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Solving a Second Order Fuzzy Initial Value Problem Using The Heaviside Function

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Abstract. In this paper, we reformulate the algorithm in [7] to find an analytical expression for α-cuts of the solution of the second order nonhomogeneous fuzzy initial value problem with fuzzy initial values and fuzzy forcing terms. Firstly, we apply Zadeh's Extension Principle to fuzzify the crisp initial value problem. Then, we use the Heaviside function and obtain the analytical form of α-cuts of the solution of the fuzzy initial value problem. Finally, we illustrate some examples using the proposed algorithm.

Keywords: Fuzzy initial value problem, fuzzy forcing function, Zadeh's extension principle, heaviside function, breaking point, structure vector.

1. Introduction

In this paper, we examine the fuzzy solution of the following second order fuzzy initial value problem through Zadeh's Extension Principle:

$$y''(x) + a_1 y'(x) + a_2 y(x) = \sum_{i=1}^{r} \overline{b}_i g_i(x);$$
$$y(0) = \overline{\gamma}_0;\ y'(0) = \overline{\gamma}_1 \tag{1.1}$$

Here a_1 and a_2 are crisp constants and g_i $(i = 1, ..., r)$ are continuous functions on the interval $[0, \infty)$. The initial values $\overline{\gamma}_0, \overline{\gamma}_1$ and forcing coefficients $\overline{b}_i$ $(i = 1, \ldots, r)$ are fuzzy numbers. Eq. (1.1) is called the second order fuzzy initial value problem.

The idea of the fuzzy number and fuzzy arithmetic was firstly introduced by Zadeh [19]. The term "fuzzy differential equation" was firstly coined in 1978 by Kandel and Byatt [8]. There were many suggestions to define the fuzzy derivative concept. One of the earliest suggestions was to generalize the Hukuhara derivative of a set-valued function. This generalization was made by Puri and Ralescu [17] and studied by Kaleva [15]. It soon appeared that the solution of the fuzzy differential equation interpreted by Hukuhara derivative has a drawback: It becomes fuzzier as time goes. Hence, the fuzzy solution behaves quite differently from the crisp solution. To alleviate the situation, Hüllermeier [14] interpreted the fuzzy differential equation as a family of differential inclusions. Another approach to solve fuzzy differential equations is to use Zadeh's extension principle [16]. The basic idea of the Zadeh's extension principle is firstly to

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consider the given fuzzy initial value problem as a crisp initial value problem and then to find its crisp solution. After getting the deterministic solution, the fuzzy solution is obtained by applying the extension principle to the deterministic solution. In [5,6] strongly generalized derivative concept was introduced. And in [3], the authors studied higher order fuzzy differential equations with the strongly generalized derivative concept. Some other researchers extensively investigated the fuzzy initial value problems [2,4,9–13,18,20]. Buckley and Feuring [7] considered the initial value problem for the n-th order fuzzy differential equations. The only fuzzy numbers were the initial values in their problem. In [1], Akın et al. investigated a similar fuzzy initial value problem, which has fuzzy initial conditions and fuzzy forcing coefficients as fuzzy numbers, according to the signs of the solution and its first and second order derivatives.

In this paper, we reformulate the approach in [7] to find an analytical form of α-cuts for the solution of the fuzzy initial value problem for the second order differential equation using the Heaviside step function. The new algorithm allows us to find the solution of the fuzzy initial value problem without considering the signs of the first and second derivatives of the solution and the sign of the solution itself [1,7].

Before we give the algorithm, let us first introduce the notation which we will be used throughout the paper. All our fuzzy sets will be fuzzy subsets of the real numbers. We place a bar over a lower-case letter to denote a fuzzy number. We also write $\overline{Y}(x)$, $\overline{A}(x)$ and $\overline{B}(x)$ for fuzzy valued functions defined on real numbers. An α-cut of a fuzzy valued function $Y(x)$ will be denoted by $Y(x)[\alpha]$, $\alpha \in (0, 1]$.

Definition 1.1. The function

$$\theta(x) := \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

is called the Heaviside step function.

Definition 1.2. Let $A_i(x)$ be real valued functions $(i = 0, ..., n)$. The vector valued function

$$\overrightarrow{\Phi}(x) = (\theta(A_0(x)), \theta(A_1(x)), ..., \theta(A_n(x)))$$

is called a structure vector of length $n + 1$.

Definition 1.3. Let $N_\delta(x^*)$ denotes the δ-neighborhood of x^*. If for any $\delta > 0$ there exists a point $x \in N_\delta(x^*)$ such that $\overrightarrow{\Phi}(x) - \overrightarrow{\Phi}(x^*) \neq 0$, then the point x^* is called a breaking point.

The rest of the paper is organized as follows. Section 2 contains the idea of obtaining the solution of a second order nonhomogeneous fuzzy initial value problem using the Heaviside function. In section 3, some numerical examples are presented for the illustration of the proposed algorithm.

2. Setting The Algorithm For Fuzzy Initial Value Problems

We try to solve the following type of the fuzzy initial value problem:

$$y''(x) + a_1 y'(x) + a_2 y(x) = \sum_{i=1}^{r} \bar{b}_i g_i(x);$$
$$y(0) = \bar{\gamma}_0;\ y'(0) = \bar{\gamma}_1 \tag{2.1}$$

Here $\bar{\gamma}_0$, $\bar{\gamma}_1$ and $\bar{b}_i$ $(i = 1, \ldots, r)$ are fuzzy numbers, a_1 and a_2 are crisp constants, and $g_i(x)$ $(i = 1, \ldots, r)$ are continuous functions on the interval $[0, \infty)$. We will firstly solve the following crisp initial value problem related to the fuzzy initial value problem in Eq. (2.1) and then apply Zadeh's Extension Principle to the solution [7]:

$$y''(x) + a_1 y'(x) + a_2 y(x) = \sum_{i=1}^{r} b_i g_i(x);$$
$$y(0) = \gamma_0;\ y'(0) = \gamma_1 \tag{2.2}$$

Here γ_0, γ_1 and b_i $(i = 1, \ldots, r)$ are real (i.e., crisp) numbers. The general solution of the differential equation (2.2) can be written as:

$$Y(x) = c_1 y_1(x) + c_2 y_2(x) + \sum_{i=1}^{r} b_i G_i(x); \tag{2.3}$$

where c_1 and c_2 are arbitrary constants; $y_1(x)$ and $y_2(x)$ are linearly independent functions satisfying the homogeneous part of Eq. (2.2) and $G_i(x)$ $(i = 1, ..., r)$ are particular solutions for the following differential equations:

$$y''(x) + a_1 y'(x) + a_2 y(x) = g_i(x); (i = 1, ..., r).$$

Let us next obtain the solution of the crisp initial value problem (2.2). To do this, we introduce the initial conditions to the general solution given by Eq. (2.3). Therefore, we obtain the following system of equations:

$$\begin{cases} c_1 y(0) + c_2 y_2(0) + \sum_{i=1}^{r} b_i G_i(0) = \gamma_0 \\ c_1 y_1'(0) + c_2 y_2'(0) + \sum_{i=1}^{r} b_i G_i'(0) = \gamma_1 \end{cases} \tag{2.4}$$

In Eq. (2.4), c_1 and c_2 are the unknowns. Here after, we use the following notations for the sake of shortness.

$$\begin{aligned} W &= \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}; \\ w_{11} &= y_1(0), w_{12} = y_2(0), w_{21} = y_1'(0), w_{22} = y_2'(0); \\ \overrightarrow{c} &= \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \overrightarrow{\gamma} = \begin{pmatrix} \gamma_0 \\ \gamma_1 \end{pmatrix}, \overrightarrow{\Delta}_i = \begin{pmatrix} \Delta_{0i} \\ \Delta_{1i} \end{pmatrix}; \\ \Delta_{0i} &= G_i(0), \Delta_{1i} = G_i'(0); i = 1, ..., r. \end{aligned}$$

According to these notations, we write (2.4) in the matrix form:

$$W\overrightarrow{c} = \overrightarrow{\gamma} - \sum_{i=1}^{r} b_i \overrightarrow{\Delta}_i.$$

Using Cramer's method, we obtain c_1 and c_2 as follows:

$$c_j = \frac{|W_{1j}|}{|W|} - \frac{|W_{2j}|}{|W|}; j = 1, 2.$$

Here

$$\begin{aligned} |W| &= \begin{vmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{vmatrix} = w_{11}w_{22} - w_{21}w_{12}; \\ |W_{11}| &= \begin{vmatrix} \gamma_0 & w_{12} \\ \gamma_1 & w_{22} \end{vmatrix} = \gamma_0 w_{22} - \gamma_1 w_{12}; \\ |W_{12}| &= \begin{vmatrix} w_{11} & \gamma_0 \\ w_{21} & \gamma_1 \end{vmatrix} = \gamma_1 w_{11} - \gamma_0 w_{21}; \\ |W_{21}| &= \begin{vmatrix} \sum_{i=1}^{r} b_i \Delta_{0i} & w_{12} \\ \sum_{i=1}^{r} b_i \Delta_{1i} & w_{22} \end{vmatrix} = \sum_{i=1}^{r} b_i(\Delta_{0i} w_{22} - \Delta_{1i} w_{12}); \\ |W_{22}| &= \begin{vmatrix} w_{11} & \sum_{i=1}^{r} b_i \Delta_{0i} \\ w_{21} & \sum_{i=1}^{r} b_i \Delta_{1i} \end{vmatrix} = \sum_{i=1}^{r} b_i(\Delta_{1i} w_{11} - \Delta_{0i} w_{21}). \end{aligned}$$

Thus, $c_1$and c_2 can be rewritten as

$$\begin{aligned} c_1 &= \frac{|W_{11}| - |W_{21}|}{|W|} = \frac{\gamma_0 w_{22} - \gamma_1 w_{12} - \sum_{i=1}^{r} b_i(\Delta_{0i} w_{22} - \Delta_{1i} w_{12})}{|W|}, \\ c_2 &= \frac{|W_{12}| - |W_{22}|}{|W|} = \frac{\gamma_1 w_{11} - \gamma 0 w_{21} - \sum_{i=1}^{r} b_i(\Delta_{1i} w_{11} - \Delta_{0i} w_{21})}{|W|}. \end{aligned}$$

To simplify the results above, c_1 and c_2 can be rewritten in the following form, respectively:

$$c_1 = \gamma_0 f_{22} - \gamma_1 f_{12} + \sum_{i=1}^{r} b_i(\Delta_{1i} f_{12} - \Delta_{0i} f_{22})$$

and

$$c_2 = \gamma_1 f_{11} - \gamma_0 f_{21} + \sum_{i=1}^{r} b_i(\Delta_{0i} f_{21} - \Delta_{1i} f_{11})$$

where $f_{ij} = \frac{w_{ij}}{|W|}; i, j = 1, 2$.
From the results for c_1 and c_2, we can now derive the classical solution of the given crisp initial value problem as follows:

$$\begin{aligned} Y(x) &= c_1 y_1(x) + c_2 y_2(x) + \sum_{i=1}^{r} b_i G_i(x) \\ &= (\gamma_0 f_{22} - \gamma_1 f_{12} + \sum_{i=1}^{r} b_i(\Delta_{1i} f_{12} - \Delta_{0i} f_{22}))y_1(x) \\ &\quad + (\gamma_1 f_{11} - \gamma_0 f_{21} + \sum_{i=1}^{r} b_i(\Delta_{0i} f_{21} - \Delta_{1i} f_{11}))y_2(x) + \sum_{i=1}^{r} b_i G_i(x). \end{aligned}$$

This solution can also be written as:

$$\begin{aligned} Y(x) &= \gamma_0 \left(f_{22} y_1(x) - f_{21} y_2(x)\right) + \gamma_1 (f_{11} y_2(x) - f_{12} y_1(x)) \\ &\quad + \sum_{i=1}^{r} b_i(G_i(x) + y_1(x)(\Delta_{1i} f_{12} - \Delta_{0i} f_{22}) + y_2(x)(\Delta_{0i} f_{21} - \Delta_{1i} f_{11})). \end{aligned}$$

Next we use the following notations for the sake of its comprehension:

$$\begin{aligned} A_0(x) &= f_{22} y_1(x) - f_{21} y_2(x),\ A_1(x) = f_{11} y_2(x) - f_{12} y_1(x); \\ B_i(x) &= G_i(x) + y_1(x)(\Delta_{1i} f_{12} - \Delta_{0i} f_{22}) + y_2(x)(\Delta_{0i} f_{21} - \Delta_{1i} f_{11}) \end{aligned} \tag{2.5}$$

where $i = 1, ..., r$. Thus the solution of the crisp initial value problem can be written as:

$$Y(x) = \gamma_0 A_0(x) + \gamma_1 A_1(x) + \sum_{i=1}^{r} b_i B_i(x) \tag{2.6}$$

It can be easily seen that the solution in Eq. (2.6) is linearly dependent only on the initial values and forcing part of Eq. (2.2). Now, we apply Zadeh's Extension Principle and write the solution of the fuzzy initial value problem as follows:

$$\bar{Y}(x) = \bar{\gamma}_0 A_0(x) + \bar{\gamma}_1 A_1(x) + \sum_{i=1}^{r} \bar{b}_i B_i(x) \tag{2.7}$$

where $\alpha-$cuts ($\alpha \in (0, 1]$) of $\bar{\gamma}_i$, $\bar{b}_i$ and $\bar{Y}(x)$ are defined as follows:

$$\begin{aligned} \bar{\gamma}_i[\alpha] &= [\gamma_{iL}(\alpha), \gamma_{iU}(\alpha)]; \\ \bar{b}_i[\alpha] &= [b_{iL}(\alpha), b_{iU}(\alpha)]; \\ \bar{Y}(x)[\alpha] &= [Y_L(x, \alpha), Y_U(x, \alpha)]. \end{aligned}$$

Here $\gamma_{iL}(\alpha)$, $b_{iL}(\alpha)$, and $Y_L(x, \alpha)$ are lower bounds and $\gamma_{iU}(\alpha)$, $b_{iU}(\alpha)$, and $Y_U(x, \alpha)$ are the upper bounds of $\alpha-$cuts, respectively.

By taking these α-cuts into account in the solution (2.7), we obtain the following result:

$$[Y_L(x, \alpha), Y_U(x, \alpha)] = \sum_{i=0}^{1} [\gamma_{iL}(\alpha), \gamma_{iU}(\alpha)] A_i(x) + \sum_{i=1}^{r} [b_{iL}(\alpha), b_{iU}(\alpha)] B_i(x)$$

where

$$Y_L(x,\alpha) = \sum_{i=0}^{1} \min\{[\gamma_{iL}(\alpha), \gamma_{iU}(\alpha)]A_i(x)\} + \sum_{i=1}^{r} min\{[b_{iL}(\alpha), b_{iU}(\alpha)]B_i(x)\},$$
$$Y_U(x,\alpha) = \sum_{i=0}^{1} max\{[\gamma_{iL}(\alpha), \gamma_{iU}(\alpha)]A_i(x)\} + \sum_{i=1}^{r} max\{[b_{iL}(\alpha), b_{iU}(\alpha)]B_i(x)\}.$$

Here the min and max are evaluated for each $x \geq 0$ and $\alpha \in (0, 1]$. Using the Heaviside function, we can write the α-cuts of the solution of $\bar{Y}(x)$ as follows:

$$\begin{aligned} Y_L(x,\alpha) = &\sum_{i=0}^{1}[\gamma_{iU}(\alpha) - (\gamma_{iU}(\alpha) - \gamma_{iL}(\alpha))\theta(A_i(x))]A_i(x) \\ &+ \sum_{i=1}^{r}[b_{iU}(\alpha) - (b_{iU}(\alpha) - b_{iL}(\alpha))\theta(B_i(x))]B_i(x) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} Y_U(x,\alpha) = &\sum_{i=0}^{1}[\gamma_{iL}(\alpha) + (\gamma_{iU}(\alpha) - \gamma_{iL}(\alpha))\theta(A_i(x))]A_i(x) \\ &+ \sum_{i=1}^{r}[b_{iL}(\alpha) + (b_{iU}(\alpha) - b_{iL}(\alpha))\theta(B_i(x))]B_i(x). \end{aligned} \tag{2.9}$$

It is known from Akın et al. [7] and Buckley et al. [1] that analytical forms of the α-cuts for the solution of a fuzzy initial value problem depends on the behavior of $Y(x)$ and derivatives of $Y(x)$. This new formulation gives us the α-cuts of the solution of the fuzzy initial value problem in Eq. (2.1) without considering the signs of the first and second derivatives of the solution and the sign of the solution itself.

3. Numerical Examples

In this section, we observe three different second order fuzzy differential equations with fuzzy initial values and fuzzy forcing coefficients using the proposed algorithm. In each case, the initial values and forcing coefficients are uncertain and modelled by triangular fuzzy numbers in the form of $\bar{a} = (a_1, a_2, a_3)$. We will find the α-cuts of the solutions by using Eq. (2.8) and Eq. (2.9). We will also find the breaking points and investigate the functions causing those breaking points by observing the structure vector $\overrightarrow{\Phi}(x)$.

3.1. **Example:** Consider the second order linear fuzzy differential equation:

$y'' + 3y' + 2y = \bar{b}_1 x^2 + \bar{b}_2 \cos x$ with fuzzy initial values $\bar{\gamma}_0 = (0, 1, 2)$, $\bar{\gamma}_1 = (4, 5, 7)$, and fuzzy forcing coefficients $\bar{b}_1 = (1, 2, 3)$, $\bar{b}_2 = (50, 100, 150)$.
Let us first solve the crisp initial value problem:

$y'' + 3y' + 2y = 2x^2 + 100 \cos x;$
$y(0) = 1,\ y\prime(0) = 5$
Since $g_1(x) = x^2$ and $g_2(x) = \cos x$, we obtain $G_1(x)$ and G_2 (x) as follows:

$\mathrm{G}_1(x) = 0.5x^2 - 1.5x + 1.75;$
$\mathrm{G}_2(x) = 0.1 \cos x + 0.3 \sin x.$

By solving the differential equation in the crisp initial value problem, we obtain the general crisp solution as:

$Y(x) = c_1e^{-2x} + c_2e^{-x} + x^2 - 3x + 3.5 + 10\cos x + 30\sin x.$

The functions $A_0(x)$, $A_1(x)$, $B_1(x)$ and $B_2(x)$ are obtained as follows:

$$\begin{aligned}
A_0(x) &= 2e^{-x} - e^{-2x},\\
A_1(x) &= e^{-x} - e^{-2x},\\
B_1(x) &= 0.5x^2 - 1.5x + 1.75 + 0.25e^{-2x} - 2e^{-x},\\
B_2(x) &= 0.1\cos x + 0.3\sin x + 0.4e^{-2x} - 0.5e^{-x}.
\end{aligned}$$

According to Eq. (2.8) and Eq. (2.9), the α-cuts of the solution can be expressed as follows:

$$\begin{aligned}
Y_L(x,\alpha) = {} & [2-\alpha-2(1-\alpha)\theta(A_0(x))]A_0(x) + [4+\alpha+3(1-\alpha)\theta(A_1(x))]A_1(x)\\
& + [1+\alpha+2(1-\alpha)\theta(B_1(x))]B_1(x) + [50+50\alpha+100(1-\alpha)\theta(B_2(x))]B_2(x),
\end{aligned}$$

$$\begin{aligned}
Y_U(x,\alpha) = {} & [\alpha+2(1-\alpha)\theta(A_0(x))]A_0(x) + [4+\alpha+3(1-\alpha)\theta(A_1(x))]A_1(x)\\
& + [1+\alpha+2(1-\alpha)\theta(B_1(x))]B_1(x) + [50+50\alpha+100(1-\alpha)\theta(B_2(x))]B_2(x).
\end{aligned}$$

We have also calculated that

$$\begin{aligned}
\overrightarrow{\Phi}(2.721) - \overrightarrow{\Phi}(2.722) &= (1,1,1,1) - (1,1,1,0) \neq \overrightarrow{0},\\
\overrightarrow{\Phi}(5.965) - \overrightarrow{\Phi}(5.966) &= (1,1,1,0) - (1,1,1,1) \neq \overrightarrow{0},\\
\overrightarrow{\Phi}(9.102) - \overrightarrow{\Phi}(9.103) &= (1,1,1,1) - (1,1,1,0) \neq \overrightarrow{0}
\end{aligned} \tag{3.1}$$

where $\overrightarrow{\Phi}(x) = (\theta(A_0(x)), \theta(A_1(x)), \theta(B_1(x)), \theta(B_2(x)))$. Hence the points $x_1^* = 2.721$, $x_2^* = 5.965$ and $x_3^* = 9.102$ are breaking points in the interval $[0, 10]$. As well, as seen from (3.1), we can conclude that

a) At the point $x_1^* = 2.721$ *the sign of the function* $B_2(x)$ *changes from positive to negative.*

b) At the point $x_2^* = 5.965$ *the sign of the function* $B_2(x)$ *changes from negative to positive.*

c) At the point $x_3^* = 9.102$ *the sign of the function* $B_2(x)$ *changes from positive to negative.*

In Figure 1 one can easily see that at the breaking points, there are corners on the α-cuts of the fuzzy solutions.

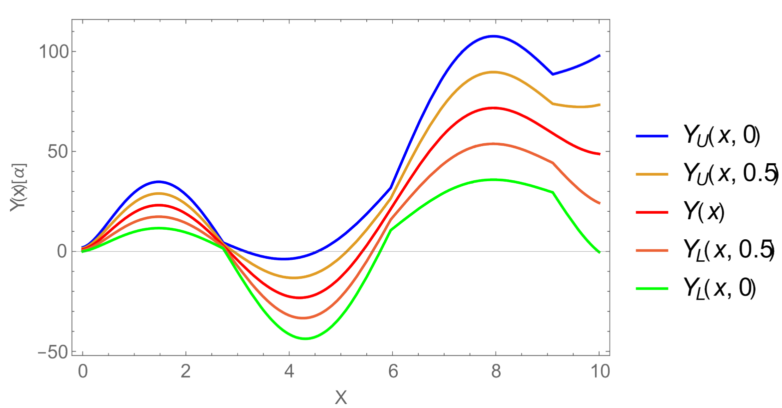

Figure 1. The $\alpha-$cuts of $\overline{Y}(x)$ for different values of α for Example 1.

In Figure 2, we illustrate the α-cuts of $\bar{Y}(x)$ for the fuzzy numbers $b_1 = (1,2,3); b_2 = (99,100,101); \gamma_0 = (0,1,2)$

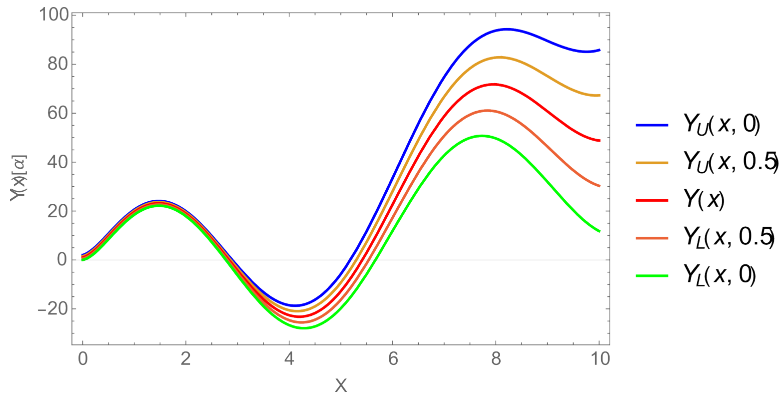

FIGURE 2. The $\alpha-$ cuts of $\overline{Y}(x)$ for different values of α for Example 1, but with $b_1 = (1, 2, 3); b_2 = (99, 100, 101); \gamma_0 = (0, 1, 2)$ and $\gamma_1 = (4, 5, 6)$.

3.2. **Example:** Consider the second order linear fuzzy differential equation :

$y'' - 3y' + 2y = \bar{b}_1 x^2 + \bar{b}_2 \cos x$ with fuzzy initial values $\bar{\gamma}_0 = (0, 1, 2)$, $\bar{\gamma}_1 = (4, 5, 7)$, and fuzzy forcing coefficients $\bar{b}_1 = (1, 2, 3)$, $\bar{b}_2 = (50, 100, 150)$.
Let us first solve the crisp initial value problem:

$y'' - 3y' + 2y = 2x^2 + 100 \cos x$;
$y(0) = 1, y\prime(0) = 5$.

Since $g_1(x) = x^2$ and $g_2(x) = \cos x$, we obtain $G_1(x)$ and G_2 (x) as follows:

$\mathrm{G}_1(x) = 0.5x^2 - 1.5x + 1.75$;
$\mathrm{G}_2(x) = 0.1 \cos x + 0.3 \sin x$.

By solving the differential equation in the crisp initial value problem, we obtain the general crisp solution as:

$Y(x) = c_1 e^x + c_2 e^{2x} + x^2 + 3x + 3.5 + 10 \cos x - 30 \sin x$

The functions $A_0(x)$, $A_1(x)$, $B_1(x)$ and $B_2(x)$ in Eq. (2.5) can be calculated as:

$$\begin{aligned}
A_0(x) &= 2e^x - e^{2x}, \\
A_1(x) &= e^{2x} - e^x, \\
B_1(x) &= 0.5x^2 + 1.5x + 1.75 - 2e^x + 0.25e^{2x}, \\
B_2(x) &= 0.1 \cos(x) - 0.3 \sin(x) - 0.5e^x + 0.4e^{2x}.
\end{aligned}$$

According to Eq. (2.8) and Eq. (2.9), the α-cuts of the solution can be expressed as follows:

$$\begin{aligned}
Y_L(x, \alpha) = {} & [2 - \alpha - 2(1-\alpha)\theta(A_0(x))]A_0(x) + [7 - 2\alpha - 3(1-\alpha)\theta(A_1(x))]A_1(x) \\
& + [3 - \alpha - 2(1-\alpha)\theta(B_1(x))]B_1(x) + [150 - 50\alpha - 100(1-\alpha)\theta(B_2(x))]B_2(x),
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
Y_U(x, \alpha) = {} & [\alpha + 2(1-\alpha)\theta(A_0(x))]A_0(x) + [4 + \alpha + 3(1-\alpha)\theta(A_1(x))]A_1(x) \\
& + [1 + \alpha + 2(1-\alpha)\theta(B_1(x))]B_1(x) + [50 + 50\alpha + 100(1-\alpha)\theta(B_2(x))]B_2(x).
\end{aligned} \tag{3.3}$$

The α-cuts of the solution for this example, obtained by using Eq. (3.2) and Eq. (3.3) for different values of α and $x \in [0, 2]$, are given in Figure 3. We observe that there are no breaking points in the interval $[0, 2]$.

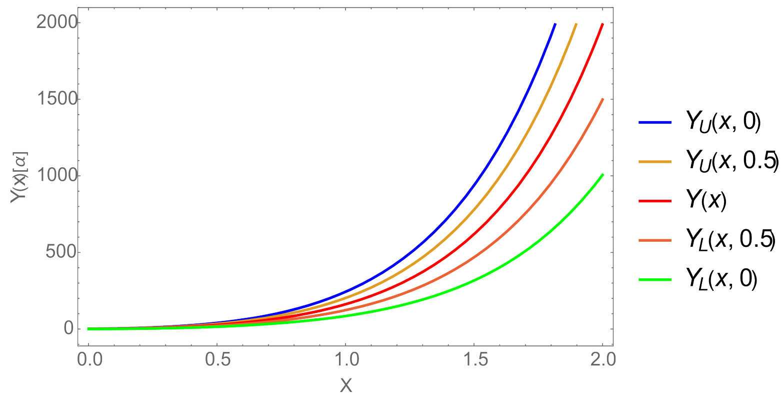

FIGURE 3. The $\alpha-$cuts of $\overline{Y}(x)$ for different values of α for Example 2.

3.3. **Example:** Consider the second order linear fuzzy differential equation:

$y'' + y = \bar{b}_1 \cos x + \bar{b}_2 x$ with fuzzy initial values $\bar{\gamma}_0 = 0/1/2$, $\bar{\gamma}_1 = 4/5/7$, and fuzzy forcing coefficients $\bar{b}_1 = 3/4/5$, $\bar{b}_2 = -3/-2/-1$.
Let us first solve the crisp initial value problem:

$y'' + y = 4\cos x - 2x;$
$y(0) = 1,\ y\prime(0) = 5.$
Since $g_1(x) = \cos x$ and $g_2(x) = x$, we obtain $G_1(x)$ and G_2 (x) as follows:
$\mathrm{G}_1(x) = 0.5\cos x + 0.5x\sin x;$
$\mathrm{G}_2(x) = x.$
Solving the differential equation in this crisp initial value problem, we obtain the general solution as:

$Y(x) = c_1 \cos x + c_2 \sin x + 2\cos x + 2x\sin x - 2x$

The functions $A_0(x)$, $A_1(x)$, $B_1(x)$ and $B_2(x)$ can be obtained as follows:

$$\begin{aligned}
A_0(x) &= \cos x,\\
A_1(x) &= \sin x,\\
B_1(x) &= 0.5x\sin x,\\
B_2(x) &= x - \sin x.
\end{aligned}$$

According to Eq. (2.8) and Eq. (2.9), the α-cuts of the solution can be expressed as follows:

$$\begin{aligned}
Y_L(x,\alpha) = {} & [2-\alpha-2(1-\alpha)\theta(A_0(x))]A_0(x) + [7-2\alpha-3(1-\alpha)\theta(A_1(x))]A_1(x)\\
& + [5-\alpha-2(1-\alpha)\theta(B_1(x))]B_1(x) + [-1-\alpha-2(1-\alpha)\theta(B_2(x))]B_2(x)
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
Y_U(x,\alpha) = {} & [\alpha+2(1-\alpha)\theta(A_0(x))]A_0(x) + [4+\alpha+3(1-\alpha)\theta(A_1(x))]A_1(x)\\
& + [3+\alpha+2(1-\alpha)\theta(B_1(x))]B_1(x) + [-3+\alpha+2(1-\alpha)\theta(B_2(x))]B_2(x)
\end{aligned} \tag{3.5}$$

The α-cuts of the solution for this example, obtained by using Eq. (3.4) and Eq. (3.5) for different values of α-cuts and $x \in [0, 10]$, are given in Figure 4.

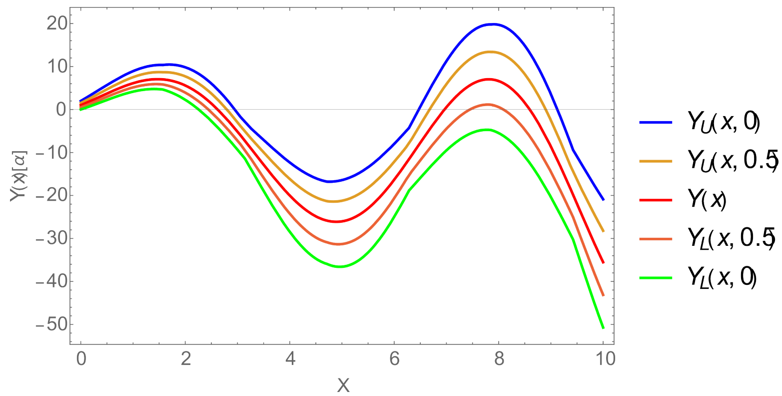

FIGURE 4. The $\alpha-$cuts of $\overline{Y}(x)$ for different values of α for Example 3.

We calculated that

$$\begin{aligned}\overrightarrow{\Phi}(3.141) - \overrightarrow{\Phi}(3.142) &= (0,1,1,1) - (0,0,0,1) \neq \overrightarrow{0},\\ \overrightarrow{\Phi}(6.283) - \overrightarrow{\Phi}(6.284) &= (1,0,0,1) - (1,1,1,1) \neq \overrightarrow{0},\\ \overrightarrow{\Phi}(9.424) - \overrightarrow{\Phi}(9.425) &= (0,1,1,1) - (0,0,0,1) \neq \overrightarrow{0},\end{aligned} \tag{3.6}$$

where $\overrightarrow{\Phi}(x) = (\theta(A_0(x)), \theta(A_1(x)), \theta(B_1(x)), \theta(B_2(x)))$. Hence the points $x_1^* = \pi$, $x_2^* = 2\pi$ and $x_3^* = 3\pi$ are breaking points in the interval $[0, 10]$. As well, as seen from (3.6), we can conclude that

a) At the point $x_1^* = \pi$ the sign of the functions $A_1(x)$ and $B_1(x)$ changes from positive to negative.

b) At the point $x_2^* = 2\pi$ the sign of the function $A_1(x)$ and $B_1(x)$ changes from negative to positive.

c) At the point $x_3^* = 3\pi$ the sign of the function $A_1(x)$ and $B_1(x)$ changes from positive to negative.

4. CONCLUSION

In this paper, we proposed an algorithm to find the fuzzy solutions of the second order non-homogeneous fuzzy initial value problems with fuzzy initial values and fuzzy forcing terms. We presented three different examples. In all examples, using the algorithm, we obtained the fuzzy solutions of the given fuzzy initial value problems. We also found the breaking points and indicated the functions which cause the breaking points by observing the structure vector. We should note that, at these points, the $\alpha-$cuts are not differentiable in the classical sense. For future research, we will be concerned with the case in which the coefficients a_1 and a_2 are also fuzzy numbers.

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