Unique Common Fixed Points For Maps With $(\psi, \alpha, \beta)$- Contractive Condition In $W^*$-Spaces

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ABSTRACT

In this paper, we introduce $W^*$-spaces which generalizes $W$-spaces introduced by Piao and Jin [10] and prove three unique common fixed point theorems in it. Some illustrative examples to highlight the results are furnished.

Key Words: $W^*$-spaces, Converse commuting maps, Common fixed points.

1. INTRODUCTION

In recent years many researchers have done much work in metric spaces, symmetric spaces[9, 3, 4], $D$-metric spaces[1, 2], $D^*$-metric spaces[6, 7], $G$-metric spaces[12, 13], Partial metric spaces[5, 8] and so on. In this direction Piao and Jin [10] introduced the concept of $W^*$-spaces in 2012, which is weaker than the notions of metric and symmetric spaces and proved some fixed point theorems.

In this paper, we introduce $W^*$-spaces to generalize $W$-spaces and proved three unique common fixed point theorems in it. We also give examples to illustrate our theorems.

Now we give the following definition.

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Definition 1.3 Let \( f \) and \( g \) be two self mappings on a non-empty set \( X \). We say that the pair \((f, g)\) satisfy Property \( (K) \) if there exists \( u \in X \) such that \( fg u = gf u \) and \( fu = gu \).

Remark 1.4 Definition 1.2 (i) and (ii) imply the Property \( (K) \) but not the converse in view of the following example.

Example 1.5 Let \( X = \{0, 1, 2\} \). \( f0 = 0, f1 = 2, f 2 = 1 \) and \( g0 = g1 = g 2 = 0 \). Clearly the pair \((f, g)\) satisfy Property \( (K) \). But \( fg 1 = gf 1 \) and \( f1 \neq g1 \).

Piao and Jin [10] proved the following theorems.

Theorem 1.6 (Theorem 1, [10]) Let \((X, d)\) be a \( W \)-space and \( f \) and \( g \) be two converse commuting self maps which have a commuting point. Suppose that \( x, y \in X \) with \( d (gx, gy) \neq 0 \) satisfy \( d (fx, gy) \leq \varphi (d (fx, g x), d (fy, g x), d (gx, gy)) \) where \( \varphi : R^3 \rightarrow R_+ \) is such that \( \varphi \) is monotone increasing for the first variable,

(i) if \( a > 0, b > 0 \) then \( a \leq \varphi (b, a, a) \) implies \( a < b \),

(ii) for any \( a > 0 \), there is \( \varphi (a, a, a) < a \) .

Then \( f \) and \( g \) have a unique common fixed point.

Theorem 1.7 (Theorem 2, [10]) Let \((X, d)\) be a \( W \)-space and \( f \) and \( g \) be two self maps which have a commuting point. Suppose that \( x, y \in X \) with \( d (fx, gy) \neq 0 \) satisfy \( d (fx, gy) \leq \psi (d (fx, gx), d (fy, gx), d (gx, gy)) \) where \( \psi : R^3 \rightarrow R_+ \) is such that

(i) \( \psi \) is monotone increasing for the first variable,

(ii) if \( a > 0, b > 0 \) then \( a \leq \psi (b, a, a) \) implies \( a < b \),

(iii) for any \( a > 0 \), there is \( \psi (a, a, a) < a \) .

Then \( f \) and \( g \) have a unique common fixed point.

Theorem 1.8 (Theorem 3, [10]) Let \((X, d)\) be a \( W \)-space and \( f_1, f_2 \) and \( g_1, g_2 \) be four self maps.

Also let \((f_1, f_2)\) and \((g_1, g_2)\) be pairs of converse commuting self mappings which have a commuting point respectively. Suppose that \( x, y \in X \) with \( d (f_2 x, g_2 y) \neq 0 \) satisfy

\[
d (f_2 x, g_2 y) \leq \varphi \left( \begin{array}{c}
d (g_1 y, f_2 x), d (g_2 y, f_2 x), d (g_1 y, g_2 y), \\
d (f_1 x, f_2 x), d (g_1 y, f_2 x)
\end{array} \right)
\]

and suppose that \( x, y \in X \) with \( d (g_1 x, f_1 y) \neq 0 \) satisfy \( d (g_1 x, f_1 y) \leq \varphi \left( \begin{array}{c}
d (f_2 y, g_2 x), d (f_2 y, g_1 x), d (g_1 x, g_2 x), \\
d (f_1 y, f_2 y), d (f_1 y, g_2 x)
\end{array} \right) \)

where \( \varphi, \varphi : R^3 \rightarrow R_+ \) satisfy \( \varphi (a, a, 0, 0) < a \) for any \( a > 0 \) and \( \varphi (a, a, 0, a) < a \) for any \( a > 0 \) . Then \( f_1, f_2, g_1 \) and \( g_2 \) have a unique common fixed point.

Now we define \( W^+ \)-spaces as follows.

Definition 1.9 Let \( X \) be a non-empty set. If a function \( d : X \times X \times X \rightarrow [0, \infty) \) satisfies the property that \( d (x, y, z) = 0 \) implies \( x = y = z \) , then \((X, d)\) is called a \( W^+ \)-space.

Example 1.10 Let \( X = [0, \infty) \) and \( d (x, y, z) = \max \{x, y, z\} \) for \( x + y + z \). Then \((X, d)\) is a \( W^+ \)-space.

Throughout this paper, let \( \psi, \alpha, \beta : [0, \infty) \rightarrow [0, \infty) \) be such that \( \psi (t) - \alpha (t) + \beta (t) > 0 \) for all \( t \geq 0 \).

Immediately it follows that \( \psi (t) - \alpha (t) + \beta (t) \leq 0 \) implies \( t = 0 \).

Now we prove our main results which are different from Theorems 1.6, 1.7 and 1.8.

2. MAIN RESULT

Theorem 2.1. Let \((X, d)\) be a \( W^+ \)-space and \( f, g : X \rightarrow X \) be satisfying

\[
\psi \left( \begin{array}{c}
d (fx, fy), d (fx, g z), \\
d (fx, fy), d (fx, g z)
\end{array} \right) \leq \alpha \max \left( \begin{array}{c}
d (fx, fy), d (fx, g z), \\
d (fx, fy), d (fx, g z)
\end{array} \right) \leq \beta \max \left( \begin{array}{c}
d (fx, fy), d (fx, g z), \\
d (fx, fy), d (fx, g z)
\end{array} \right)
\]

for all \( x, y, z \in X \) with \( d (gx, gy, g z) \neq 0 \) and \( d (gx, gy, g z) \neq 0 \).

(2.1.2) the pair \((f, g)\) satisfies Property \( (K) \).

Then \( f \) and \( g \) have a unique common fixed point.

Proof. From (2.1.2), there exists \( u \in X \) such that \( fg u = gf u \) and \( fu = gu \).

Hence

\[
ff u = fg u = gf u = gu .
\]

Suppose \( d (gu, gu, gu) \neq 0 \).
Putting \( x = u, \ y = u \) and \( z = gu \) in (2.1.1) and using (1), we obtain
\[
\psi(d(gu, gu, ggu)) \leq \alpha(d(gu, gu, ggu)) - \beta(d(gu, gu, ggu))
\]
which in turn yields that \( d(gu, gu, ggu) = 0 \). It is a contradiction. Hence \( gu = gu \).

From (1), it follows that \( gu \) is a common fixed point of \( f \) and \( g \).

Suppose \( x \) and \( y \) are two common fixed points of \( f \) and \( g \). Then \( d(x, y, y) = d(gx, gy, gy) \neq 0 \).

Then using (2.1.1) with \( x = x, \ y = y \) and \( z = y \) we obtain
\[
\psi(d(x, y, y)) \leq \alpha(d(x, y, y)) - \beta(d(x, y, y))
\]
which in turn yields that \( d(x, y, y) = 0 \). It is a contradiction. Hence \( x = y \). Thus \( f \) and \( g \) have a unique common fixed point.

**Example 2.2** Let \( X = \{0,1,2\} \) and \( d(x, y, z) = x + y + z \). Let \( g0 = g1 = 0, g2 = 1 \) and \( f0 = 0, f1 = 1, f2 = 2 \). Let \( \psi, \alpha, \beta : [0, \infty) \rightarrow [0, \infty) \)
be defined by \( \psi(t) = 1, \alpha(t) = \frac{3t}{4} \) and \( \beta(t) = \frac{t}{4} \).

Then clearly (2.1.1) and (2.1.2) are satisfied and \( 0 \) is the unique common fixed point of \( f \) and \( g \).

Next we give the following theorem without using the converse commuting condition.

**Theorem 2.3**. Let \( (X, d) \) be a \( W^* \)-space and \( f, g : X \rightarrow X \) be satisfying

\[
\psi\left( \max \left\{ \frac{d(fx, gy, gz)}{d(gx, fy, fz)} \right\} \right) \leq \alpha\left( \max \left\{ \frac{d(gx, fx, fy)}{d(gy, fz, fx)}, \frac{d(fx, gx, gy)}{d(fy, gz, gx)} \right\} \right) - \beta\left( \max \left\{ \frac{d(gx, fx, fy)}{d(gy, fz, fx)}, \frac{d(fx, gx, gy)}{d(fy, gz, gx)} \right\} \right)
\]
for all \( x, y, z \in X \) with \( \max\{d(fx, gy, gz), d(gx, fy, fz)\} \neq 0 \)

(2.3.2) the pair \( \langle f, g \rangle \) has a commuting point in \( X \).

In addition to these, assume that \( \alpha \) is monotonically increasing and \( \beta \) is monotonically decreasing.

Then \( f \) and \( g \) have a unique common fixed point in \( X \).

**Proof.** Let \( u \) be a commuting point of \( f \) and \( g \), i.e. \( fgu = gfu \) for some \( u \in X \).

Suppose that \( fu \neq gu \).

From (2.3.1), we have

\[
\psi\left( \max \left\{ \frac{d(fu, gu, fu)}{d(gu, fu, fu)} \right\} \right) \leq \alpha\left( \max \left\{ \frac{d(fu, fu, fu)}{d(gu, fu, fu)} \right\} \right) - \beta\left( \max \left\{ \frac{d(fu, gu, fu)}{d(gu, fu, fu)} \right\} \right)
\]
which in turn yields that \( \max\{d(fu, gu, fu), d(gu, fu, fu)\} = 0 \).

It is a contradiction. Hence \( fu = gu \). (2)

Hence from (2), we have \( ffu = fg = gfu = ggu \). (3)

Suppose that \( ffu \neq fu \).

Putting \( x = fu, \ y = u \), \( z = u \) in (2.3.1), we have

\[
\psi\left( \max \left\{ \frac{d(ffu, gu, fu)}{d(gu, fu, fu)} \right\} \right) \leq \alpha\left( \max \left\{ \frac{d(fu, fu, fu)}{d(gu, fu, fu)} \right\} \right) - \beta\left( \max \left\{ \frac{d(ffu, gu, fu)}{d(gu, fu, fu)} \right\} \right)
\]

Hence

\[
\psi\left( \max \left\{ \frac{d(fu, fu, fu)}{d(gu, fu, fu)} \right\} \right) \leq \alpha\left( \max \left\{ \frac{d(fu, fu, fu)}{d(gu, fu, fu)} \right\} \right) - \beta\left( \max \left\{ \frac{d(ffu, gu, fu)}{d(gu, fu, fu)} \right\} \right)
\]

(4)

Put \( x = u, \ y = fu, \ z = u \) and \( x = u, \ y = u, \)
\( z = fu \) in (2.3.1), we have

\[
\psi\left( \max \left\{ \frac{d(fu, fu, fu)}{d(fu, fu, fu)} \right\} \right) \leq \alpha\left( \max \left\{ \frac{d(ffu, fu, fu)}{d(ffu, fu, fu)} \right\} \right) - \beta\left( \max \left\{ \frac{d(fu, fu, fu)}{d(fu, fu, fu)} \right\} \right)
\]

(5)

\[
\psi\left( \max \left\{ \frac{d(fu, fu, fu)}{d(fu, fu, fu)} \right\} \right) \leq \alpha\left( \max \left\{ \frac{d(ffu, fu, fu)}{d(ffu, fu, fu)} \right\} \right) - \beta\left( \max \left\{ \frac{d(fu, fu, fu)}{d(fu, fu, fu)} \right\} \right)
\]

(6)

From (4), (5) and (6), using monotonically increasing and decreasing properties of \( \alpha \) and \( \beta \) respectively, we get
\[ \psi \left( \max \left\{ \frac{d(fu, fu),}{d(fu, fu, fu)}, \right\} \right) = \max \left\{ \frac{d(gu, fu),}{d(gu, fu, fu)}, \right\} \]  
\[ \leq \alpha \max \left\{ \frac{d(gu, fu),}{d(gu, fu, fu)}, \right\} - \beta \max \left\{ \frac{d(gu, fu),}{d(gu, fu, fu)}, \right\} \]

which in turn yields that \( ffu = fu \).

Thus \( fu \) is a common fixed point of \( f \) and \( g \).

Suppose \( v \) and \( v' \) are common fixed points of \( f \) and \( g \).

Taking \( x = v \), \( y = v', \ z = v; \ x = v, \ y = v', \ z = v \) and \( x = v, \ y = v, \ z = v' \) in (2.3.1) and using monotonically increasing of \( \alpha \) and decreasing of \( \beta \) we can show that \( v = v' \).

Thus \( f \) and \( g \) have a unique common fixed point.

Finally we give a unique common fixed point theorem for two pairs of mappings satisfying Property (K).

**Theorem 2.4** Let \( (X, d) \) be a \( W^* \)-space and \( f, g, S, T : X \rightarrow X \) be satisfying

\[ (2.4.1) \quad \psi (d(fx, gy, Sz)) \leq \alpha \max \left\{ \frac{d(gx, fy, Sz),}{d(gx, fy, Ty)}, \right\} \]

\[ - \beta \max \left\{ \frac{d(gx, fy, Sz),}{d(gx, fy, Ty)}, \right\} \]

for all \( x, y, z \in X \) with \( d(fx, gy, Sz) \neq 0 \) and (2.4.2) the pairs \((f, g)\) and \((S, T)\) satisfy the Property (K).

Then \( f, g, S \) and \( T \) have a unique common fixed point in \( X \).

**Proof.** From (2.4.2), there exist \( u \) and \( v \) in \( X \) such that \( fu = gu \), \( (7) \)

\[ fgu = gu \]  
(8)

and \( Sv = Tv \), \( (9) \)

\[ STv = TSv. \]  
(10)

Hence \( ffu = fgu = gfu = ggu \) \( (11) \)

and \( SSv = STv = TSv = TTv. \)  
(12)

Now suppose that \( fu \neq Sv \). Then \( d(fu, fu, Sv) \neq 0 \).

From (2.4.1), we have

\[ \psi (d(fu, fu, Sv)) = \psi (d(fu, gu, Sv)) \]

\[ \leq \alpha \max \left\{ \frac{d(gu, fu, Sv),}{d(gu, fu, Tv)}, \right\} - \beta \max \left\{ \frac{d(gu, fu, Sv),}{d(gu, fu, Tv)}, \right\} \]

which in turn yields that \( d(fu, fu, Sv) = 0 \). Hence \( fu = Sv \).

Thus

\[ gu = fu = Sv = Tv. \]  
(13)

Suppose that \( ffu \neq fu \). Then \( d(fffu, fu, fu) \neq 0 \).

From (2.4.1) and (13), we have

\[ \psi (d(fffu, fu, fu)) = \psi (d(fgu, gu, Sv)) \]

\[ \leq \alpha \max \left\{ \frac{d(gu, fu, Sv),}{d(gu, fu, Tv)}, \right\} - \beta \max \left\{ \frac{d(gu, fu, Sv),}{d(gu, fu, Tv)}, \right\} \]

which in turn yields that \( d(fffu, fu, fu) = 0 \). Hence \( ffu = fu \). \( (14) \)

Now from (11) and (14) we have

\[ gfu = fu. \]  
(15)

Also from (13) and (10), we get

\[ Tfu = TSv = STv = Sfu. \]  
(16)

Suppose that \( fuv \neq Sfu \).

Again from (2.4.1), we have

\[ \psi (d(fu, fu, Sfu)) = \psi (d(fu, gu, Sfu)) \]

\[ \leq \alpha \max \left\{ \frac{d(gu, fu, Sfu),}{d(gu, fu, Tfuu)}, \right\} - \beta \max \left\{ \frac{d(gu, fu, Sfu),}{d(gu, fu, Tfuu)}, \right\} \]

which gives that \( d(fu, fu, Sfu) = 0 \). Hence \( Sfu = fu \). \( (17) \)

Hence from (16) and (17)

\[ Tfu = fu. \]  
(18)

Thus from (14), (15), (17) and (18) \( fu \) is a common fixed point of \( f, g, S \) and \( T \).

If \( p \) and \( q \) are common fixed points of \( f, g, S \) and \( T \), by (2.4.1) one can easily prove that \( p = q \).

Thus \( f, g, S \) and \( T \) have a unique common fixed point in \( X \).

**Example 2.5** Let \( X = [0, 1] \) and \( d(x, y, z) = x + y + z \).

Define \( fx = gx = 0, \ Sx = \frac{x}{8} \) and \( Tx = \frac{x}{4}, \forall x \in X \).
Let \( \psi(t) = t \), \( \alpha(t) = \frac{3t}{4} \) and \( \beta(t) = \frac{t}{4} \). Then clearly

the pairs \((f, g)\) and \((S, T)\) satisfy the Property (K) respectively.

Now consider for \( d(fx, gy, Sz) \neq 0 \),

\[
\psi(d(fx, gy, Sz)) = \frac{z}{8} = \frac{1}{2} d(fx, gy, Tz) \\
\leq \frac{1}{2} \max \left\{ \frac{d(gx, fy, Sz)}{d(gx, fy, Tz)}, \frac{d(gx, fy, Tz)}{d(fx, gy, Tz)} \right\} \\
= \alpha \left( \frac{d(gx, fy, Sz)}{d(fx, gy, Tz)} \right) \\
- \beta \left( \frac{d(gx, fy, Tz)}{d(fx, gy, Tz)} \right)
\]

Hence the condition (2.4.1) is satisfied and 0 is the unique common fixed point of \( f, g, S \) and \( T \).

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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