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i}\big(\frac{c_3}{c_4}\big)^{l_i}L^{-(k_i+l_i)}, \tag{2.20}$$

eventually.
By using (2.14) and (2.20), we have, for all large x and y,

$$\begin{aligned}
R(x,y)\leq& c_4\prod_{x_0<x_m<x+a}\big(1+L_m\big)-\frac{c_1}{c_4}\sum_{i=1}^{\upsilon}q_ie^{-(k_i+l_i-2)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i-1}\big(\frac{c_3}{c_4}\big)^{l_i-1}\\
&\times\prod_{j=1}^{k_i-1}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+2a}\big(1+L_m\big)^{-l_i}\\
&-\frac{c_2}{c_4}\sum_{i=1}^{\upsilon}q_ie^{-(k_i+l_i-1)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i-1}\big(\frac{c_3}{c_4}\big)^{l_i}\\
&\times\prod_{j=1}^{k_i-1}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+2a}\big(1+L_m\big)^{-(l_i+1)}\\
&-\frac{c_3}{c_4}\sum_{i=1}^{\upsilon}q_ie^{-(k_i+l_i-1)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i}\big(\frac{c_3}{c_4}\big)^{l_i-1}\\
&\times\prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l_i}\\
\leq& c_4L-\frac{c_1}{c_4}c_4L^2\big(1-e^{\lambda^*}\big)e^{2\lambda^*}\frac{c_4^2}{c_2c_3}\\
&-\frac{c_2}{c_4}c_4L\big(1-e^{\lambda^*}\big)e^{\lambda^*}\frac{c_4}{c_2}-\frac{c_3}{c_4}c_4L\big(1-e^{\lambda^*}\big)e^{\lambda^*}\frac{c_4}{c_3}\\
=&c_4L\Big[1-\big(1-e^{\lambda^*}\big)e^{\lambda^*}\Big(\frac{c_1c_4}{c_2c_3}Le^{\lambda^*}+2\Big)\Big].
\end{aligned}$$

Hence

(2.21)

$$T(x,y) \geq \frac{1}{c_4L\Big[1-(1-e^{\lambda^*})e^{\lambda^*}\Big(\frac{c_1c_4}{c_2c_3}Le^{\lambda^*}+2\Big)\Big]}\Big\{\frac{c_1}{c_4}c_4L^2(1-e^{\lambda^*})e^{2\lambda^*}\frac{c_4^2}{c_2c_3}$$

$$+\frac{c_2}{c_4}c_4L(1-e^{\lambda^*})e^{\lambda^*}\frac{c_4}{c_2}+\frac{c_3}{c_4}c_4L(1-e^{\lambda^*})e^{\lambda^*}\frac{c_4}{c_3}\Big\}+\frac{1}{c_4}\sum_{i=1}^{\upsilon}Q_i(x,y)e^{-(k_i+l_i)\lambda^*}$$

$$\times\Big(\frac{c_2}{c_4}\Big)^{k_i}\Big(\frac{c_3}{c_4}\Big)^{l_i}\prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}(1+L_m)^{-1}\Big)\prod_{x_0<x_m<x+a}(1+L_m)^{-(l_i+1)}$$

$$=\frac{(1-e^{\lambda^*})e^{\lambda^*}\Big(\frac{c_1c_4}{c_2c_3}Le^{\lambda^*}+2\Big)}{1-(1-e^{\lambda^*})e^{\lambda^*}\Big(\frac{c_1c_4}{c_2c_3}Le^{\lambda^*}+2\Big)}+\frac{1}{c_4}\sum_{i=1}^{\upsilon}Q_i(x,y)e^{-(k_i+l_i)\lambda^*}\Big(\frac{c_2}{c_4}\Big)^{k_i}\Big(\frac{c_3}{c_4}\Big)^{l_i}$$

$$\times\prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}(1+L_m)^{-1}\Big)\prod_{x_0<x_m<x+a}(1+L_m)^{-(l_i+1)}$$

Corollary 2.3. *If*

$$\limsup_{x,y\to\infty}\frac{1}{c_4}\sum_{i=1}^{\upsilon}Q_i(x,y)e^{-(k_i+l_i)\lambda^*}\Big(\frac{c_2}{c_4}\Big)^{k_i}\Big(\frac{c_3}{c_4}\Big)^{l_i}$$

$$\times\prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}(1+L_m)^{-1}\Big)\prod_{x_0<x_m<x+a}(1+L_m)^{-(l_i+1)} \tag{2.22}$$

$$>\frac{1-2(1-e^{\lambda^*})e^{\lambda^*}\Big(\frac{c_1c_4}{c_2c_3}Le^{\lambda^*}+2\Big)}{1-(1-e^{\lambda^*})e^{\lambda^*}\Big(\frac{c_1c_4}{c_2c_3}Le^{\lambda^*}+2\Big)},$$

then every solution of (1.1)-(1.2) *is oscillatory.*

From (2.18),

$$T(x,y)\geq\frac{(1-e^{\lambda^*})e^{\lambda^*}\Big(\frac{c_1c_4}{c_2c_3}Le^{\lambda^*}+2\Big)}{1-(1-e^{\lambda^*})e^{\lambda^*}\Big(\frac{c_1c_4}{c_2c_3}Le^{\lambda^*}+2\Big)}+(1-e^{\lambda^*}), \tag{2.23}$$

for all large x and y.

Corollary 2.4. *If*

$$\frac{(1-e^{\lambda^*})e^{\lambda^*}\Big(\frac{c_1c_4}{c_2c_3}Le^{\lambda^*}+2\Big)}{1-(1-e^{\lambda^*})e^{\lambda^*}\Big(\frac{c_1c_4}{c_2c_3}Le^{\lambda^*}+2\Big)}+(1-e^{\lambda^*})>1, \tag{2.24}$$

then every solution of (1.1)-(1.2) *is oscillatory.*

Remark that the special case of (1.1): $\upsilon = 1$, $k = l = 0$, $c_1 = c_2 = c_3 = c_4 = 0$. From Theorem 9 of [2], if

$$\limsup_{x,y\to\infty} Q(x,y) > L$$

then every solution of (1.1)-(1.2) is oscillatory.
In this case, $e^{\lambda^*} = 1 - \frac{q}{L}$. By Corollary 2.3, if

$$\limsup_{x,y\to\infty} Q(x,y) > L\frac{1 - 2\frac{q}{L}\left(1-\frac{q}{L}\right)\left(L-q+2\right)}{1 - \frac{q}{L}\left(1-\frac{q}{L}\right)\left(L-q+2\right)}, \tag{2.25}$$

then every solution of (1.1)-(1.2) is oscillatory.
It is clear that the right-hand side of (2.25) is less than L. So our result is sharper than that in [2] in this case.

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