The Smallest Dimension Submanifolds of Para $\beta$-Kenmotsu Manifold

Aysel TURGUT VANLI$^{1, *}$, Ramazan SARI$^2$

$^1$Gazi University, Faculty of Arts and Sciences, Department of Mathematics, 06100, Ankara-TURKEY
$^2$Amasya University, Merzifon Vocational School, Amasya, TURKEY

ABSTRACT

In this paper, we have studied the smallest dimensional submanifold of para $\beta$-Kenmotsu manifold. Necessary and sufficient conditions are given on 3-dimensional submanifolds of a 5-dimensional para $\beta$-Kenmotsu manifold to be a slant submanifold. After that, we have studied the 3-dimensional minimal slant submanifolds of para $\beta$-Kenmotsu manifold.

Key words: Para $\beta$-Kenmotsu manifold, smallest dimension, slant submanifold

1. INTRODUCTION

As a generalization of invariant submanifold and anti-invariant submanifolds, B.Y. Chen introduced slant submanifolds of almost Hermitian manifold in 1990 [5], [6]. On the other hand A. Lotta introduced the notion of slant immersion of a Riemannian manifold into an almost contact manifold [9]. He also studied 3-dimensional slant submanifolds K-contact manifold [10]. Recently, Cabrerizo et al. [2] studied slant submanifold of Sasakian manifold and general view about slant immersions can be founds in [3]. Khan et al. studied slant submanifold of Kenmotsu manifold [7], [8].

In 1976, Sato defined the notion of an almost para contact Riemannian manifold [11]. After [12], Olszak introduced para $\beta$-Kenmotsu manifold. Many authors studied smallest dimension submanifolds [4], [8].

The purpose of present paper is to study slant submanifolds of para $\beta$-Kenmotsu manifolds with the smallest dimension. The paper organized as follows. In section 2, we give basic formula and definition of para $\beta$-Kenmotsu manifold. We review, in section 3, formulas and definitions for para $\beta$-Kenmotsu manifolds and their submanifolds, which we use later. In section 4, we obtain the smallest dimension slant submanifold of para $\beta$-
Kenmotsu manifold. Necessary and sufficient conditions are given on a 3-dimensional submanifolds of 5-dimensional para $\beta$-Kenmotsu manifold to be slant submanifold after studied 3-dimensional minimal submanifolds of para $\beta$-Kenmotsu manifold.

2. PRELIMINARIES

Let $M$ be a $(2n+1)$-dimensional differentiable manifold endowed with a quadruplet $(\varphi, \xi, \eta, g)$, where $\varphi$ is a $(1,1)$-tensor field, $\xi$ is a vector field, $\eta$ is a 1-form, and $g$ is a pseudo-Riemannian such that

$$\varphi^2 X = \mu (X - \eta(X) \xi) \quad \eta(\xi) = 1$$

(1)

$$g(\varphi X, \varphi Y) = - \mu (g(X,Y) - \eta(X)\eta(Y))$$

(2)

for all $X, Y \in \Gamma(TM)$. Here, $\mu \in \{1, -1\}$. In addition, we have

$$\varphi(\xi) = 0, \quad \eta \varphi = 0, \quad \eta(X) = \varepsilon g(X, \xi).$$

(3)

The manifold $M$ will be called almost para contact metric, and the quadruplet $(\varphi, \xi, \eta, g)$ will be called the almost para contact metric structure on $M$.

When $\mu = 1$, then the manifold $M$ is an almost contact metric manifold. In this case the metric $g$ is assumed to be pseudo-Riemannian in general, including Riemannian. Thus, if $\varepsilon = 1$, the signature of $g$ is equal to $2p$, where $0 \leq p \leq n$ and if $\varepsilon = 1$, the signature of $g$ is equal to $2p+1$, where $0 \leq p \leq n$.

When $\mu = 1$, then the manifold $M$ is an almost paracontact metric manifold. In this case, the metric $g$ is pseudo-Riemannian, and its signature is equal to $n$ when $\varepsilon = 1$, or $n+1$ when $\varepsilon = -1$. One notes that in this case, the eigenspaces of the linear operator $\varphi$ corresponding to the eigenvalues 1 and -1 are both $n$-dimensional at every point of the manifold [12].

Then a 2-form $\Phi$ is defined by $\Phi(X, Y) = g(X, \varphi Y)$, for any $X, Y \in \Gamma(TM)$, called the fundamental 2-form.

Moreover, an almost para contact metric manifold is normal if $$[\varphi, \varphi] - 2d\eta \otimes \xi = 0.$$ where $[\varphi, \varphi]$ is denoting the Nijenhuis tensor field associated to $\varphi$ [12]. A normal almost para contact metric manifold is called para contact metric manifold.

the almost para contact metric structure on $M$.

Proposition 1 Let $(M, \varphi, \xi, \eta, g)$ be an almost para contact manifold. Then, the Levi-Civita connection $\nabla$ satisfies the following equality, for any $X, Y, Z \in \Gamma(TM)$,

$$2g((\nabla_X \varphi)Y, Z) = 3d\Phi(X, \varphi Y, \varphi Z) - 3d\Phi(X, Y, Z) + g(N(Y, Z), \varphi X) + \mu N^2(Y, Z)\eta(X) + 2\mu \eta(\varphi X, Y)\eta(Z) - 2\mu \eta(\varphi Z, X)\eta(Y)$$

where $N^2(Y, X) = 2d\eta(\varphi X, Y) - 2d\eta(\varphi Y, X)$.

Definition 1 Let $M$ be an almost para contact metric manifold of dimension $(2n+1)$, with $(\varphi, \xi, \eta, g)$. $M$ is said to be an almost para $\beta$-Kenmotsu manifold if 1-form $\eta$ are closed and $d\Phi = 2\beta \eta \wedge \Phi$. A normal almost para $\beta$-Kenmotsu manifold $M$ is called a para $\beta$-Kenmotsu manifold.

Theorem 1 Let $(\tilde{M}, \varphi, \xi, \eta, g)$ be an almost para contact metric manifold. $\tilde{M}$ is a para $\beta$-Kenmotsu manifold if and only if

$$(\tilde{\nabla}_X \varphi)Y = \beta (g(\varphi X, Y)\xi - \eta(Y)\varphi X)$$

(4)

for all $X, Y \in \Gamma(TM)$ where $\tilde{\nabla}$ is Levi-Civita connection on $\tilde{M}$.

Proof. Let $\tilde{M}$ be a para $\beta$-Kenmotsu manifold. From Proposition 1, for $X, Y \in \Gamma(TM)$, we have

$$2g((\tilde{\nabla}_X \varphi)Y, Z) = 3d\Phi(X, \varphi Y, \varphi Z) - 3d\Phi(X, Y, Z).$$

Then, we have

$$g((\tilde{\nabla}_X \varphi)Y, Z) = - \beta \eta(X)g(\varphi Y, \varphi Z) + \beta \eta(X)g(Y, \varphi Z) - \beta \eta(Y)g(Z, \varphi X) - \beta \eta(X)g(Y, \varphi Y) = \beta (g(\varphi X, Y)\xi - \eta(Y)\varphi X, Z).$$

Conversely, firstly, using (4), we get

$$\varphi \tilde{\nabla}_X \xi = \beta (g(\varphi X, \xi)\xi - \eta(\xi)\varphi X)$$

hence, we get

$$\tilde{\nabla}_X \xi = \beta \varphi^2 X.$$
\[ d\eta(X,Y) = \frac{1}{2} \{ g(Y, -\varphi^2 X) - g(X, -\varphi^2 Y) \} = 0 \]

for all \( X, Y \in \Gamma(TM) \). In addition, we know

\[ 3d\Phi(X,Y,Z) = g(Y, (\nabla_X \varphi) Z) - g(Z, (\nabla_Y \varphi) X) - g(X, (\nabla_Z \varphi) Y) \]

From hypothesis, we have

\[ 3d\Phi(X,Y,Z) = \beta(g(\varphi X, Z)g(Y, \xi) - \eta(Z)g(Y, \varphi X) - g(\varphi Y, Z)g(X, \xi) + \eta(Z)g(X, \varphi Y) + g(\varphi Z, Y)g(X, \xi) - \eta(Y)g(X, \varphi Z)) \]

\[ = 2\beta(\Phi(Z,X)\eta(Y) + \Phi(X,Y)\eta(Z)) + \Phi(Y,Z)\eta(X). \]

Then, we obtain

\[ d\Phi = 2\beta \eta \wedge \Phi. \]

Moreover, the Nijenhuis torsion of \( \varphi \) is obtained

\[ N_\varphi(X,Y) = \varphi(-\beta(g(\varphi(X),\varphi(Y))\xi - \eta(Y)\varphi X) + \beta(g(\varphi(Y),X)\xi - \eta(X)\varphi Y)) + \beta(g(\varphi^2 X, Y)\xi - \eta(Y)\varphi^2 X) - \beta(g(\varphi^2 Y, X)\xi - \eta(X)\varphi^2 Y) = 0. \]

Hence, we have

\[ [\varphi, \varphi] - 2d\eta \otimes \xi = 0. \]

The proof is completed.

**Corollary 1** Let \( \bar{M} \) be \((2n+1)\)-dimensional a para \( \beta \)-Kenmotsu manifold with structure \( (\varphi, \xi, \eta, g) \). Then we have

\[ \nabla_X \xi = \beta \varphi^2 X \]

for all \( X \in \Gamma(TM) \).

**3 SUBMANIFOLDS OF PARA \( \beta \)-KENMOTSU MANIFOLD**

Now, let \( M \) be a submanifold of the \((2n+1)\)-dimensional a para \( \beta \)-Kenmotsu manifold \( \bar{M} \). Let \( \nabla \) be the Levi-Civita connection of \( M \) with respect to the induced metric \( g \). Then Gauss and Weingarten formulas are given by

\[ \nabla_X Y = \nabla_X Y - h(X, Y) \]

\[ \nabla_X V = \nabla_X ^\perp Y - A_\nu X \]

for any \( X, Y \in \Gamma(TM) \) and \( V \in \Gamma(TM)^\perp \). \( \nabla^\perp \) is the connection in the normal bundle, \( h \) is the second fundamental from of \( M \) and \( A_\nu \) is the Weingarten endomorphism associated with \( V \). The second fundamental form \( h \) and the shape operator \( A \) related by

\[ g(h(X,Y),V) = g(A_\nu X, Y). \]

The mean curvature tensor \( H \) is defined by

\[ H = \frac{1}{m} \sum_{k=1}^m h(e_k, e_k) \]

where \( \{e_1, \ldots, e_m\} \) is a local orthonormal basis of \( TM \). \( M \) said to be minimal if \( H \) vanishes identically.

Now, let \( \{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\} \) be local orthonormal basis of \( TM \) such that the vector fields \( \{e_1, \ldots, e_n\} \) are tangential to \( M \) and \( \{e_{n+1}, \ldots, e_m\} \) are normal to \( M \). Then for any \( X \in \Gamma(TM) \)

\[ \nabla_X e_i = \sum_{j=1}^n w_i^j e_j + \sum_{k=n+1}^m w_i^k e_k \]

where \( i=1, \ldots, n \) and \( r=n+1, \ldots, m \) and \( w_i^j = g(\nabla e_i, e_j) \). The 1-forms \( w_i^j, w_i^k \) can called connection forms of \( M \).

On the other hand, the mix second fundamental form in the direction \( e_r \) is defined

\[ h_{ij}^r = g(h(e_i, e_j), e_r) \]

For every tangent vector field \( X \) we write

\[ \varphi X = TX + NX \]

where \( TX \) (resp. \( NX \)) denotes the tangential (resp. normal) component of \( \varphi X \) and \( NX \) is the normal one. Moreover for every normal vector field \( V \),

\[ \varphi V = tV + nV \]

where \( tV \) in the tangential component and \( nV \) is the normal one.

Now, for later use, we establish proposition for submanifolds of para \( \beta \)-Kenmotsu manifold.

**Proposition 2** Let \( M \) be submanifold of para \( \beta \)-
Kenmotsu manifold $\tilde{M}$. Then,
\[(\nabla_X T)Y = A_{NY}X + \text{th}(X,Y) + \beta(g(TX,Y)\xi - \eta(Y)TX) \quad (12)\]
\[(\nabla_X N)Y = nh(X,Y) - h(X,TY) - \beta\eta(Y)NX \quad (13)\]
for all $X,Y \in \Gamma(TM)$.

Proof. For any $X,Y \in \Gamma(TM)$
\[\nabla_X \phi Y = \nabla_X \phi Y - \phi \nabla_X Y.\]
Then, using (4), (6) and (7)
\[\beta(g(TX + NX,Y)\xi - \eta(Y)(TX + NX))\]
\[= \nabla_X (TY + NY) - \phi(\nabla_X Y + h(X,Y))\]
\[= \nabla_X TY + h(X,TY) - A_{NY}X + \nabla_X NY - \nabla_Y X - nh(X,Y)\]
\[= (\nabla_X T)Y + (\nabla_X N)Y + h(X,TY) - A_{NY}X\]
\[- h(X,Y) - nh(X,Y)\]
or
\[(\nabla_X T)Y + (\nabla_X N)Y = \beta(g(TX + NX,Y)\xi - \eta(Y)TX\]
\[= h(X,TY) + A_{NY}X + \text{th}(X,Y) - nh(X,Y).\]

**Proposition 3** Let $M$ be a submanifold of para $\beta$-Kenmotsu manifold $\tilde{M}$, tangent to the structure vector field. Then,
\[\nabla_X \xi = \beta \phi^2 X\]
and
\[h(X,\xi) = 0\]
for any $X,Y \in \Gamma(TM)$.

Now, we defined slant submanifold of para $\beta$-Kenmotsu manifold.

**Definition 2** Let $M$ be a submanifold of a para $\beta$-Kenmotsu manifold $\tilde{M}$. $M$ is a slant submanifold if for any $x \in M$ and $X \in T_x M$ linearly independent of $\{\xi\}$, the angle between $\phi X$ and $T_x M$ is a constant $\theta \in \left[0, \frac{\pi}{2}\right]$. Then $\theta$ called the slant angle of $M$ in $\tilde{M}$.

**Theorem 2** Let $M$ be a submanifold of para $\beta$-Kenmotsu manifold $\tilde{M}$, tangent to the structure vector fields. Then, $M$ is a slant submanifold if and only if there exists a constant $\lambda \in \left[0, \frac{\pi}{2}\right]$, such that
\[T^2 = \lambda(I - \eta \otimes \xi) \quad (14)\]
Furthermore in such case, if $\theta$ is the slant angle of $M$ it satisfies that $\lambda = \cos^2 \theta$.

**Corollary 2** Let $M$ be a slant submanifold of para $\beta$-Kenmotsu manifold $\tilde{M}$, with slant angle $\theta$. Then, for any $X,Y \in \Gamma(TM)$ we have
\[g(TX,TY) = -\cos^2 \theta(g(X,Y) - \epsilon \eta(X)\eta(Y))\]
\[g(TX,TY) = -\sin^2 \theta(g(X,Y) - \epsilon \eta(X)\eta(Y)).\]

### 4 Submanifolds of Smallest Dimension in Para $\beta$-Kenmotsu Manifold

Let $M$ be 3-dimensional slant submanifold of 5-dimensional para contact manifold $\tilde{M}$ and $\{e_1, e_2, e_3, e_4, \xi\}$ be local orthonormal basis of $T\tilde{M}$. Let $e_1$ be unit vector field. $\phi$ is para contact structure,
\[g(e_1, \phi e_1) = 0.\]
Then, we can choose
\[e_2 = \sec \theta e_1.\]
Then
\[\{-\sec \theta e_2, -\sec \theta e_1, \xi\}\]
is a local orthonormal basis of $TM$.

On the other hand,
\[\{\csc N e_1, \csc N e_2\}\]
is a local orthonormal basis of $TM^\perp$.

**Proposition 4** Let $M$ be a 3-dimensional non-invariant slant submanifold of a 5-dimensional para contact manifold $\tilde{M}$. Let $e_1$ be an unit vector field and tangent to $M$. If
\[e_1 = -\sec \theta e_2, \quad e_2 = -\sec \theta e_1, \quad e_3 = \csc \theta N e_1, \quad e_4 = \csc \theta N e_2.\]
Then $\{e_1, e_2, e_3, e_4, \xi\}$ be a local orthonormal basis of $T\tilde{M}$, where $\{e_1, e_2, \xi\}$ are tangent to $M$ and $\{e_3, e_4\}$ are normal to $M$. Moreover, we have
\[te_3 = -\sin \theta e_1, \quad ne_3 = -\cos \theta e_4.\]
\[ te_4 = -\sin \theta e_2, \quad ne_4 = -\cos \theta e_3. \]

**Proof.** It is easy that \( \{e_1, e_2, e_3, e_4, \xi\} \) is local orthonormal basis off \( T\bar{M} \). We only show that last section
\[ \varphi e_3 = \varphi (\csc \theta N e_1) \]
\[ te_3 + ne_3 = \csc \theta (\varphi (\varphi e_1 - Te_2)) \]
\[ = \csc \theta (e_1 - \varphi (\cos \theta e_2)) \]
\[ = \csc \theta (e_1 - \cos \theta (Te_2 + Ne_2)) \]
\[ = \csc \theta (e_1 - \cos \theta (\cos \theta e_1 + \sin \theta e_4)) \]
\[ = \frac{1}{\sin \theta} e_1 - \frac{\cos^2 \theta}{\sin \theta} e_1 - \cos \theta e_4. \]

Then
\[ te_3 = \sin \theta e_1 \]
and
\[ ne_3 = -\cos \theta e_4. \]

Similarly
\[ te_4 = -\sin \theta e_2, \quad ne_4 = -\cos \theta e_3. \]

**Theorem 3** Let \( M \) be 3-dimensional submanifold of para \( \beta \)-Kenmotsu manifold \( \bar{M} \) Then \( M \) is slant submanifold if and only if
\[ (\nabla_X)Y = \beta (g(TX,Y)e_1 - \eta(Y)TX) \]  \hspace{1cm} (15)

for all \( X, Y \in \Gamma(TM) \).

**Proof.** Let \( M \) be slant submanifold. We can choose local orthonormal basis \( \{e_1, e_2, \xi\} \) of \( TM \), where
\[ e_1 = \sec \theta e_2 \quad \text{and} \quad e_2 = \sec \theta e_1. \]
Then for all \( X, Y \in \Gamma(TM) \)
\[ (\nabla_X)Y = \nabla_X e_1 - \nabla_X e_2 = \nabla_X (\sec \theta e_2) - \nabla_X e_1 \]
\[ = \sec \theta \nabla_X e_2 - \nabla_X e_1 \]
\[ = \sec \theta \nabla_X T \cdot e_2 - \nabla_X e_1 \]
from (14)
\[ (\nabla_X) e_1 = \cos \theta \nabla_X e_2 - \nabla_X e_1. \]

Then using (9)
\[ (\nabla_X) e_1 = \cos \theta \sum_{i=1}^{3} \beta g(X,T e_2) \xi_i \]
\[ = \sum_{i=1}^{3} \beta g(X,T^2 e_1) \xi_i \]
\[ = -\cos^2 \theta \sum_{i=1}^{3} \beta g(X,e_1) \xi_i \]  \hspace{1cm} (16)

Similarly,
\[ (\nabla_X) e_2 = \nabla_X e_2 - \nabla_X e_1 \]
\[ = -\cos \theta \sum_{i=1}^{3} w_i(X) \xi_i \]  \hspace{1cm} (17)

and
\[ (\nabla_X) \xi = -T(\cos^2 \theta \beta (T^2 X)) \]
\[ = -\cos^2 \theta \beta (TX). \]  \hspace{1cm} (18)

On the other hand, for any \( Y \in \Gamma(TM) \) writing
\[ Y = c_1 e_1 + c_2 e_2 + \eta(Y) \xi. \]

Then
\[ \nabla_X TY = c_1 \nabla_X T e_1 + c_2 \nabla_X T e_2 + g(Y,\xi) \nabla_X e_2 \]  \hspace{1cm} (19)

and
\[ \nabla_X Y = c_1 T \nabla_X e_1 + c_2 T \nabla_X e_2 + g(Y,\xi) T \nabla_X e_1. \]  \hspace{1cm} (20)

Finally, using (19) and (20)
\[ (\nabla_X) Y = c_1 (\nabla_X) e_1 + c_2 (\nabla_X) e_2 + \eta(Y)(\nabla_X) \xi. \]  \hspace{1cm} (21)

Then
\[ (\nabla_X) Y = \beta (g(TX,Y)e_1 - \eta(Y)TX). \]

**Corollary 3** Let \( M \) be 3-dimensional submanifold of para \( \beta \)-Kenmotsu manifold \( \bar{M} \) Then \( M \) is slant submanifold if and only if
\[ A_{XY}X = A_{XK}Y \]
for all \( X, Y \in \Gamma(TM) \).

**Proposition 5** Let \( M \) be 3-dimensional proper slant submanifold of 5-dimensional para \( \beta \) - Kenmotsu manifold \( \bar{M} \) and let \( \{e_1, e_2, e_3, e_4, e_5 = \xi\} \) be basis of \( T\bar{M} \). Then
\[ h^3_{12} = h^4_{11}, \quad h^3_{22} = h^4_{12} \]  \hspace{1cm} (22)
and the other mixed second fundamental forms are zero.

Proof. Firstly,
\[ h^3_{12} = g(h(e_1,e_2),e_3) = g(h(e_1,e_2),\csc\theta Ne_1) = \csc\theta g(h(e_1,e_2),Ne_1) \]

using (8),
\[ h^3_{12} = \csc\theta g(A_{Ne_1}e_2,e_1) \]

from Corollary 3,
\[ h^3_{12} = \csc\theta g(A_{Ne_1}e_1,e_1) = \csc\theta g(h(e_1,e_1),Ne_2) = g(h(e_1,e_1),e_4) = h^4_{11}. \]

Similarly
\[ h^4_{22} = h^4_{12}. \]

Theorem 4 Let \( M \) be 3-dimensional submanifold of 5-dimensional para-\( \beta \)-Kenmotsu manifold \( \bar{M} \). Then \( M \) is proper slant submanifold of para-\( \beta \)-Kenmotsu manifold \( \bar{M} \) if and only if

\[ (\nabla_X N)Y = -\beta \eta(Y)NX. \]

Proof. Let \( \{e_1,e_2,e_3,e_4,e_5\} = \xi \) be basis of \( T\bar{M} \). Using (13)
\[ (\nabla_X N)Y = nh(X,Y) - h(X,TY) - \beta \eta(Y)NX \]

and from (22),
\[ (\nabla_X N)Y = -\beta \eta(Y)NX. \]

Conversely, let (23) hold. Then, \( \forall X,Y \in \Gamma(TM) \)
\[ nh(X,Y) = h(X,TY). \]

On the other hand, from (8)
\[ g(A_{Ne_1}e_2,X) = g(h(e_2,X),Ne_1). \]

Then
\[ g(A_{Ne_1}e_2,X) = g(h(\sec\theta Te_1,X),\sin\theta e_3) = \sin\theta g(h(e_1,X),e_4) \]

On the other hand,
\[ g(A_{Ne_1}e_2,X) = g(h(e_2,X),Ne_1) = 0. \]

In that case, \( M \) is slant submanifold of corollary 2.

Moreover,
\[ h^3_{12} = g(h(e_1,e_1),e_2) = -g(h(e_1,e_2),e_4) = \sec\theta g(h(Te_2,e_2),e_4) = -g(h(e_2,e_2),e_3) = -h^4_{22}. \]

Similarly
\[ h^4_{22} = -h^4_{22}. \]

Then \( M \) is minimal slant submanifold.

Example 1 In what follows, \( (\mathbb{R}^{2n+1},\varphi,\xi,\eta,g) \) will denote the manifold \( \mathbb{R}^{2n+1} \) with its usual \( \beta \)-Kenmotsu structure given by
\[ \varphi(X_1,\ldots,X_n,Y_1,\ldots,Y_n,\xi) = (Y_1,\ldots,Y_n,-X_1,\ldots,-X_n) \]
\[ \xi = \frac{\partial}{\partial z}, \quad \eta = dz \]
\[ g = e^{-2\beta} \sum_{i=1}^{n} (dx_i \otimes dx_i + dy_i \otimes dy_i) - edz \otimes dz \]

where \( \beta = e^{-2\beta} \). The consider a submanifold of \( \mathbb{R}^5 \) defined by
\[ M = X(u,v,t) = (u \cos\theta, u \sin\theta, v, 0, t). \]

Then the local frame of \( TM \)
\[ e_1 = \cos\theta \frac{\partial}{\partial x_1} + \sin\theta \frac{\partial}{\partial x_2}, \quad e_2 = \frac{\partial}{\partial y_1}, \quad e_3 = \xi = \frac{\partial}{\partial t}. \]

On the other hand
\[ (\nabla_X N)e_1 = 0, \quad (\nabla_X N)e_2 = 0, \quad (\nabla_X N)e_3 = \beta NX. \]

For any \( Y \in \Gamma(TM) \) writing
\[ Y = c_1 e_1 + c_2 e_2 + \eta(Y) e_3. \]
In that case,
\[(\nabla_X N)Y = c_1(\nabla_X N)e_1 + c_2(\nabla_X N)e_2 + \eta(Y)(\nabla_X N)e_3.\]

Then \(M\) is a minimal slant submanifold.

**CONFLICT OF INTEREST**
No conflict of interest was declared by the authors.

**REFERENCES**


Soc. 41 (1990), 135-147.


