A New Regular Matrix Defined By Fibonacci Numbers
And Its Applications

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Abstract
The main goal of this paper is to define a new infinite Toeplitz matrix and to examine some algebraic and topological properties of the sequence spaces \( l_p, l_\infty, c \) and \( c_0 \) where \( 1 \leq p < \infty \) by means of this matrix.

Keywords: Regular matrix, Fibonacci numbers, Sequence space

1. Introduction

By \( w \), we shall denote the space of all real valued sequences. Each linear subspace of \( w \) is called a sequence space. Let \( l_\infty, c, c_0 \) and \( l_p (1 \leq p < \infty) \) be the linear spaces of bounded, convergent, null sequences and \( p \)-absolutely convergent series, respectively.

Suppose \( A = (a_{nk}) \) is an infinite matrix of real numbers \( a_{nk} \), where \( n, k \in IN \) and \( x = (x_k) \in w \). We write \( Ax = (A_n(x)) \) if \( A_n(x) = \sum_k a_{nk}x_k \) converges for each \( n \in IN \). If \( Ax = (A_n(x)) \in Y \) for each \( x = (x_k) \in X \), then \( A \) defines a matrix mapping from \( X \) into \( Y \) and we denote it by \( A : X \rightarrow Y \). \( (X : Y) \) is the class of all matrices \( A \) such that \( A : X \rightarrow Y \). The domain \( X_A \) is defined by

\[
X_A = \{ x \in w : Ax \in X \} \tag{1.1}
\]

which is a sequence space. If \( A \) is triangle, then it can be easily shown that the sequence spaces \( X_A \) and \( X \) are linearly isomorphic, i.e., \( X_A \cong X \) [1].

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A sequence space $X$ with a linear topology is called a $K$-space provided each of the maps $p_n: X \rightarrow C$ defined by $p_n(x) = x_n$ is continuous for all $n \in IN$, where $C$ denotes the complex field and $IN = \{0,1,2,...\}$. A $K$-space $X$ is called an $FK$-space provided $X$ is a complete linear metric space. An $FK$-space whose topology is normable is called a $BK$-space [2]. The spaces $l_{\infty}, c, c_0$ are $BK$-spaces with the sup-norm $\|x\| = \sup_k |x_k|$ and the space $l_p(1 \leq p < \infty)$ is $BK$-space with $\|x\|_p = \left( \sum_{k=0}^{\infty} |x_k|^p \right)^{1/p}$.

The Fibonacci numbers are famous for possessing wonderful and amazing properties. Some of these properties are well-known. For instance, the sums and differences of Fibonacci numbers are Fibonacci numbers, and the ratios of Fibonacci numbers converge to the golden section, $\tau = \frac{1 + \sqrt{5}}{2}$, which is important in Architecture, Nature and Art, physics [3].

The Fibonacci numbers $f_n$ are the terms of the sequence $0,1,1,2,3,5,...$ where in each term is the sum of the preceding terms, beginning with the values $f_0 = 0$ and $f_1 = 1$. However, some fundamental properties of Fibonacci numbers are given as follows [4]:

\[
\sum_{k=1}^{n} f_k = f_{n+2} - 1; n \geq 1
\]

\[
\sum_{k=1}^{n} f_k^2 = f_n f_{n+1}
\]

\[
\{f_k\}_{k=1}^{\infty} \text{converges}
\]

In the present study, we define the matrix $F = (f_{nk})_{n,k=1}^{\infty}$ using Fibonacci numbers $f_n$ and establish the sequence spaces $l_p(F), l_{\infty}(F), c(F)$ and $c_0(F)$ where $1 \leq p < \infty$. These spaces were also studied by different matrix in [5].

2. Main Results

Now, we state the well-known Toeplitz theorem which gives the necessary and sufficient conditions for regularity of a matrix.

**Theorem 2.1** [6, Lemma 2.1]. A matrix $A = (a_{nk})_{n,k=1}^{\infty}$ is regular if and only if the following three conditions hold:

i. There exists $M > 0$ such that for every $n = 1,2,3,...$ the inequality $\sum_{k=1}^{\infty} |a_{nk}| \leq M$ holds;

ii. $\lim_{n \rightarrow \infty} a_{nk} = 0$ for every $k = 1,2,...$;

iii. $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} a_{nk} = 1$.

In consideration of the above information, we define the Fibonacci matrix $F = (f_{nk})_{n,k=1}^{\infty}$ as follows:
\[ f_{nk} = \begin{cases} f_{2k} / f_{2n+1} - 1, & 1 \leq k \leq n, \\ 0, & \text{otherwise} \end{cases} \]

\[ F = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 0 & 0 & 0 & 0 & \cdots \\ 4 & 4 & 8 & 0 & 0 & 0 & \cdots \\ 12 & 12 & 12 & 21 & 0 & 0 & \cdots \\ 33 & 33 & 33 & 33 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]

In connection with \( f_{nn} \neq 0 \) and \( f_{nk} = 0 \) for \( k > n \), the above matrix \( F \) is triangle and also it can be easily seen by the Toeplitz theorem that the method \( F \) is regular.

Hereby, we introduce the following Fibonacci sequence space where the sequence

\[ y = (y_k) = F_k(x) = \frac{1}{f_{2k+1} - 1} \sum_{i=1}^{k} f_{2i} x_i \]  \hspace{1cm} (2.1)

is the \( F \)-transform of a sequence \( x = (x_k) \) for all \( k \in \mathbb{N}^0 \):

\[ X(F) = \{ x \in w : Fx = y = (y_k) \in X \} \]

Here and in the sequel, \( X \) denotes any of the sequence spaces \( l_\infty, c, c_0 \) and \( l_p (1 \leq p < \infty) \). We can redefine the space \( X(F) \) with the notation (1.1) as follows:

\[ X(F) = X_F. \]  \hspace{1cm} (2.2)

**Theorem 2.2.** The space \( X(F) \) is a BK space with the norm

\[ \|x\|_{X(F)} = \|Fx\|_X = \|y\|_X = \sup_k |y_k| \quad \text{for} \quad X \in \{l_\infty, c, c_0 \} \]  \hspace{1cm} (2.3)

and also

\[ \|x\|_{X(F)} = \|Fx\|_X = \|y\|_X = \left( \sum_{k=1}^{\infty} |y_k|^p \right)^{1/p} \quad \text{for} \quad X = l_p (1 \leq p < \infty). \]  \hspace{1cm} (2.4)

**Proof:** Since the matrix \( F \) is triangle, (2.2) and Theorem 4.3.12 of Wilansky [7] gives the fact that the space \( X(F) \) is BK-space with the above norms.

**Theorem 2.3.** The Fibonacci sequence space \( X(F) \) is isometrically isomorphic to space \( X \).

**Proof:** We should show the existence of an isometric isomorphism between the spaces \( X(F) \) and \( X \). Let us take in consideration the transformation \( P \) defined from \( X(F) \) to \( X \) by

\[ P : X(F) \rightarrow X, \quad x \rightarrow Px = y, \quad y = (y_k) = F_k(x) = \frac{1}{f_{2k+1} - 1} \sum_{i=1}^{k} f_{2i} x_i. \]

In that case, for every \( x \in X(F) \) we have \( Px = y = F(x) \in X \). In addition, it is clear that \( P \) is linear. Then, it can be easily seen that \( Px = 0 \Rightarrow x = 0 \) and so \( P \) is injective.

Besides, let us define the sequence \( x = (x_k) \) as follows:
\[ x_k = \frac{f_{2k+1}}{f_{2k}} y_k - \frac{f_{2k-1}}{f_{2k}} y_{k-1}, \quad k \in \mathbb{N}^0, \quad y = (y_k) \in X. \]  

(2.5)

Then, for every \( k \in \mathbb{N}^0 \) the following equality is obtained from (2.1) and (2.5):

\[ F_k(x) = \frac{1}{f_{2k+1}} \sum_{i=1}^{k} f_{2i} x_i = \frac{1}{f_{2k+1}} \sum_{i=1}^{k} [(f_{2i+1} - 1)y_i - (f_{2i-1} - 1)y_{i-1}] = y_k. \]

It means that \( Fx = y \) and thus we get that \( Fx \in X \) as \( y \in X \). By this way, we conclude that \( x \in X(F) \) and \( Px = y \). As a consequence, \( P \) is surjective. Additionally, it follows from (2.3) and (2.4) that \( P \) is norm preserving, that is,

\[ \|Px\|_X = \|y\|_X = \|F(x)\|_X = \|x\|_{X(F)} \]

for any \( x \in X(F) \). Hence \( P \) is isometry. Accordingly, the spaces \( X(F) \) and \( X \) are isometrically isomorphic, that is, \( X(F) \cong X \).

**Lemma 2.4.** Let \( \{f_k\}_{k=1}^\infty \) be Fibonacci number sequence. If the sequence \( \left( \frac{1}{f_{2k+1} - 1} \right) \) is in \( l_1 \), then

\[ \sup_i \left( f_{2i} \sum_{k=1}^{\infty} \frac{1}{f_{2k+1} - 1} \right) < \infty. \]

**Proof:** It can be easily seen that the sequence \( \left( \frac{1}{f_{2k+1} - 1} \right) \) is in \( l_1 \). So, the result follows from Lemma 4.11 of Mursaleen and Noman [8].

**Theorem 2.5.** For \( X = c_0, c, l_\infty \) the inclusion \( c_0(F) \subset c(F) \subset l_\infty(F) \) strictly holds.

**Proof:** It is clear that the inclusion \( c_0(F) \subset c(F) \subset l_\infty(F) \) holds. Consider the sequence \( x = (x_i) \) defined by \( x_i = 1 \) for all \( i \in \mathbb{N}^0 \). Then we have for every \( k \in \mathbb{N}^0 \),

\[ F_k(x) = \frac{1}{f_{2k+1} - 1} \sum_{i=1}^{k} f_{2i} = 1. \]

Hence, it is obvious that \( Fx \in c \) but it is not in \( c_0 \). So the sequence \( x \) is in \( c(F) \) but \( x \not\in c_0(F) \). Consequently, the inclusion \( c_0(F) \subset c(F) \) is strict. Now, let us consider the sequence

\[ x_i = \frac{(-1)^i (f_{2i+1} + f_{2i-1} - 1)}{f_{2i}} \]

for all \( i \in \mathbb{N}^0 \). By this way, we have

\[ F_k(x) = \frac{1}{f_{2k+1} - 1} \sum_{i=1}^{k} f_{2i} x_i = (-1)^k \]

for every \( k \in \mathbb{N}^0 \). This shows that \( Fx \in l_\infty \) but not in \( c \). Thus, it is clear that \( x \in l_\infty(F) \) but \( x \not\in c(F) \). Hereby, the inclusion \( c(F) \subset l_\infty(F) \) is strict.

**Theorem 2.6.** The inclusion \( X \subset X(F) \) holds.

**Proof:** Since the matrix \( F \) is regular, the inclusion is obvious for \( X = c_0, c \). If we take \( x = (x_i) \in l_\infty \), then there is a constant \( M > 0 \) such that \( |x_i| \leq M \) for all \( i \in \mathbb{N}^0 \). Thus, we obtain the following inequality which gives that \( Fx \in l_\infty \).
\[
|F_k(x)| \leq \frac{1}{f_{2k+1}} \frac{1}{1} \sum_{i=1}^{k} f_{2i} |x_i| \leq \frac{M}{f_{2k+1}} \frac{1}{1} \sum_{i=1}^{k} f_{2i} = M.
\]

Hence, we conclude that \( x = (x_i) \in l_p \Rightarrow x = (x_i) \in l_p (F) \). Now let us take \( x = (x_i) \in l_p \), \( 1 < p < \infty \). By using the Hölder’s inequality, we have for every \( k \in IN^0 \) the following inequality:

\[
|F_k(x)|^p \leq \left[ \sum_{i=1}^{k} f_{2i} \right]^{p-1} \left[ \sum_{i=1}^{k} f_{2i} |x_i|^p \right] \left[ \sum_{i=1}^{k} f_{2i} |x_i|^p \right]^{p-1} = \frac{1}{f_{2k+1}^{p-1}} \sum_{i=1}^{k} f_{2i} |x_i|^p. \tag{2.6}
\]

The inequality (2.6) gives the fact that

\[
\sum_{k=1}^{n} |F_k(x)|^p \leq \sum_{k=1}^{n} \frac{1}{f_{2k+1}^{p-1}} \sum_{i=1}^{k} f_{2i} |x_i|^p = \sum_{k=1}^{n} \sum_{i=1}^{k} f_{2i} |x_i|^p \sum_{i=1}^{k} f_{2i} \frac{1}{f_{2k+1}^{p-1}}.
\]

For a given \( p \), \( \sum_{i=1}^{k} f_{2i} \frac{1}{f_{2k+1}^{p-1}} < \infty \), it follows from lemma 2.4 that

\[
\|x\|^p = \sum_{k=1}^{n} |x_k|^p = M \left\| x \right\|_p^p.
\]

Hence, we have \( x \in l_p (F) \) and \( l_p \subset l_p (F) \) for \( 1 < p < \infty \). For \( p = 1 \), it can be similarly shown that (2.7) holds. To prove that the converse of Theorem 2.6 holds, we’ll use the matrix \( A = \left( a_{nk} \right) \) defined by

\[
a_{nk} = \left\{ \begin{array}{ll}
\frac{\lambda_k - \lambda_{k-1}}{\lambda_k} & (1 \leq k \leq n) \\
0 & (k > n)
\end{array} \right.
\]

positive reals tending to infinity in [9]. In the special case \( \lambda_n = f_{2n+1} - 1 \), we have \( \lambda_k - \lambda_{k-1} = f_{2k} \) and so \( F = A \) for every \( k \in IN^0 \). In these premises, we have that

\[
\lim_{n \to \infty} \frac{\lambda_{n+1}}{\lambda_n} = \lim_{n \to \infty} \frac{f_{2n+3} - 1}{f_{2n+1} - 1} = \lim_{n \to \infty} \left( 1 + \frac{f_{2n+2}}{f_{2n+1} - 1} \right) = 1 + \lim_{n \to \infty} \frac{f_{2n+2}}{f_{2n+1} - 1} > 1.
\]

Consequently, we obtain from [9, corollary 4.7] that \( X(F) \subset X \) for \( X = \left\{ c_0, c, l_p \right\} \) where \( 1 \leq p \leq \infty \).

Since the inclusions \( X(F) \subset X \) and \( X \subset X(F) \) hold, we can give the following result:

**Corollary 2.7.** \( X = X(F) \).

**References**


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