J. Algebra Comb. Discrete Appl.
3(3) • 195–200

Received: 04 November 2015
Accepted: 08 April 2016

Journal of Algebra Combinatorics Discrete Structures and Applications

Quasisymmetric functions and Heisenberg doubles

Research Article

Jie Sun

Abstract: The ring of quasisymmetric functions is free over the ring of symmetric functions. This result was previously proved by M. Hazewinkel combinatorially through constructing a polynomial basis for quasisymmetric functions. The recent work by A. Savage and O. Yacobi on representation theory provides a new proof to this result. In this paper, we proved that under certain conditions, the positive part of a Heisenberg double is free over the positive part of the corresponding projective Heisenberg double. Examples satisfying the above conditions are discussed.

2010 MSC: 16T05, 05E05

Keywords: Quasisymmetric function, Heisenberg double, Tower of algebras, Hopf algebra, Fock space

1. Introduction

Symmetric functions are formal power series which are invariant under every permutation of the indeterminates ([14]). Let Sym denotes the ring of symmetric functions over integers. Then the elementary symmetric functions form a polynomial basis of Sym. The existence of comultiplication and counit gives Sym a Hopf algebra structure. It's well-known (see [4]) that as Hopf algebras Sym is isomorphic to the Grothendieck group of the abelian category $\mathbb{C}[\mathfrak{S}_n]$-mod, where $\mathfrak{S}_n$ is the n-th symmetric group.

Among different generalizations of symmetric functions, there are noncommutative symmetric functions and quasisymmetric functions. As an algebra, NSym is the free algebra $\mathbb{Z}\langle \mathbf{h}_1, \mathbf{h}_2, \cdots \rangle$. As a Hopf algebra, NSym is isomorphic to the Grothendieck group of the abelian category $H_n(0)$-pmod ([3], [9], [15]), where $H_n(0)$ is the 0-Hecke algebra of degree n. The ring of quasisymmetric functions QSym $\subset \mathbb{Z}[[x_1, x_2, \cdots]]$ consists of shift invariant formal power series of bounded degrees. Let Comp(n) denote the set of compositions of n. Then the monomial quasisymmetric functions M_α, where $\alpha \in \bigcup_{n\in\mathbb{N}} \mathrm{Comp}(n)$, form an additive basis for QSym ([14]). As Hopf algebras, NSym and QSym are dual to each other under the bilinear form $\langle \mathbf{h}_\alpha, M_\beta \rangle = \delta_{\alpha,\beta}$.

Polynomial freeness of QSym was conjectured by E. Ditters in 1972 in his development of the theory of formal groups ([2]). Ditters conjecture was proved by M. Hazewinkel ([7], [6]) combinatorially through

Jie Sun; Department of Mathematical Sciences, Michigan Technological University, Houghton, MI 49931, USA (email: sjie@mtu.edu).

http://dx.doi.org/10.13069/jacodesmath.27877 ISSN 2148-838X

constructing a polynomial basis for quasisymmetric functions. The explicit basis constructed by M. Hazewinkel contains the elementary symmetric functions, hence QSym is free over Sym.

From the representation theory point of view, A. Savage and O. Yacobi [12] provide a new proof to the freeness of QSym over Sym. For each dual pair of Hopf algebras (H^+, H^-), one can construct the Heisenberg double $\mathfrak{h} = \mathfrak{h}(H^+, H^-)$ of H^+. The algebra $\mathfrak{h}$ has a natural representation on H^+, called the Fock space representation $\mathcal{F}$. In [12] it is proved that any representation of $\mathfrak{h}$ generated by a lowest weight vacuum vector is isomorphic to $\mathcal{F}$. As an application of this Stone-von Neumann type theorem to $\mathfrak{h}(\mathrm{QSym}, \mathrm{NSym})$, A. Savage and O. Yacobi gave a different proof that QSym is free as a Sym-module (Proposition 9.2 in [12]).

In this paper, after review some basic definitions and examples, we proved our main result Theorem 3.2 which says that under certain conditions, the positive part of a Heisenberg double is free over the positive part of the corresponding projective Heisenberg double. Examples satisfying the conditions of Theorem 3.2 are discussed.

2. Definitions and examples

In this section, we will recall some basic definitions and examples. Most of the definitions can be found in [12]. Fix a commutative ring $\mathbb{K}$. We use Sweedler notation $\triangle(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$ for coproducts.

Definition 2.1. *(Dual pair). We say that (H^+, H^-) is a dual pair of Hopf algebras if H^+ and H^- are both graded connected Hopf algebras and $H^\pm$ is graded dual to $H^\mp$ via a perfect Hopf pairing $\langle \cdot, \cdot \rangle : H^+ \times H^- \to \mathbb{K}$.*

From a dual pair of Hopf algebras, one can construct the Heisenberg double.

Definition 2.2. *(The Heisenberg double, [13]). We define $\mathfrak{h}(H^+, H^-)$ to be the Heisenberg double of H^+. As $\mathbb{K}$-modules $\mathfrak{h}(H^+, H^-) \cong H^+ \otimes H^-$ and we write $a\sharp x$ for $a \otimes x$, $a \in H^+$, $x \in H^-$, viewed as an element of $\mathfrak{h}(H^+, H^-)$. Multiplication is given by $(a\sharp x)(b\sharp y) = \sum_{(x)} a\ {}^R x^*_{(1)}(b)\sharp x_{(2)} y = \sum_{(x),(b)} \langle x_{(1)}, b_{(2)} \rangle a b_{(1)} \sharp x_{(2)} y$, where ${}^R x^*_{(1)}(b)$ is the left-regular action of H^- on H^+.*

For the Heisenberg double $\mathfrak{h} = \mathfrak{h}(H^+, H^-)$, there is a natural representation, called the Fock space representation.

Definition 2.3. *(Fock space representation). The algebra $\mathfrak{h}$ has a natural representation on H^+ given by $(a\sharp x)(b) = a\ {}^R x^*(b)$, $\quad a, b \in H^+, x \in H^-$, which is called the lowest weight Fock space representation of $\mathfrak{h}$ and is denoted by $\mathcal{F} = \mathcal{F}(H^+, H^-)$. It is generated by the lowest weight vacuum vector $1 \in H^+$.*

Examples of the Heisenberg double include the usual Heisenberg algebra and the quasi-Heisenberg algebra. When we take $H^+ = H^- = \mathrm{Sym}$, the Heisenberg double $\mathfrak{h} = \mathfrak{h}(\mathrm{Sym}, \mathrm{Sym})$ is the usual Heisenberg algebra. Indeed, we can take $p_1, p_2, \cdots$ to be the power sums in H^+ and $p_1^*, p_2^*, \cdots$ to be the power sums in H^-, then the multiplication in $\mathfrak{h} = \mathfrak{h}(\mathrm{Sym}, \mathrm{Sym})$ gives the usual presentation of the Heisenberg algebra: $p_m p_n = p_n p_m$, $\quad p_m^* p_n^* = p_n^* p_m^*$, $\quad p_m^* p_n = p_n p_m^* + m\delta_{m,n}$. When we take $H^+ = \mathrm{QSym}$ and $H^- = \mathrm{NSym}$, the Heisenberg double $\mathfrak{h} = \mathfrak{h}(\mathrm{QSym}, \mathrm{NSym})$ is the quasi-Heisenberg algebra. The natural action on QSym is the Fock space representation. Both the Heisenberg algebra and the quasi-Heisenberg algebra can be regarded as the Heisenberg double associated to a tower of algebras. We now recall the definition of tower of algebras in the following.

Definition 2.4. *Let $A = \oplus_{n \in \mathbb{N}} A_n$ be a graded algebra over a field $\mathbb{F}$ with multiplication $\rho : A \otimes A \to A$. Then A is called a tower of algebras if the following conditions are satisfied:*

(TA1) Each graded piece A_n, $n \in \mathbb{N}$, is a finite dimensional algebra (with a different multiplication) with a unit 1_n. We have $A_0 = \mathbb{F}$.

(TA2) The external multiplication $\rho_{m,n} : A_m \otimes A_n \to A_{m+n}$ is a homomorphism of algebras for all $m, n \in \mathbb{N}$ (sending $1_m \otimes 1_n$ to 1_{m+n}).

(TA3) We have that A_{m+n} is a two-sided projective $A_m \otimes A_n$-module with the action defined by $a \cdot (b \otimes c) = a\rho_{m,n}(b \otimes c)$ and $(b \otimes c) \cdot a = \rho_{m,n}(b \otimes c)a$, for all $m, n \in \mathbb{N}$, $a \in A_{m+n}$, $b \in A_m$, $c \in A_n$.

(TA4) For each $n \in \mathbb{N}$, the pairing $\langle \cdot, \cdot \rangle : K_0(A_n) \times G_0(A_n) \to \mathbb{Z}$, given by $\langle [P], [M] \rangle = \dim_{\mathbb{F}} \mathrm{Hom}_{A_n}(P, M)$, is perfect. (Note this condition is automatically satisfied if $\mathbb{F}$ is an algebraically closed field.)

In the above definition the notation $G_0(A_n) = \mathcal{K}_0(A_n\text{-mod})$ denotes the Grothendieck group of the abelian category A_n-mod, and $K_0(A_n) = \mathcal{K}_0(A_n\text{-pmod})$ denotes the Grothendieck group of the abelian category A_n-pmod. For the rest of this section we assume that A is a tower of algebras and let

$$\mathcal{G}(A) = \oplus_{n\in\mathbb{N}} G_0(A_n) \quad \text{and} \quad \mathcal{K}(A) = \oplus_{n\in\mathbb{N}} K_0(A_n).$$

We have a perfect pairing $\langle \cdot, \cdot \rangle : \mathcal{K}(A) \times \mathcal{G}(A) \to \mathbb{Z}$ given by $\langle [P], [M] \rangle = \dim_{\mathbb{F}} \mathrm{Hom}_{A_n}(P, M)$ if $P \in A_n$-pmod and $M \in A_n$-mod for some $n \in \mathbb{N}$, and 0 otherwise.

Definition 2.5. *(Strong tower of algebras) A tower of algebras A is strong if induction is conjugate right adjoint to restriction and a Mackey-like isomorphism relating induction and restriction holds.*

For the technical definition of the Mackey-like isomorphism in Definition 2.5 we refer the reader to [12] (Definition 3.4 and Remark 3.5).

Definition 2.6. *(Dualizing tower of algebras) A tower of algebras A is dualizing if $\mathcal{K}(A)$ and $\mathcal{G}(A)$ are dual pair Hopf algebras.*

Strong dualizing towers of algebras categorify the Heisenberg double ([1]) and its Fock space representation ([12] Theorem 3.18). To a dualizing tower of algebras, we can define the associated Heisenberg double.

Definition 2.7. *(Heisenberg double associated to a tower) Suppose A is a dualizing tower of algebras. Then $\mathfrak{h}(A) = \mathfrak{h}(\mathcal{G}(A), \mathcal{K}(A))$ is the associated Heisenberg double and $\mathcal{F}(A) = \mathcal{F}(\mathcal{G}(A), \mathcal{K}(A))$ is the Fock space representation of $\mathfrak{h}(A)$.*

When A_n is the Hecke algebra at a generic value of q, the associated Heisenberg double is the usual Heisenberg algebra. When A_n is the 0-Hecke algebra, the associated Heisenberg double is the quasi-Heisenberg algebra.

Fix a dualizing tower of algebras A. For each $n \in \mathbb{N}$, A_n-pmod is a full subcategory of A_n-mod. The inclusion functor induces the Cartan map $\mathcal{K}(A) \to \mathcal{G}(A)$. Let $\mathcal{G}_{\mathrm{proj}}(A)$ denote the image of the Cartan map. Let

$$H^- = \mathcal{K}(A),\ H^+ = \mathcal{G}(A),\ H^+_{\mathrm{proj}} = \mathcal{G}_{\mathrm{proj}}(A),\ \mathfrak{h} = \mathfrak{h}(A),\ \mathcal{F} = \mathcal{F}(A).$$

Proposition 3.12 in [12] showed that H^+_{proj} is a subalgebra of H^+ that is invariant under the left-regular action of H^-. We next recall the projective Heisenberg double and its Fock space in the following.

Definition 2.8. *(The projective Heisenberg double $\mathfrak{h}_{proj}$) The subalgebra $\mathfrak{h}_{proj} = \mathfrak{h}_{proj}(A) := H^+_{proj} \sharp H^-$ (the subalgebra of $\mathfrak{h}$ generated by H^+_{proj} and H^-) is called the projective Heisenberg double associated to A.*

Definition 2.9. *(Fock space $\mathcal{F}_{proj}$ of $\mathfrak{h}_{proj}$) The action of the algebra $\mathfrak{h}_{proj}$ on H^+_{proj} is called the lowest weight Fock space representation of $\mathfrak{h}_{proj}$ and is denoted by $\mathcal{F}_{proj} = \mathcal{F}_{proj}(A)$. It is generated by the lowest weight vacuum vector $1 \in H^+_{proj}$.*

In [12] a Stone-von Neumann type theorem (Theorem 2.11 in [12]) was proved. A consequence of this theorem tells that any representation of $\mathfrak{h}_{\mathrm{proj}}$ generated by a lowest weight vacuum vector is isomorphic to $\mathcal{F}_{\mathrm{proj}}$ (see Proposition 3.15 in [12]).

3. Main result

In this section, we will first prove a lemma about the complete reducibility of an $\mathfrak{h}_{\text{proj}}$-module generated by a finite set of lowest weight vacuum vectors. Then we will prove our main result which says that under certain conditions, the positive part of a Heisenberg double is free over the positive part of the corresponding projective Heisenberg double. We will use the notations from section 2.

Lemma 3.1. *Suppose V is an $\mathfrak{h}_{\text{proj}}$-module which is generated as an $\mathfrak{h}_{\text{proj}}$-module by a finite set of lowest weight vacuum vectors. Then V is a direct sum of copies of lowest weight Fock space $\mathcal{F}_{\text{proj}}$.*

Proof. Let $\{v_i\}_{i\in I}$ denote a finite generating set of V consisting of lowest weight vacuum vectors. Suppose that I has minimal cardinality. Suppose that $\mathbb{Z}v_i \cap \mathbb{Z}v_j \neq \{0\}$, for some $i \neq j$. Then $n_i v_i = n_j v_j$ for some $n_i, n_j \in \mathbb{Z}$. Let $m = \gcd(n_i, n_j)$. Then there exists $a_i, a_j \in \mathbb{Z}$ such that $m = a_i n_i + a_j n_j$. Let $w = a_j v_i + a_i v_j$. Then w is a lowest weight vacuum vector. Calculation shows that

$$\frac{n_i}{m} w = \frac{1}{m}(a_j n_i v_i + a_i n_i v_j) = \frac{1}{m}(a_j n_j v_j + a_i n_i v_j) = v_j,$$

$$\frac{n_j}{m} w = \frac{1}{m}(a_j n_j v_i + a_i n_j v_j) = \frac{1}{m}(a_j n_j v_i + a_i n_i v_i) = v_i.$$

Therefore, $\{v_k\}_{k\in I\setminus\{i,j\}} \cup \{w\}$ is also a generating set of V consisting of lowest weight vacuum vectors. This contradicts the minimality of the cardinality of I. Thus

$$\mathbb{Z}v_i \cap \mathbb{Z}v_j = \{0\}, \text{ for all } i \neq j.$$

By Proposition 3.15(c) in [12], any representation of $\mathfrak{h}_{\text{proj}}$ generated by a lowest weight vacuum vector is isomorphic to $\mathcal{F}_{\text{proj}}$. Thus $\mathfrak{h}_{\text{proj}} \cdot v_i \cong \mathcal{F}_{\text{proj}}$ as $\mathfrak{h}_{\text{proj}}$-modules. By Propostion 3.15(a) in [12], the only submodules of $\mathcal{F}_{\text{proj}}$ are those submodules of the form $n\mathcal{F}_{\text{proj}}$ for $n \in \mathbb{Z}$. Therefore $\mathfrak{h}_{\text{proj}} \cdot v_i \cap \mathfrak{h}_{\text{proj}} \cdot v_j = \{0\}$ for $i \neq j$. The complete reducibility of V follows. □

Theorem 3.2. *Let A be a dualizing tower of algebras. Suppose there exists an increasing filtration $\{0\} \subset H^+_{proj} = (H^+)^{(0)} \subset (H^+)^{(1)} \subset (H^+)^{(2)} \subset \cdots$ of $\mathfrak{h}_{proj}$-submodules of H^+ such that $(H^+)^{(n)}/(H^+)^{(n-1)}$ is generated by a finite set of vacuum vectors, then H^+ is free as an H^+_{proj}-module.*

Proof. Let $V_n = (H^+)^{(n)}/(H^+)^{(n-1)}$, then by Lemma 3.1, $V_n = \oplus_{v\in L_n} H^+_{\text{proj}} \cdot v$, where L_n is some collection of vacuum vectors in V_n. Consider the short exact sequence

$$0 \to (H^+)^{(n-1)} \to (H^+)^{(n)} \to V_n \to 0.$$

Since V_n is a free H^+_{proj}-module, the above short exact sequence split. By induction on n, we know that all $(H^+)^{(n)}$ $(n \in \mathbb{N})$ is free over H^+_{proj}. Thus we can choose nested sets of vectors in H^+:

$$\widetilde{L_0} \subset \widetilde{L_1} \subset \widetilde{L_2} \subset \cdots$$

such that for each $n \in \mathbb{N}$, we have $(H^+)^{(n)} = \oplus_{\widetilde{v}\in\widetilde{L_n}} H^+_{\text{proj}} \cdot \widetilde{v}$. Let $\widetilde{L} = \bigcup_{n\in\mathbb{N}} \widetilde{L_n}$, then $H^+ = \oplus_{v\in\widetilde{L}} H^+_{\text{proj}} \cdot v$ and H^+ is free over H^+_{proj}. □

The main ideas used in proving Lemma 3.1 and Theorem 3.2 follow from Lemma 9.1 and Proposition 9.2 in [12]. Observing that the existence of a special filtration of $\mathfrak{h}_{\text{proj}}$-submodules of H^+ is the key to the polynomial freeness of H^+ over H^+_{proj}, we see that Theorem 3.2 generalizes the result of Proposition 9.2 in [12] (see the discussion of Example 4.1 in section 4).

4. Applications

In this section, we will discuss some applications of our main theorem.

Example 4.1. *(Tower of 0-Hecke algebras)*

Let $A = \oplus_{n\in\mathbb{N}} H_n(0)$, where $H_n(0)$ is the 0-Hecke algebra of degree n. Then $H^+ = \mathcal{G}(A) = \mathrm{QSym}$ and $H^- = \mathcal{K}(A) = \mathrm{NSym}$. The associated Heisenberg double is the quasi-Heisenberg algebra $\mathfrak{q} = \mathfrak{h}(\mathrm{QSym}, \mathrm{NSym})$. The projective quasi-Heisenberg algebra $\mathfrak{q}_{\mathrm{proj}}$ is the subalgebra generated by $H^+_{\mathrm{proj}} = \mathrm{Sym} \subset \mathrm{QSym}$ and $H^- = \mathrm{NSym}$. For $n \in \mathbb{N}$, let

$$\mathrm{QSym}^{(n)} := \sum_{l(\alpha)\le n} \mathfrak{q}_{\mathrm{proj}} \cdot M_\alpha,$$

where $\alpha \in \bigcup_{n\in\mathbb{N}} \mathrm{Comp}(n)$ and $l(\alpha)$ is the number of nonzero parts of α. Then $\{0\} \subset \mathrm{Sym} = \mathrm{QSym}^{(0)} \subset \mathrm{QSym}^{(1)} \subset \mathrm{QSym}^{(2)} \subset \cdots$ defines an increasing filtration of $\mathfrak{q}_{\mathrm{proj}}$-submodules of QSym. For $\alpha \in \bigcup_{n\in\mathbb{N}} \mathrm{Comp}(n)$ such that $l(\alpha) = n$, we have ${}^R\mathbf{h}^*_m(M_\alpha) \in \mathrm{QSym}^{(n-1)}$ for any $m > 0$. So all M_α with $l(\alpha) = n$ are lowest weight vacuum vectors in the quotient $V_n = \mathrm{QSym}^{(n)}/\mathrm{QSym}^{(n-1)}$ and generate V_n. The condition of Theorem 3.2 is satisfied. Therefore, $H^+ = \mathrm{QSym}$ is free over $H^+_{\mathrm{proj}} = \mathrm{Sym}$. This recovers Proposition 9.2 in [12].

Example 4.2. *(Tower of* 0*-Hecke-Clifford algebras)*

Let $A = \oplus_{n\in\mathbb{N}} HCl_n(0)$, where $HCl_n(0)$ is the 0-Hecke-Clifford algebra of degree n. Then $H^+ = \mathcal{G}(A)$ and $H^- = \mathcal{K}(A)$ form a dual pair of Hopf algebras. There are two main ideas used in [10] (section 3.3) to prove that $(\mathcal{G}(A), \mathcal{K}(A))$ is a dual pair of Hopf algebras. First, the Mackey property of 0-Hecke-Clifford algebras guarantees that $\mathcal{G}(A)$ and $\mathcal{K}(A)$ are Hopf algebras. Second, it is shown that $HCl_n(0)$ is a Frobenius superalgebra which satisfies the conditions of Proposition 6.7 in [11]. The associated Heisenberg double is $\mathfrak{h}(\mathcal{G}(A), \mathcal{K}(A))$ and the corresponding projective Heisenberg double is $\mathfrak{h}_{\mathrm{proj}}$. Here $H^+ = \mathrm{Peak}^*$ is the space of peak quasisymmetric functions. Let $\theta : \mathrm{QSym} \to \mathrm{Peak}^*$ be the descent-to-peak map and let $N_\alpha = \theta(M_\alpha)$. Then H^+ is spanned by N_α, where $\alpha \in \bigcup_{n\in\mathbb{N}} \mathrm{Comp}(n)$. For $n \in \mathbb{N}$, let

$$(\mathrm{Peak}^*)^{(n)} := \sum_{l(\alpha)\le n} \mathfrak{h}_{\mathrm{proj}} \cdot N_\alpha.$$

Then $\{0\} \subset H^+_{\mathrm{proj}} = (\mathrm{Peak}^*)^{(0)} \subset (\mathrm{Peak}^*)^{(1)} \subset (\mathrm{Peak}^*)^{(2)} \subset \cdots$ defines an increasing filtration of $\mathfrak{h}_{\mathrm{proj}}$-submodules of Peak^*. We know that all N_α with $l(\alpha) = n$ are lowest weight vacuum vectors in the quotient $V_n = (\mathrm{Peak}^*)^{(n)}/(\mathrm{Peak}^*)^{(n-1)}$ and generate V_n. The condition of Theorem 3.2 is satisfied. Therefore, $H^+ = \mathrm{Peak}^*$ is free over H^+_{proj}, where H^+_{proj} is the subring of symmetric functions spanned by Schur's Q-functions. This recovers Proposition 4.2.2 in [10].

Example 4.3. *(Towers of algebras related to the symmetric groups and their Hecke algebras)*

In [8], the algebra $H\mathfrak{S}_n$ is defined to be the subalgebra of $\mathrm{End}(\mathbb{C}\mathfrak{S}_n)$ generated by both sets of operators from $\mathbb{C}[\mathfrak{S}_n]$ and $H_n(0)$. This algebra $H\mathfrak{S}_n$ has interesting combinatorial properties. For examples, the dimensional formula calculated in [8] finds applications in the study of symmetric functions associated with stirling permutations [5]. The author conjectures that Theorem 3.2 applies to the tower of algebras $\oplus_{n\in\mathbb{N}} H\mathfrak{S}_n$. The proof of this conjecture is the work in progress of the author in a forthcoming paper.

Acknowledgment: The author would like to thank Alistair Savage for explaining the paper [12]. The author would also like to thank Rafael S. González D'León for pointing out the algebra $H\mathfrak{S}_n$ in [8].

References

[1] N. Bergeron, H. Li, Algebraic structures on Grothendieck groups of a tower of algebras, J. Algebra 321(8) (2009) 2068–2084.

[2] E. J. Ditters, Curves and formal (Co)groups, Invent. Math. 17(1) (1972) 1–20.

[3] G. Duchamp, D. Krob, B. Leclerc, J-Y. Thibon, Fonctions quasi-symmétriques, fonctions symmétriques noncommutatives, et algèbres de Hecke à $q = 0$ (French) [Quasisymmetric functions, noncommutative symmetric functions and Hecke algebras at $q = 0$], C. R. Acad. Sci. Paris Sér. I Math. 322(2) (1996), 107–112.

[4] L. Geissinger, Hopf algebras of symmetric functions and class functions. Combinatoire et représentation du group symétrique (Actes Table Ronde C.N.R.S., Univ. Louis-Pasteur Strasbourg, Strasbourg, 1976), pp. 168–181. Lecture Notes in Math., Vol. 579, Springer, Berlin, 1977.

[5] R. S. González D'León, A family of symmetric functions associated with stirling permutations, preprint, 2015.

[6] M. Hazewinkel, The algebra of quasi-symmetric functions is free over the integers, Adv. Math. 164(2) (2001) 283–300.

[7] M. Hazewinkel, N. Gubareni, V. V. Kirichenko, Algebras, rings and modules. Lie algebras and Hopf algebras. Mathematical Surveys and Monographs, 168. American Mathematical Society, Providence, RI, 2010.

[8] F. Hivert, N. M. Thiéry, Representation theories of some towers of algebras related to the symmetric groups and their Hecke algebras, Formal Power Series and Algebraic Combinatorics, San Diego, California, 2006.

[9] D. Krob, J-Y. Thibon, Noncommutative symmetric functions. IV. Quantum linear groups and Hecke algebras at $q = 0$, J. Algebraic Combin. 6(4) (1997) 339–376.

[10] Y. Li, Representation theory of 0-Hecke-Clifford algebras, J. Algebra 453 (2016) 189–220.

[11] D. Rosso, A. Savage, Towers of graded superalgebras categorify the twisted Heisenberg double, J. Pure Appl. Algebra 219(11) (2015) 5040–5067.

[12] A. Savage, O. Yacobi, Categorification and Heisenberg doubles arising from towers of algebras, J. Combin. Theory Ser. A 129 (2015) 19–56.

[13] M. A. Semenov-Tian-Shansky, Poisson Lie groups, quantum duality principle, and the quantum double. Mathematical aspects of conformal and topological field theories and quantum groups (South Hadley, MA, 1992), 219–248, Contemp. Math., 175, Amer. Math. Soc., Providence, RI, 1994.

[14] R. Stanley, Enumerative Combinatorics, Vol. 2, Cambridge Studies in Advanced Mathematics 62, Cambridge University Press, 1999.

[15] J-Y. Thibon, B-C-V. Ung, Quantum quasi-symmetric functions and Hecke algebras, J. Phys. A: Math. Gen. 29(22) (1996) 7337–7348.