On product of Fuzzy Semiprime ideals in $\Gamma$-LA-Semigroups

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ABSTRACT
The purpose of this paper is to introduce the notion of a weakly fuzzy quasi-semiprime ideals in $\Gamma$-LA-semigroups, we study direct product of fuzzy semiprime, fuzzy weakly completely semiprime, weakly fuzzy semiprime and weakly fuzzy quasi-semiprime ideals in $\Gamma$-LA-semigroups. Some characterizations of weakly fuzzy semiprime and weakly fuzzy quasi-semiprime ideals are obtained. Moreover, we investigate relationships between fuzzy weakly completely semiprime and weakly fuzzy quasi-semiprime ideals in $\Gamma$-LA-semigroups

Key words: fuzzy semiprime, fuzzy quasi-semiprime, fuzzy weakly completely semiprime, weakly fuzzy semiprime, quasi-semiprime

1. INTRODUCTION
A left almost semigroup (LA-semigroup) is a generalization of semigroup theory with wide range of usages in theory of flocks [23]. The fundamentals of this non-associative algebraic structure were first discovered by Kazim and Naseeruddin (1972). A groupoid $S$ is called an LA-semigroup if it satisfies the left invertive law:

$$(ab)c = (cb)a$$

for all $a, b, c \in S$. It is interesting to note that an LA-semigroup with right identity becomes a commutative monoid [21]. This structure is closely related to a commutative semigroup. Because of containing a right identity, an LA-semigroup becomes a commutative monoid [21]. A left identity in an LA-semigroup is unique [21]. It lies between a groupoid and a commutative semigroup with wide range of applications in theory of flocks [23]. Ideals in LA-semigroups have been discussed in [22]. Now we define the concepts that we will used. Let $S$ be an LA-semigroup. By an LA-subsemigroup of [20], we means a non-empty subset $A$ of $S$ such that $A^2 \subseteq A$. A non-empty subset $A$ of an LA-semigroup $S$ is called a left (right) ideal of [18] if

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SA ⊆ A(AS ⊆ A). By two-sided ideal or simply ideal, we mean a non-empty subset of an LA-semigroup S which is both a left and a right ideal of S. In 1981, the notion of Γ -semigroups was introduced by M. K. Sen. A groupoid is called an Γ -LA-semigroup if it satisfies the left invertive law:

\[(aγb)αc = (cγb)αa\]

for all \(a, b, c \in S\) and \(γ, α \in Γ\) [26]. This structure is also known as an Γ -Abel-Grassmann's groupoid (Γ -AG-groupoid). In this paper, we are going to investigate some interesting properties of recently discovered classes, namely Γ -LA-semigroup always satisfies the Γ -medial law:

\[(aγb)α(cβd) = (aγc)α(bβd)\]

for all \(a, b, c, d \in S\) and \(γ, α, β \in Γ\) [26], while an Γ -LA-semigroup with left identity always satisfies Γ -paramedial law:

\[(aγb)α(cβd) = (dβc)α(bβa)\]

for all \(a, b, c, d \in S\) and \(γ, α, β \in Γ\) [26]. Recently T. Shah and I. Rehman have discussed Γ -ideals and Γ -Bi-Ideals in Γ -LA-semigroups. An ideal \(P\) of an Γ -LA-semigroup \(S\) is called semiprime if \(A^2 \subseteq P\) implies that either \(A \subseteq P\), for all ideal \(A\) in \(S\). Q. Mushtaq and M. Khan defined the direct product of left (resp, right) ideals, prime ideals, maximal ideals and investigate the properties of such ideals [19].

The fundamental concept of fuzzy sets was first introduced by Zadeh [28] in 1965. Given a set \(S\), a fuzzy subset of \(S\) is, by definition an arbitrary mapping \(f : S \rightarrow [0, 1]\), where \([0, 1]\) is the unit interval.

A fuzzy subset \(f\) of \(S\) is called a fuzzy sub Γ -LA-semigroup of \(S\) if

\[f(xγy) \geq \min\{f(x), f(y)\}\]

for all \(x, y \in S, γ \in Γ\), and is called a fuzzy left (right) Γ -ideal of \(S\) if

\[f(xγy) \geq f(y)(f(xγy) \geq f(x))\]

Kuroki initiated the theory of fuzzy bi ideals in semigroups [15]. The thought of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset was defined by Murali [17]. Recently, M. Khan et al. introduced the concept of fuzzy ideals and anti fuzzy ideals of LA-semigroups in this papers [27]. There are many mathematicians who added several results to the theory fuzzy Γ -LA-semigroups, see [2, 3, 26]. In this paper we characterize the fuzzy subset in Γ -LA-semigroup. We investigate the relationships between fuzzy weakly completely semiprime and weakly fuzzy quasi-semiprime Γ -ideals in Γ -LA-semigroups.

2. PRELIMINARIES

Let \(S\) be an Γ -LA-semigroup. A nonempty subset \(A\) of \(S\) is called a left Γ -ideal of \(S\) if \(STA \subseteq A\). \(A\) is called a right Γ -ideal of \(S\) if \(AΓ S \subseteq A\) and \(A\) is called an Γ -ideal of \(S\) if \(A\) is both a left and a right Γ -ideal of \(S\). A function \(f\) from \(S\) to the unit interval \([0, 1]\) is a fuzzy subset of \(S\). The Γ -LA-semigroup \(S\) itself is a fuzzy subset of \(S\) such that \(S(x) = 1\) for all \(x \in S\), denoted also by \(S\). Let \(f\) and \(g\) be two fuzzy subsets of \(S\). Then the inclusion relation \(f \subseteq g\) is defined \(f(x) ≤ g(x)\), for all \(x \in S\). \(f \cap g\) and \(f \cup g\) are fuzzy subsets of \(S\) defined by

\[(f \cap g)(x) = \min\{f(x), g(x)\},\]

\[(f \cup g)(x) = \max\{f(x), g(x)\}\]

for all \(x \in S\).

The product \(fΓ g\) is defined as follows;

\[(fΓ g)(y) = \begin{cases} \sup\{\min\{f(y), g(z)\}\} & \text{if there exist } y, z \in S, \text{such that } x = yz \\ 0 & \text{otherwise.} \end{cases}\]

for all \(x, y \in S, γ \in Γ\), if \(f\) is both fuzzy left and right Γ -ideal of \(S\), then \(f\) is called a fuzzy Γ -ideal of \(S\) [24]. It is easy that \(f\) is a fuzzy Γ -ideal of \(S\) if and only if \(f(xγy) ≥ \max\{f(x), f(y)\}\) for all \(x, y \in S, γ \in Γ\) and any fuzzy left (right) Γ -ideal of \(S\) is a fuzzy sub Γ -LA-semigroup of \(S\). Equivalently, We can prove easily that \(A\) is a (left, right) Γ -ideal of \(S\) if and only if the characteristic function \(f\) of \(A\) is a
fuzzy (left, right) $\Gamma$-ideal of $S$ [6].

Lemma 2.1. [6, 24] If $S$ is an $\Gamma$-LA-semigroup and $f, g, h$ are fuzzy subsets of $S$, then $(f \Gamma g) \Gamma h = (h \Gamma g) \Gamma f$.

Proof. The proof is available in [6, 24].

Lemma 2.2. [6, 24] If $S$ is an $\Gamma$-LA-semigroup with left identity and $f, g, h, k$ are fuzzy subsets of $S$, then

1. $f \Gamma (g \Gamma h) = g \Gamma (f \Gamma h)$;
2. $(f \Gamma g) \Gamma (h \Gamma k) = (k \Gamma h) \Gamma (g \Gamma f)$.

Proof. The proof is available in [6, 24].

Lemma 2.3. [6, 24] Let $f$ be a fuzzy subset of an $\Gamma$-LA-semigroup $S$. Then the following properties hold.

1. $f$ is a fuzzy sub-$\Gamma$-LA-semigroup of $S$ if and only if $f \Gamma f \subseteq f$.
2. $f$ is a fuzzy left $\Gamma$-ideal of $S$ if and only if $S \Gamma f \subseteq f$.
3. $f$ is a fuzzy right $\Gamma$-ideal of $S$ if and only if $f \Gamma S \subseteq f$.
4. $f$ is a fuzzy $\Gamma$-ideal of $S$ if and only if $S \Gamma f \subseteq f$ and $f \Gamma S \subseteq f$.

Proof. The proof is available in [6, 24].

Lemma 2.4. [6] Let $f$ be a fuzzy left ideal of an $\Gamma$-LA-semigroup $S$. Then

1. $S \Gamma f = S$.
2. $S \Gamma f = f$.

Proof. The proof is available in [6].

Definition 2.5. A fuzzy subset $f$ of an $\Gamma$-LA-semigroup $S$ is called fuzzy quasi-semiprime if for any fuzzy left $\Gamma$-ideal $g$ of $S$ such that $g \Gamma g \subseteq f$ implies $g \subseteq f$.

Definition 2.6. A fuzzy subset $f$ of an $\Gamma$-LA-semigroup $S$ is called fuzzy semiprime of $S$ if for any fuzzy $\Gamma$-ideal $g$ of $S$ such that $g \Gamma g \subseteq f$ implies $g \subseteq f$.

It is easy to see that every fuzzy semiprime $\Gamma$-ideal is fuzzy quasi-semiprime.

Definition 2.7. A fuzzy subset $f$ of an $\Gamma$-LA-semigroup of $S$ is called fuzzy weakly completely semiprime if $f(x) \geq f(x^2)$, for all $x \in S$.

Lemma 2.8. A fuzzy $\Gamma$-ideal $f$ of an $\Gamma$-LA-semigroup of $S$ is fuzzy weakly completely semiprime if and only if $f(x) = f(x^2)$, for all $x \in S$.

Proof. It is straightforward by Definition 2.7.

Theorem 2.9. Let $S$ be an $\Gamma$-LA-semigroup. Then $f$ is fuzzy sub-$\Gamma$-LA-semigroup of $S$ if and only if $1 - f$ is fuzzy weakly completely semiprime.

Proof. $(\Rightarrow)$ Assume that $f$ is a fuzzy sub-$\Gamma$-LA-semigroup of $S$. Since $f(x^2) \geq f(x)$, we have $1 - f(x^2) \leq 1 - f(x)$, for all $x \in S$. Then $1 - f$ is fuzzy weakly completely semiprime.

$(\Leftarrow)$ Suppose that $1 - f$ is fuzzy weakly completely semiprime of $S$. Since $1 - f(x) \geq 1 - f(x^2)$, we have $f(x^2) \geq f(x)$, for all $x \in S$. Hence $f$ is a fuzzy sub-$\Gamma$-LA-semigroup of $S$.

Theorem 2.10. Let $S$ be an $\Gamma$-LA-semigroup. If $P_i, i \in I$ are fuzzy weakly completely semiprime subsets of $S$, then $\bigcup_{i \in I} P_i$ is fuzzy weakly completely semiprime subset of $S$.

Proof. Suppose that $P_i, i \in I$ are fuzzy weakly completely semiprime subset of $S$. Then $P_i(x^2) \leq P_i(x)$, for all $x \in S$, and for $i \in I$. Since $\bigcup_{i \in I} P_i(x) \geq P_i(x^2)$, for all $i \in I$, we get
Theorem 2.11. [24] Let $I$ be a non-empty subset of an $\Gamma$-$\text{LA}$-semigroup $S$ and $f_I : S \to [0,1]$ be a fuzzy subset of $S$ such that
\[ f_I(x) = \begin{cases} 1; & x \in I \\ 0; & x \notin I. \end{cases} \]
Then $I$ is a left $\Gamma$-ideal (right $\Gamma$-ideal, $\Gamma$-ideal) of $S$ if and only if $f_I$ is a fuzzy left $\Gamma$-ideal (resp. fuzzy right $\Gamma$-ideal, fuzzy $\Gamma$-ideal) of $S$.

Proof. The proof is available in [24].

Theorem 2.12. Let $I$ be an $\Gamma$-ideal (left, right $\Gamma$-ideal) of an $\Gamma$-$\text{LA}$-semigroup $S$, $m \in (0,1]$. If $f_I$ is fuzzy set of $S$ such that
\[ f_I(x) = \begin{cases} m; & x \in I \\ 0; & x \notin I, \end{cases} \]
then $f_I$ is a fuzzy $\Gamma$-ideal (fuzzy left, fuzzy right $\Gamma$-ideal) $S$.

Proof. It is straightforward by Theorem 2.11.

Definition 2.13. [24] Let $S$ be an $\Gamma$-$\text{LA}$-semigroup, $x \in S$ and $t \in [0,1]$. A fuzzy point $x_t$ of $S$ is defined by the rule that
\[ x_t(y) = \begin{cases} t; & x = y \\ 0; & x \neq y. \end{cases} \]
It is accepted that $x_t$ is a mapping from $S$ into $[0,1]$, then a fuzzy point of $S$ is a fuzzy subset of $S$. For any fuzzy subset $f$ of $S$, we also denote $x_t \subseteq f$ by $x_t \in f$ in sequel. Let $tf_A$ be a fuzzy subset of $S$ defined as follows:
\[ tf_A(x) = \begin{cases} t \in (0,1]; & x \in A \\ 0; & x \notin A. \end{cases} \]

Lemma 2.14. Let $A$ be a subset of an $\Gamma$-$\text{LA}$-semigroup $S$ and $f$ be a fuzzy set of $S$. Then the following statements are equivalent
1. $tg_A \subseteq f, t \in [0,1]$
2. $A \subseteq f_t, t \in [0,1]$.

Proof. It is straightforward by Definition 2.13.

Definition 2.15. A fuzzy subset $f$ of $S$ is said to be a weakly fuzzy semiprime if $tg_A \Gamma tg_A \subseteq f$ implies $tg_A \subseteq f$, for the $\Gamma$-ideal $A$ in $S$ and for all $t \in (0,1]$.

Definition 2.16. A fuzzy subset $f$ of $S$ is said to be a weakly fuzzy quasi-semiprime if $tg_A \Gamma tg_A \subseteq f$ implies $tg_A \subseteq f$, for the left $\Gamma$-ideal $A$ in $S$ and for all $t \in (0,1]$.

It is easy to see that every weakly fuzzy semiprime is weakly fuzzy quasi-semiprime.

3. FUZZY QUASI-SEMIPRIME $\Gamma$-IDEALS OF $\Gamma$-SEMIGROUPS

The results of the following lemmas seem to play an important role to study fuzzy semiprime $\Gamma$-ideals in $\Gamma$-$\text{LA}$-semigroups; these facts will be used frequently and normally we shall make no reference to this lemma.

Lemma 3.1. Let $A, B$ be any non-empty subset of an $\Gamma$-$\text{LA}$-semigroup $S$. Then for any $t \in (0,1]$ the following statements are true.
1. $tf_A \Gamma tf_B = tf_{A \cap B}$.
2. $tf_A \cap tf_B = tf_{A \cap B}$.
3. $tf_A \cup tf_B = tf_{A \cup B}$.
4. $tf_A = \bigcup_{a \in A} a_t$.
5. $S \Gamma tf_A = tf_{t S A}$, $S \Gamma S = tf_{A S}$ and $S \Gamma (tf_A \Gamma S) = tf_{(A S)}$.
6. If $A$ is a left $\Gamma$-ideal (right, $\Gamma$-ideal) of
$S$, then $tf_A$ is a fuzzy left $\Gamma$-ideal (fuzzy left, fuzzy $\Gamma$-ideal) of $S$.

**Proof.** 1. If $x \in A \Gamma B$, then $tf_{A \Gamma B}(x) = t$, and $x = ayb$, for some $a \in A, b \in B$ and $\gamma \in \Gamma$. Thus $tf_A \Gamma tf_B(x) = sup(min\{tf_A(a), tf_B(b)\}) = sup(min\{t, t\}) = t$.

If $x \notin A \Gamma B$, then $f_{A \Gamma B}(x) = 0$. We now prove that $(tf_A \Gamma tf_B)(x) = 0$. If $x \neq y \gamma z$, then

$$(tf_A \Gamma tf_B)(x) = 0,$$

and $(tf_A \Gamma tf_B)(x) = tf_{A \Gamma B}(x)$.

If $x = y \gamma z$ and $y \in A$ and $z \in B$, then $y \gamma z \in A \Gamma B$, so $x \in A \Gamma B$, which is impossible. Thus $y \notin A$ or $z \notin B$. If $y \notin A$, then $tf_A(y) = 0$. Since $tf_B(z) \geq 0$, we have $min\{tf_A(y), tf_B(z)\} = 0$. If $z \notin B$ then, as in the previous case, we also have $min\{tf_A(y), tf_B(z)\} = 0$. Therefore,

$$(tf_A \Gamma tf_B)(x) = min\{tf_A(y), tf_B(z)\} = 0.$$

2. We will show that

$$(tf_A \cap tf_B)(x) = tf_{A \cap B}(x),$$

for all $x \in S$. If $x \in A \cap B$, then $tf_{A \cap B}(x) = t$.

Since $x \in A$ and $x \in B$, we have

$$tf_A(x) = tf_B(x) = t,$$

so that

$$(tf_A \cap tf_B)(x) = tf_A(x) \cap tf_B(x) = t.$$

If $x \notin A \cap B$, then $tf_{A \cap B}(x) = 0$. Suppose that $x \notin A$. Then $(tf_A \cap tf_B)(x) \leq tf_A(x) = 0$. Thus we obtain that $(tf_A \cap tf_B)(x) = tf_{A \cap B}(x)$, for all $x \in S$.

3. The proof is similar to the proof of 1 with suitable modification by using the definition.

4. If $x \in A$, then

$$\bigcup_{a \in A} a_i(x) = sup_{a \in A} a_i(x) = t = tf_A(x).$$

If $x \notin A$, then $tf_A(x) = 0$. Since $x \notin A$, we have $x \neq a$, for all $a \in A$, and so $a_i(x) = 0$. It implies that

$$\bigcup_{a \in A} a_i(x) = sup_{a \in A} a_i(x) = t.$$

5. The proof is similar to the proof of 1 with a slight modification.

6. Suppose that $A$ is a left $\Gamma$-ideal of $S$. Then $tf_A(x \gamma y) \geq tf_A(y)$, for all $x, y \in S, \gamma \in \Gamma$.

If $y \notin A$, then $tf_A(y) = 0$. Since $tf_A$ is a fuzzy subset of $S$, we have $tf_A(x \gamma y) \geq 0 = tf_A(y)$. If $y \in A$, then $tf_A(y) = t$. Since $A$ is a left $\Gamma$-ideal of $S$ and $x \in S, y \in A, \gamma \in \Gamma$, we then have $x \gamma y \in A$. Thus, $tf_A(x \gamma y) = t = tf_A(y)$.

**Theorem 3.2.** Let $P$ be a fuzzy left $\Gamma$-ideal of an $\Gamma$-LA-semigroup with left identity $S$. Then the following statements are equivalent:

1. $P$ is a weakly fuzzy quasi-semiprime of $S$.

2. For any $x \in S$ and $t \in (0, 1]$, if $x_i \Gamma (S \Gamma x_i) \subseteq P$, then $x_i \in P$.

3. For any $x \in S$ and $t \in (0, 1]$, if $tf_A \Gamma tf_x \subseteq P$, then $x_i \in P$.

4. If $A$ is a left $\Gamma$-ideal of $S$ such that $tf_A \Gamma tf_A \subseteq P$, then $tf_A \subseteq P$.

**Proof.** ($1 \Rightarrow 2$) Let $P$ be a weakly fuzzy quasi-semiprime of $S$. For any $x \in S$ and $t \in (0, 1]$, if $x_i \Gamma (S \Gamma x_i) \subseteq P$, then $tf_{S \Gamma \Gamma S} \Gamma tf_{S \Gamma \Gamma S}$$

$$= (S \Gamma (x_i \Gamma S)) \Gamma (S \Gamma (x_i \Gamma S))$$

$$= (S \Gamma S) \Gamma ((x_i \Gamma S) \Gamma (x_i \Gamma S))$$

$$= (S \Gamma S) \Gamma ((x_i \Gamma S) \Gamma (x_i \Gamma S))$$

$$= (S \Gamma S) \Gamma ((S \Gamma S) \Gamma (x_i \Gamma x_i))$$

$$\subseteq S \Gamma (S \Gamma (x_i \Gamma x_i))$$

$$= S \Gamma (x_i \Gamma (S \Gamma x_i))$$

$$\subseteq S \Gamma P$$

$$\subseteq P.$$
Since $P$ is a weakly fuzzy quasi-semiprime, we get $tf_x^2 \subseteq tf_x \subseteq P$. Hence $x_i \in tf_x \subseteq P$.

(2 $\Rightarrow$ 3) Let $x \in S$, $t \in (0,1]$ and $tf_x \subseteq P$. Then

$$x_i \Gamma (S \Gamma x_i) \subseteq tf_x \Gamma (S \Gamma tf_x)$$

$$= S \Gamma (tf_x \Gamma tf_x)$$

$$\subseteq S \Gamma P$$

$$\subseteq P.$$ 

Thus, by hypothesis $x_i \in P$.

(3 $\Rightarrow$ 4) Let $A$ be a left $\Gamma$-ideal of $S$.

Then, by Lemma 3.1, we get $tf_A$ is a fuzzy left $\Gamma$-ideal of $S$. Suppose that $tf_A \subseteq P$ and $tf_A \not\subseteq P$, then there exists $x \in A$ such that $x_i \not\in P$. By Lemma 3.1 and hypothesis, we have

$$tf_x \Gamma tf_x = tf_x^2$$

$$\subseteq tf_A \Gamma A$$

$$= tf_A \Gamma tf_A$$

$$\subseteq P.$$ 

Since $x_i \not\in P$, which implies $tf_x \not\subseteq P$. But this leads to a contradiction.

(4 $\Rightarrow$ 1) By Definition 2.16, the following corollary is obvious.

**Corollary 3.3.** Let $P$ be a fuzzy $\Gamma$-ideal of an $\Gamma$-LA-semigroup with left identity $S$. Then the following statements are equivalent:

1. $P$ is a weakly fuzzy semiprime $\Gamma$-ideal of $S$.

2. For any $x \in S$ and $t \in (0,1]$, if $x_i \Gamma (S \Gamma x_i) \subseteq P$, then $x_i \in P$.

3. For any $x \in S$ and $t \in (0,1]$, if $tf_x \Gamma tf_x \subseteq P$, then $x_i \in P$.

4. If $A$ is an $\Gamma$-ideal of $S$ such that $tf_A \Gamma tf_A \subseteq P$, then $tf_A \subseteq P$.

**Proof.** This follows from Theorem 3.2.

**Theorem 3.4.** Let $S$ be an $\Gamma$-LA-semigroup with left identity. If $supf(a \Gamma (S \Gamma a)) = f(a)$, for all $a \in S$, then $f$ is a fuzzy quasi-semiprime of $S$.

**Proof.** Let $g$ be a fuzzy left $\Gamma$-ideal of $S$ such that $g \Gamma g \subseteq f$. If $g \not\subseteq f$, then there exist $a \in S$ such that $f(a) < g(a)$. Since

$$f(a) = supf(a \Gamma (S \Gamma a)),$$

there exists $s \in S$, $\gamma, \alpha \in \Gamma$ such that

$$f(\alpha \gamma(s a a)) \leq f(a).$$

Then $f(\alpha \gamma(s a a)) < g(a)$ so that

$$g(a) > f(\alpha \gamma(s a a))$$

$$\geq g \Gamma g(\alpha \gamma(s a a))$$

$$\geq sup[\min\{g(a), g(s a a)\}]$$

$$\geq min\{g(a), g(s a a)\}$$

$$= g(a)$$

since $g$ is fuzzy left $\Gamma$-ideal of $S$. But this leads to a contradiction.

**Theorem 3.5.** Let $S$ be an $\Gamma$-LA-semigroup with left identity. If $f$ is a fuzzy quasi-semiprime of $S$, then $inf\{f(a \Gamma (S \Gamma a))\} = f(a)$, for all $a \in S$.

**Proof.** Suppose that $inf\{f(a \Gamma (S \Gamma a))\} \neq f(a)$, for some $a \in S$. Since $f$ is fuzzy left $\Gamma$-ideal of $S$, we get $f(\alpha \gamma(s a a)) \geq f(s a a) \geq f(a)$, for all $s \in S$, $\gamma, \alpha \in \Gamma$. Then

$$f(a) < inf\{f(a \Gamma (S \Gamma a))\}.$$

Let $inf\{f(a \Gamma (S \Gamma a))\} = m$ and $g_{a \Gamma S}$ be fuzzy subset of $S$ such that

$$g_{a \Gamma S}(x) = \begin{cases} m; x \in a \Gamma S \\ 0; x \not\in a \Gamma S. \end{cases}$$
Then by above Theorem 2.13, \( g_{aS} \) is a fuzzy left \( \Gamma \)-ideal of \( S \). If \( g_{aS} = \Gamma g_{aS} \), then

\[
m = \text{sup}_{x \in S} \left[ \text{min} \left\{ g_{aS}(y), g_{aS}(z) \right\} \right].
\]

This means there exist some \( u, v \in a\Gamma S \) such that \( u \gamma v = x \). Put \( u = a \alpha t, v = a \beta k \). Then

\[
f(x) = f(u \gamma v) = f((a \alpha t) \gamma (a \beta k)) = f((a \alpha \alpha) \gamma (t \beta k)) = f(k \beta t) \gamma (a \alpha \alpha)) \geq f(a \alpha a) = f(a \alpha (e \delta a)) \geq \text{inf} (f(a \Gamma (a \Gamma S))) = m
\]

so that \( g_{aS} \Gamma g_{aS} \subseteq f \) and hence \( g_{aS} \subseteq f \). Thus \( g_{aS}(a) = g_{aS}(a \gamma e) = m \). But from

\[
m = g_{aS}(a) \leq f(a) = \text{inf} (f(a \Gamma (a \Gamma S))) = m,
\]

we have a contradiction.

**Corollary 3.6.** Let \( S \) be an \( \Gamma \)-LA-semigroup with left identity. If \( f \) is a fuzzy semiprime of \( S \), then \( \text{inf} (f(a \Gamma (a \Gamma S))) = f(a) \), for all \( a \in S \).

**Proof.** This follows from Theorem 3.5.

**Theorem 3.7.** Let \( S \) be an \( \Gamma \)-LA-semigroup with left identity. A fuzzy \( \Gamma \)-ideal \( P \) of an \( \Gamma \)-LA-semigroup \( S \) is weakly fuzzy quasi-semiprime \( \Gamma \)-ideal if and only if \( P(x^2) = P(x) \), for all \( x \in S \).

**Proof.** \((\Rightarrow)\) Suppose that \( P \) is a fuzzy \( \Gamma \)-ideal of \( S \). Then \( P(x^2) \geq f(x) \), for all \( x \in S \). On the other hand, if \( P(x^2) > P(x) \), then there exists \( t \in (0,1) \) such that \( P(x^2) > t > P(x) \). Thus

\[
x_i \Gamma (S \Gamma x_i) = S \Gamma (x_i \Gamma x_i) \subseteq S \Gamma (x_i^2), \in S \Gamma P \subseteq P,
\]

for all \( x \in S \). Since \( P \) is a weakly fuzzy quasi-semiprime \( \Gamma \)-ideal of \( S \), we get \( x_i \notin P \), which is impossible. Therefore, \( P(x^2) = P(x) \), for all \( x \in S \).

\((\Leftarrow)\) Suppose that \( x_i(t \in (0,1)) \) are the fuzzy point of \( S \) such that \( x_i \Gamma (S \Gamma x_i) \subseteq P \). Since

\[
S \Gamma (x_i^2) = S \Gamma (x_i \Gamma x_i) = x_i \Gamma (S \Gamma x_i) \subseteq P
\]

and \( P(x^2) = P(x) \), we have \( P(x^2) \geq t \), which implies that \( P(x) \geq t \). Then \( x_i \notin P \).

**Corollary 3.8.** Let \( S \) be an \( \Gamma \)-LA-semigroup with left identity. If \( P \) is a fuzzy weakly completely semi-prime, then \( P \) is weakly fuzzy quasi-semiprime of \( S \).

**Proof.** One can easily show by induction method.

4. **PRODUCT OF FUZZY \( \Gamma \)-IDEALS OF \( \Gamma \)-SEMIGROUPS**

We start with the following theorem that gives a relation between product of fuzzy \( \Gamma \)-ideal and fuzzy \( \Gamma \)-ideal in \( \Gamma \)-LA-semigroup. Our starting points are the following definitions:

Let \( S_1 \) and \( S_2 \) be two \( \Gamma \)-LA-semigroups. Then

\[
S_1 \times S_2 := \{(x, y) \in S_1 \times S_2 \mid x \in S_1, y \in S_2 \}
\]

and for any \( (a, b), (c, d) \in S_1 \times S_2, \gamma \in \Gamma \) we define \( (a, b) \gamma (c, d) := (a \gamma c, b \gamma d) \), then \( S_1 \times S_2 \) is an \( \Gamma \)-LA-semigroup as well. Let \( f : S_1 \to [0,1] \) and \( g : S_2 \to [0,1] \) be two fuzzy subsets of \( \Gamma \)-LA-semigroups \( S_1 \) and \( S_2 \) respectively. Then the product of fuzzy subsets is denoted by \( f \times g \) and defined as

\[
(f \times g)(x, y) = \min \{f(x), g(y)\}.
\]

**Lemma 4.1.** If \( f \) and \( g \) are fuzzy sub \( \Gamma \)-LA-semigroups of \( S_1 \) and \( S_2 \) respectively, then \( f \times g \) is a fuzzy sub \( \Gamma \)-LA-semigroup of \( S_1 \times S_2 \).

**Proof.** Let \((x_1, y_1), (x_2, y_2) \in S_1 \times S_2 \) and \( \gamma \in \Gamma \). Then \((f \times g)((x_1, y_1), (x_2, y_2)) \)}
\[(f \times g)(x_1, y_1, x_2, y_2) = \min \{f(x_1, y_1, x_2, y_2), g(y_1, y_2)\}\]

\[\leq \min \{f(x_1), f(x_2), g(y_1), g(y_2)\}\]

\[\geq \min \{f(x_1), f(x_2), g(y_1), g(y_2)\}\]

\[\geq \min \{\min \{f(x_2), g(y_2)\}, \min \{f(x_2), g(y_2)\}\}\]

\[= \min \{(f \times g)(x_1, y_1), (f \times g)(x_2, y_2)\}\]

Therefore \(f \times g\) is a fuzzy sub-\(\Gamma\)-LA-semigroup of \(S_1 \times S_2\).

**Lemma 4.2.** If \(f\) and \(g\) are fuzzy left \(\Gamma\)-ideals (fuzzy right \(\Gamma\)-ideals) of \(S_1\) and \(S_2\) respectively, then \(f \times g\) is a fuzzy left \(\Gamma\)-ideal (fuzzy right \(\Gamma\)-ideal, fuzzy \(\Gamma\)-ideal) of \(S_1 \times S_2\).

**Proof.** Let \((x_1, y_1), (x_2, y_2) \in S_1 \times S_2\) and \(\gamma \in \Gamma\). Then \((f \times g)((x_1, y_1)\gamma (x_2, y_2))\)

\[= (f \times g)(x_1, y_1, x_2, y_2)\]

\[= \min \{f(x_1, y_1, x_2, y_2), g(y_1, y_2)\}\]

\[\geq \min \{f(x_2), g(y_2)\}\]

\[= (f \times g)(x_2, y_2)\]

Therefore \(f \times g\) is a fuzzy left \(\Gamma\)-ideal of \(S_1 \times S_2\).

**Corollary 4.3.** Let \(f_1, f_2, f_3, \ldots, f_n\) be a fuzzy subsets of \(\Gamma\)-LA-semigroups \(S_1, S_2, S_3, \ldots, S_n\) respectively.

1. If \(f_1, f_2, \ldots, f_n\) are fuzzy sub-\(\Gamma\)-LA-semigroups of \(S_1, S_2, \ldots, S_n\) respectively, then \(\prod_{i=1}^{n} f_i\) is fuzzy sub-\(\Gamma\)-LA-semigroup of \(\prod_{i=1}^{n} S_i\).

2. If \(f_1, f_2, f_3, \ldots, f_n\) are fuzzy left \(\Gamma\)-ideals (fuzzy right \(\Gamma\)-ideals, fuzzy \(\Gamma\)-ideals) of \(S_1, S_2, S_3, \ldots, S_n\) respectively, then \(\prod_{i=1}^{n} f_i\) is fuzzy left \(\Gamma\)-ideal (fuzzy right \(\Gamma\)-ideal, fuzzy \(\Gamma\)-ideal) of \(\prod_{i=1}^{n} S_i\).

**Proof.** This follows from Lemma 4.1 and Lemma 4.2.

**Lemma 4.4.** Let \(f, g\) be fuzzy subsets of \(\Gamma\)-LA-semigroup with left identity \(S_1, S_2\) respectively such that \(f \times g\) is a fuzzy sub-\(\Gamma\)-LA-semigroup of \(S_1 \times S_2\). Then \(f\) or \(g\) is fuzzy sub-\(\Gamma\)-LA-semigroup of \(S_1\) or \(S_2\) respectively.

**Proof.** We know that

\[\min \{f(e_1), g(e_2)\} = (f \times g)(e_1, e_2)\]

\[\geq (f \times g)(x, y)\]

\[= \min \{f(x), g(y)\}\]

for all \((x, y) \in S_1 \times S_2\). Then \(f(x) \leq f(e_1)\) or \(g(y) \leq g(e_2)\). If \(f(x) \leq f(e_1)\), then

\[f(x) \leq g(e_2)\]

Let \(f(x) \leq g(e_2)\). Then \((f \times g)(x, e_2) = f(x)\) so that

\[f(xy, y) = (f \times g)(xy, y, e_2)\]

\[= (f \times g)((x, e_2)\gamma (y, e_2))\]

\[\geq \min \{(f \times g)(x, e_2), (f \times g)(y, e_2)\}\]

\[= \min \{f(x), g(y)\}\]

Therefore \(f\) is a fuzzy sub-\(\Gamma\)-LA-semigroup of \(S_1\).

Now suppose that \(f(x) \leq g(e_2)\) is not true for all \(x \in S_1\). If \(f(x) > g(e_2)\) for some \(x \in S_1\), then \(g(y) \leq g(e_2)\), for all \(y \in S_2\). Therefore \((f \times g)(e_1, y) = g(y)\), for all \(y \in S_2\). Similarly

\[g(xy, y) = (f \times g)(e_1, xy, y)\]

\[= (f \times g)((e_1, x)\gamma (e_1, y))\]

\[\geq \min \{(f \times g)(e_1, x), (f \times g)(e_1, y)\}\]

\[= \min \{g(x), g(y)\}\]

Hence \(g\) is a fuzzy sub-\(\Gamma\)-LA-semigroup of \(S_2\).
Lemma 4.5. Let $f, g$ be fuzzy subsets of $\Gamma$-LA-semigroups with left identity $S_1, S_2$ respectively such that $f \times g$ be a fuzzy left $\Gamma$-ideal (fuzzy right $\Gamma$-ideal, fuzzy $\Gamma$-ideal) of $S_1 \times S_2$. Then $f$ or $g$ is fuzzy left $\Gamma$-ideal (fuzzy right $\Gamma$-ideal, fuzzy $\Gamma$-ideal) of $S_1$ or $S_2$ respectively.

Proof. We know that

$$\min\{f(e_1), g(e_2)\} = (f \times g)(e_1, e_2) \geq (f \times g)(x, y) = \min\{f(x), g(y)\},$$

for all $(x, y) \in S_1 \times S_2$. Then $f(x) \leq f(e_1)$ or $g(y) \leq g(e_2)$. If $f(x) \leq f(e_1)$, then $f(x) \leq g(e_2)$ or $g(y) \leq g(e_2)$.

Let $f(x) \leq g(e_2).$ Then $(f \times g)(x, e_2) = f(x)$ so that

$$f(xy, y) = (f \times g)(xy, y, e_2) = (f \times g)((x, e_2)g(y, e_2)) \geq (f \times g)(y, e_2) = f(y).$$

Therefore $f$ is a fuzzy left $\Gamma$-ideal of $S_1$. Now suppose that $f(x) \leq g(e_2)$ is not true for all $x \in S_1$.

If $f(x) > g(e_2)$ for some $x \in S_1$, then $g(y) \leq g(e_2)$, for all $y \in S_2$. Therefore $(f \times g)(e_1, y) = g(y)$, for all $y \in S_2$. Similarly

$$g(xy, y) = (f \times g)(e_1, xy, y) = (f \times g)((e_1, x)(e_1, y)) \geq (f \times g)(e_1, y) = g(y).$$

Hence $g$ is fuzzy left $\Gamma$-ideal of $S_2$.

Corollary 4.6. Let $f_1, f_2, f_3, \ldots, f_n$ be a fuzzy subsets of $\Gamma$-LA-semigroups $S_1, S_2, S_3, \ldots, S_n$ respectively.

1. If $\prod_{i=1}^n f_i$ is a fuzzy sub $\Gamma$-LA-semigroup of $\prod_{i=1}^n S_i$, then $f_1$ or $f_2$ or $f_3$ or $\ldots$ or $f_n$ is a fuzzy sub $\Gamma$-LA-semigroup of $S_1, S_2, S_3, \ldots, S_n$ respectively.

Proof. This follows from Lemma 4.5.

Lemma 4.7. Let $f, g$ be fuzzy subsets of $\Gamma$-LA-semigroups $S_1, S_2$ respectively and $t \in [0,1]$. Then $(f \times g)_t = f_t \times f_t$.

Proof. Let $f, g$ be fuzzy subsets of $\Gamma$-LA-semigroup $S_1, S_2$ respectively and $t \in [0,1]$. Then

$$(x, y) \in f_t \times g_t \iff x \in f_t \text{ and } y \in g_t,$$

$$\iff f(x) \geq t \text{ and } g(y) \geq t,$$

$$\iff \min\{f(x), g(y)\} \geq t,$$

$$\iff (f \times g)(x, y) \geq t,$$

$$\iff (x, y) \in (f \times g)_t,$$

for all $x \in S_1$, $y \in S_2$. Hence $(f \times g)_t = f_t \times f_t$.

Corollary 4.8. Let $f_1, f_2, f_3, \ldots, f_n$ be a fuzzy subsets of $\Gamma$-LA-semigroups $S_1, S_2, S_3, \ldots, S_n$ respectively and $t \in [0,1]$. Then $\prod_{i=1}^n (f_i)_t = \prod_{i=1}^n (f_i)_t$.

Proof. This follows from Lemma 4.7.

Theorem 4.9. Let $f$ and $g$ be two fuzzy weakly
completely semiprime (fuzzy semiprime, quasi-semiprime) \( \Gamma \)-ideals of an \( \Gamma \)-LA-semigroups \( S_1, S_2 \) respectively. Then \((f \times g)\) is a fuzzy weakly completely semiprime (fuzzy semiprime, quasi-semiprime) \( \Gamma \)-ideal of \( S_1 \times S_2 \).

**Proof.** Let \((a, b) \in S_1 \times S_2\). Since \( f \) and \( g \) are fuzzy weakly completely semiprime \( \Gamma \)-ideals of \( S \), we get

\[
(f \times g)(a, b)^2 = (f \times g)(a^2, b^2) = \min \{f(a^2), g(b^2)\} = \min \{f(a), g(b)\} = (f \times g)(a, b).
\]

Hence \((f \times g)\) is a fuzzy weakly completely semiprime \( \Gamma \)-ideal of \( S_1 \times S_2 \).

**Theorem 4.10.** Let \( f, g \) be fuzzy subsets of \( \Gamma \)-LA-semigroup with left identity \( S_1, S_2 \) respectively such that \( f \times g \) is a fuzzy weakly completely semiprime (fuzzy semiprime \( \Gamma \)-ideal, quasi-semiprime \( \Gamma \)-ideal) of \( S_1 \times S_2 \). Then \( f \) or \( g \) is fuzzy weakly completely semiprime (fuzzy semiprime \( \Gamma \)-ideal, quasi-semiprime \( \Gamma \)-ideal) of \( S_1 \) or \( S_2 \) respectively.

**Proof.** We know that

\[
\min \{f(e_1), g(e_2)\} = (f \times g)(e_1, e_2) \geq (f \times g)(x, y) = \min \{f(x), g(y)\},
\]

for all \((x, y) \in S_1 \times S_2\). Then \( f(x) \leq f(e_1) \) or \( g(y) \leq g(e_2) \). If \( f(x) \leq f(e_1) \), then

\[
f(x) \leq g(e_2) \quad \text{or} \quad g(y) \leq g(e_2).
\]

Let \( f(x) \leq g(e_2) \). Then \((f \times g)(x, e_2) = f(x) \) so that

\[
f(x^2) = (f \times g)(x^2, e_2) = (f \times g)(x, e_2)^2 \leq (f \times g)(x, e_2)
\]

Therefore \( f \) is a fuzzy weakly completely semiprime of \( S_1 \). Now suppose that \( f(x) \leq g(e_2) \) is not true for all \( x \in S_1 \). If \( f(x) > g(e_2) \) for some \( x \in S_1 \), then \( g(y) \leq g(e_2) \), for all \( y \in S_2 \). Therefore \((f \times g)(e_1, y) = g(y)\), for all \( y \in S_2 \). Similarly

\[
\min \{f(e_1), g(e_2)\} \geq (f \times g)(e_1, y^2) = (f \times g)(e_1, y) \geq (f \times g)(e_1, y)
\]

Hence \( g \) is fuzzy weakly completely semiprime of \( S_2 \).

**Theorem 4.11.** Let \( f_1, f_2 \) be a fuzzy subsets of \( \Gamma \)-LA-semigroups \( S_1, S_2 \) respectively. Then \( f \times g \) is a fuzzy weakly completely semiprime \( \Gamma \)-ideal of \( S_1 \times S_2 \) if and only if the level subset \((f \times g), t \in \text{Im}(f \times g)\) of \( f \times g \) is a weakly completely semiprime \( \Gamma \)-ideal of \( S_1 \times S_2 \), for every \( t \in [0,1] \).

**Proof.** (\( \Rightarrow \)) Suppose that \( f \times g \) is a fuzzy weakly completely semiprime \( \Gamma \)-ideal of \( S_1 \times S_2 \). Let \((x, y) \in S_1 \times S_2 \) such that \((x, y)^2 \in (f \times g), t \in \text{Im}(f \times g)\). Then \((f \times g)(x, y)^2 \geq t \) so that

\[
(f \times g)(x^2, y^2) \geq t.
\]

Since \( f \times g \) is a fuzzy weakly completely semiprime \( \Gamma \)-ideal of \( S_1 \times S_2 \), we have

\[
(f \times g)(x, y)^2 = (f \times g)(x, y).
\]

Then \( t \leq (f \times g)(x, y) \), so \((x, y) \in (f \times g), t \in [0,1] \). Let \((x, y) \in S_1 \times S_2 \). By Definition fuzzy subset, we
get \((f \times g)(x, y)^2 \geq 0\). Since
\[
(x, y)^2 \in (f \times g)(f \times g)(x, y)^2
\]
by hypothesis, we have \((x, y) \in (f \times g)(f \times g)(x, y)^2\).
Thus \((f \times g)(x, y) \geq (f \times g)(x, y)^2\).

**Corollary 4.12.** Let \(f_1, f_2, f_3, \ldots, f_n\) be a fuzzy subsets of \(\Gamma\)-LA-semigroups \(S_1, S_2, S_3, \ldots, S_n\) respectively and and \(t \in [0, 1]\). Then \(\prod_{i=1}^{n} f_i\) is a fuzzy weakly completely semiprime \(\Gamma\)-ideal of \(\prod_{i=1}^{n} S_i\) ifand only if the level subset \((\prod_{i=1}^{n} f_i), t \in Im(\prod_{i=1}^{n} S_i)\) is a weakly completely semiprime \(\Gamma\)-ideal of \(\prod_{i=1}^{n} S_i\).

**Proof.** This follows from Theorem 4.11.

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